Electrically nonneutral ground states of stars

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To approximately compute the nonrelativistic ground state of an electrically nonneutral star, an exactly solvable model was recently introduced, and partly solved, in Krivoruchenko *et al.* [Phys. Rev. D **97,** 083016 (2018)]. The model generalizes the well-known Lane-Emden equation of a polytropic gas ball of index $n = 1$ to a two-fluid setting. Here its complete solution is presented in terms of simple elementary functions; it is also generalized to a more-than-two-fluid setting where it remains exactly solvable. It is shown that, given the number of nuclei, a maximal negatively charged solution and a maximal positively charged solution exist, plus a continuous family of solutions which interpolates between these extremes. Numerical comparisons show that this exactly solvable model captures the qualitative behavior of the more physical model it is supposed to approximate. Furthermore, it correctly answers the question: how nonneutral can the star be? The answer is independent of the speed of light c and the Planck quantum h. It supports Penrose's weak cosmic censorship hypothesis, in the sense that the bounds on the excess charge are compatible with the bound on the charge of a Reissner-Weyl-Nordström black hole.

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I. INTRODUCTION

In the theory of stellar structure [1–3], it is common practice to work with an effective two-fluid approximation and to reduce it further to an effective single-density model by invoking a local neutrality approximation. The two-fluid approximation simply means that charge and mass densities are computed with the density function $\nu_e(\boldsymbol{s})$ of the electrons and a single effective density function $\nu_+(\boldsymbol{s})$ for all the species of positively charged nuclei, with $\nu_e(\boldsymbol{s})$ and $\nu_+(\boldsymbol{s})$ normalized to the number of electrons, N_e, and nucleons, N_n, in the star; here, $\boldsymbol{s}$ is the space point at which the densities are considered. The charge density is then given by $\sigma(\boldsymbol{s}) = -e\nu_e(\boldsymbol{s}) + e\bar{z}\nu_+(\boldsymbol{s})$, where $\bar{z}$ denotes the average number of elementary charges per nucleon in the star. Neglecting small differences between the proton mass and the average mass per nucleon, the mass density is essentially given by $\mu(\boldsymbol{s}) = m_e\nu_e(\boldsymbol{s}) + m_p\nu_+(\boldsymbol{s})$ (the mass m_e of the electrons is usually neglected here, due to its smallness relative to the nuclear masses). Imposing on this the local-neutrality approximation $\sigma(\boldsymbol{s}) = 0 \ \forall \ \boldsymbol{s}$, based on the argument that the electrical coupling between electron and proton is about 10^{39} times stronger than their gravitational coupling so that any local electric imbalance must be negligible for the purpose of computing the overall mass density function $\mu(\boldsymbol{s})$, one eliminates $\nu_+(\boldsymbol{s})$ in favor of $\nu_e(\boldsymbol{s})$, say. Further arguments are still needed to obtain a closed equation for $\nu_e(\boldsymbol{s})$.

For example, we recall Chandrasekhar's theory of nonrotating white dwarfs [2]. Based on Fowler's insight [4] that white dwarfs are stabilized against their gravitational inward pull by the Pauli principle for electrons [5], modeled in the form of the gradients of the pressure $p_e(\boldsymbol{s})$ of a degenerate ideal gas of electrons, Chandrasekhar computed first the nonrelativistic and subsequently also the special-relativistic relationship between $\nu_e(\boldsymbol{s})$ and $p_e(\boldsymbol{s})$ for a *completely* degenerate Fermi gas, expected to characterize the fate of the electrons in a white dwarf after it has radiated away all its available energy and settled into a black dwarf. (Chandrasekhar himself often spoke of models for black dwarfs rather than white dwarfs.) In the locally neutral approximation to the two-fluid approximation, for a star whose angular momentum vanishes the hydrostatic force balance reads $-\mu(\boldsymbol{s})\nabla\phi_N(\boldsymbol{s}) - \nabla p_e(\boldsymbol{s}) = 0$, with $\phi_N(\boldsymbol{s})$ the gravitational Newton potential; it satisfies Poisson's equation $\Delta\phi_N(\boldsymbol{s}) = 4\pi G\mu(\boldsymbol{s})$. With $p_e(\boldsymbol{s})$ given in terms of $\nu_e(\boldsymbol{s})$, and with $\mu(\boldsymbol{s})$ also given in terms of $\nu_e(\boldsymbol{s})$ through the locally neutral approximation to the two-fluid model, it is clear that a closed equation for $\nu_e(\boldsymbol{s})$ ensues. In the nonrelativistic setting it has a radially symmetric solution for any finite mass $M > 0$, but in the special-relativistic setting this is true only if $M < M_{\mathrm{Ch}} = C\sqrt{3\pi}(N_e/N_n)^2(\hbar c/G)^{3/2}/m_p^2$, the critical mass discovered by Chandrasekhar, with $C \approx 1.01$; cf. [6]. When $M \to M_{\mathrm{Ch}}$, the mass density function degenerates into a Dirac δ function concentrated at a point.

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In general relativity it is not possible to continuously shrink a mass density to a δ function with finite mass $M > 0$; before that could happen, a dynamical instability sets in and causes the collapse of the so-modeled star, forming a black hole in the process. The critical mass $M_{\mathrm{GR}} < M_{\mathrm{Ch}}$, but not by much; it corresponds to the smallest nonzero radius which such a star could have.

The critical mass is affected also by the finite size of the nucleons, and their strong and weak interactions. In particular, inverse β decay causes electrons to be absorbed by nuclei (converting their protons into neutrons), when the central density exceeds a critical value.

All these investigations have not challenged the local neutrality approximation, which seems to have been perceived as so compelling (see [7]; cf. [8], Chap. 16, Sec. 9.5) that research into the large scale electric structure of stars has for a long time lived a life in the shadows by comparison; one of the few early papers on the subject is [9]. However, since the local neutrality approximation trivially implies global neutrality, $Q = 0$, it cannot be used to address the following important problems, which in the past dozen or so years have rekindled the interest in nonneutral stars. Thus there is a desire to better understand the formation of charged black holes, such as the Reissner-Weyl-Nordström black holes or, if angular momentum is included, the Kerr-Newman black holes, through the collapse of charged stars [10,11]. In particular, since there is a limit as to how nonneutral a charged black hole can be, given its ADM mass M (named after Arnowitt, Deser, and Misner) and angular momentum aM, it is important to find out whether this limit is also obeyed in models of nonneutral stars, or whether stars could be more nonneutral, in which case Penrose's weak cosmic censorship hypothesis [12] could be in jeopardy. Also certain collapse-unrelated questions, concerning hypothetical quark and strange stars, seem to require an understanding of their large-scale electrostatic fields for answers [13]. Their charge densities then become interesting subjects of research; see [10,13,14].

In this paper we pick up on the recent publication [13] where a two-fluid model of a star was studied without invoking the local neutrality approximation. One fluid component represents the electrons, and the other fluid component represents the mix of positively charged nuclei in the star, in the spirit of the reasoning recalled above. For both fluids a pressure-density relation in the form of a polytropic power law is assumed, with special emphasis on the power $\gamma = 5/3$ which is predicted by nonrelativistic quantum mechanics for a completely degenerate gas of spin-1/2 fermions, in the spirit of Chandrasekhar's pioneering calculations [2]. We note already that while the assumption of a polytropic $\gamma = 5/3$ pressure-density relation is, of course, compelling for the electrons and inherited by the mix-of-nuclei fluid if the local neutrality approximation is made in the two-fluid model, without the local neutrality approximation the assumption of a polytropic $\gamma = 5/3$ pressure-density relation for the nuclei fluid would need to be justified separately; we will come back to this point below. But first, we summarize what is done about this two-fluid model in [13], and which new results our paper contributes.

The nonlinear system of equations of this nonneutral stellar Thomas-Fermi model are more complicated than those of its neutral approximation, and so the authors of [13] have looked for other approximations which facilitate the study of the nonneutral models. Since the ratio of gravitational to electrical coupling constants are fantastically tiny numbers, e.g., $Gm_{\mathrm{e}}^2/e^2 \approx 2.40 \times 10^{-43}$ for two electrons, $Gm_{\mathrm{p}}m_{\mathrm{e}}/e^2 \approx 4.41 \times 10^{-40}$ for an electron proton system, and $Gm_{\mathrm{p}}^2/e^2 \approx 8.09 \times 10^{-37}$ for two protons, one approach has been to utilize these small numbers for a first-order perturbative expansion in their powers to access, and assess, the nonneutral neighborhood of a neutral stellar model; cf. [13]. Unfortunately, as noted in [13], such a perturbation is singular: one effectively perturbs around the zero-gravity case, but without gravity there are no nontrivial stellar equilibrium configurations. Thus, instead of simplifying matters, such a singular expansion introduces artificial new difficulties, which are absent from the nonlinear Thomas-Fermi equations.

To gain further qualitative insights into their nonlinear two-fluid model, the authors of [13] also introduced a linear proxy model; see Sec. IV.B of [13]. The approximation consists in changing the polytropic power $\gamma = 5/3$ in the pressure-density relation of a nonrelativistic, completely degenerate ideal Fermi gas to $\gamma = 6/3$. This alteration is small, but it has the advantage that the structure equations become exactly solvable in terms of elementary functions, as already noted in [13]. The solutions of this "6/3 model" can then be compared with numerically computed solutions of the "5/3 model" which it is meant to approximate.

The discussion in [13] is, however, confined to the structure in the interior (the "bulk") of the star where both two-fluid density functions are nonzero. A complete understanding of the model requires also a discussion of what we call the "atmospheric region" where one or the other density function, but not both, vanishes. It is the interplay between the bulk region and the atmospheric region which selects the admissible solutions.

In this paper we present the complete set of finite mass solutions of this exactly solvable 6/3 two-fluid model, covering bulk and atmosphere. In the Appendix A we explain that the model can be generalized to an arbitrary number of fluid components while remaining exactly solvable in terms of simple elementary functions.

We compare our exact solutions for the 6/3 model with numerical evaluations of the 5/3 model which it is meant to approximate, and also with numerical solutions of a Chandrasekhar-type special-relativistic model. We also supply rigorous arguments to back up the numerical results. An important by-product of our results is that the bounds on

how nonneutral a star can be support Penrose's weak cosmic censorship hypothesis [12].

Thus, the exactly solvable 6/3 model introduced in [13] can serve several purposes. First of all, it provides insight into the qualitative structure of the possible solutions to the physically more realistic 5/3 model. Second, it can serve as a test case for numerical methods designed to solve these physically more realistic sets of nonlinear equations that cannot be solved in closed form. Third, there seem to be some "universal" electrical facts that are largely independent of the details of the stellar ground state model, and these are readily reproduced by the exactly solvable 6/3 model. Finally, a power 2 pressure-density relation is, of course, well known in the general theory of stellar structure [1–3] and yields a polytrope of index $n = 1$. Aside from being discussed in the astrophysical literature, polytropes also appear in pedagogical papers; e.g., they are found in [15–19]. In this vein we believe that this exactly solvable model could also be incorporated in a course on stellar structure.

We have reached the point where we need to come back to the question of how realistic the two-fluid 5/3 model is, which is approximated by the 6/3 model. We already recalled that the treatment of the electrons as a completely degenerate Fermi gas is justified for stars in their ground state, and a reasonable approximation for white dwarf stars which are energetically near their ground state, an insight which goes back to Fowler. On the other hand, each and every noncollapsed star in the heavens contains more than one species of nuclei, presumably, and with the electrons treated as one of the fluid components, a two-fluid approximation for a star is a plausible approximation only as long as the various positive nuclei species are sufficiently mixed by convective and turbulent motions so that throughout the star any species of nuclei with z elementary charges per nucleus and mass $A_z m_\mathrm{p}$, where A_z is the mass number (for each z we only take the dominant isotope into account, for simplicity), has a number density in essentially constant proportion to the number density of free protons $\nu_\mathrm{p}(r)$, viz. $\nu_z(r) \approx C_z \nu_\mathrm{p}(r)$ (with "=" instead of "≈" when $z = 1$, and $C_1 = 1$). Since nuclei of type $_z$ carry positive integer multiples z of the elementary charge e, the positive charge density function $e\bar{z}\nu_+(r) \coloneqq e\sum_z z\nu_z(r) \approx e(\sum_z zC_z)\nu_\mathrm{p}(r)$, and the two-fluid approximation consists of replacing "≈" by "=," so one can work with the electron density ν_e, and either ν_+ or ν_p, as one pleases. For the mass density, one then has $\mu(r) \coloneqq \sum_z m_z\nu_z(r) + m_\mathrm{e}\nu_\mathrm{e}(r) \approx (\sum_z m_z C_z)\nu_\mathrm{p}(r) + m_\mathrm{e}\nu_\mathrm{e}(r)$ and "≈" is replaced by "=" in the two-fluid approximation (and m_e may be neglected).

However, white dwarfs are stars where nuclear fusion processes have expired, and the distribution of nuclei is no longer mixed up by convection and other processes, featuring instead an onion layer structure, as addressed long ago by Hamada and Salpeter [7]. Moreover, while the protons are fermions, the other important species in a low-to-medium mass white dwarf are bosons (for instance, α particles, ^{12}C, and ^{16}O nuclei). Treating such a segregated ensemble of nuclei species as a single completely degenerate Fermi fluid with effective mass and charge parameters is not a compelling approximation.

The only special case of the two-fluid 5/3 model in [13] which is not subject to the just leveled criticism is an idealized model star with only one species of nuclei: protons, which *are* fermions. Even though no astronomer will presumably ever see a star made of only electrons and protons, in particular not one that is in its energetic ground state, it is certainly not absurd to contemplate this model as a valid simplifying approximation to a model for a first generation star with such a low mass that it failed to ignite (a "failed star," like a brown dwarf [20]) and essentially cooled down to its lowest energy state: a black dwarf with a very low mass, between ≈13 and ≈80 Jupiter masses. To emphasize that the star never ignited, we prefer to rather speak of a "failed white dwarf." As per the cosmological standard model [21], on average such a star's nuclei composition would consist of ≈92% free protons and ≈8% α particles, and the suggestive approximation which leads to the 5/3 model of two completely degenerate Fermi gases consists in replacing the ≈8% α particles by protons. By some statistical fluke there may well be regions in the early universe where the percentages are even more lopsided toward the protons. A nonrelativistic model suffices, with Newton's gravity and Coulomb's electricity stabilized by the gradients of the degeneracy pressures. The latter are then approximated by replacing $\gamma = 5/3$ with $\gamma = 6/3 (= 2)$.

Furthermore, as we pointed out in [22] already, though without giving any details, the 6/3 model gives the same answer to the question how many electrons per proton fit on a failed white dwarf as does the 5/3 model, and it allows one to demonstrate by explicitly writing down the solution pairs, that the bounds on $N_\mathrm{e}/N_\mathrm{p}$ can be saturated. To come to the same conclusions in the 5/3 model requires more work. In the special-relativistic model one has to stay away from the critical Chandrasekhar mass, but this, of course, is implicitly understood because the critical mass beyond which degeneracy pressure no longer stabilizes against collapse is far greater than the critical mass beyond which a star's nuclear fuel ignites.

In Sec. II we recall the Thomas-Fermi equations of a failed white dwarf star made of protons and electrons that are treated as ideal Fermi gases of spin-$\frac{1}{2}$ particles. This section is essentially identical with Sect. IV of [22].

Then, in Sec. III, we will apply the $5/3 \to 6/3$ approximation to this two-species model. We solve the approximate model explicitly in terms of elementary functions and compute the allowed interval of $N_\mathrm{e}/N_\mathrm{p}$ values.

Section IV explains that the allowed interval of $N_\mathrm{e}/N_\mathrm{p}$ values is the same also in more physical models; cf. [22].

In Sec. V we explain that in the more physical models the electron and proton numbers, N_e and N_p, can be computed in terms of the zeros of the particle densities and the derivatives of the densities at the zeros.

In Sec. VI we illustrate our findings and compare the 6/3 model with the physical 5/3 model, and also with the Chandrasekhar-type special-relativistic model, for which we have carried out numerical evaluations.

Section VII has the Kepler problem of charged binaries.

In Sec. VIII we convert the bounds on N_e/N_p into bounds on the total charge Q which imply a bound on Q^2 proportional to M^2, valid also for Reissner-Weyl-Nordström black holes. In this sense our results support Penrose's weak cosmic censorship hypothesis.

The conclusions are presented in Sec. IX.

In Appendix A we formulate the generalization of the exactly solvable two-species model to more than two species. We leave its solution to some future work.

In Appendix B we present some exact solutions to the two-species polytropic $n = 5$ and the isothermal model, and also for the 5/3 model.

In Appendix C we invoke the usual local neutrality approximation which yields the single-density model discussed in [1–3]. We will take the opportunity, in a subsection in that Appendix, to explain our $5/3 \to 6/3$ approximation for the locally neutral single-density model, which produces the Lane-Emden polytrope of index $n = 1$ and its elementary solution.

II. THE THOMAS-FERMI EQUATIONS OF A FAILED WHITE DWARF STAR

The basic equations of the structure of a nonrotating white dwarf star composed of electrons and nuclei can be found in Chandrasekhar's original publications composed into his classic book [2], in [3], and also in [16,17], for instance. For a nonrotating star one may assume spherical symmetry, so all the basic structure functions are then functions only of the radial distance r from the star's center, and the differential equations involved in the discussion reduce to the ordinary type.

We specialize the discussion to a failed star composed only of protons and electrons, both of which are spin-$\frac{1}{2}$ fermions. Each species is treated as an ideal Fermi gas. The number density functions $\nu_p(r) \geq 0$ and $\nu_e(r) \geq 0$ are assumed to integrate to the total number of protons and electrons, respectively, viz.

$$\int_{\mathbb{R}^3} \nu_p(r) d^3r = N_p, \tag{1}$$

$$\int_{\mathbb{R}^3} \nu_e(r) d^3r = N_e. \tag{2}$$

The protons have rest mass m_p and charge $+e$, and the electrons have rest mass m_e and charge $-e$. Thus the mass density of the star is given by

$$\mu(r) = m_p \nu_p(r) + m_e \nu_e(r) \tag{3}$$

and its charge density by

$$\sigma(r) = e\nu_p(r) - e\nu_e(r). \tag{4}$$

The star is overall neutral if $N_p = N_e$; otherwise it carries an excess charge which may have either sign.

The electrons and protons jointly produce a Newtonian gravitational potential $\phi_N(r)$ and an electric Coulomb potential $\phi_C(r)$. The Newton potential ϕ_N is related to the mass density μ by a radial Poisson equation,

$$(r^2 \phi_N'(r))' = 4\pi G \mu(r) r^2, \tag{5}$$

where G is Newton's constant of universal gravitation. Similarly, the Coulomb potential ϕ_C is related to the charge density σ by a radial Poisson equation,

$$-(r^2 \phi_C'(r))' = 4\pi \sigma(r) r^2. \tag{6}$$

As usual, the primes in (5) and (6) mean derivative with respect to the displayed argument, in this case r.

Each species, the electrons and the protons, satisfies an Euler-type mechanical force balance equation,

$$\nu_p(r)[-m_p \phi_N'(r) - e\phi_C'(r)] - p_p'(r) = 0, \tag{7}$$

$$\nu_e(r)[-m_e \phi_N'(r) + e\phi_C'(r)] - p_e'(r) = 0. \tag{8}$$

Here, p_p and p_e are the degeneracy pressures of the ideal proton and electron gases, respectively. For a nonrelativistic gas of spin-$\frac{1}{2}$ fermions (subscript $_f$) of mass m_f and number density ν_f one has (see e.g., [2,8,23])

$$p_f(r) = \frac{\hbar^2}{m_f} \frac{(3\pi^2)^{2/3}}{5} \nu_f^{5/3}(r); \tag{9}$$

here, $_f$ stands for either $_p$ or $_e$, and $\hbar$ is the reduced Planck constant. We remark that (9) is of the type $p = K_\gamma \nu^\gamma$ for some constant K_γ, called a *polytropic* law of power γ, here with $\gamma = 5/3$. Associated with γ is a polytropic index $n = 1/(\gamma - 1)$; here $n = 3/2$.

The system of structure equations can be reduced to a closed system of equations for the densities $\nu_p(r)$ and $\nu_e(r)$ alone, but one needs to distinguish three regions:

(a) $\nu_p(r) > 0$ and $\nu_e(r) > 0$ (the bulk region),
(b) $\nu_p(r) > 0$ and $\nu_e(r) = 0$ (positive atmosphere),
(c) $\nu_p(r) = 0$ and $\nu_e(r) > 0$ (negative atmosphere).

We begin with the bulk region, where both Eqs. (7) and (8) are nontrivial. We use (3) and (4) to express μ and σ in terms of ν_p and ν_e in (5) and (6); now we multiply (5) by $-m_p$ and (6) by e and add the resulting two equations, then use (7) to replace $-m_p \phi_N'(r) - e\phi_C'(r)$ in terms of $\nu_p(r)$,

and next we use (9) to express p_p in terms of ν_p. Similarly, we multiply (5) by $-m_\mathrm{e}$ and (6) by $-e$ and also add these equations; then we use (8) to replace $-m_\mathrm{e}\phi_N'(r)+e\phi_C'(r)$ in terms of $\nu_\mathrm{e}(r)$ and $p_\mathrm{e}'(r)$, and next we use (9) to express p_e in terms of ν_e. This yields

$$-\varepsilon\zeta\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\nu_\mathrm{p}^{2/3}(r)\right) = -\left(1-\frac{Gm_\mathrm{p}^2}{e^2}\right)\nu_\mathrm{p}(r)+\left(1+\frac{Gm_\mathrm{p}m_\mathrm{e}}{e^2}\right)\nu_\mathrm{e}(r), \qquad (10)$$

$$-\zeta\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\nu_\mathrm{e}^{2/3}(r)\right) = \left(1+\frac{Gm_\mathrm{p}m_\mathrm{e}}{e^2}\right)\nu_\mathrm{p}(r)-\left(1-\frac{Gm_\mathrm{e}^2}{e^2}\right)\nu_\mathrm{e}(r), \qquad (11)$$

a system of nonlinear second-order differential equations valid wherever both $\nu_\mathrm{p}(r)>0$ and $\nu_\mathrm{e}(r)>0$; here, $\varepsilon:=m_\mathrm{e}/m_\mathrm{p}$ and $\zeta:=(3^{2/3}\pi^{1/3}/8)\hbar^2/m_\mathrm{e}e^2$ is approximately 50 reduced Compton wavelengths of the electron.

Coming to the atmospheric regimes, a positive atmosphere is governed by (10) with $\nu_\mathrm{e}(r)=0$, while a negative atmosphere is governed by (11) with $\nu_\mathrm{p}(r)=0$.

Each equation is of second order and requires two initial conditions. At the bulk-atmosphere interface at $r=r_0$ the density of the species which forms the atmosphere needs to be continuously differentiable. In the bulk, conditions are posed at $r=0$. Naturally $\nu_\mathrm{p}'(0)=0=\nu_\mathrm{e}'(0)$. The values of $\nu_\mathrm{p}(0)$ and $\nu_\mathrm{e}(0)$ are to be chosen such that Eqs. (1) and (2) hold.

This system of coupled differential equations for the density functions ν_p and ν_e in bulk and atmosphere regions generalizes the single Lane-Emden equation for the polytrope of index $n=\frac{3}{2}$, which has only a bulk interior; see [2,3,16,17].

As for the numerical values of the parameters, $\varepsilon\approx 1/1836\approx 5.54\times10^{-4}$ and $\zeta\approx 52.185\,\frac{\hbar}{m_\mathrm{e}c}$, where $\frac{\hbar}{m_\mathrm{e}c}\approx 3.86\times10^{-13}$ m is the electron's reduced Compton wavelength. The three ratios of gravitational-to-electrical coupling constants which appear in the coefficient matrix on the right-hand sides of Eqs. (10) and (11) are fantastically tiny numbers, viz. $Gm_\mathrm{e}^2/e^2\approx 2.40\times10^{-43}$, $Gm_\mathrm{p}m_\mathrm{e}/e^2\approx 4.41\times10^{-40}$, and $Gm_\mathrm{p}^2/e^2\approx 8.09\times10^{-37}$. All the same, the three tiny ratios of coupling constants are the only places where Newton's constant of universal gravitation, G, enters the equations, and since it is gravity, not electricity, which binds the ideal Fermi gases together to form a star, one cannot neglect these tiny numbers versus 1 in the cofficients—this would result in a singular coefficient matrix, and there would not be any nontrivial solution pair $\nu_\mathrm{p},\nu_\mathrm{e}$.

The nonlinearity of Eqs. (10) and (11), coupled to each other and to their atmospheric counterparts, stands in the way of solving them generally in closed form, although one special elementary solution can be found (see further below). In principle one can evaluate them numerically on a computer, but the tiny ratios of the coupling constants do create problems. Also the small ratio of the masses, $m_\mathrm{e}/m_\mathrm{p}\approx 1/1836$, is a source of numerical trouble. In this situation it definitely is prudent to look for a solvable model, to which we turn next.

III. THE 5/3 → 6/3 APPROXIMATION

Note that we cannot simply replace $\nu_\mathrm{f}^{5/3}$ by $\nu_\mathrm{f}^{6/3}$, for ν_f is not dimensionless. This can be overcome by switching to dimensionless densities with the help of some reference density. In the astrophysical literature one often finds the central density as the reference density, a choice motivated by seeking a definite initial value problem for the numerical integration of the Lane-Emden equation on a computer: the so-normalized dimensionless density takes the value 1 at $r=0$, and its derivative vanishes there. We will be able to solve the 6/3 model equations explicitly, so we have no need for such a normalization. Instead, since the fermionic degeneracy pressure already is expressed with the microscopic constants $\hbar, m_\mathrm{p}, m_\mathrm{e}$, we may as well now choose as the reference length the electron's reduced Compton length $\hbar/m_\mathrm{e}c$, where c is the speed of light in vacuum. While this is somewhat unconventional, it is not unnatural and the resulting formulas are easy to interpret. Thus we set $r=:(\hbar/m_\mathrm{e}c)\rho$ and $\nu(r)=:(m_\mathrm{e}c/\hbar)^3\upsilon(\rho)$, and we also set $\nu_\mathrm{p}(r)=:(m_\mathrm{e}c/\hbar)^3\upsilon_\mathrm{p}(\rho)$ and $\nu_\mathrm{e}(r)=:(m_\mathrm{e}c/\hbar)^3\upsilon_\mathrm{e}(\rho)$. Inserted into the formulas for the degeneracy pressures, we find $p_\mathrm{p}(r)\propto\upsilon_\mathrm{p}(\rho)^{5/3}$ and $p_\mathrm{e}(r)\propto\upsilon_\mathrm{e}(\rho)^{5/3}$, and *now* we can replace $\upsilon_\mathrm{p}^{5/3}$ by $\upsilon_\mathrm{p}^{6/3}$ and $\upsilon_\mathrm{e}^{5/3}$ by $\upsilon_\mathrm{e}^{6/3}$.

This hurdle cleared, we may for the sake of completeness also introduce dimensionless potential functions through $\phi_N(r)=:c^2\psi_N(\rho)$ and $\phi_C(r)=:c^2\frac{m_\mathrm{e}}{e}\psi_C(\rho)$, but we will not need this, given we already have the system of Eqs. (10) and (11), plus their atmospheric specializations.

As in the 5/3 model we distinguish the regions:
(a) $\upsilon_\mathrm{p}(\rho)>0$ and $\upsilon_\mathrm{e}(\rho)>0$ (the bulk region),
(b) $\upsilon_\mathrm{p}(\rho)>0$ and $\upsilon_\mathrm{e}(\rho)=0$ (positive atmosphere),
(c) $\upsilon_\mathrm{p}(\rho)=0$ and $\upsilon_\mathrm{e}(\rho)>0$ (negative atmosphere).

A. The bulk region

In the bulk region we now have the following coupled system of linear second-order differential equations for the density functions υ_p and υ_e, which generalizes the single Lane-Emden equation for the polytrope of index $n=1$, (C7), in the common interior of the charged gases where both $\upsilon_\mathrm{p}(\rho)>0$ and $\upsilon_\mathrm{e}(\rho)>0$:

$$-\varepsilon\varsigma\frac{1}{\rho^2}(\rho^2 v_{\rm p}'(\rho))' = -\left(1-\frac{Gm_{\rm p}^2}{e^2}\right)v_{\rm p}(\rho)+\left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)v_{\rm e}(\rho), \tag{12}$$

$$-\varsigma\frac{1}{\rho^2}(\rho^2 v_{\rm e}'(\rho))' = \left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)v_{\rm p}(\rho)-\left(1-\frac{Gm_{\rm e}^2}{e^2}\right)v_{\rm e}(\rho). \tag{13}$$

Here, $\varsigma := \frac{3^{2/3}\pi^{1/3}}{10}\frac{\hbar c}{e^2} \approx 41.74766$.

We now solve the system of Eqs. (12) and (13) explicitly. A nonsingular system of linear second-order differential equations has four linearly independent solutions, from which we have to select the ones compatible with our physical problem. This is done as follows.

We remark that similar to the Lane-Emden equation for the polytrope of index $n = 1$, Eq. (C7), a change of dependent variables $v_{\rm p}(\rho) \mapsto \rho v_{\rm p}(\rho) =: \chi_{\rm p}(\rho)$ and $v_{\rm e}(\rho) \mapsto \rho v_{\rm e}(\rho) =: \chi_{\rm e}(\rho)$ transforms Eqs. (12) and (13) into a linear second-order system with constant coefficients for $\chi_{\rm p}(\rho), \chi_{\rm e}(\rho)$, and such a system (when not singular) can always be solved by the ansatz $\chi_{\rm f}(\rho) \propto \exp(\kappa\rho)$, with $_f$ standing for either $_{\rm p}$ or $_e$. In terms of $v_{\rm p}, v_{\rm e}$ this means that the ansatz $v_{\rm p}(\rho) = B_{\rm p}\exp(\kappa\rho)/\rho$ and $v_{\rm e}(\rho) = B_{\rm e}\exp(\kappa\rho)/\rho$, with the same κ, will transform the system of differential equations (12) and (13) into a linear system of algebraic equations. Indeed, away from $\rho = 0$ we have

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\frac{\exp(\kappa\rho)}{\rho}\right) = \kappa^2\frac{\exp(\kappa\rho)}{\rho}, \tag{14}$$

and so we obtain the matrix problem

$$\begin{pmatrix} 1-\frac{Gm_{\rm p}^2}{e^2}-\kappa^2\varepsilon\varsigma; & -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2} \\ -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2}; & 1-\frac{Gm_{\rm e}^2}{e^2}-\kappa^2\varsigma \end{pmatrix}\begin{pmatrix} B_{\rm p} \\ B_{\rm e}\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}; \tag{15}$$

here we have placed semicolons in the matrix to facilitate the identification of the matrix elements. The solvability condition for Eq. (15) is the characteristic equation

$$\det\begin{pmatrix} 1-\frac{Gm_{\rm p}^2}{e^2}-\kappa^2\varepsilon\varsigma; & -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2} \\ -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2}; & 1-\frac{Gm_{\rm e}^2}{e^2}-\kappa^2\varsigma \end{pmatrix} = 0, \tag{16}$$

which yields

$$\left(1-\frac{Gm_{\rm e}^2}{e^2}-\kappa^2\varsigma\right)\left(1-\frac{Gm_{\rm p}^2}{e^2}-\kappa^2\varepsilon\varsigma\right) - \left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)^2 = 0, \tag{17}$$

a quadratic problem in κ^2, viz. $a\kappa^4 + b\kappa^2 + c = 0$, with $a = \varepsilon\varsigma^2 > 0$, $b = -\varsigma(1+\varepsilon - G(\varepsilon m_{\rm e}^2 + m_{\rm p}^2)/e^2) < 0$, and $c = -G(m_{\rm e}+m_{\rm p})^2/e^2 < 0$. By the quadratic formula we have two real solutions,

$$(\kappa^2)_\pm = -\frac{b}{2a}\left(1 \pm \sqrt{1-4\frac{ac}{b^2}}\right), \tag{18}$$

one of which is positive and the other one negative, with $(\kappa^2)_+ \approx 44.0025$, and $(\kappa^2)_- \approx -1.94025\times 10^{-38}$. This now yields the *hyperbolic* $\kappa_h := \sqrt{(\kappa^2)_+} \approx 6.63344$ and the *trigonometric* $\kappa_t := |\sqrt{(\kappa^2)_-}| \approx 1.3929\times 10^{-19}$. The latter step obviously generates two imaginary κ values. Converted to real solutions by taking appropriate linear combinations, the set of four linear independent solutions consists of one exponentially growing mode, one exponentially decaying mode, one sine mode, and one cosine mode, each of them divided by the independent variable ρ.

Next we recall the well-known fact that Newton's and Coulomb's $1/r$ potentials correspond to a point source at $r = 0$, and this we need to rule out. This means that the mode $\cos(\kappa\rho)/\rho$ is not admissible, while $\sin(\kappa\rho)/\rho$ is. Similarly, only the linear combination of the exponential modes into the hyperbolic $\sinh(\kappa\rho)/\rho$ mode is admissible, while all other linear combinations are not; in particular the hyperbolic $\cosh(\kappa\rho)/\rho$ mode is not admissible.

Thus, the physically admissible general solution of Eqs. (12) and (13) is of the form (cf. Sec. IV. B in [13])

$$v_{\rm p}(\rho) = B_{\rm p}^h\frac{\sinh(\kappa_h\rho)}{\rho} + B_{\rm p}^t\frac{\sin(\kappa_t\rho)}{\rho}, \tag{19}$$

$$v_{\rm e}(\rho) = B_{\rm e}^h\frac{\sinh(\kappa_h\rho)}{\rho} + B_{\rm e}^t\frac{\sin(\kappa_t\rho)}{\rho}, \tag{20}$$

where we have added superscripts ^{h} and ^{t} at the bulk region coefficients $B_{\rm p}$ and $B_{\rm e}$ to match with the hyperbolic and trigonometric modes. Here, the pairs $(B_{\rm p}^h, B_{\rm e}^h)$ and $(B_{\rm p}^t, B_{\rm e}^t)$ are eigenvectors of the coefficient matrix on the left-hand side of Eq. (15) for the corresponding eigenvalues $(\kappa^2)_\pm$, respectively, and so only two of the four bulk coefficients are independent in the general physical solution. Linear algebra yields the relationships between $B_{\rm p}^h$ and $B_{\rm e}^h$, respectively, between $B_{\rm p}^t$ and $B_{\rm e}^t$, with the results

$$\frac{B_{\rm e}^h}{B_{\rm p}^h} = \frac{1-\frac{Gm_{\rm p}^2}{e^2}-\varepsilon\varsigma\kappa_h^2}{1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}} \approx -5.45\times 10^{-4}, \tag{21}$$

$$\frac{B_{\rm e}^t}{B_{\rm p}^t} = \frac{1-\frac{Gm_{\rm p}^2}{e^2}+\varepsilon\varsigma\kappa_t^2}{1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}} \approx 1-8.09\times 10^{-37}. \tag{22}$$

We pause for a moment to take in the results obtained. The trigonometric parts of the general solution obviously correspond to the $n = 1$ polytrope of the Lane-Emden

equation for the single-density approximation, with $\kappa_t \approx \kappa$ given by Eq. (C8) to a high degree of accuracy, which in concert with Eq. (22) confirms that the positive and negative *large scale* densities are very well approximated by the single-density model almost all the way up to the bulk radius. This confirms by explicit example what everyone knows already, that the locally neutral single-density approximation suffices to obtain the bulk structure of the white dwarf star.

In addition we now have information on the charge separation effects, which are accounted for by the hyperbolic parts of the general solution. These vary significantly on a very short scale by comparison, and so their amplitudes must be very tiny. Interestingly, the hyperbolic modes of the positive and negative species have significantly different bulk amplitudes, roughly corresponding in ratio to the ratio of the rest masses of electrons and protons.

The remaining two independent bulk amplitudes, say B^h_{p} and B^t_{p}, cannot be fixed with the bulk densities alone; this requires also the atmospheric densities. By inspecting the general bulk solution formulas (19) and (20) it is easy to see, though, that $\upsilon_{\mathrm{p}}(\rho) > 0$ can only be achieved with $B^t_{\mathrm{p}} > 0$, while B^h_{p} can take either sign. The analogous conclusion holds therefore for $B^t_{\mathrm{e}} > 0$ and B^h_{e}. As soon as one or the other density reaches zero, the system of equations changes to describe the atmospheric region, unless it happens that both densities reach zero simultaneously (the case of no atmosphere; it will be addressed separately). An atmosphere can be populated either purely with protons or purely with electrons, yet either version is determined in a similar manner. We next turn to these atmospheric cases.

B. The positive atmosphere

In the positive atmosphere the electron density vanishes, $\upsilon_{\mathrm{e}}(\rho) = 0$, while the proton density is still positive, $\upsilon_{\mathrm{p}}(\rho) > 0$, so the electrons' Eulerian force balance equation is trivially satisfied, while that for the protons now is given by (12) with $\upsilon_e = 0$, viz.

$$-\varepsilon\varsigma\frac{1}{\rho^2}(\rho^2\upsilon'_{\mathrm{p}}(\rho))' = -\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)\upsilon_{\mathrm{p}}(\rho), \tag{23}$$

valid for $\rho > \rho_0$, where $\rho_0 = \sup\{\rho : \upsilon_{\mathrm{e}}(\rho) > 0\}$ is the radius of the bulk region. If Gm_{p}^2/e^2 would be greater than 1, Eq. (23) would be a Lane-Emden equation of the $n = 1$ polytrope (mathematically speaking). However, since Gm_{p}^2/e^2 is the tiny number it happens to be, Eq. (23) differs from this Lane-Emden equation by the sign of its right-hand side. Analogous to solving for the bulk region, the general solution of (23) now reads

$$\upsilon_{\mathrm{p}}(\rho) = A^+_{\mathrm{p}}\frac{\exp(\varkappa_{\mathrm{p}}\rho)}{\rho} + A^-_{\mathrm{p}}\frac{\exp(-\varkappa_{\mathrm{p}}\rho)}{\rho}, \tag{24}$$

where $\varkappa_{\mathrm{p}} > 0$ is the positive root of

$$\varkappa_{\mathrm{p}}^2 = \frac{10}{3^{2/3}\pi^{1/3}}\frac{e^2}{\hbar c}\frac{m_{\mathrm{p}}}{m_{\mathrm{e}}}\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right). \tag{25}$$

Note that in this expression one may approximate the last parenthetical factor by 1. Note furthermore that $\varkappa_{\mathrm{p}} \approx 6.63$ is essentially determined by the electrical coupling.

A few comments are in order right now.

First, it could seem reasonable to throw out the exponentially growing mode, but note that a small negative A^+_{p} in concert with a large positive A^-_{p} will result in a $\upsilon_{\mathrm{p}}(\rho)$ which rapidly goes to zero in the positive atmosphere region, so an exponentially growing mode is not a problem because it would be terminated as soon as the proton density vanishes.

Second, since $\rho > \rho_0$, there is no reason now to only allow the linear combination of the exponential modes into the hyperbolic sine, as was the case in the bulk region where there would otherwise be a problem at the origin $\rho = 0$. Incidentally, equivalent to (24) we may write the general solution of the positive atmosphere as

$$\upsilon_{\mathrm{p}}(\rho) = A^+_{\mathrm{p}}\frac{\cosh(\varkappa_{\mathrm{p}}\rho)}{\rho} + A^-_{\mathrm{p}}\frac{\sinh(\varkappa_{\mathrm{p}}\rho)}{\rho}. \tag{26}$$

Third, the two atmospheric amplitudes A^+_{p} and A^-_{p} are constrained by the requirement that the proton density $\upsilon_{\mathrm{p}}(\rho)$ be continuously differentiable at the boundary $\rho = \rho_0$ of the bulk region, so both are needed in general. We will get to this shortly.

C. The negative atmosphere

The discussion of the negative atmosphere region mirrors the one for the positive atmosphere region, so we may be brief. While $\upsilon_{\mathrm{p}}(\rho) = 0$, the structure equation for $\upsilon_{\mathrm{e}}(\rho)$ is given by (13) with $\upsilon_{\mathrm{p}} = 0$, viz.

$$-\varsigma\frac{1}{\rho^2}(\rho^2\upsilon'_{\mathrm{e}}(\rho))' = -\left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)\upsilon_{\mathrm{e}}(\rho), \tag{27}$$

valid for $\rho > \rho_0$, where now the radius of the bulk region is $\rho_0 = \sup\{\rho : \upsilon_{\mathrm{p}}(\rho) > 0\}$.

The general solution of (27) reads

$$\upsilon_{\mathrm{e}}(\rho) = A^+_{\mathrm{e}}\frac{\exp(\varkappa_{\mathrm{e}}\rho)}{\rho} + A^-_{\mathrm{e}}\frac{\exp(-\varkappa_{\mathrm{e}}\rho)}{\rho}, \tag{28}$$

where $\varkappa_{\mathrm{e}} > 0$ is the positive root of

$$\varkappa_{\mathrm{e}}^2 = \frac{10}{3^{2/3}\pi^{1/3}}\frac{e^2}{\hbar c}\left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right). \tag{29}$$

Note that $\varkappa_e^2 \approx \frac{m_e}{m_p}\varkappa_p^2$, where the "≈" is due to some slight differences beginning to show 36 decimal places after the leading digit. Again, also in (29) one may approximate the last parenthetical factor by 1. Note that also $\varkappa_e \approx 0.155$ is essentially determined by the electrical coupling.

Of course, equivalent to (28) we may also write the general solution of the negative atmosphere as

$$\upsilon_e(\rho) = A_e^+ \frac{\cosh(\varkappa_e\rho)}{\rho} + A_e^- \frac{\sinh(\varkappa_e\rho)}{\rho}. \tag{30}$$

The two atmospheric amplitudes A_e^+ and A_e^- are constrained by the requirement that the electron density $\upsilon_e(\rho)$ be continuously differentiable at the boundary $\rho = \rho_0$ of the bulk region.

We will now address this matching of a positive or negative atmosphere to the bulk region.

D. The bulk-atmosphere interface

Having obtained the general physical solution type in the bulk region and the general physical solution type in the atmosphere region, which can be either an electron or a proton atmosphere, we now match these general solutions at their common bulk-atmosphere interface. Both cases, positive and negative atmosphere, can be discussed in parallel.

In the bulk region the two density functions together feature four amplitudes, but Eqs. (21) and (22) express the two electron amplitudes in terms of the two pertinent proton amplitudes, or the other way round. The density function of the atmosphere-forming species features two further amplitudes in the atmosphere region. It has to vary continuously differentiably across the boundary ρ_0 of the bulk region, where the other density reaches zero. In each case, whether the atmosphere consists of protons or of electrons, the requirement that the atmosphere-forming density function $\upsilon_f(\rho)$ is continuously differentiable at the boundary $\rho = \rho_0$ of the bulk region allows us to express the two amplitudes of the density function $\upsilon_f(\rho)$ in the atmosphere region in terms of its two amplitudes in the bulk region.

We explain the procedure using the positive atmosphere case. The negative atmosphere case is completely analogous, and we will only state its final formulas.

The boundary ρ_0 of the bulk region of a white dwarf star with positive atmosphere is determined by the vanishing of the right-hand side of (20), and canceling $1/\rho_0$ this yields

$$B_e^h \sinh(\kappa_h\rho_0) + B_e^t \sin(\kappa_t\rho_0) = 0, \tag{31}$$

where $B_e^h \propto B_p^h$ and $B_e^t \propto B_p^t$; see Eqs. (21) and (22). This is an implicit equation for ρ_0, given B_e^h and B_e^t (equivalently, given B_p^h and B_p^t), which can easily be solved numerically on a computer, but generally not in a closed form. It should be noted, though, that Eq. (31) permits B_e^h to vanish [in which case also B_p^h vanishes by (21)], given any $B_e^t > 0$ (equivalently, given $B_p^t > 0$), namely when $\rho_0 = \pi/\kappa_t$. This is perhaps the only case in which ρ_0 is explicitly obtained from the bulk amplitudes, i.e., from $B_e^h = 0$. We have already remarked that only positive trigonometric bulk amplitudes are permitted, due to the requirement that the bulk densities must not be negative.

At this point, a change of perspective will yield a decisive simplification: From Eq. (31) we obtain

$$\frac{B_e^h}{B_e^t} = -\frac{\sin(\kappa_t\rho_0)}{\sinh(\kappa_h\rho_0)}. \tag{32}$$

We will think of (32) as yielding the ratio B_e^h/B_e^t (equivalently, B_p^h/B_p^t) explicitly as a function of ρ_0, and hence treat the interface location ρ_0 as an independent parameter.

Coming now to the matching of atmospheric amplitudes with the bulk amplitudes, we note that for the protons we have, first of all, the continuity of their density function $\upsilon_p(\rho)$ at $\rho = \rho_0$, which (after canceling $1/\rho_0$) yields

$$\begin{aligned} &B_p^h \sinh(\kappa_h\rho_0) + B_p^t \sin(\kappa_t\rho_0) \\ &\quad = A_p^+ e^{\varkappa_p\rho_0} + A_p^- e^{-\varkappa_p\rho_0}, \end{aligned} \tag{33}$$

equivalently,

$$\begin{aligned} &B_p^h \sinh(\kappa_h\rho_0) + B_p^t \sin(\kappa_t\rho_0) \\ &\quad = A_p^+ \cosh(\varkappa_p\rho_0) + A_p^- \sinh(\varkappa_p\rho_0). \end{aligned} \tag{34}$$

Second, we need the continuity of the derivative of their density function $\upsilon_p(\rho)$ at $\rho = \rho_0$. By the product rule, the ρ-derivative of each term in the general solution is a sum of the ρ-derivative of the numerator, divided by ρ, plus the numerator times the derivative of $1/\rho$. Yet all terms proportional to the derivative of $1/\rho$ can be grouped together and, with the help of (33), this group can be seen to vanish by itself. Thus, and after canceling the remaining overall factor $1/\rho_0$, continuity of the ρ-derivative of $\upsilon_p(\rho)$ at $\rho = \rho_0$ yields

$$\begin{aligned} &B_p^h\kappa_h \cosh(\kappa_h\rho_0) + B_p^t\kappa_t \cos(\kappa_t\rho_0) \\ &\quad = A_p^+\varkappa_p e^{\varkappa_p\rho_0} - A_p^-\varkappa_p e^{-\varkappa_p\rho_0}, \end{aligned} \tag{35}$$

equivalently

$$\begin{aligned} &B_p^h\kappa_h \cosh(\kappa_h\rho_0) + B_p^t\kappa_t \cos(\kappa_t\rho_0) \\ &\quad = A_p^+\varkappa_p \sinh(\varkappa_p\rho_0) + A_p^-\varkappa_p \cosh(\varkappa_p\rho_0). \end{aligned} \tag{36}$$

Using either the pair of equations (33) and (35) or the pair (34) and (36), we can write a linear transformation from the pair of B_p amplitudes to the pair of A_p amplitudes. We choose the pair (33) and (35) and obtain

$$\begin{pmatrix} \sinh(\kappa_h\rho_0); & \sin(\kappa_t\rho_0) \\ \kappa_h\cosh(\kappa_h\rho_0); & \kappa_t\cos(\kappa_t\rho_0) \end{pmatrix}\begin{pmatrix} B^h_{\rm p} \\ B^t_{\rm p} \end{pmatrix} = \begin{pmatrix} \exp(\varkappa_{\rm p}\rho_0); & \exp(-\varkappa_{\rm p}\rho_0) \\ \varkappa_{\rm p}\exp(\varkappa_{\rm p}\rho_0); & -\varkappa_{\rm p}\exp(-\varkappa_{\rm p}\rho_0) \end{pmatrix}\begin{pmatrix} A^+_{\rm p} \\ A^-_{\rm p} \end{pmatrix}. \quad (37)$$

This linear transformation is valid as long as the left- (and therefore the right-)hand side of Eq. (33) is strictly positive, as required for having a positive atmosphere.

We note that the determinant of the coefficient matrix at the right-hand side of Eq. (37) equals $-2\varkappa_{\rm p} < 0$, and therefore the matrix is always invertible and the pair $(A^+_{\rm p}, A^-_{\rm p})$ is uniquely given by (37) in terms of the pair $(B^h_{\rm p}, B^t_{\rm p})$, for any physically meaningful choice of $\rho_0 > 0$. How to choose the physically meaningful ρ_0 we work out in the next subsection. But first we list the analogous formulas for the case of a star with a negative atmosphere.

The pertinent formulas are easily obtained from the formulas of the positive atmosphere setting. Thus, given $B^h_{\rm p}/B^t_{\rm p}$ (equivalently, given $B^h_{\rm e}/B^t_{\rm e}$), from the vanishing of the right-hand side of (19), and after canceling $1/\rho_0$, we obtain

$$\frac{B^h_{\rm p}}{B^t_{\rm p}} = -\frac{\sin(\kappa_t\rho_0)}{\sinh(\kappa_h\rho_0)}. \quad (38)$$

Moreover, we now obtain the linear relationship

$$\begin{pmatrix} \sinh(\kappa_h\rho_0); & \sin(\kappa_t\rho_0) \\ \kappa_h\cosh(\kappa_h\rho_0); & \kappa_t\cos(\kappa_t\rho_0) \end{pmatrix}\begin{pmatrix} B^h_{\rm e} \\ B^t_{\rm e} \end{pmatrix} = \begin{pmatrix} \exp(\varkappa_{\rm e}\rho_0); & \exp(-\varkappa_{\rm e}\rho_0) \\ \varkappa_{\rm e}\exp(\varkappa_{\rm e}\rho_0); & -\varkappa_{\rm e}\exp(-\varkappa_{\rm e}\rho_0) \end{pmatrix}\begin{pmatrix} A^+_{\rm e} \\ A^-_{\rm e} \end{pmatrix} \quad (39)$$

between the $B_{\rm e}$ and $A_{\rm e}$ amplitudes. This linear transformation is valid as long as $\upsilon_{\rm e}(\rho_0) > 0$, as required for having a negative atmosphere.

E. Two intervals of admissible ρ_0 values

By now we have determined the density functions $\upsilon_{\rm p}(\rho)$ and $\upsilon_{\rm e}(\rho)$ of the two-species 6/3 model uniquely in terms of three parameters: (i) a choice of sign, as to whether the positive or negative species defines the bulk radius, (ii) the location ρ_0 of the interface between the bulk region and atmosphere, and (iii) the positive trigonometric bulk amplitude B^t of the species defining the bulk radius. However, the resulting solution may not be integrable to yield the finite total number of particles $N_{\rm p}$ and $N_{\rm e}$. The requirement that it should determines the physically allowed interval of ρ_0 values in the positive and negative amplitude situation. We note that similar to the $n = 1$ polytropic single-density model, the value of the trigonometric amplitude $B^t > 0$ is chosen independently of ρ_0.

Again, having the answer worked out for the case of a star with a positive atmosphere, the answer for a star with a negative atmosphere will follow analogously.

Therefore, assume that the star has a positive atmosphere. Then ρ_0 is the point where the electron bulk density $\upsilon_{\rm e}(\rho)$ has declined to zero. We already know from our discussion that the trigonometric mode of the bulk regime essentially captures the density distribution, so $B^t_{\rm e} > 0$. Moreover, from (32) we see that $B^h_{\rm e} < 0$ if $\rho_0 < \pi/\kappa_t$, and $B^h_{\rm e} > 0$ if $\rho_0 > \pi/\kappa_t$, with $B^h_{\rm e} = 0$ if $\rho_0 = \pi/\kappa_t$. By (21) and (22), then also $B^t_{\rm p} > 0$, while $B^h_{\rm p}$ and $B^h_{\rm e}$ have opposite signs, except when both vanish.

Of course, the case $\rho_0 = \pi/\kappa_t$ which leads to $B^h_{\rm e} = 0 = B^h_{\rm p}$ is the case without atmosphere at all, and the bulk densities $\upsilon_{\rm p}(\rho)$ and $\upsilon_{\rm e}(\rho)$ are then given by essentially the same Lane-Emden $n = 1$ polytrope as in the single-density approximation, Eq. (C9), except for minute differences in the parameter values. Therefore, to have a nonempty atmosphere we need to consider $\rho_0 \neq \pi/\kappa_t$. In fact, we will need $\rho_0 < \pi/\kappa_t$.

Indeed, if $\rho_0 < \pi/\kappa_t$, then since $B^t_{\rm e} > 0$ we have $B^h_{\rm e} < 0$ by (32), and therefore now both $B^t_{\rm p} > 0$ and $B^h_{\rm p} > 0$, by (21) and (22). Now, by assumption $\upsilon_{\rm e}(\rho_0) = 0$, but $\upsilon_{\rm p}(\rho_0)$ is the same linear combination of the $B_{\rm p}$ amplitudes as $\upsilon_{\rm e}(\rho_0)$ is of the $B_{\rm e}$ amplitudes, with $B^t_{\rm p} \approx B^t_{\rm e} > 0$ yet $B^h_{\rm p} > 0$ while $B^h_{\rm e} < 0$, and so we conclude that $\upsilon_{\rm p}(\rho_0) > 0$, as claimed.

Proceeding analogously when $\rho_0 > \pi/\kappa_t$, we find that now both $B^t_{\rm e} > 0$ and $B^h_{\rm e} > 0$ by (32), and therefore now $B^t_{\rm p} > 0$ while $B^h_{\rm p} < 0$. Thus, since by assumption $\upsilon_{\rm e}(\rho_0) = 0$ with two positive amplitudes, the left-hand side of (19) with one positive and one negative amplitude evaluated at ρ_0 is actually negative, in violation of the requirement that particle densities cannot be negative. Thus a positive atmosphere is not possible with $\rho_0 > \pi/\kappa_t$, which cannot be a zero of $\upsilon_{\rm e}(\rho)$ in the bulk.

Next, since $\upsilon_{\rm p}(\rho_0) > 0$ in the case of a positive-atmosphere star, it is clear that $A^+_{\rm p}$ and $A^-_{\rm p}$ cannot both be (strictly) positive or both be negative: two negative $A_{\rm p}$ amplitudes cannot produce a strictly positive particle atmospheric density. Two strictly positive $A_{\rm p}$ amplitudes do yield a positive particle density, but this density grows rapidly beyond any upper bound and cannot integrate to a finite particle number. On the other hand, the combination $A^+_{\rm f} \leq 0$ and $A^-_{\rm f} > 0$ is manifestly admissible, for it will always lead to an atmospheric density function $\upsilon_{\rm f}(\rho)$ which is integrable.

We now rule out the combination $A^+_{\rm f} > 0$ and $A^-_{\rm f} < 0$. It suffices to discuss one of these cases, for the other follows by analogy.

Thus, consider the positive atmosphere. Suppose $A^+_{\rm p} > 0$ and $A^-_{\rm p} < 0$. Recall that $\rho > \rho_0$ in the atmosphere, and that $\rho_0 \approx \pi/\kappa_t \gg 1$ is huge. Now $\varkappa_{\rm p} \approx 6.6$, so $\varkappa_{\rm p}\rho \gg 1$ is also

huge, and therefore the function $\rho \mapsto (A_{\mathrm{p}}^{+} e^{\varkappa_{\mathrm{p}}\rho} + A_{\mathrm{p}}^{-} e^{-\varkappa_{\mathrm{p}}\rho})/\rho$ is increasing for all $\rho > \rho_0$. Since it has to be positive at $\rho = \rho_0$, it cannot be integrable over $\rho > \rho_0$, which finishes the argument.

A similar reasoning rules out the combination $A_{\mathrm{e}}^{+} > 0$ and $A_{\mathrm{e}}^{-} < 0$. Even though $\varkappa_{\mathrm{e}} \approx 0.155$ is smaller than 1, $\varkappa_{\mathrm{e}}\rho$ is still so huge for $\rho > \rho_0$ that the function $\rho \mapsto (A_{\mathrm{e}}^{+} e^{\varkappa_{\mathrm{e}}\rho} + A_{\mathrm{e}}^{-} e^{-\varkappa_{\mathrm{e}}\rho})/\rho$ is increasing for all $\rho > \rho_0$.

This proves that $A_{\mathrm{f}}^{+} > 0$ & $A_{\mathrm{f}}^{-} < 0$ is not allowed.

Thus the only possible combinations are $A_{\mathrm{f}}^{+} \leq 0$ and $A_{\mathrm{f}}^{-} > 0$. The extremal case $A_{\mathrm{p}}^{+} = 0$ and $A_{\mathrm{p}}^{-} > 0$ defines the lower limit ρ_0^{-} of the bulk boundary ρ_0 if ρ_0 is the zero of $v_e(\rho)$. It is straightforward to work out the equation defining ρ_0^{-}, and while it contains only simple elementary functions, it is transcendental and cannot be solved in closed form. However, because of the fantastically tiny ratios of the gravitational to electric coupling constants, a very accurate approximate expression for ρ_0^{-} can be found in terms of simple elementary functions (see our subsection to this subsection below). It reads

$$\rho_0^{-} \approx \frac{\pi}{\kappa_t} - \frac{1}{(1-q)\varkappa_{\mathrm{p}} - q\kappa_h}, \tag{40}$$

where

$$q = \frac{1 - \frac{Gm_{\mathrm{p}}^2}{e^2} + \varepsilon\varsigma\kappa_t^2}{1 - \frac{Gm_{\mathrm{p}}^2}{e^2} - \varepsilon\varsigma\kappa_h^2} \approx -1836. \tag{41}$$

Note that $\kappa_t\rho_0^{+}$ is just barely smaller than π.

The discussion for a negative-atmosphere star mirrors the one for the positive-atmosphere star. Thus the only allowed combinations are $A_{\mathrm{e}}^{+} \leq 0$ and $A_{\mathrm{e}}^{-} > 0$. Analogously to our computation in the positive atmosphere case we now find (see below)

$$\rho_0^{+} \approx \frac{\pi}{\kappa_t} - \frac{q}{(q-1)\varkappa_{\mathrm{e}} - \kappa_h}; \tag{42}$$

also $\kappa_t\rho_0^{+}$ is just barely smaller than π.

We summarize: The bulk radii $\rho_0^{\pm}$ are defined as the smallest possible zeros of the positive, respectively, negative species in a solution pair. The ranges $[\rho_0^{\pm}, \pi/\kappa_t]$ of possible bulk radii are very tiny intervals to the left of the no-atmosphere value $\rho_0 = \pi/\kappa_t$, relative to that value. No bulk radius is bigger than π/κ_t. Since κ_t agrees nearly perfectly with the single-density model value κ given by (C8), the bulk radii of all the failed white dwarf stars in the $5/3 \to 6/3$ approximation are essentially given by (C10). However, the atmosphere of a star can nevertheless be very extended. In particular, in the two extreme cases the atmosphere extends all the way out to infinity, yet with its density approaching zero exponentially fast.

1. Computing $\rho_0^{\pm}$

In the case of an extreme negative atmosphere, $\rho_0^{+} < \pi/\kappa_t$ is determined by the matching of the bulk part of $v_{\mathrm{e}}(\rho)$ with its atmospheric part in the limiting case where $A_{\mathrm{e}}^{+} = 0$. So from (39) we obtain

$$\begin{pmatrix} \sinh(\kappa_h\rho_0); & \sin(\kappa_t\rho_0) \\ \kappa_h\cosh(\kappa_h\rho_0); & \kappa_t\cos(\kappa_t\rho_0) \end{pmatrix} \begin{pmatrix} B_{\mathrm{e}}^{h} \\ B_{\mathrm{e}}^{t} \end{pmatrix} = A_{\mathrm{e}}^{-} e^{-\varkappa_{\mathrm{e}}\rho_0} \begin{pmatrix} 1 \\ -\varkappa_{\mathrm{e}} \end{pmatrix}, \tag{43}$$

and these are two different equations for $A_{\mathrm{e}}^{-}\exp(-\varkappa_{\mathrm{e}}\rho_0)$. Elimination of $A_{\mathrm{e}}^{-}\exp(-\varkappa_{\mathrm{e}}\rho_0)$ now yields, after some simple manipulations,

$$\frac{B_{\mathrm{e}}^{h}}{B_{\mathrm{e}}^{t}}\sinh(\kappa_h\rho_0) + \sin(\kappa_t\rho_0) = -\frac{\kappa_h}{\varkappa_{\mathrm{e}}}\frac{B_{\mathrm{e}}^{h}}{B_{\mathrm{e}}^{t}}\cosh(\kappa_h\rho_0) - \frac{\kappa_t}{\varkappa_{\mathrm{e}}}\cos(\kappa_t\rho_0). \tag{44}$$

With the help of Eqs. (21), (22), and (38) we find

$$\frac{B_{\mathrm{e}}^{h}}{B_{\mathrm{e}}^{t}} = -\frac{1}{q}\frac{\sin(\kappa_t\rho_0)}{\sinh(\kappa_h\rho_0)}, \tag{45}$$

with q given in (41). Note that (45) is not in contradiction to (32), for (45) holds for the extreme negative atmosphere, while (32) holds for any positive atmosphere. Substituting (45) in (44), dividing by $\sin(\kappa_t\rho_0)$, and reshuffling now yields

$$(q-1)\varkappa_{\mathrm{e}} = \kappa_h\coth(\kappa_h\rho_0) - q\kappa_t\cot(\kappa_t\rho_0) \tag{46}$$

for the lower limit $\rho_0^{+} < \pi/\kappa_t$ of the zero of the bulk density $v_{\mathrm{p}}(\rho)$. Since coth is a monotonic decreasing function on the positive real line and cot is a monotonic decreasing function on its first positive period, and since $q < 0$, we see that the right-hand side of (46) is a strictly monotonic decreasing function in the interval $0 < \kappa_t\rho_0 < \pi$; thus it has a unique solution ρ_0^{+}. With the values of the parameters q, κ_t, κ_h, and $\varkappa_{\mathrm{e}}$ as given, this solution is in the left vicinity of $\rho_0 = \pi/\kappa_t$. Recall that $\kappa_h \approx 6.6$ and $\kappa_t \approx 2 \times 10^{-19}$. Thus, if $\kappa_t\rho_0^{+} \approx \pi$, then $\kappa_h\rho_0^{+} \gg \pi$ is huge, and then $\coth(\kappa_h\rho_0^{+}) \approx 1$ asymptotically exact, with exponentially small corrections. Moreover, in the left vicinity of $\rho_0 = \pi/\kappa_t$ we have $\cot(\kappa_t\rho_0) \approx 1/(\kappa_t\rho_0 - \pi) < 0$ asymptotically exact, and this yields (42).

Analogously we handle the case of an extreme positive atmosphere, where $\rho_0^{-} < \pi/\kappa_t$ is determined by the matching of the bulk part of $v_{\mathrm{p}}(\rho)$ with its atmospheric part in the limiting case where $A_{\mathrm{p}}^{+} = 0$. This time

$$\frac{B_{\mathrm{p}}^{h}}{B_{\mathrm{p}}^{t}} = -q\frac{\sin(\kappa_t\rho_0)}{\sinh(\kappa_h\rho_0)}, \tag{47}$$

with q given in (41). Also (47) is not in contradiction to (38), for (47) holds for the extreme positive atmosphere, while (38) holds for any negative atmosphere. We find

$$(1-q)\varkappa_{\mathrm{p}} = q\kappa_h\coth(\kappa_h\rho_0) - \kappa_t\cot(\kappa_t\rho_0) \tag{48}$$

for the lower limit $\rho_0^- < \pi/\kappa_t$ of the zero of the bulk $\upsilon_{\mathrm{e}}(\rho)$. The right-hand side of (48) is a strictly monotonic increasing function in ρ_0 in the first positive period of the cot function, with a solution in the left vicinity of $\rho_0 = \pi/\kappa_t$. Using that $\kappa_h\rho_0 \gg \pi$ is huge we again can set $\coth(\kappa_h\rho_0)\approx 1$ asymptotically exact, with exponentially small corrections. Moreover, in the left vicinity of $\rho_0 = \pi/\kappa_t$ we have $\cot(\kappa_t\rho_0) \approx 1/(\kappa_t\rho_0 - \pi) < 0$ asymptotically exact, and this now yields (40).

F. Computing the ratio $N_{\mathrm{e}}/N_{\mathrm{p}}$ as function of ρ_0

As in any set of homogeneous linear equations, so also in the 6/3 model there is an amplitude invariance; i.e., if $(\nu_{\mathrm{p}}, \nu_{\mathrm{e}})$ is a solution pair, then so is $(\lambda\nu_{\mathrm{p}}, \lambda\nu_{\mathrm{e}})$ for any λ (with $\lambda > 0$ to be meaningful). Therefore there is no such thing as the number of protons N_{p} and the number of electrons N_{e} associated with a solution. Incidentally, although any total number of particles is mathematically allowed in this linear model, as explained in the Introduction, a failed white dwarf is a low-mass star, and since the mass is essentially given by the number of protons, N_{p} should be restricted to about 1.5×10^{55} to 9×10^{55} protons to be physically meaningful.

The ratio $N_{\mathrm{e}}/N_{\mathrm{p}}$ is a well-defined quantity associated with any solution pair $(\nu_{\mathrm{p}}, \nu_{\mathrm{e}})$, though. Given the choice of either a positive or a negative atmosphere, the ratio $N_{\mathrm{e}}/N_{\mathrm{p}}$ is uniquely determined by the allowed values of the bulk boundary location ρ_0. Its computation as a function of ρ_0 can be effected by directly integrating the explicit solutions parametrized by ρ_0. Yet it is easier to work directly with the differential equations.

Starting with the case of a negative atmosphere, we multiply Eq. (13) with $4\pi\rho^2$ and integrate from 0 to ρ_e, obtaining

$$\varsigma 4\pi\rho_e^2\upsilon_{\mathrm{e}}'(\rho_e) = \left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)N_{\mathrm{e}} - \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)N_{\mathrm{p}}, \tag{49}$$

where $\upsilon_{\mathrm{e}}'(\rho_e)$ is the left derivative of $\upsilon_{\mathrm{e}}(\rho)$ at $\rho = \rho_e$. We next complement (49) by deriving its counterpart for the positive species. Thus we multiply Eq. (12) with $4\pi\rho^2$ and integrate from 0 to ρ_0, obtaining

$$-\varepsilon\varsigma 4\pi\rho_0^2\upsilon_{\mathrm{p}}'(\rho_0) = -\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)N_{\mathrm{p}} + \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\int_0^{\rho_0}\upsilon_{\mathrm{e}}(\rho)4\pi\rho^2 d\rho. \tag{50}$$

Here, $\upsilon_{\mathrm{p}}'(\rho_0)$ is the left derivative of $\upsilon_{\mathrm{p}}(\rho)$ at $\rho = \rho_0$. Noting that $\int_0^{\rho_0}\upsilon_{\mathrm{e}}(\rho)4\pi\rho^2 d\rho + \int_{\rho_0}^{\rho_e}\upsilon_{\mathrm{e}}(\rho)4\pi\rho^2 d\rho = N_{\mathrm{e}}$, we multiply Eq. (27) with $4\pi\rho^2$ and integrate from ρ_0 to ρ_e, the point where $\upsilon_{\mathrm{e}}(\rho)$ has decreased to zero. This yields

$$4\pi\varsigma(\rho_0^2\upsilon_{\mathrm{e}}'(\rho_0) - \rho_e^2\upsilon_{\mathrm{e}}'(\rho_e)) = -\left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)\int_{\rho_0}^{\rho_e}\upsilon_{\mathrm{e}}(\rho)4\pi\rho^2 d\rho. \tag{51}$$

Now we multiply (51) by $(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2})/(1-\frac{Gm_{\mathrm{e}}^2}{e^2})$ and subtract the result from (50), which yields

$$4\pi\varsigma\left[(\rho_0^2\upsilon_{\mathrm{e}}'(\rho_0) - \rho_e^2\upsilon_{\mathrm{e}}'(\rho_e))\frac{1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}}{1-\frac{Gm_{\mathrm{e}}^2}{e^2}} + \varepsilon\rho_0^2\upsilon_{\mathrm{p}}'(\rho_0)\right] = \left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)N_{\mathrm{p}} - \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)N_{\mathrm{e}}. \tag{52}$$

Equations (49) and (52) form a linear system for N_{p} and N_{e} in terms of their coefficients and their left-hand sides. This linear system is easily solved formally for N_{p} and N_{e}, from which we obtain $N_{\mathrm{e}}/N_{\mathrm{p}}$. Symbolically,

$$\begin{pmatrix} N_{\mathrm{p}} \\ N_{\mathrm{e}} \end{pmatrix} = 4\pi\varsigma\begin{pmatrix} 1-\frac{Gm_{\mathrm{p}}^2}{e^2}; & -1-\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2} \\ -1-\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}; & 1-\frac{Gm_{\mathrm{e}}^2}{e^2} \end{pmatrix}^{-1}\begin{pmatrix} (\rho_0^2\upsilon_{\mathrm{e}}'(\rho_0) - \rho_e^2\upsilon_{\mathrm{e}}'(\rho_e))\frac{1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}}{1-\frac{Gm_{\mathrm{e}}^2}{e^2}} + \varepsilon\rho_0^2\upsilon_{\mathrm{p}}'(\rho_0) \\ \rho_e^2\upsilon_{\mathrm{e}}'(\rho_e) \end{pmatrix}, \tag{53}$$

and the inverse matrix is easily computed as

$$\begin{pmatrix} 1-\frac{Gm_{\rm p}^2}{e^2}; & -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2} \\ -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2}; & 1-\frac{Gm_{\rm e}^2}{e^2} \end{pmatrix}^{-1} = \frac{e^2}{G(m_{\rm p}+m_{\rm e})^2} \begin{pmatrix} -1+\frac{Gm_{\rm e}^2}{e^2}; & -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2} \\ -1-\frac{Gm_{\rm p}m_{\rm e}}{e^2}; & -1+\frac{Gm_{\rm p}^2}{e^2} \end{pmatrix}. \tag{54}$$

This gives $(N_{\rm p}, N_{\rm e})$ uniquely in terms of the zeros of the densities and the derivatives at the zeros. Recall, though, that a choice of the sign of the atmosphere (negative in this case) plus a choice of ρ_0 do not uniquely determine a solution pair, by the linearity of the equations. If $(\upsilon_{\rm p}, \upsilon_{\rm e})$ is a solution pair, then so is $(\lambda\upsilon_{\rm p}, \lambda\upsilon_{\rm e})$, and this changes the derivatives $(\upsilon'_{\rm p}, \upsilon'_{\rm e})$ to $(\lambda\upsilon'_{\rm p}, \lambda\upsilon'_{\rm e})$ everywhere, and hence also $(N_{\rm p}, N_{\rm e})$ to $(\lambda N_{\rm p}, \lambda N_{\rm e})$. Therefore (53) does not yield $(N_{\rm p}, N_{\rm e})$ uniquely as a function of ρ_0 and the sign of the atmosphere. However, the fraction $N_{\rm e}/N_{\rm p}$ is scaling-invariant, and uniquely given as a function of ρ_0 and the sign of the atmosphere. It reads

$$\frac{N_{\rm e}}{N_{\rm p}} = \left[1 - \frac{\rho_e^2\upsilon'_{\rm e}(\rho_e)\frac{G(m_{\rm p}+m_{\rm e})^2}{e^2}}{\rho_0^2\upsilon'_{\rm e}(\rho_0)(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2})^2 + \varepsilon\rho_0^2\upsilon'_{\rm p}(\rho_0)(1-\frac{Gm_{\rm e}^2}{e^2})(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2})}\right]\frac{1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}}{1-\frac{Gm_{\rm e}^2}{e^2}}. \tag{55}$$

Next we compute the pertinent derivatives at ρ_0 and $\rho_e = (1/2\varkappa_{\rm e})\ln(-A_{\rm e}^-/A_{\rm e}^+)$. We find

$$\upsilon'_{\rm e}(\rho_e) = 2A_{\rm e}^+\varkappa_{\rm e}\frac{\exp(\varkappa_{\rm e}\rho_e)}{\rho_e}, \tag{56}$$

$$\upsilon'_{\rm e}(\rho_0) = \left(\varkappa_{\rm e}-\frac{1}{\rho_0}\right)A_{\rm e}^+\frac{\exp(\varkappa_{\rm e}\rho_0)}{\rho_0} - \left(\varkappa_{\rm e}+\frac{1}{\rho_0}\right)A_{\rm e}^-\frac{\exp(-\varkappa_{\rm e}\rho_0)}{\rho_0}, \tag{57}$$

$$\upsilon'_{\rm p}(\rho_0) = B_{\rm p}^h\kappa_h\frac{\cosh(\kappa_h\rho_0)}{\rho_0} + B_{\rm p}^t\kappa_t\frac{\cos(\kappa_t\rho_0)}{\rho_0}. \tag{58}$$

Since the derivatives of the densities enter linearly at the numerator and at the denominator of (55), the expression (55) is a manifestly amplitude-scaling invariant. Thus (55) is an explicit formula for $N_{\rm e}/N_{\rm p}$ as a function of ρ_0 in the negative atmosphere regime. We note that the term in square parentheses is smaller than 1, yet it converges upward to 1 when $\rho_e^2\upsilon'_{\rm e}(\rho_e) \to 0$ and $\rho_0 \searrow \rho_0^+$. In that case $N_{\rm e}/N_{\rm p}$ reaches its upper limit given by the right-hand side of (66).

In a similar manner we can treat the case of a positive atmosphere and find

$$\frac{N_{\rm e}}{N_{\rm p}} = \left[1 - \frac{\varepsilon\rho_{\rm p}^2\upsilon'_{\rm p}(\rho_{\rm p})\frac{G(m_{\rm p}+m_{\rm e})^2}{e^2}}{\varepsilon\rho_0^2\upsilon'_{\rm p}(\rho_0)(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2})^2 + \rho_0^2\upsilon'_{\rm e}(\rho_0)(1-\frac{Gm_{\rm p}^2}{e^2})(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2})}\right]^{-1}\frac{1-\frac{Gm_{\rm p}^2}{e^2}}{1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}}. \tag{59}$$

The pertinent derivatives at ρ_0 and $\rho_{\rm p} = (1/2\varkappa_{\rm p})\ln(-A_{\rm p}^-/A_{\rm p}^+)$ read

$$\upsilon'_{\rm p}(\rho_{\rm p}) = 2A_{\rm p}^+\varkappa_{\rm p}\frac{\exp(\varkappa_{\rm p}\rho_{\rm p})}{\rho_{\rm p}}, \tag{60}$$

$$\upsilon'_{\rm p}(\rho_0) = \left(\varkappa_{\rm p}-\frac{1}{\rho_0}\right)A_{\rm p}^+\frac{\exp(\varkappa_{\rm p}\rho_0)}{\rho_0} - \left(\varkappa_{\rm p}+\frac{1}{\rho_0}\right)A_{\rm p}^-\frac{\exp(-\varkappa_{\rm p}\rho_0)}{\rho_0}, \tag{61}$$

$$\upsilon'_{\rm e}(\rho_0) = B_{\rm e}^h\kappa_h\frac{\cosh(\kappa_h\rho_0)}{\rho_0} + B_{\rm e}^t\kappa_t\frac{\cos(\kappa_t\rho_0)}{\rho_0}. \tag{62}$$

Again all amplitudes are proportional to $B_{\rm p}^t$ (equivalently, $B_{\rm e}^t$), which actually cancels out from (59). Thus (59) is an explicit formula for $N_{\rm e}/N_{\rm p}$ as a function of ρ_0 in the positive atmosphere regime. We note that the term in square parentheses is smaller than 1, yet it converges upward to 1 when $\rho_{\rm p}^2\upsilon'_{\rm p}(\rho_{\rm p}) \to 0$ and $\rho_0 \searrow \rho_0^-$. In that case $N_{\rm e}/N_{\rm p}$ reaches its lower limit given by the left-hand side of (66).

Consistency check: when $\rho_0 = \pi/\kappa_t$, then $\rho_e = \rho_{\rm p} = \rho_0$, and both (59) and (55) reduce to

$$\frac{N_{\rm e}}{N_{\rm p}} = \frac{\upsilon'_{\rm e}(\frac{\pi}{\kappa_t})(1-\frac{Gm_{\rm p}^2}{e^2}) + \varepsilon\upsilon'_{\rm p}(\frac{\pi}{\kappa_t})(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2})}{\upsilon'_{\rm e}(\frac{\pi}{\kappa_t})(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}) + \varepsilon\upsilon'_{\rm p}(\frac{\pi}{\kappa_t})(1-\frac{Gm_{\rm e}^2}{e^2})}, \tag{63}$$

with the derivatives reducing to $v_{\rm e}'(\frac{\pi}{\kappa_t}) = -B_{\rm e}^t \kappa_t^2/\pi$ and $v_{\rm p}'(\frac{\pi}{\kappa_t}) = -B_{\rm p}^t \kappa_t^2/\pi$. Now factoring out $v_{\rm e}'(\frac{\pi}{\kappa_t})$ from both numerator and denominator produces the ratio $v_{\rm p}'(\frac{\pi}{\kappa_t})/v_{\rm e}'(\frac{\pi}{\kappa_t}) = B_{\rm p}^t/B_{\rm e}^t$ [cf. (78)], which can be read from (22). From this expression one then finds that in this special no-atmosphere case the ratio $N_{\rm e}/N_{\rm p} < 1$.

Equations (55) and (59) are easy to implement on a computer. We used them to generate Figs. 3 and 5.

G. The interval of allowed $N_{\rm e}/N_{\rm p}$ ratios

We now ask: "What does the 6/3 model say about the possible numbers of electrons per proton, $N_{\rm e}/N_{\rm p}$, in a failed white dwarf?" (cf. [22]).

Since the successful single-density models are based on the local neutrality approximation, which implies $N_{\rm p} = N_{\rm e}$, one should expect that any nonneutral pair $(N_{\rm p}, N_{\rm e})$ will have a ratio $N_{\rm e}/N_{\rm p} \approx 1$ to a high degree of precision.

It is clear that the extreme values of the ratio $N_{\rm e}/N_{\rm p}$ will be obtained by inserting the extremal values $\rho_0^{\pm}$ for ρ_0 into the $N_{\rm e}/N_{\rm p}$ formulas, which we have already computed as an elementary function of ρ_0. However, to obtain these extreme ratios we can resort to a simpler argument (cf. [22]) for which we here can use that we have full knowledge of the solution family of the 6/3 model.

Namely, consider the extreme case of a star with negative atmosphere, i.e., $\rho_0 = \rho_0^+$. We multiply Eq. (13) by $4\pi\rho^2$ and integrate over ρ from 0 to ∞. [Strictly speaking, Eq. (13) is *a priori* only valid inside the bulk region, but comparison with the atmospheric equation (27) reveals that we can extend (13) to all ρ by noting that $v_{\rm p}(\rho) = 0$ for $\rho \geq \rho_0^+$.] Using that $\int v_{\rm p}(\rho) d^3\rho = N_{\rm p}$ and $\int v_{\rm e}(\rho) d^3\rho = N_{\rm e}$, and using that $v_{\rm e}'(0) = 0$ and that $v_{\rm e}'(\rho) \to 0$ exponentially fast when $\rho \to \infty$, we obtain

$$0 = \left(1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}\right) N_{\rm p}^- - \left(1 - \frac{G m_{\rm e}^2}{e^2}\right) N_{\rm e}^-, \quad (64)$$

where the negative superscript at $N_{\rm p}$ and $N_{\rm e}$ indicates an extreme negative atmosphere case. Similarly, consider the extreme case of a star with positive atmosphere, i.e., $\rho_0 = \rho_0^-$. We multiply Eq. (12) by $4\pi\rho^2$ and integrate over ρ from 0 to ∞; again we extend also Eq. (12) to all ρ by noting that $v_{\rm e}(\rho) = 0$ for $\rho \geq \rho_0^-$. Using once again that $\int v_{\rm p}(\rho) d^3\rho = N_{\rm p}$ and $\int v_{\rm e}(\rho) d^3\rho = N_{\rm e}$, and using now that $v_{\rm p}'(0) = 0$ and that $v_{\rm p}'(\rho) \to 0$ exponentially fast when $\rho \to \infty$, we obtain

$$0 = -\left(1 - \frac{G m_{\rm p}^2}{e^2}\right) N_{\rm p}^+ + \left(1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}\right) N_{\rm e}^+, \quad (65)$$

where the positive superscript at $N_{\rm p}$ and $N_{\rm e}$ indicates an extreme positive atmosphere case. From Eqs. (64) and (65) we now obtain the allowed range of ratios $N_{\rm e}/N_{\rm p}$ in the 6/3 model as

$$\boxed{\frac{1 - \frac{G m_{\rm p}^2}{e^2}}{1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}} \leq \frac{N_{\rm e}}{N_{\rm p}} \leq \frac{1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}}{1 - \frac{G m_{\rm e}^2}{e^2}}}. \quad (66)$$

We have boxed formula (66), for it will turn out to be "universal," in a sense we will explain next.

Note that the bounds (66) are independent of $\hbar$ and c.

IV. "UNIVERSALITY" OF THE $N_{\rm e}/N_{\rm p}$ BOUNDS

We will present a compelling argument for why (66) is the correct $N_{\rm e}/N_{\rm p}$ interval for a failed white dwarf star made of protons and electrons, and not merely in the nonrelativistic theory.

In the 6/3 model the two extreme values of $N_{\rm e}/N_{\rm p}$ are attained by the only two solutions which extend all the way out to spatial infinity, and the densities of the infinitely extended extremal atmospheres decay faster than exponentially to zero when the radial variable goes to infinity. All other solutions have density function pairs $(\nu_{\rm p}(r), \nu_{\rm e}(r))$ which have finite radial extent and a $N_{\rm e}/N_{\rm p}$ ratio sandwiched between the bounds in (66). This suggests that also among all the solutions of the structure equations of the physically more realistic models the solutions with an extreme surplus of charge are those which have one of their two density functions extend to spatial infinity, approaching zero sufficiently rapidly together with its radial derivative so that some surface integrals vanish in the limit—note that we cannot expect a decay to zero to be exponentially fast or even faster; this is a model-specific detail. We will now confirm this. The gist of the discussion can also be found in [22].

A. Proof that an atmospheric density has to reach zero with zero slope to saturate the bounds (66)

For simplicity we present the proof for the 5/3 model, but it will be clear from the proof how to adjust it to also apply to the special-relativistic failed white dwarf model.

Starting with the case of a negative atmosphere, we multiply Eq. (11) with $4\pi r^2$ and integrate from 0 to r_e, the point where the density $\nu_{\rm e}(r)$ reaches 0, obtaining

$$4\pi\zeta r_e^2 \nu_{\rm e}^{\frac{2}{3}\prime}(r_e) = \left(1 - \frac{G m_{\rm e}^2}{e^2}\right) N_{\rm e} - \left(1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}\right) N_{\rm p}. \quad (67)$$

Since $\nu_{\rm e}'(r_e)$ is the left derivative of $\nu_{\rm e}(r)$ at the point r_e, and since an otherwise positive function cannot reach 0 with a positive slope, it follows that $\nu_{\rm e}^{\frac{2}{3}\prime}(r_e) \leq 0$, and so

$$\frac{N_{\rm e}}{N_{\rm p}} \leq \frac{1 + \frac{G m_{\rm p} m_{\rm e}}{e^2}}{1 - \frac{G m_{\rm e}^2}{e^2}}, \quad (68)$$

which is the upper bound on $N_{\rm e}/N_{\rm p}$ given in (66). We abbreviate the right-hand side of (68) by $(N_{\rm e}/N_{\rm p})^-$. In the

limit in which the upper bound is saturated, from (67) we now obtain

$$\lim_{\frac{N_e}{N_p}\nearrow(\frac{N_e}{N_p})^-} r_e^2 \nu_e^{\frac{2}{3}\prime}(r_e) = 0, \tag{69}$$

where we consider r_e as a function of N_e/N_p. Since here $r_e > r_p > 0$, it follows that $\nu_e^{\frac{2}{3}\prime}(r_e) \to 0$ in the limit.

In a completely analogous manner we obtain in the case of a positive atmosphere that

$$4\pi\varepsilon\zeta r_p^2 \nu_p^{\frac{2}{3}\prime}(r_p) = \left(1 - \frac{Gm_p^2}{e^2}\right) N_p - \left(1 + \frac{Gm_p m_e}{e^2}\right) N_e, \tag{70}$$

from which we deduce that

$$\frac{N_e}{N_p} \geq \frac{1 - \frac{Gm_p^2}{e^2}}{1 + \frac{Gm_p m_e}{e^2}}, \tag{71}$$

which is the lower bound on N_e/N_p given in (66). Abbreviating the right-hand side of (71) by $(N_e/N_p)^+$, in the limit in which the lower bound is saturated, from (70) we now obtain

$$\lim_{\frac{N_e}{N_p}\searrow(\frac{N_e}{N_p})^+} r_p^2 \nu_p^{\frac{2}{3}\prime}(r_p) = 0, \tag{72}$$

where we consider r_p as a function of N_e/N_p. Since now $r_p > r_e > 0$, it follows that $\nu_p^{\frac{2}{3}\prime}(r_p) \to 0$ in the limit.

B. Proof that the atmosphere of an extremely surcharged solution is infinitely extended

Consider an extremal solution with negative atmosphere. We have just seen that both $\nu_e(r_e) = 0$ and $\nu_e'(r_e) = 0$. Now suppose $r_e < \infty$. Then by a familiar uniqueness result for (11), and using that $\nu_p(r) = 0$ for $r_p < r < r_e$ in a negative-atmosphere star, it now follows that $\nu_e(r) = 0$ for all $r > r_p$. But this violates the negative-atmosphere hypothesis which says that $\nu_e(r)$ is strictly positive for $r_p \leq r < r_e$. Hence an extremal negative atmosphere extends to infinity.

The analogous conclusion holds for an extremal positive atmosphere.

C. Existence of extremely surcharged solutions

The arguments presented in the previous two subsections establish that any extremely surcharged solution must be infinitely extended, and that in the limit where the radial variable goes to infinity, the derivative of the atmospheric density must vanish very rapidly [see (69) and (72)]. It remains to show that such extremely surcharged solutions do exist. Of course, we are only interested in solutions with a finite total mass.

Here is the argument, which involves continuous dependence of solutions on the data, plus *a priori* bounds. We already established the existence of the no-atmosphere solution, where both densities go to zero at the same distance from the center. Both densities, in this case, are rescaled standard $n = 3/2$ polytropes.

Note that the central electron density is smaller than the central proton density. Now, keeping the central density of the protons fixed, start lowering the central density of the electrons. Considering the system of ordinary differential equations for the densities as an initial value problem at the origin, with vanishing slope, one can extract the information that the electron density function decreases together with its central density, and so its zero now moves to the left, while the proton density increases and its zero moves to the right. Thus N_e decreases and N_p increases. The ratio, of course, can never violate the *a priori* bounds in (66), and so, given the opposite monotonicity of the particle numbers, N_p in particular cannot increase to infinity, and N_e cannot decrease to zero. How far can one push this? Answer: As long as both densities hit zero with a finite slope, one can continue into the neighborhood of the solution to find a new solution. This process can therefore be continued until the slope of the proton density vanishes when the proton density reaches zero. As shown already, this can happen only if the proton density vanishes only at infinity.

In a similar manner one can proceed keeping the central electron density fixed and lowering the central proton density. This leads to a sequence of decreasing N_p and increasing N_e, which can be continued until the electron density extends all the way to infinity, with a vanishing slope at infinity.

We still need to show that in either of these borderline cases the density vanishes sufficiently rapidly so that (69) and (72) hold. So suppose this would not hold. Then (and assuming convergence here, for simplicity) in the case of the negative atmosphere we necessarily have that $r^2 \frac{d}{dr} \nu_e^{2/3}(r) \to C < 0$ for $r \to \infty$; this implies that $\nu_e(r) \sim 1/r^{3/2}$ for $r \to \infty$, but such a $\nu_e(r)$ is not integrable at ∞, in violation of the fact that we know that $N_e < \infty$ and $N_p < \infty$. Similarly one can rule out that $r^2 \frac{d}{dr} \nu_p^{2/3}(r) \to C < 0$ for $r \to \infty$ for the positive atmosphere case. This shows that (69) and (72) do hold.

This establishes the existence of two extremal atmosphere solutions in the 5/3 model, satisfying (69), respectively, (72).

But then we can multiply Eq. (11) by $4\pi r^2$ and integrate over r from 0 to ∞, with the understanding that $\nu_p(r) = 0$ for $r \geq r_p$. Using that $r^2 \frac{d}{dr} \nu_e^{2/3}(r) \to 0$ for $r \to \infty$, the result of this integration is again Eq. (64). Similarly we can

proceed in the case of an extreme positive atmosphere, and once again find Eq. (65).

Thus our bounds (66) are also valid when working with the proper 5/3 power law of the nonrelativistic degeneracy pressures of the protons and the electrons, as claimed.

D. Relativity

Our discussion of the 5/3 model can be adapted to the special-relativistic setting in the manner done by Chandrasekhar [2] for the single-density model, which in the structure equations (10) and (11) changes the $\nu_f^{2/3}$ into some nonlinear function of ν_f that interpolates continuously between $\nu_f^{2/3}$ and $\nu_f^{1/3}$. Explicitly, introducing $P_{\mathrm{f}} = (1/24\pi^2)m_f^4c^5/\hbar^3$ and $\ell_f = (3\pi^2)^{1/3}\hbar/m_f c$, and $k_f = m_f c^2/4\pi e^2$, the nonrelativistic pressure law (9) gets replaced with the somewhat intimidating expression

$$p_{\mathrm{f}}(r) = P_{\mathrm{f}}\ell_f\nu_f^{1/3}(r)(2\ell_f^2\nu_f^{2/3}(r) - 3) \times \sqrt{1+\ell_f^2\nu_f^{2/3}(r)} + \sinh^{-1}(\ell_f\nu_f^{1/3}(r)). \quad (73)$$

Wherever both $\nu_{\mathrm{p}}(r) > 0$ and $\nu_{\mathrm{e}}(r) > 0$ one can follow the same steps used in the derivation of (10) and (11) to get

$$-k_{\mathrm{p}}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\sqrt{1+\ell_{\mathrm{p}}^2\nu_{\mathrm{p}}^{2/3}(r)}\right) = -\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)\nu_{\mathrm{p}}(r) + \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\nu_{\mathrm{e}}(r), \quad (74)$$

$$-k_{\mathrm{e}}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\sqrt{1+\ell_e^2\nu_{\mathrm{e}}^{2/3}(r)}\right) = \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\nu_{\mathrm{p}}(r) - \left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)\nu_{\mathrm{e}}(r). \quad (75)$$

All the same, the integration of the pertinent structure equations will always produce Eqs. (64) and (65), and therefore (66).

There is one caveat to what we just said, and that is that we have tacitly assumed that we stay away from the Chandrasekhar mass $\propto (\hbar c/G)^{\frac{3}{2}}/m_{\mathrm{p}}^2$. However, since we are only discussing failed white dwarfs, which are low-mass stars, with N_{p} restricted to about 1.5×10^{55}–9×10^{55} protons, we certainly are on the safe side.

All our results so far are based on Newtonian gravity, though. We suspect that (66) also holds general relativistically, again for failed white dwarfs whose mass is far away from any critical mass beyond which no stellar equilibrium is possible in a general-relativistic setting. To show this in the detailed manner as done for the nonrelativistic, and by analogy special-relativistic models, is a more complicated problem which requires the discussion of the Einstein field equations coupled with both the matter equations for the Fermi gases and the Maxwell equations of the electrostatic field in curved spacetime; cf. [9,10,14]. We plan to do this in a future work. Here we are content with the remark that the key argument in our derivation of (66) is the behavior of the atmospheric densities at spatial infinity, and in an asymptotically flat spacetime this is the region where the general-relativistic equations are expected to go over into the nonrelativistic equations of Newtonian physics—hence the independence of $\hbar$ and c, and our conjecture that (66) is truly universally valid for *failed white dwarfs* which never ignited, and assumed to consist of electrons and protons.

More realistic models of ground states of failed white dwarfs and white dwarfs (black dwarfs) require other compositions of particles, not just electrons and protons, and this will, of course, change the bounds on the excess charge in terms of nuclear-chemical composition. It is an interesting question whether they will be independent of $\hbar$ and c, all the way up to Chandrasekhar's critical mass, $\propto (N_{\mathrm{e}}/N_n)^2(\hbar c/G)^{\frac{3}{2}}/m_{\mathrm{p}}^2$, where N_n is the number of nucleons in the star.

E. Other pressure-density relations

It is clear from our discussion so far that the key to (66) is the existence of infinitely extended density solutions which vanish rapidly at infinity such that (analogs of) (69), respectively, (72) hold, where the $\nu_f^{2/3}$ at the left-hand side is replaced by some nonlinear function of ν_f obtained from any pressure-density law which leads to solutions which are integrable. This is a large class of models which all lead to the same surcharge bounds for a failed star, which then is not necessarily considered to be in the white dwarf stage already.

There are also many laws which do not produce solutions with finite N_{p} and N_{e}; e.g., polytropic laws with index $n > 5$. Also the isothermal pressure law with finite temperature will *not* lead to integrable density functions. Yet there is an analog of (66); see Appendix B.

V. DETERMINING N_{p} AND N_{e} OF NONEXTREMAL SOLUTIONS

Having numerically computed a solution pair for the nonlinear 5/3 model or the special-relativistic model, the number of protons N_{p} and electrons N_{e} of the solution can, of course, be obtained by integrating $4\pi r^2\nu_{\mathrm{p}}(r)$ over r from 0 to r_{p}, and $4\pi r^2\nu_{\mathrm{e}}(r)$ over r from 0 to r_e. However, there is a simpler way to get to these numbers directly after integrating the differential equations, in the manner done earlier for the 6/3 model.

Thus, for the 5/3 model, we obtain $(N_{\mathrm{p}}, N_{\mathrm{e}})$ uniquely in terms of the zeros of the densities and the derivatives at the zeros by simply replacing $\upsilon_f'(\rho)$ by $\frac{5}{4}(\upsilon_f^{2/3})'(\rho)$ in Eq. (53), $_f = _{\mathrm{p}}$ or $_e$, where we have tacitly switched to the dimensionless variables of the 6/3 model. So when solving the

system of equations for $(\upsilon_{\rm p}, \upsilon_{\rm e})$ as an initial value problem with prescribed central densities and vanishing central radial derivatives, all one needs to compute are the zeros of the densities and their left derivatives at the zeros. This reduces the computational effort.

In an analogous manner one can compute $(N_{\rm p}, N_{\rm e})$ uniquely in terms of the zeros of the densities and the derivatives at the zeros for the special-relativistic Chandrasekhar-type setup.

VI. COMPARISON OF THE MODELS

Having a complete set of solution formulas for the 6/3 model one can generate figures which illustrate the findings, and compare these with the results of numerical evaluations of the 5/3 model and also with the Chandrasekhar-type special-relativistic model. As emphasized earlier, the 6/3 model serves also as a test case for the numerical algorithm, which has to reproduce the exact solutions to the degree of accuracy demanded.

There are two compromises to be made, though.

Namely, the fantastically tiny ratios of the gravitational to electrical coupling constants between electron and proton are definitely a numerical problem, but also the small mass ratio $m_{\rm e}/m_{\rm p} \approx 1/1836$ is a source of trouble. Both these small numbers taken together make it sheer impossible to produce any useful graphs at all.

For instance, let us try to resolve the interval of the allowed values of the ratio $N_{\rm e}/N_{\rm p}$. From (66) we see that $N_{\rm e}/N_{\rm p}$ varies between about $1 - 8.1 \times 10^{-37}$ and about $1 + 4.4 \times 10^{-40}$. This can be ameliorated a little bit by centering the $N_{\rm e}/N_{\rm p}$ axis at 1 and scaling up the units by a factor of $(1/4.4) \times 10^{40}$. Incidentally, the construction just described is equivalent to using a rescaled $\ln(N_{\rm e}/N_{\rm p})$ as the base variable. This has eliminated the problems with the tininess of the coupling constant ratios. However, the small mass ratio $m_{\rm e}/m_{\rm p}$ still poses a hurdle, for the negatively charged stars will occupy about 1 positive unit in the allowed interval of $\ln(N_{\rm e}/N_{\rm p})$ and the positive stars 1836 negative units. To resolve such a lopsided asymmetry graphically without introducing otherwise obscuring transformations is impossible. We therefore decided to work with the SciFi value $m_{\rm e}/m_{\rm p} = 1/10$.

Furthermore, while the discussion just given shows that for certain questions the tiny ratios of the coupling constants can be dealt with and only the small ratio of $m_{\rm e}/m_{\rm p}$ is a problem, when one wants to plot both electron and proton density functions in one panel, they will appear indistinguishable when attempted with the actual values of $Gm_{\rm p}^2/e^2$, etc. To illustrate this, we plot $N_{\rm e}/N_{\rm p}$ for the no-atmosphere solution of the 6/3 model, computed analytically with formula (63), versus $\log_{10}(Gm_{\rm p}^2/e^2)$ for the actual $m_{\rm e}/m_{\rm p} = 1/1836$ and for the SciFi value $m_{\rm e}/m_{\rm p} = 1/10$; see Fig. 1.

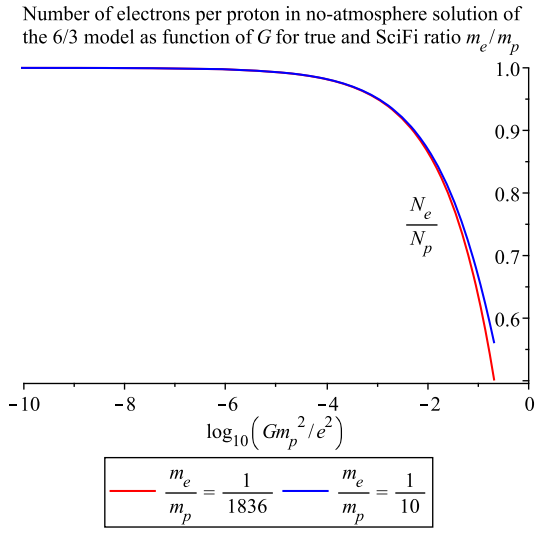

FIG. 1. Shown is $N_{\rm e}/N_{\rm p}$ computed analytically as a function of $Gm_{\rm p}^2/e^2$ for the no-atmosphere solutions of the 6/3 model [formula (63)], once for the physical ratio $m_{\rm e}/m_{\rm p} = 1/1836$ and once for the science fiction ratio $m_{\rm e}/m_{\rm p} = 1/10$. Note that the physical value of $\log_{10}(Gm_{\rm p}^2/e^2)$ is located on the horizontal axis at ≈ -36.

Figure 1 shows that to qualitatively visualize the difference of the particle densities one needs to replace the actual value of $Gm_{\rm p}^2/e^2$ with science fiction values. In this vein, in the following we illustrate our findings for the SciFi values $Gm_{\rm p}^2/e^2 = 1/2$ and $m_{\rm e}/m_{\rm p} = 1/10 (= \varepsilon)$; for consistency, therefore, $Gm_{\rm p}m_{\rm e}/e^2 = \varepsilon/2$ and $Gm_{\rm e}^2/e^2 = \varepsilon^2/2$. The other physical constant, $\alpha_s = 1/137.036$.

We have tested our numerical algorithm (essentially a Runge-Kutta 45 scheme) by comparing the plots of the exact solution formulas with those produced by numerically solving the 6/3 model for the science fiction values of the constants; see Fig. 2 for a representative error plot.

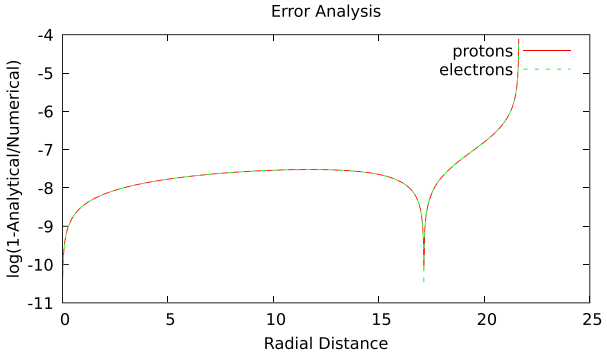

FIG. 2. Shown is $\log_{10}|1 - \upsilon_f^{(a)}/\upsilon_f^{(n)}|$, where $\upsilon_f^{(a)}$ and $\upsilon_f^{(n)}$ denote the analytically and the numerically computed scaled densities for the no-atmosphere solutions of the 6/3 model, for electrons ($_f =_e$) and for protons ($_f =_{\rm p}$), as functions of the scaled radial distance ρ (for the units, see Sec. III, first paragraph), though for science fiction values $Gm_{\rm p}^2/e^2 = 1/2$ and $m_{\rm e}/m_{\rm p} = 1/10$.

Figure 2 demonstrates that the numerical algorithm approximates the exact analytical solutions with a relative error of less than 10^{-4}, and even less than 10^{-7} over most of the bulk region. This indicates that our Runge-Kutta 45 scheme also computes the solutions to the physically more realistic models accurately, where we do not have analytical solutions to compare.

We next graph the bulk radius ρ_0 which the star adapts in response to N_e/N_p, in Fig. 3 for the 6/3 model (analytical) and in Fig. 4 for the 5/3 model (numerical).

In the 6/3 model, thanks to the amplitude scaling invariance of its linear set of structure equations, there is only one equal-N_p curve representing all solutions; this is, of course, a degenerate situation. Each point on the curve corresponds to a whole scaling family of solution pairs (ν_p, ν_e) with the same ratio N_e/N_p. The nonlinear set of structure equations of the 5/3 model breaks the amplitude scaling invariance of the 6/3 model. To each N_p there now corresponds a separate curve representing solution pairs. On each such curve, every point belongs to a unique solution with the given N_p and an associated N_e, which varies along the curve. It is, however, too time consuming to push all the way to the extreme solutions, which is noticeable by comparing Figs. 3 and 4.

Next we show the sets of equal-N_p curves in the plane of radii, in Fig. 5 for the analytical 6/3 solutions, and in Fig. 6 for the numerical 5/3 solutions. The no-atmosphere solutions are situated on the diagonal in these two diagrams.

We next compare the numerically computed particle density functions of the 5/3 model with the analytically computed ones of its 6/3 approximation, with the same SciFi values given to the physical constants. The central

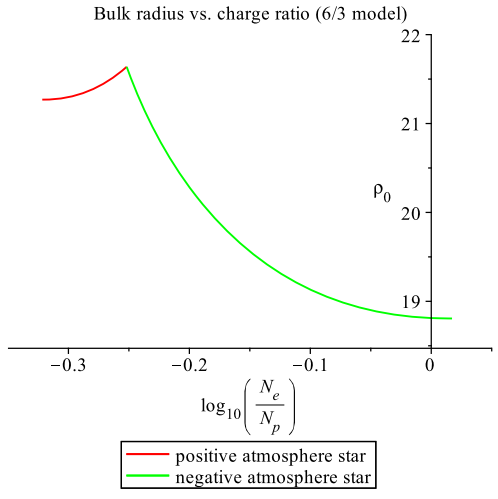

FIG. 3. Shown is the bulk radius ρ_0 (units in Sec. III, first paragraph) vs $\log_{10}(N_e/N_p)$, computed analytically, for the full range of allowed values, though for SciFi values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. Most solutions carry a positive surcharge, but only a small fraction of them has a positive atmosphere. This strong twofold asymmetry is caused by $m_e/m_p \ll 1$.

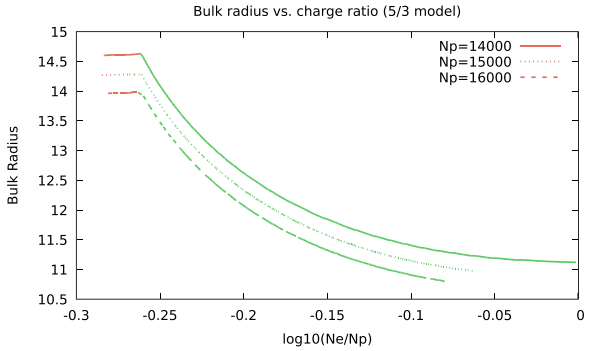

FIG. 4. Shown is the bulk radius ρ_0 (units in Sec. III, first paragraph) vs $\log_{10}(N_e/N_p)$, for an (not the full) interval of allowed N_e/N_p values, computed numerically, for three different values of N_p. Along each of the curves N_p is constant. We use science fiction values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$.

proton bulk density in the 5/3 model is the same in all examples. In the comparisons of 5/3 with pertinent 6/3 densities, the two solutions have the same N_p.

We begin with the distinguished pair of solutions consisting of the densities of a star without atmosphere, when both $\upsilon_p(\rho)$ and $\upsilon_e(\rho)$ vanish at the same dimensionless bulk radius ρ_0. We show the density functions of both the 5/3 and the 6/3 models; see Fig. 7. The no-atmosphere solutions in the two models behave qualitatively similar; however, note the difference in the scales. The no-atmosphere solutions of the 6/3 model with equal proton number N_p have a much more spread-out bulk than those of the 5/3

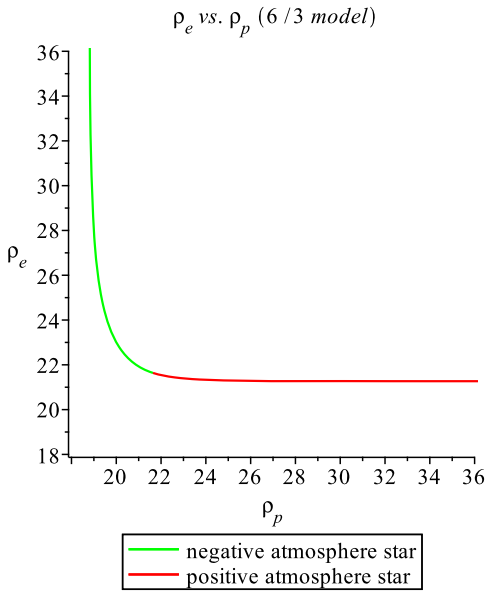

FIG. 5. Shown is the radius ρ_e of the electron density vs the radius ρ_p of the proton density (for the units, see Sec. III, first paragraph), computed analytically, for science fiction values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. Any point on the curve is a scaling family of solution pairs with fixed ratio N_e/N_p.

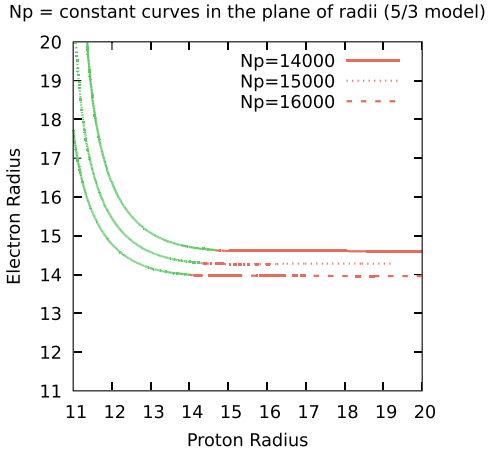

FIG. 6. Shown is the radius ρ_e of the electron density vs the radius ρ_p of the proton density (for the units, see Sec. III, first paragraph), computed numerically, for science fiction values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. Each point on a given curve represents a unique solution pair with the same N_p.

model, and the central densities are much smaller in the 6/3 model than in the 5/3 model.

In the 6/3 model a star without an atmosphere has $\rho_0 = \pi/\kappa_t$, as we discussed earlier, and both densities then are scaled $n = 1$ polytropes. For the proper 5/3 model it can also be shown that both densities are scaled polytropes, though for $n = 3/2$, of course.

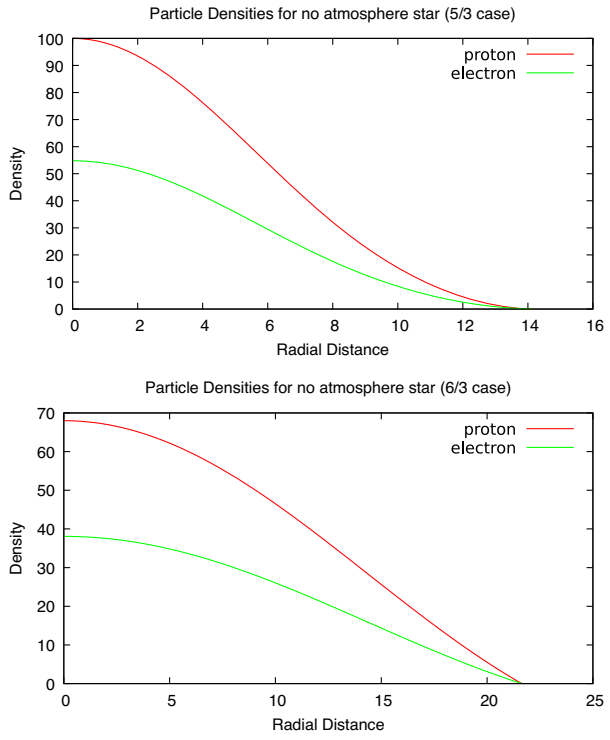

FIG. 7. Shown are the density functions $\upsilon_p(\rho)$ and $\upsilon_e(\rho)$ (units in Sec. III, first paragraph), computed numerically for the 5/3 model and analytically for the 6/3 model, of a star without atmosphere, for SciFi values $Gm_p^2/e^2 = \frac{1}{2}$ and $m_e/m_p = \frac{1}{10}$.

Note that in the 6/3 model one has $\upsilon_e(\rho) = \lambda\upsilon_p(\rho)$ when $\rho_0 = \pi/\kappa_t$, with λ given by the right-hand side of Eq. (22). Let us instead make the ansatz $\upsilon_e(\rho) = \lambda\upsilon_p(\rho)$ in Eqs. (12) and (13) of the 6/3 model. We then obtain two equations for $\upsilon_p(\rho)$ (say), and this generally overdetermines the problem. Their compatibility condition is the quadratic problem $a\lambda^2 + b\lambda + c = 0$, with $a = (1 + Gm_em_p/e^2)/\varepsilon > 0$, $b = -(1 - Gm_p^2/e^2)/\varepsilon + (1 - Gm_e^2/e^2) < 0$, and $c = -(1 + Gm_em_p/e^2) < 0$. The "quadratic formula" yields two real solutions,

$$\lambda_\pm = -\frac{b}{2a}\left(1 \pm \sqrt{1 - 4\frac{ac}{b^2}}\right), \tag{76}$$

one of which is positive and the other one negative. Now a particle density cannot be negative, so $\lambda = \lambda_+$, and this is *precisely* the right-hand side of Eq. (22).

Similarly one can insert the ansatz $\nu_e(r) = \lambda\nu_p(r)$ also into the equations of the 5/3 model, i.e., Eqs. (10) and (11), and now the compatibility condition is the vanishing of the degree-5 polynomial $a\eta^5 + b_e\eta^3 + b_p\eta^2 + d = 0$, where $\eta := \lambda^{1/3}$, and $a = (1 + Gm_em_p/e^2)/\varepsilon > 0$, $b_e = (1 - Gm_e^2/e^2) > 0$, $b_p = -(1 - Gm_p^2/e^2)/\varepsilon < 0$, and $c = -(1 + Gm_em_p/e^2) < 0$. There generally does not exist a solution in closed form, but from the signs of the coefficients in this polynomial one can deduce right away that there exists a unique positive solution η_+, say, very close to 1, and for $\lambda = \eta_+^3$ both (10) and (11) reduce to the equation (cf. [22])

$$\varepsilon\zeta\frac{1}{r^2}(r^2\nu_p^{\frac{2}{3}\prime}(r))' = \left[1 - \frac{Gm_p^2}{e^2} - \lambda\left(1 + \frac{Gm_pm_e}{e^2}\right)\right]\nu_p(r), \tag{77}$$

which is equivalent (not identical) to the polytropic equation of index $n = 3/2$, Eq. (C4). Indeed, setting $\nu_p^{\frac{2}{3}}(r) = \nu_p^{\frac{2}{3}}(0)\theta(\xi)$ and scaling $r = C\xi$ appropriately converts (77) into the standardized format $-\frac{1}{\xi^2}(\xi^2\theta'(\xi))' = \theta_+^{3/2}(\xi)$ (cf. [1,2]), and thus the no-atmosphere densities are obtained by rescaling the standardized $n = 3/2$ polytrope.

We remark that inserting the dimensionless bulk radius of a star without an atmosphere, $\rho_0 = \pi/\kappa_t$, into our formula for the ρ_0-dependent number of electrons per proton in the 6/3 model yields

$$\frac{N_e}{N_p} = \frac{\lambda(1 - \frac{Gm_p^2}{e^2}) + \varepsilon(1 + \frac{Gm_pm_e}{e^2})}{\lambda(1 + \frac{Gm_pm_e}{e^2}) + \varepsilon(1 - \frac{Gm_e^2}{e^2})}, \tag{78}$$

with $\lambda = B_e^t/B_p^t$ given by (22). Alternatively, knowing that $\upsilon_e = \lambda\upsilon_p$ in this case, Eq. (78) follows directly from multiplying Eq. (12) by $\lambda 4\pi\rho^2$ and Eq. (13) by $\varepsilon 4\pi\rho^2$,

then integrating over ρ, and then subtracting the first result from the second, followed by simple algebra.

Similarly the number of electrons per proton of the no-atmosphere solution of a failed white dwarf star as computed with the physical 5/3 model is obtained from Eqs. (10) and (11). With $\lambda = \eta_+^3$ one finds [22]

$$\frac{N_e}{N_p} = \frac{\lambda^{2/3}(1 - \frac{Gm_p^2}{e^2}) + \varepsilon(1 + \frac{Gm_p m_e}{e^2})}{\lambda^{2/3}(1 + \frac{Gm_p m_e}{e^2}) + \varepsilon(1 - \frac{Gm_e^2}{e^2})}. \tag{79}$$

Finally we turn to the extremely surcharged stars, whose densities are shown in Figs. 8 and 9.

It is manifest that the extreme solutions in the 6/3 model also behave qualitatively similar to those in the 5/3 model. The extreme solutions of the 6/3 model have a much more spread-out bulk than those of the 5/3 model with equal proton number N_p, but their central densities are much smaller than those in the 5/3 model. Interestingly, the ratio of the two central densities in the 6/3 model seems to roughly equal the one in the 5/3 model.

In all density function plots the central proton density is larger than the central electron density. In Fig. 10 we display the ratio of the central proton density over the central electron density as a function of the ratio N_e/N_p, or

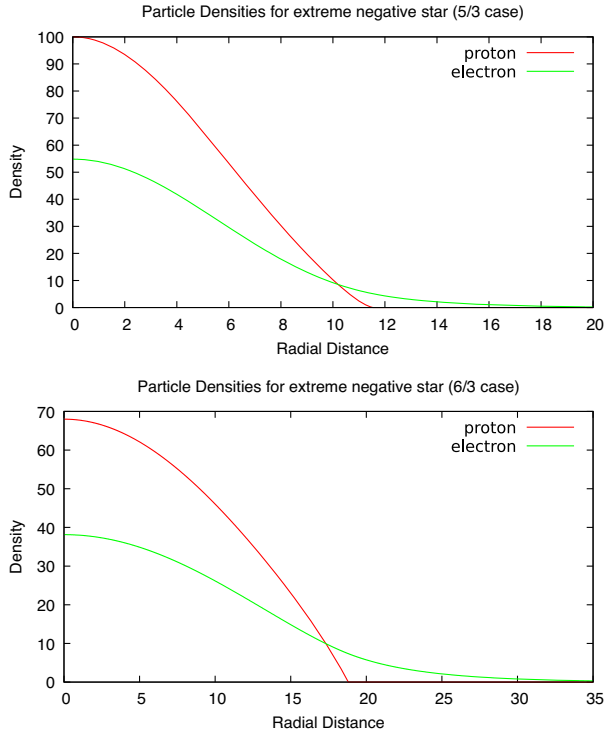

FIG. 8. Shown are the density functions $\upsilon_p(\rho)$ and $\upsilon_e(\rho)$ (for the units, see Sec. III, first paragraph) of the upper extreme ratio $N_e/N_p = (1 + \frac{Gm_p m_e}{e^2})/(1 - \frac{Gm_e^2}{e^2})$, with science fiction values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. The top graph is computed numerically, the bottom graph analytically.

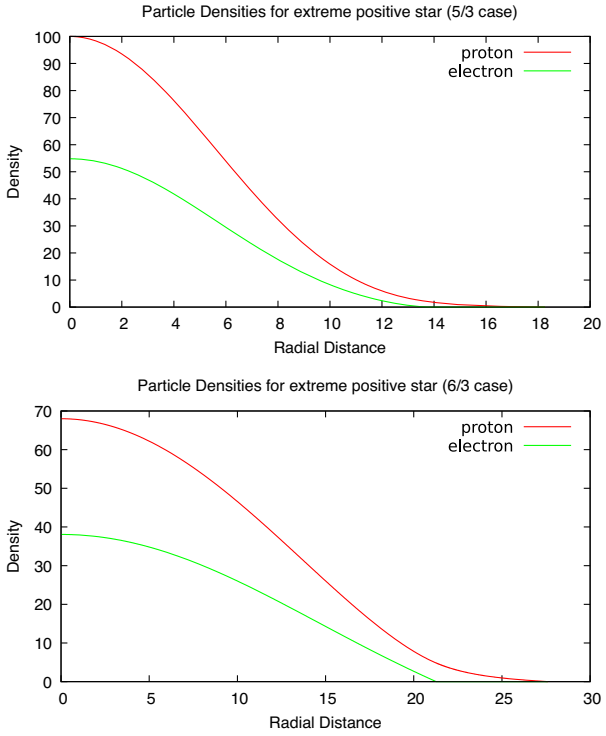

FIG. 9. Shown are the density functions $\upsilon_p(\rho)$ and $\upsilon_e(\rho)$ (for the units, see Sec. III, first paragraph) of the lower extreme ratio $N_e/N_p = (1 - \frac{Gm_p^2}{e^2})/(1 + \frac{Gm_p m_e}{e^2})$, with science fiction values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. The top graph is computed numerically, the bottom graph analytically.

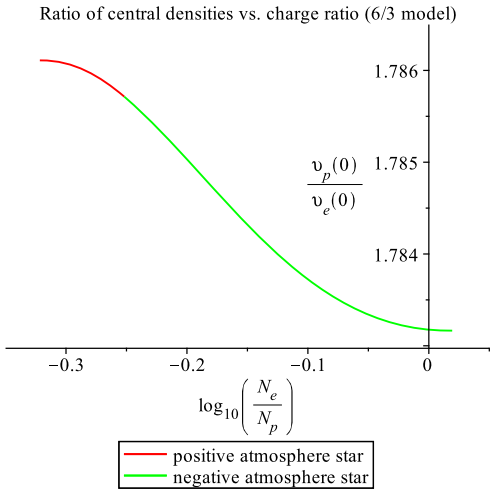

FIG. 10. Shown is the ratio of the central proton density over central electron density vs $\log_{10}$, computed analytically, of the number of electrons per proton, N_e/N_p, covering the full range of allowed N_e/N_p ratios, though for SciFi values $Gm_p^2/e^2 = 1/2$ and $m_e/m_p = 1/10$. Note that the central proton density is always larger than the central electron density. Also this asymmetry is caused by the mass ratio $m_e/m_p < 1$.

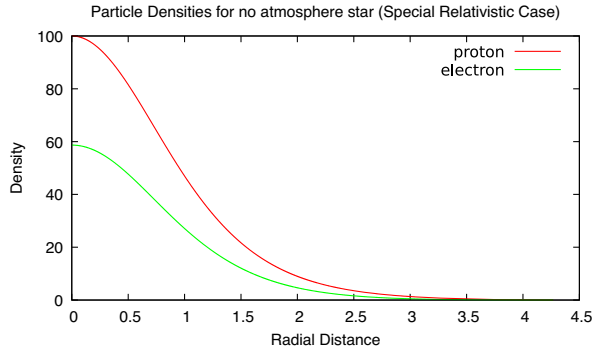

FIG. 11. Shown are the density functions $\upsilon_{\mathrm{p}}(\rho)$ and $\upsilon_{\mathrm{e}}(\rho)$ (units in Sec. III, first paragraph), computed numerically, of a star without atmosphere, for SciFi values $Gm_{\mathrm{p}}^2/e^2 = 1/2$ and $m_{\mathrm{e}}/m_{\mathrm{p}} = 1/10$.

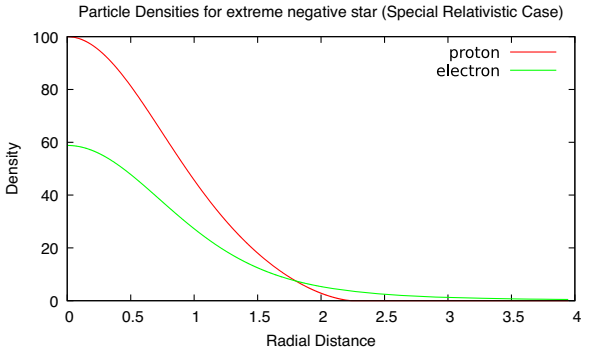

FIG. 13. Shown are the density functions $\upsilon_{\mathrm{p}}(\rho)$ and $\upsilon_{\mathrm{e}}(\rho)$ (for the units, see Sec. III, first paragraph), computed numerically, of the upper extreme ratio $N_{\mathrm{e}}/N_{\mathrm{p}} = (1 + \frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2})/(1 - \frac{Gm_{\mathrm{e}}^2}{e^2})$, with science fiction values $Gm_{\mathrm{p}}^2/e^2 = 1/2$ and $m_{\mathrm{e}}/m_{\mathrm{p}} = 1/10$.

rather its decadic logarithm, in the $6/3$ model. Note the strong asymmetry caused by $m_{\mathrm{e}}/m_{\mathrm{p}} \ll 1$. In the $5/3$ model there will be such a curve for each value of N_{p} separately; all such curves coincide in the $6/3$ model.

Overall our algorithm worked sufficiently accurately so that we decided to trust it also for the special-relativistic model of the Chandrasekhar type. Figures 11, 12, 13, and 14 are the Chandrasekhar-type special relativistic model equivalents of Figs. 7, 8, and 9; see Sec. IV for a theoretical discussion and the presentation of the equations of this model. The figures indicate that the solutions of this model behave qualitatively in the same way as the $6/3$ and $5/3$ models, although we have done no rigorous analysis to estimate the similarity. It is difficult to see a difference between the plots with no atmosphere and with an extremely positive atmosphere when viewing the entire density curves. We have therefore included an extra plot (Fig. 15) of the positive atmospheric model at a smaller scale so that one can see that the behavior is indeed the same qualitatively as in the other models.

Here we can make some observations about the numerics of this model. As in the case of comparing the $6/3$ to the $5/3$ model, the special relativistic model has bulk radius smaller than the $5/3$ model. This makes sense as the pressure law interpolates between $5/3$ and $4/3$.

Another characteristic of these solutions which cannot easily be shown in figures is that the decay rate of the extreme solutions is smaller than those of the $5/3$ model. Again, this makes sense when one considers the exact atmospheric solution for the $5/3$ model presented in Appendix B.3. An ultrarelativistic $4/3$ model would have an exact atmospheric solution with a (nonintegrable) decay rate of $\eta = -3$ compared to $\eta = -6$ for the $5/3$ model.

A final interesting observation is that it appears that the possible masses for the special-relativistic model exists in a small band; that is, one cannot scale the solutions and increase the total number of protons and electrons as in the $6/3$ or $5/3$ models. Figure 12 shows that as the central densities of the model are increased, the radius decreases. The result is that the total mass changes by a small amount, hardly at all.

Of course, these observations are only based on numerical results, and therefore need to be verified mathematically

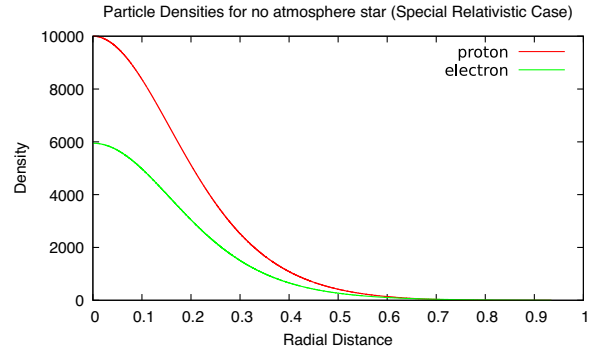

FIG. 12. Shown are the density functions $\upsilon_{\mathrm{p}}(\rho)$ and $\upsilon_{\mathrm{e}}(\rho)$ (units in Sec. III, first paragraph), computed numerically, of a star without atmosphere and higher central densities, for science fiction values $Gm_{\mathrm{p}}^2/e^2 = 1/2$ and $m_{\mathrm{e}}/m_{\mathrm{p}} = 1/10$.

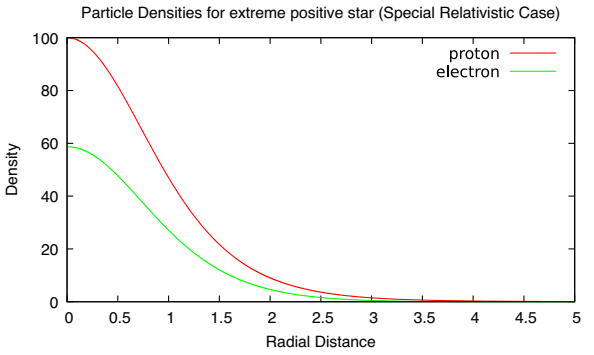

FIG. 14. Shown are the density functions $\upsilon_{\mathrm{p}}(\rho)$ and $\upsilon_{\mathrm{e}}(\rho)$ (units in Sec. III, first paragraph), computed numerically, of the lower extreme ratio $N_{\mathrm{e}}/N_{\mathrm{p}} = (1 - \frac{Gm_{\mathrm{p}}^2}{e^2})/(1 + \frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2})$, with science fiction values $Gm_{\mathrm{p}}^2/e^2 = 1/2$ and $m_{\mathrm{e}}/m_{\mathrm{p}} = 1/10$.

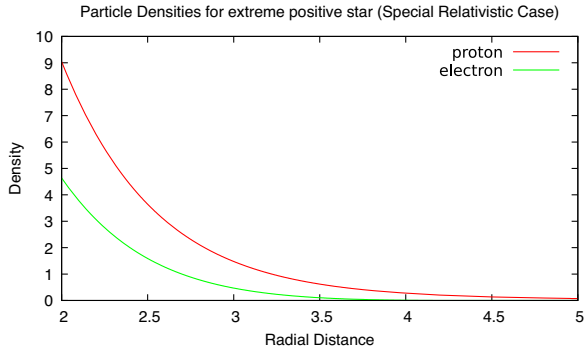

FIG. 15. Zoom-in of the density functions $\upsilon_p(\rho)$ and $\upsilon_e(\rho)$ of Fig. 14, revealing the positive atmosphere of the star.

to make any definitive statement. This will be the subject of our future work.

VII. CAN EXCESS CHARGE HAVE A NOTICEABLE EFFECT ON THE ORBITS OF A BINARY SYSTEM?

As an application of our surcharge bounds (66), consider the following scenario. Suppose a maximal negatively charged and a maximal positively charged failed white dwarf have formed a binary system of two equal mass components, each with mass M. The binary system is supposed to be sufficiently separated to vindicate the spherical approximation for their shapes. Moreover, the atmospheric densities, which are rapidly decaying to zero, will be treated as having an effectively finite radius compared to the separation distance. The maximal charge imbalance is tiny, true, but since the microscopic electric coupling constants are so much stronger than the gravitational ones, it is in principle conceivable that even a tiny surcharge could be influencing the dynamics in a significant way. So let us find out by doing a calculation.

Note that $N_e \approx N_p := N$ to high accuracy. From (66) we obtain for the Coulomb coupling coefficient of a maximal oppositely surcharged binary

$$-\frac{Gm_pm_e}{e^2}\frac{Gm_p^2}{e^2}N^2e^2 \approx -\frac{Gm_pm_e}{e^2}GM^2, \tag{80}$$

where we have used that the mass M of each binary component is $\approx m_pN$. Since GM^2 is the gravitational coupling coefficient between the two binaries, Eq. (80) reveals that the electrical attraction between the two binary components is still 10^{-40} times smaller than their gravitational attraction.

Thus astronomers can relax. The validity of the determination of the masses of binary components based on their orbital data with the help of the gravitational Kepler problem is not in question.

VIII. COSMIC CENSORSHIP

Formula (66) is equivalent to the two inequalities

$$(N_p - N_e)e^2 \le G(N_pm_p + N_em_e)m_p \tag{81}$$

and

$$(N_p - N_e)e^2 \ge -G(N_pm_p + N_em_e)m_e. \tag{82}$$

Noting that $(N_p - N_e)e = Q$ is the net charge of the star and $N_pm_p + N_em_e = M$ its mass, these yield the interval

$$\boxed{-\frac{GMm_e}{e} \le Q \le \frac{GMm_p}{e}} \tag{83}$$

for the total charge a star made of electrons and protons can carry. This interval is "universal" in the same sense as formula (66) is; recall Sec. IV. Furthermore, Ref. [24], the left inequality in (83) is also "universal" in a wider sense; namely it holds also in a Thomas-Fermi-Hartree model for the ground state of a star which consists of electrons, protons, and several species of heavier nuclei that are bosons (recall our discussion in the Introduction); the right inequality is possibly no longer true in the presence of bosons. This, of course, does not follow from our derivation here, in which nuclei heavier than protons are absent.

From (83) we can derive an important inequality for Q^2, as follows. Considering first the left inequality in (83), we multiply through with $-N_ee$, which yields $-N_eeQ \le GMN_em_e$. Considering next the right inequality in (83), we multiply through with N_pe, which yields $N_peQ \le GMN_pm_p$. Adding these then yields $(N_p - N_e)eQ \le GM(N_pm_p + N_em_e)$, viz.

$$\boxed{Q^2 \le GM^2}. \tag{84}$$

Inequality (84), here derived from a Thomas-Fermi model for the ground state of a star made of protons and electrons, is also valid for a Reissner-Weyl-Nordström black hole. A Reissner-Weyl-Nordström spacetime which violates (84) features a naked singularity, i.e., a singularity which is not hidden from an infinitely remote observer behind a closed event horizon. In Sec. IV D we explained why our bounds (66) are to be expected to be valid also when Newtonian gravity is replaced by Einsteinian gravity; hence a general-relativistic treatment of a two-species Thomas-Fermi model of a failed white dwarf should also obey the bounds (84) on the stellar charge. This implies that if the quantum mechanical stabilization was magically turned off, such a star could not collapse to a charged naked singularity but would turn into a charged black hole.

Thus we have arrived at an important result in support of Penrose's weak cosmic censorship hypothesis.

IX. CONCLUSIONS

In this paper we have presented the complete solution of an approximate model of a failed white dwarf star, here for simplicity assumed to consist of electrons and protons only, in which the polytropic power $5/3$ of the pressure-density relation, predicted by nonrelativistic quantum mechanics, is replaced with the nearby $6/3$, and which was introduced in [13]. Based on the availability of the elementary exact solutions of this model we were able to discuss the whole solution family thoroughly. The model captures the qualitative behavior of the solutions of the physical $5/3$ and special relativistic models correctly, and even gets the quantitative answer to the question of the maximal relative surcharge exactly right [see (66)]; this we have shown in Sec. IV (more on that below). As was demonstrated with the $5/3$ model, it can serve as a test case for computer algorithms which tackle physically more realistic many species models; see also the Appendix for a brief discussion of how the model generalizes to more than two species. The approximate model also can easily be incorporated in an introductory astrophysics course which covers the basic equations of stellar structure, in particular for white and brown dwarf stars.

In this vein we can compare the $6/3$ model of [13] with examples from statistical mechanics that come to mind. The most prominent ones, perhaps, are the two-dimensional Ising model [25] and the two-dimensional ice models [26,27], which have provided valuable qualitative insights into the behavior of the more realistic physical models that require heavy use of numerical methods.

The equations for two-species models corresponding to different polytropic laws than those discussed above can easily be written down, although the arguments presented here concerning the structure of solutions may not hold. For example, the reasoning behind the saturation of the N_{p}/N_e bounds does not hold for polytropic laws of index larger than $n = 3$, the ultrarelativistic white dwarf case. But the $n = 5$ case is exactly solvable and represents a border case of Lane-Emden equations, so it may be of some interest what can be said about the corresponding two species model. While not related to the structure of white dwarfs, the $n = 5$ polytrope does find uses in stellar dynamics and some fluid sphere solutions in general relativity. We have included in the Appendix an analytic solution of the two species model corresponding to the $n = 5$ polytrope. Of course, this is just the beginning of a description of the solutions of such a model.

Interesting by-products of our investigation are the $\hbar$- and c-independent bounds (66) on $N_{\mathrm{e}}/N_{\mathrm{p}}$. We have presented compelling arguments for why our bounds (66) are the correct bounds for a failed white dwarf star made of electrons and individual protons, not only nonrelativistically but also in the special- and general-relativistic theories, because one is far away from the Chandrasekhar mass. A more pedagogical account of these findings is presented in [22]. As we explained in Sec. VIII, our bounds support Penrose's weak cosmic censorship hypothesis.

After submitting this paper we started to investigate more realistic stellar ground state models that, inevitably, are no longer exactly solvable; cf. [24] for a nonrelativistic Thomas-Fermi-Hartree model of electrons, protons, and several species of nuclei that are bosons (such as α particles). The bounds on $N_{\mathrm{e}}/N_{\mathrm{p}}$ will then be replaced by bounds on the ratios N_z/N_{e}, where N_z is the number of nuclei in the star with z elementary charges. In regard to the question of the maximal and minimal electric charges on a star, we found that the lower bound expressed in (83) is valid also in the Thomas-Fermi-Hartree model; the upper bound in (83) is modified, though. Thus our negative surcharge bound obtained in this paper with a highly simplified model is very robust.

We also plan to investigate the special-relativistic formulation of the problem all the way up to near to the critical Chandrasekhar mass, taking a mixture of different nuclei species into account with a mix of special-relativistic Thomas-Fermi and Hartree type equations. We expect the left inequality in (83) to remain valid.

We also want to investigate the general-relativistic problem, with its effects on the critical mass; an interesting question is whether (84) will hold.

In the pursuit of more realism also the weak and strong nuclear forces should eventually be taken into account if one considers stellar ground states with masses close to the above-mentioned critical Chandrasekhar mass, respectively, the general-relativistic critical mass, for then the central densities exceed the threshold for inverse β decay, which will turn a certain percentage of electrons and protons (bound in the nuclei) into neutrons, thus changing the composition of the star and affecting the critical mass. Since inverse β decay preserves the total charge involved in the process, it should not affect the allowed surplus charge Q on a star.

So much on the excess charges of stellar ground states. We close this discussion by reminding the reader that the question of electrical surplus of charge on a star is mostly meaningful for the ground state. Real stars in the universe are estimated not yet to be in their ground state, and since finite temperature effects include the phenomenon of solar and/or stellar winds, real stars which constantly evaporate render the question of their electrical surcharge pointless.

Back to the exactly solvable model, very much of interest is to extend it, by including magnetism and rotation. This will complicate the problem considerably, for the spherical symmetry of the problem will be broken both by rotation, due to centrifugal effects (obviously), and by magnetism's anisotropy; cf. [28–32].

ACKNOWLEDGMENTS

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APPENDIX A: THE 6/3 MODEL FOR MORE THAN TWO SPECIES

The $6/3$ model can easily be generalized to an arbitrary number of fermion species without affecting its exact solubility. It is, of course, to be seen as a $5/3 \to 6/3$ approximation to a more-than-two species $5/3$ model, and such a model has the physical deficiency that all species are fermions, whereas both the primordial nucleosynthesis [21] and also nuclear fusion in stars [3] essentially produce effectively bosonic heavier nuclei, in particular ^{4}He (both primordial nucleosynthesis and stellar fusion), ^{12}C and ^{16}O (the latter predominantly only in stellar fusion). A multispecies fermion model is surely to be taken with some grain of salt. All the same, the usual local neutrality approximation, traditionally used in astrophysical works on stellar structure, throws all these differences out the window also, so that a multispecies fermionic model is presumably not worse.

Of course, the combinatorial complexity increases, and for more than four species the κ and $\varkappa$ eigenvalues can no longer be expressed in closed form, but even for three and four species, when one can, the closed form expressions are not very illuminating. Fortunately this is not necessary, since the fantastic tininess of the ratios of the coupling constants of the various species allow a very efficient evaluation with approximate expressions which are more accurate than any typical numerical approximation on a machine.

Instead of presenting here the generalization to an arbitrary number of fermion species, we present the three-species version, pretending that because of some unlikely fluke the primordial nucleosynthesis [21] has, in some corner of the universe, produced a mix of only protons and ^{3}He, a spin-$1/2$ fermion with two elementary charges known as *helion*, which together with the protons and the electrons now constitutes our failed white dwarf, if the total mass remains below $\approx$80 Jupiter masses. One may also contemplate modeling a low mass white dwarf (no longer failed) if the mass is a bit above the threshold for the onset of nuclear fusion, but not too high so that no fusion into heavier nuclei than helium happened; one has to pretend that by some even more unlikely statistical fluke, also in the star only ^{3}He is produced. (Obviously, this narrative is not meant to be taken literally.)

Choosing $_h$ as subscript for ^{3}He we now have the following coupled system of three linear second-order differential equations for the dimensionless density functions υ_h, υ_p, and υ_e in the bulk region,

$$-\varepsilon_h \varsigma \frac{1}{\rho^2}(\rho^2 \upsilon_\mathrm{h}'(\rho))' = -\left(4 - \frac{G m_\mathrm{h}^2}{e^2}\right)\upsilon_\mathrm{h}(\rho) - \left(2 - \frac{G m_\mathrm{h} m_\mathrm{p}}{e^2}\right)\upsilon_\mathrm{p}(\rho) + \left(2 + \frac{G m_\mathrm{h} m_\mathrm{e}}{e^2}\right)\upsilon_\mathrm{e}(\rho), \tag{A1}$$

$$-\varepsilon\varsigma \frac{1}{\rho^2}(\rho^2 \upsilon_\mathrm{p}'(\rho))' = -\left(2 - \frac{G m_\mathrm{h} m_\mathrm{p}}{e^2}\right)\upsilon_\mathrm{h}(\rho) - \left(1 - \frac{G m_\mathrm{p}^2}{e^2}\right)\upsilon_\mathrm{p}(\rho) + \left(1 + \frac{G m_\mathrm{p} m_\mathrm{e}}{e^2}\right)\upsilon_\mathrm{e}(\rho), \tag{A2}$$

$$-\varsigma \frac{1}{\rho^2}(\rho^2 \upsilon_\mathrm{e}'(\rho))' = \left(2 + \frac{G m_\mathrm{h} m_\mathrm{e}}{e^2}\right)\upsilon_\mathrm{h}(\rho) + \left(1 + \frac{G m_\mathrm{p} m_\mathrm{e}}{e^2}\right)\upsilon_\mathrm{p}(\rho) - \left(1 - \frac{G m_\mathrm{e}^2}{e^2}\right)\upsilon_\mathrm{e}(\rho), \tag{A3}$$

valid where $\upsilon_\mathrm{p}(\rho) > 0$, $\upsilon_\mathrm{e}(\rho) > 0$, and $\upsilon_\mathrm{h}(\rho) > 0$. Here, $m_\mathrm{h} \approx 3m_\mathrm{p}$ is the mass of ^{3}He and $\varepsilon_h := m_\mathrm{e}/m_\mathrm{h} \approx \varepsilon/3$.

Sandwiched between the three-species bulk and the single-species atmosphere regions there is now an intermediate region where exactly one of the densities vanishes and two of the densities are nonzero. The density functions in the intermediate region satisfy precisely the bulk equations of the two-species model, except perhaps that we need to allow for the possibility that it is not a proton-electron system now but a ^{3}He-electron system, even though astrophysical stellar models suggest that the helium zone resides inside the hydrogen zone.

Different from the discussion of the bulk region of the two-species model, though, the two-species intermediate zone does not require using only $\sin(\kappa_l\rho)/\rho$ and $\sinh(\kappa_h\rho)/\rho$ linear combinations, because one stays away from the center of the star.

As for the three-species bulk region, it is clear that the ansatz $\upsilon_f(\rho) = A_f \frac{\exp(\kappa\rho)}{\rho}$ will once again lead to an eigenvalue problem for κ, this time it is a cubic equation in κ^2. Initial conditions at $\rho = 0$ are posed, namely the vanishing of the radial derivatives, while the central densities are to be chosen such as to satisfy the constraints that the particle densities integrate to N_h, N_p, and N_e; recall, N_p is the number of individual protons, not bound in nuclei with $Z > 1$ elementary charges. At the interface between bulk and intermediate regions, the two nonvanishing densities of the intermediate region have to go over continuously differentiable into the bulk region, and at the intermediary-atmosphere interface, the earlier continuous differentiability conditions for the atmospheric density are imposed.

It is clear that the combinatorial possibilities are already daunting for this three-species setup, but it is also clear that it can be worked out completely, and analogously one can proceed with an arbitrary number of fermion species, in principle at least.

APPENDIX B: SOME EXACT SOLUTIONS TO RELATED MODELS

1. An elementary no-atmosphere solution for the two-species 6/5 model

As has been discussed, the multispecies 6/3 model is a generalization of the Lane-Emden equation of index $n = 1$, and its no-atmosphere solutions are obtained by rescaling the elementary solution of the Lane-Emden equation of index $n = 1$. Aside from the less interesting index $n = 0$ case, there is also an elementary no-atmosphere solution to the index $n = 5$ case, which corresponds to a polytropic pressure law with power $\gamma = 6/5$. Following the procedure in Sec. III, we can derive the two-species equivalent of (10) and (11) for the index $n = 5$ case:

$$-\varepsilon\tau\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}v_{\mathrm{p}}^{1/5}(\rho)\right) = -\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)v_{\mathrm{p}}(\rho) + \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)v_{\mathrm{e}}(\rho), \tag{B1}$$

$$-\tau\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}v_{\mathrm{e}}^{1/5}(\rho)\right) = \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)v_{\mathrm{p}}(\rho) - \left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)v_{\mathrm{e}}(\rho); \tag{B2}$$

here, $\tau = \frac{\hbar c}{e^2}\frac{\pi^{1/3}3^{5/3}}{50}$, and these equations are valid where both densities are positive. Let $\theta_f(\rho) = v_f^{1/5}(\rho)$ on the set where v_f is positive. Then these equations become

$$-\varepsilon\tau\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\theta_{\mathrm{p}}(\rho)\right) = -\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)\theta_{\mathrm{p}}^5(\rho) + \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\theta_e^5(\rho), \tag{B3}$$

$$-\tau\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\theta_e(\rho)\right) = \left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\theta_{\mathrm{p}}^5(\rho) - \left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)\theta_e^5(\rho). \tag{B4}$$

Let us try $\theta_f(\rho) = \alpha_f(1+\kappa_f\frac{\rho^2}{3})^{-1/2}$, adapting the Lane-Emden index $n = 5$ solution similarly to how we adapted the Lane-Emden index $n = 1$ solution for the $\gamma = 6/3$ case. Then this system reduces to

$$\varepsilon\tau\alpha_{\mathrm{p}}\kappa_{\mathrm{p}}\left(1+\kappa_{\mathrm{p}}\frac{\rho^2}{3}\right)^{-5/2} = -\alpha_{\mathrm{p}}\left(1-\frac{Gm_{\mathrm{p}}^2}{e^2}\right)\left(1+\kappa_{\mathrm{p}}\frac{\rho^2}{3}\right)^{-5/2} + \alpha_e\left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\left(1+\kappa_e\frac{\rho^2}{3}\right)^{-5/2}, \tag{B5}$$

$$\tau\alpha_e\kappa_e\left(1+\kappa_e\frac{\rho^2}{3}\right)^{-5/2} = \alpha_{\mathrm{p}}\left(1+\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}\right)\left(1+\kappa_{\mathrm{p}}\frac{\rho^2}{3}\right)^{-5/2} - \alpha_e\left(1-\frac{Gm_{\mathrm{e}}^2}{e^2}\right)\left(1+\kappa_e\frac{\rho^2}{3}\right)^{-5/2}. \tag{B6}$$

To have equality, we then must take $\kappa_{\mathrm{p}} = \kappa_e = \kappa$. So as in the 6/3 case, we obtain a matrix problem:

$$\begin{pmatrix} 1-\frac{Gm_{\mathrm{p}}^2}{e^2}+\kappa\varepsilon\tau; & -1-\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2} \\ -1-\frac{Gm_{\mathrm{p}}m_{\mathrm{e}}}{e^2}; & 1-\frac{Gm_{\mathrm{e}}^2}{e^2}+\kappa\tau \end{pmatrix}\begin{pmatrix}\alpha_{\mathrm{p}}\\ \alpha_e\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{B7}$$

Again, the determinant must be zero, and we obtain the following quadratic in κ, viz.: $a\kappa^2 + b\kappa + c = 0$, with $a = \varepsilon\tau^2 > 0$, $b = \tau(1+\varepsilon - G(\varepsilon m_{\mathrm{e}}^2 + m_{\mathrm{p}}^2)/e^2) > 0$, and $c = -G(m_{\mathrm{e}}+m_{\mathrm{p}})^2/e^2 < 0$. We find $\kappa_+ \approx -1837$ and $\kappa_- \approx 3.2329\times10^{-47}$. We cannot use κ_+ since $\theta_f(\rho)$ would have a singularity at $\sqrt{3/(-\kappa_{\mathrm{p}})}$, and so we let $\theta_f(\rho) = \alpha_f(1+\kappa_-\frac{\rho^2}{3})^{-1/2}$. The matrix (B7) also gives us that

$$\alpha_{\mathrm{p}} = \frac{1+G\frac{m_e m_{\mathrm{p}}}{e^2}}{1-\frac{Gm_{\mathrm{p}}^2}{e^2}+\kappa\varepsilon\tau}\alpha_e. \tag{B8}$$

We then obtain a one-parameter family of solutions to (B5) and (B6) such that θ_{p} and θ_e are in proportion as given by (B8) and have unbounded support. The same can therefore be said about v_{p} and v_{e}, solutions to (B1) and (B2).

2. An elementary solution for the two-species isothermal model

One of the earliest self-gravitating models, together with Homer Lane's polytropes, was Zöllner's isothermal self-gravitating ideal classical gas ball model. Its basic equations were later studied much more thoroughly by Emden [1] and are nowadays named in his honor. We recall that the pressure-density relation of the isothermal ideal classical gas reads $p = k_B T\nu$, where k_B is Boltzmann's constant.

Treating both electrons and protons as isothermal classical perfect gases, with equal temperature $T > 0$, yields the following system of nonlinear second-order differential equations for the density functions $\nu_{\rm p}$ and $\nu_{\rm e}$, valid wherever both $\nu_{\rm p}(r) > 0$ and $\nu_{\rm e}(r) > 0$:

$$-\varpi\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\ln\nu_{\rm p}(r)\right) = -\left(1-\frac{Gm_{\rm p}^2}{e^2}\right)\nu_{\rm p}(r)+\left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)\nu_{\rm e}(r), \tag{B9}$$

$$-\varpi\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\ln\nu_{\rm e}(r)\right) = \left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)\nu_{\rm p}(r)-\left(1-\frac{Gm_{\rm e}^2}{e^2}\right)\nu_{\rm e}(r); \tag{B10}$$

here, $\varpi := k_B T/4\pi e^2$. As the single-species Emden equation for the isothermal self-gravitating classical gas ball, also the system (B9) and (B10) is not generally solvable in closed form. However, following Zöllner's treatment of the single-species model, we can find one elementary solution to this two-species model by making the Ansatz $\nu_f(r) = 2\varpi A_f/r^2$ with $A_f > 0$, and $_f$ standing for either $_{\rm p}$ or $_e$, as before. This Ansatz turns (B9) and (B10) into

$$1 = -\left(1-\frac{Gm_{\rm p}^2}{e^2}\right)A_{\rm p}+\left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)A_e, \tag{B11}$$

$$1 = \left(1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\right)A_{\rm p}-\left(1-\frac{Gm_{\rm e}^2}{e^2}\right)A_e. \tag{B12}$$

Thus

$$\begin{pmatrix}A_{\rm p}\\ A_{\rm e}\end{pmatrix} = \begin{pmatrix}-1+\frac{Gm_{\rm p}^2}{e^2}; & 1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}\\ 1+\frac{Gm_{\rm p}m_{\rm e}}{e^2}; & -1+\frac{Gm_{\rm e}^2}{e^2}\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix}, \tag{B13}$$

and the inverse matrix is the negative of (54), so

$$\begin{pmatrix}A_{\rm p}\\ A_{\rm e}\end{pmatrix} = \frac{e^2}{G(m_{\rm p}+m_{\rm e})^2}\begin{pmatrix}2+\frac{G(m_{\rm p}-m_{\rm e})m_{\rm e}}{e^2}\\ 2-\frac{G(m_{\rm p}-m_{\rm e})m_{\rm p}}{e^2}\end{pmatrix}. \tag{B14}$$

Both $A_{\rm p} > 0$ and $A_{\rm e} > 0$ thanks to the smallness of the ratio of gravitational to electric coupling constants, and hence we have found an exact solution pair to (B9) and (B10).

These $1/r^2$ densities are singular at the origin, but locally integrable. Of course, they are not globally integrable, so $N_{\rm p} = \infty = N_{\rm e}$. Interestingly, though, the number of particles of species $_f$ inside a sphere of radius r, i.e., $\mathcal{N}_f(r) := 4\pi\int_0^r \nu_f(s)s^2 ds = (2k_BT/e^2)A_f r$, yields the r-independent ratio

$$\frac{\mathcal{N}_e(r)}{\mathcal{N}_{\rm p}(r)} = \frac{A_{\rm e}}{A_{\rm p}} = \frac{2-\frac{G(m_{\rm p}-m_{\rm e})m_{\rm p}}{e^2}}{2+\frac{G(m_{\rm p}-m_{\rm e})m_{\rm e}}{e^2}}. \tag{B15}$$

For general solution pairs of (B9) and (B10) one may proceed analogously. Since Emden's isothermal gas ball solutions all tend asymptotically for large r to a $1/r^2$ behavior, we expect that $\lim_{r\to\infty}\mathcal{N}_e(r)/\mathcal{N}_{\rm p}(r)$ exists for each pair and plays the role of $N_{\rm e}/N_{\rm p}$ for such infinite-mass solutions. Moreover, whenever $\lim_{r\to\infty}\mathcal{N}_e(r)/\mathcal{N}_{\rm p}(r)$ exists, a small modification of our arguments in Sec. IV shows that the limit obeys the bounds (66) without saturation.

3. An exact atmospheric solution of the two-species 5/3 model

The nonlinearity of Eqs. (10) and (11) stands in the way of solving them generally in closed form, yet one atmospheric density solution actually can be obtained explicitly. We show this for the negative atmosphere case.

Consider (11) with $v_{\rm p}(\rho) = 0$ for $\rho > \rho_0^+$; it does not matter where ρ_0^+ is located, all we use is that it is a finite distance. We now make the ansatz $v_{\rm e}(\rho) = A_{\rm e}\rho^\eta$ and find $\eta = -6$ and $A_{\rm e} = 12\zeta/(1-Gm_{\rm e}^2/e^2)$. While this is slower than the exponential decay to zero, it still is fast enough to be integrable at $\rho\to\infty$, viz. $\rho^2 v_{\rm e}'(\rho)\to 0$ as $\rho\to\infty$. This solution would still have to be matched to the bulk interior, which may or may not be possible.

APPENDIX C: THE LOCAL NEUTRALITY APPROXIMATION

So suppose temporarily that $\nu_{\rm p}(r) = \nu_{\rm e}(r) =: \nu(r)$ for all r. Then $\sigma = 0$ by Eq. (4), and Eq. (6) is then solved by $\phi_C = 0$. Moreover, by Eq. (3) we now have $\mu(r) = (m_{\rm p}+m_{\rm e})\nu(r)$. This is usually approximated further by neglecting the electron mass versus the proton mass, yet technically this does not yield a simplification.

A subtler step is the next one. We still have to deal with Eqs. (7) and (8), but having set $\nu_{\rm p} = \nu_{\rm e} =: \nu$, we then have two different equations for one unknown, $\nu(r)$, and this overdetermines the problem, strictly speaking. What this shows is that the strict local neutrality approximation cannot be exactly correct, but, of course, it was never assumed to be exactly correct. Therefore, to proceed in the spirit of the approximation, one needs to mold the two equations (7) and (8) into one. This is done by replacing them by their sum, which in concert with $\phi_C = 0$ yields the mechanical force balance equation

$$-\mu(r)\phi_N'(r) - p'(r) = 0, \tag{C1}$$

where the pressure function $p(r) = p_{\rm p}(r) + p_{\rm e}(r)$ reads

$$p(r) = \hbar^2 \left(\frac{1}{m_\mathrm{p}} + \frac{1}{m_\mathrm{e}}\right) \frac{(3\pi^2)^{2/3}}{5} \nu^{5/3}(r). \tag{C2}$$

This is usually approximated further by neglecting $1/m_\mathrm{p}$ versus $1/m_\mathrm{e}$, yet again technically this does not yield a simplification either.

Since $\mu(r) = (m_\mathrm{p} + m_\mathrm{e})\nu(r)$, Eq. (C1) with $p(r)$ given by (C2) can be integrated once to yield ϕ_N as a function of ν, which can be inverted to yield

$$\nu(r) = \left(\frac{2}{(3\pi^2)^{2/3}} \frac{m_\mathrm{p} m_\mathrm{e}}{\hbar^2} [\phi_N^* - \phi_N(r)]_+\right)^{3/2}; \tag{C3}$$

here, the notation $[g]_+$ means "positive part," i.e., $[g]_+(r) = g(r) > 0$ for $0 < r < R$, where R is the smallest r-value for which $g(r) = 0$, and $[g]_+(r) = 0$ for $r \geq R$. Furthermore, ϕ_N^* is a constant of integration determined by $\int \nu(r) d^3 r = N_\mathrm{p}$. Inserting this relation into the Poisson equation (5) yields the familiar Lane-Emden equation of the polytropic gas ball for $\gamma = 5/3$, equivalently of index $n := 1/(\gamma - 1) = 3/2$,

$$\frac{1}{r^2}(r^2 \phi_N'(r))' = C[\phi_N^* - \phi_N(r)]_+^{3/2}, \tag{C4}$$

$$C = \frac{2^{7/2}}{3\pi} \frac{G}{\hbar^3} (m_\mathrm{p} + m_\mathrm{e})(m_\mathrm{p} m_\mathrm{e})^{3/2}; \tag{C5}$$

see [1–3]. By shifting and scaling, Eq. (C4) can easily be brought into the dimensionless standardized format $-\frac{1}{\xi^2}(\xi^2 \theta'(\xi))' = \theta_+^{3/2}(\xi)$, complemented with the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$; cf. [1,2,16]. The equations for the polytropic gas balls, or gas spheres as they are often called, have been studied extensively in the astrophysical literature in dependence on their parameter γ, respectively, n. For $\gamma = \infty$, $\gamma = 2$, and $\gamma = 6/5$ ($n = 0$, $n = 1$, and $n = 5$) the polytropic gas ball equation can be solved in terms of elementary functions, and in all other cases the equation itself defines the polytropic density functions. In particular, the case $\gamma = 5/3$ has been studied thoroughly due to its importance in the theory of white dwarf structure [2].

For our purposes the case $\gamma = 2$, viz. $n = 1$, is of particular interest because of our $5/3 \to 6/3$ approximation. As a primer we briefly discuss this approximation in the context of the single-density model.

1. The $\frac{5}{3} \to \frac{6}{3}$ approximation in the single-density model

We again set $r =: (\hbar/m_\mathrm{e}c)\rho$ and $\nu(r) =: (m_\mathrm{e}c/\hbar)^3 \upsilon(\rho)$. Inserted into the formula for the degeneracy pressure, we find $p(r) \propto \upsilon(\rho)^{5/3}$, and since $\upsilon(\rho)$ is dimensionless, we may *now* replace $\upsilon^{5/3}$ by $\upsilon^{6/3} (= \upsilon^2)$. We also set $\phi_N(r) =: c^2 \psi_N(\rho)$ and proceed analogously to how we arrived at the polytropic equation with index $n = \frac{3}{2}$; this time it is index $n = 1$, except that there is little incentive now to invert the linear relationship between ψ_N and υ, which results from the force balance equation (C1) wherever $\upsilon(\rho) > 0$,

$$-\psi_N'(\rho) = \varepsilon K \upsilon'(\rho). \tag{C6}$$

Here we introduced $\varepsilon := m_\mathrm{e}/m_\mathrm{p} \approx 1/1836$ and $K := 2(3\pi^2)^{\frac{2}{3}}/5$. We can even avoid the step of integrating (C6) and instead use it directly to eliminate $\psi_N'(\rho)$ [viz. $\phi_N'(r)$] from Eq. (5) in favor of $\upsilon'(\rho)$ to get

$$-\frac{1}{r^2}(r^2 \upsilon'(\rho))' = \kappa^2 \upsilon(\rho), \tag{C7}$$

$$\kappa^2 = \frac{10}{3^{2/3}\pi^{1/3}} \frac{G m_\mathrm{p}(m_\mathrm{p} + m_\mathrm{e})}{\hbar c}. \tag{C8}$$

Note that the Lane-Emden equation of index $n = 1$, Eq. (C7), is valid until $\upsilon(\rho)$ runs into its first zero.

Several observations are in order.

First, we note that $G m_\mathrm{p}(m_\mathrm{p} + m_\mathrm{e})/\hbar c \approx 6 \times 10^{-39}$ is a gravitational analog of Sommerfeld's fine structure constant $e^2/\hbar c := \alpha_S \approx 1/137.036$; it is much smaller, though. This means that to see any appreciable effect in a solution of Eq. (C7) the variable ρ has to reach very large values. But this is only to be expected, for our unit of length is the reduced Compton length of the electron, and sure enough the structure of a star varies on scales which are gigantic in terms of these units.

Second, the Lane-Emden equation of index $n = 1$, Eq. (C7), is not only linear, it is one of the three special cases which can be solved in terms of elementary functions. It is a special case of a Bessel-type differential equation, and the solution relevant to our discussion is given by a spherical Bessel function, explicitly

$$\upsilon(\rho) = B \frac{\sin(\kappa\rho)}{\rho}, \qquad \rho \in (0, \pi/\kappa); \tag{C9}$$

the bulk amplitude B is determined by $\int \upsilon(\rho) d^3\rho = N_\mathrm{p}$.

Third, the radius of the star in this approximate single-density model is $R = \frac{\pi}{\kappa} \frac{\hbar}{m_\mathrm{e} c}$. Inserting the values for the physical and mathematical constants yields

$$R \approx 2.2566 \times 10^{19} \frac{\hbar}{m_\mathrm{e} c} \approx 8,714 \text{ km}, \tag{C10}$$

i.e., $\approx 3/2$ earth radii, compatible with the accepted radius of white dwarf stars with half the mass of the sun.

Fourth, note that R is *independent* of N_p (or N_e for this matter). This, of course, is not physically reasonable. However, we note that the physical range of acceptable values for N_p (hence, N_e) is very narrow. Indeed, to have the interior of a gravitational object accurately modeled as an ideal Fermi gas, the mass needs to be sufficiently big, say $N_\mathrm{p} > 1.5 \times 10^{55}$ (13 Jupiter masses), and to be allowed to work with the nonrelativistic approximation, it cannot be

too big either, say $N_p < 10^{57}$ (a solar mass). Furthermore, we also assumed that the white dwarf failed to ignite, yet surely our sun did not. This assumption reduces the allowed range of N_p to $N_p < 9 \times 10^{55}$. For such a narrow range of N_p values it is not too unrealistic to have the model predict an N_p-independent radius and a central density which increases proportional to N_p.

The $5/3$ model breaks the scaling invariance, and then the radii are N_p-dependent, as visible in our Figs. 4 and 6.

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