


Schwinger pair production in $SL(2, \mathbb{C})$ topologically nontrivial fields via non-Abelian worldline instantons

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Schwinger pair production is analyzed in a Belavin, Polyakov, Schwarz and Tyupkin (BPST) instanton background and in its $SL(2, \mathbb{C})$ complex extension for complex scalar particles. A non-Abelian extension of the worldline instanton method is utilized, wherein Wong's equations in a coherent state picture adopted for $SL(2, \mathbb{C})$ are solved in Euclidean spacetime. While pair production is not predicted in the BPST instanton, a complex extension of the BPST instanton, existing as parallel fields in Minkowski spacetime, is shown to decay via the Schwinger effect.

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I. INTRODUCTION

The quantum field theoretic (QFT) vacuum in a strong electric field is thought unstable against the production of particle anti-particle pairs in what is known as the Schwinger mechanism [1]. Observation of the Schwinger effect could be impactful for the understanding of non-perturbative QFTs, and is actively being or to be sought not only in strong quantum electrodynamics (QED) in high-power laser facilities (e.g., ELI beam lines, etc. reviewed in [2]), but also in analog condensed matter settings, whereby the effect is facilitated through Landau-Zener transitions [3]. Yet, due to a strong exponential suppression, (i.e., $m^2 c^3 \pi / e E \hbar$ for homogeneous electric field, E , and scalar/fermion mass, m), the effect still has not been seen. While smallish in high-power lasers, the Schwinger mechanism is thought to be a prominent feature of non-Abelian chromo-electric flux tube breaking in heavy-ion collisions [4], during which, in topologically nontrivial fields, chiral transport phenomena can develop.

The Schwinger mechanism has been argued to underlie the chiral anomaly for finite fermion mass systems [5], and has been confirmed numerically [6]. The axial Ward identity at operator level in QED reads $\partial_\mu j_5^\mu = -(e^2/16\pi^2)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + 2m\bar{\psi}i\gamma_5\psi$, with j_5^μ the axial current density and ψ the fermion field [7]. Then in homogeneous fields with nonzero Chern-Pontryagin density, the

matrix element vacuum polarization (i.e., in-out expectation values), of the axial Ward identity indicates an anomaly cancellation; however in-in expectation values of the identity, properly accounting for the Schwinger effect, restore the anomaly [8]. And, a pertinent and intuitive question one may ask is how does the anomaly behave under the Schwinger mechanism in non-Abelian and nontrivial topological fields? To address this we must first understand the Schwinger effect for massive particles under such fields. The case of massless fermions under a isotropic and homogeneous $SU(2)$ gauge field background with a nonvanishing Chern-Pontryagin density leading to the chiral anomaly via the Schwinger effect was explored in [9]. And massive pair production has also been studied in axion- $SU(2)$ field during inflation [10].

The index theorem is well known to relate the fermionic left and right zero modes to the Pontryagin number [11]. A key example [12] is provided through the Dirac operator in a Belavin, Polyakov, Schwarz, and Tyupkin (BPST) instanton [13] background, and therefore the instanton is an intuitive choice of a topologically nontrivial background under which to study Schwinger pair production. A measure of pair production is provided through the vacuum non-persistence, namely the appearance of an imaginary part in the background field effective action; the nonpersistence predicts the vacuum instability sum over any number of pair permutations [14]. One may characterize the nonpersistence as arising for the condition, $|\langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle|^2 \neq 1$, for in and out asymptotic vacuum states. Then, as the BPST instanton interpolates between differing asymptotic winding numbers in Euclidean time, one should not dismiss out of hand a role played by the Schwinger mechanism. Nevertheless, an imaginary part of the fermion/boson determinant in a BPST instanton is not seen; they have been evaluated exactly in [15]. Not only do we seek to explain this, but moreover

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we also seek to explore under what conditions does Schwinger pair production occur.

Pair production under inhomogeneous background fields can be analytically cumbersome, but has been well studied using the nonperturbative worldline instanton (WI) method [16–18]. Thus, it is instructive to study the more complicated topological fields through a non-Abelian extension of the WI method. To avoid confusion with the WIs, let us refer to disparate BPST instantons compactly as Yang-Mills instantons (YMI); anti-instantons are simply anti-YMI. The WI reminiscent Lorentz force equation is to an Abelian system what Wong’s equations [19–21] are to a non-Abelian system. And Wong’s equations describe the classical evolution of a particle in a Yang-Mills field. To arrive at Wong’s equations we employ a coherent state formalism [22] on the Wilson loop, converting the proper-time ordered matrix weighted exponential into a path integral over the Haar measure. This process is known for the non-Abelian Stokes theorem [23,24] as well as for the chiral kinetic theory [25], with non-Abelian degrees of freedom [26]. The chiral kinetic theory has also been explored on the worldline [27–29]. And worldline techniques under a coherent state formalism have proved valuable for the calculation of polarized deeply inelastic scattering [30]. Let us last point out that color degrees of freedom in the Wilson loop have also equivalently been described with auxiliary fields; see [31,32] for the more general case.

What is novel in our approach is the extension of the coherent state formalism for the Wilson loop to a non-compact $SL(2, \mathbb{C})$ group. There is a topological equivalence between the coherent states and coset elements, here $SL(2, \mathbb{C})/SU(2)$, that is manifest in the construction of the Hilbert space, H . Group complexifications of the type H^C/H —most remarkably $SU(2)$ —have played a stellar role in various fields. For instance, precise formulation of the AdS/CFT correspondence requires Euclidean anti-de Sitter space AdS_3 string theory topologically equivalent $SL(2, \mathbb{C})/SU(2)$ [33,34]. Also Chern-Simons gauge theory with complex gauge group $SL(2, \mathbb{C})$ [35] has been found to exhibit many interesting connections with three-dimensional quantum gravity and the geometry of a hyperbolic three-manifold; see [36] and references within. Let us also comment that most spinfoam models for 4D gravity have been constructed as discretized path integrals for constrained background field theories with $SL(2, \mathbb{C})$ [37]. In this paper we consider $SU(2)^C$, extending the WI formalism to explore non-Abelian topologically nontrivial fields, and also through analytical continuation we parametrize the effective action.

The WI method has also been extended to finite temperature [38] and to worldline sphalerons [39]. However, we treat the effective action to one-loop at zero temperature, and we also negate backreaction effects. Last, we focus on complex scalar production. In a YMI background the

fermion functional determinant is proportional to that of a complex scalar [40], since the spectrums are similar apart from a multiplicity factor and zero modes. Furthermore, apart from a prefactor the fermion effective action only differs from the complex scalar one through a spin factor, whose contributions can be safely neglected in performing the WI method for homogeneous, Sauter-type, or sinusoidal potentials [17].

This work is organized as follows: we begin with a cursory examination of the Schwinger effect, and its absence in $SU(2)$ Euclidean fields in Sec. II. Then we develop the worldline formalism for non-Abelian field in Sec. III. Then as a demonstration of the absence, we examine pair production in a YMI in Sec. IV. And its complex extension, which does yield pair production, is sought in Sec. V. Conclusions are finally presented in Sec. VI.

II. MINKOWSKI ELECTRIC FIELDS AND THE SCHWINGER EFFECT

Let us begin our discussion of pair production (or the lack thereof) in a YMI by examining the large instanton limit, $R^2 \gg x^2$ for instanton parameter R ; this is for the field strength of the YMI in the regular gauge,

$$-\frac{4R^2}{g(x^2 + R^2)^2} \sigma_{\mu\nu} \rightarrow -\frac{4}{gR^2} \sigma_{\mu\nu}; \quad (1)$$

we use conventional notations that are listed below. The large instanton limit was explored in [41]. The key point here is that in the large instanton limit, the YMI resembles a homogeneous field, one with (in a Minkowski spacetime picture) magnetic fields, $B_j = -(2/g)[R^2/(\rho^2 + R^2)^2]\sigma_j$, and imaginary electric fields, $E_j = i(2/g)[R^2/(\rho^2 + R^2)^2]\sigma_j$. As is well understood for the Abelian homogeneous case, there can be no pair production in sole constant magnetic fields [42]. And the imaginary electric fields too act as a magnetic field, giving rise to no poles in Schwinger proper time. Therefore, it is anticipated there should be no pair production in the YMI.

It is simple to see why this should be the case. Pair production is governed by the nonpersistence criteria, which goes as, for a complex scalar, $|\langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle|^2 = |\det(-\mathcal{D}^2 + m^2)|$. The determinant can be written as a product of eigenvalues of the scalar operator. However, in a Euclidean metric, since the scalar operator under a YMI background is Hermitian, its squared operator eigenvalues must be semipositive definite. And one would not expect to see a complex phase emerging from the determinant. Put another way, one must have the sign problem in order to see Schwinger pair production, that is the sign problem in a worldline path integral formalism in a Euclidean metric, which is unambiguous. After Wick rotation, real Minkowski electric fields become imaginary in Euclidean spacetime, giving rise to the sign problem in the worldline action.

Through similar reasoning, we would anticipate a similar pair production absence for a number of topological Yang Mills solutions in $SU(2)$, such as for Wu-Yang monopoles [43], merons [44], among others [45]. Sphalerons [46], however, exist at finite temperature, to which the above arguments may not apply. But it is curious to ask to what topological objects may Schwinger pair production play a role. In light of the large instanton limit an intuitive surmise would be objects with real electric fields in Minkowski spacetime. We explore such objects below in a natural extension to the YMI. However, a real $SU(2)$ background field in Minkowski spacetime after Wick rotation becomes an $SL(2, \mathbb{C})$ field in Euclidean spacetime [47]. We apply the WI method to study pair production, and to make the extension to the most general complex field we apply the coherent state method to noncompact groups, doing so furthermore furnishes us with a worldline action amenable to Wong's equations and WIs. Even if one were to not employ a Euclidean metric the extension to a complexified group is expected, indeed, even for the Abelian case, complex WIs are important [48].

III. NON-ABELIAN WORLDLINE INSTANTONS

To more fully explore the Schwinger effect in nontrivial topological fields let us build on the WI method [16–18]. Our starting point is the one-loop effective action for a complex scalar particle with mass, m , in a non-Abelian Euclidean background field,

$$\Gamma[A] = -\frac{1}{2} \log \det(-\mathcal{D}^2 + m^2), \quad (2)$$

where in the fundamental representation we have $\mathcal{D}_\mu = \partial_\mu - igA_\mu(x)$ with $A_\mu(x) = A_\mu^a(x)T^a$. $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. Also we use the Euclidean convention, for metric $g_{\mu\nu} = \text{diag}[+, +, +, +]$, such that our magnetic and electric fields in Minkowski spacetime read $G_{ij} = \epsilon_{ijk}B_k$ and $G_{4i} = -iE_i$, respectively. In the worldline path integral formalism the non-Abelian effective action may be written as [49]

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint \mathcal{D}x e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} \mathcal{W}. \quad (3)$$

Here the coordinate boundary conditions, $x(0) = x(T) = x'$, are periodic with path integral measure $\oint \mathcal{D}x := \int dx' \int \mathcal{D}x$. $\dot{x} := dx/d\tau$. Spacetime indices to be summed over are understood and suppressed for readability. The path ordered Wilson loop reads

$$\mathcal{W} := \text{tr} \mathcal{P} e^{ig \int_0^T d\tau A_{\dot{x}}}. \quad (4)$$

The challenge here in contrast to an Abelian gauge for the application of steepest descents is a matrix weighted

worldline action. However, with application of a coherent state approach, we will show that the Wilson loop may be cast as a path integral with two merits: (1) the worldline action becomes a c number amenable to a worldline instanton approach. (2) The path ordering is negated. Furthermore, for the most general configuration we extend the application of the coherent state formalism to the Wilson loop in $SL(2, \mathbb{C})$.

A. Coherent state formalism in $SL(2, \mathbb{C})$

To cast the Wilson loop as a coherent state path integral we follow the approach used in [22,23], whereby we extend applicability of the coherent state Wilson loop to the noncompact group, $G = SL(2, \mathbb{C})$. The essence of the approach entails one find a Haar measure leading to a resolution of the identity, to which one may insert into the infinitesimally segmented Wilson loop.

Let us first go over relevant or basic details of G ; G is described through the algebra, $[l^i, l^i] = ie^{ijk}l^k$, $[l^i, k^i] = ie^{ijk}k^k$, and $[k^i, k^i] = -ie^{ijk}k^k$, for $l^i = \sigma^i/2$ and $k^i = i\tilde{l}^i$ corresponding to the generators of $SU(2)$ and $SU(1,1)$, respectively, and so both l and k transform as vectors under $SU(2)$.

One can make use of the coherent state formalism [22] through an exploitation of the *one-to-one* correspondence between coherent states $|\alpha\rangle$, representing elements of the group G , and points α in the complex plane. The map is a continuous manifestation of the topological equivalence between these two spaces. In this way, distances are determined by the intrinsic metric associated to the inner product, of which this Hilbert space is endowed with.

The Lie algebra of G , $\mathfrak{sl}(2, \mathbb{C})$, is semisimple, and so it is more convenient to rewrite it in its standard Cartan basis: $\{H_\beta, E_\beta, E_{-\beta}\}$, with the usual diagonal, H_β , and off diagonal, E_β , shift operators. For the construction of coherent states of a dynamical group G , we will require a normalized reference state $|\Lambda\rangle \in \mathcal{H}^\Lambda$ with Hilbert space \mathcal{H}^Λ corresponding to the unitary irreducible representation of G . And we will also require the maximum stability subgroup H and coset G/H [22]. Here we have $H = SU(2)$ pointing towards $G/H = SL(2, \mathbb{C})/SU(2)$ as the target space. The $\mathfrak{sl}(2, \mathbb{C})$ algebra is obtained through complexification of $SU(2)$ group algebra, e.g., $\mathfrak{su}(2) \otimes i\mathfrak{su}(2)$, and so its topology is $S^3 \times H^3$. G/H is topologically equivalent to the upper sheet of a three-dimensional mass hyperboloid H_3^+ , and is thus endowed with a hyperbolic metric as we will see. The coset element can be generated through the action of the displacement operator, with $u \in SL(2, \mathbb{C})$ where we write $|\Lambda, u\rangle := u|\Lambda\rangle$,

$$|\Lambda, u\rangle = \exp(\eta_\beta E_\beta - \bar{\eta}_\beta E_{-\beta})|\Lambda\rangle, \quad (5)$$

where a sum over the roots of the algebra—above denoted with β —is implicit, with angle $\eta_\beta \in \mathbb{C}$. The states are in

one-to-one correspondence and topologically equivalent to G/H . Then making use of Baker-Campbell-Hausdorff formula one can show the displacement operator becomes [22]

$$|\Lambda, u\rangle = e^{z_\alpha E_\alpha} e^{\gamma_\sigma H_\sigma} e^{-\bar{z}_\beta E_{-\beta}} |\Lambda, z\rangle, \quad (6)$$

$$= N(z, \bar{z})^{-1/2} \exp(z_\alpha E_\alpha) |\Lambda, z\rangle. \quad (7)$$

All powers of $e^{-\bar{z}_\beta E_{-\beta}}$ vanish after acting on the extremal state. The action of $e^{\gamma_\sigma H_\sigma}$ gives rise to the normalization factor $N(z, \bar{z})$ identified as the Kähler potential $K := \log N(z, \bar{z})$. The invariant metric then can be found as the second derivative of the potential $g_{\alpha\beta} = \partial\bar{\partial}K$, with Kähler two-form $\omega = ig_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$. And the invariant measure is used in the construction of the Haar measure; to find the measure it is convenient to make use of a complex projective map.

Let us then consider the Cartan decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{p}$ of the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra, with \mathfrak{q} the Lie algebra of $\mathfrak{su}(2)$ and $\mathfrak{p} = \eta_\beta E_\beta - \bar{\eta} E_{-\beta}$ its orthogonal complement, such that $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{q}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{q}$ as outlined in [22]. Then we can see that for noncompact groups, such as our case, the matrix representations of generators $R(\mathfrak{q})$, $R(\mathfrak{p})$ are skew symmetric and symmetric, respectively; therefore the coset $\text{SL}(2, \mathbb{C})/\text{SU}(2)$ is a symmetric space. Let us look at a matrix representation of the noncompact coset group [50],

$$\begin{pmatrix} \sqrt{1+w\bar{w}} & w \\ \bar{w} & \sqrt{1+w\bar{w}} \end{pmatrix}, \quad (8)$$

with $w = \eta \sinh(|\eta|)/|\eta|$. Likewise, our coset group defined as the set of all Hermitian two-by-two matrices with determinant one, can be parametrized by the convenient global coordinates of H_3^+ as

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{z} & 1 \end{pmatrix} \\ = \begin{pmatrix} e^{\frac{\gamma}{2}} + |z|^2 e^{-\frac{\gamma}{2}} & z e^{-\frac{\gamma}{2}} \\ \bar{z} e^{-\frac{\gamma}{2}} & e^{-\frac{\gamma}{2}} \end{pmatrix}. \quad (9)$$

Equivalently, one may arrive at this expression by explicitly working from the displacement operator, Eq. (5). Equating these two parametrizations for the coset group leads us to the complex projective map: $z = w(1+w\bar{w})^{1/2}$ and $e^{\gamma/2} = (1+w\bar{w})^{-1/2}$. It follows that any group element $u \in \text{SL}(2, \mathbb{C})$ (parametrized by coefficients $\alpha, \beta, \gamma, \delta$) acting on a coset element z is a holomorphic Möbius transformation, i.e., $T_u(z) = (\alpha z + \beta)/(\gamma z + \delta)$.

Equipped with this chart, we can rewrite the left hand side of Eq. (9) purely in terms of z and express it in the form of Eq. (7). The normalization of these states follows from an explicit computation of the exponential operator on the reference state.

$$|\Lambda, u\rangle = \frac{1}{(1-z\bar{z})^\Lambda} \exp(z_\alpha E_\alpha) |\Lambda, z\rangle. \quad (10)$$

At this point the Kähler potential, and hence its invariant Haar coset metric, can be recognized:

$$g_{\alpha\bar{\beta}} = -\Lambda \partial\bar{\partial} \log(1-z\bar{z}), \quad (11)$$

$$= \frac{\Lambda}{(1-z\bar{z})^2} (\delta_{\alpha\bar{\beta}}(1-z\bar{z}) + z^\alpha \bar{z}^\beta). \quad (12)$$

The Haar measure eventually becomes

$$d\mu_\Lambda(z, \bar{z}) = \frac{2\Lambda + 1}{4\pi} \frac{dz d\bar{z}}{(1-z\bar{z})^2}, \quad (13)$$

for the coset $\text{SL}(2, \mathbb{C})/\text{SU}(2)$, that is the complex projection of upper sheet of the three-dimensional hyperboloid H_3^+ , the Poincare disk metric.

We can in this way expand any arbitrary state $|\Psi\rangle \in \mathcal{H}^\Lambda$ Hilbert space into coherent space whose coefficients are smooth functions in u defined over the coset space. However, due to the overcompleteness of coherent states, the expansion is not unique. And so the expansion coefficients may be deformed. This set of functions will on the other hand allow us to construct the basis for function space $L^2(\text{SL}(2, \mathbb{C})/\text{SU}(2))$ [33]:

$$\mathcal{H} \equiv L^2(H_3^+) = \int_{s \in \mathbb{R}^+}^\oplus ds s^2 \mathcal{H}_{-1/2+is}, \quad (14)$$

through the principal series decomposition of the group, where the spin j takes on complex values, $j = -\frac{1}{2} + is$, $s \in \mathbb{R}$, coming from the Casimir having a continuous spectrum. For a detailed account on $\text{SL}(2, \mathbb{C})$ we refer the reader to [51].

Equipped with the Haar measure and hence the resolution of identity we may transform the Wilson loop into a path integral over coherent states. To cast Eq. (4) into its path integral form we must both expand the path ordered exponential of parallel transporter $\mathcal{W}_L(\tau_0, \tau)$ and evaluate its trace to recover the Wilson loop. This is a standard procedure; we first partition the path $L: \tau_0 \rightarrow \tau$ into infinitesimal segments,

$$\mathcal{P} \exp \left[ig \int_{\tau_0}^\tau d\tau A \dot{x} \right] = \mathcal{P} \prod_{k=0}^{N-1} (1 + ig\epsilon A \dot{x}), \quad (15)$$

where $\epsilon = (\tau - \tau_0)/N$ and take proper limits at the end. To evaluate trace of this object we are required to choose a set of states resolving the identity $1 = \int |\Lambda, u_k\rangle d\mu(u_k) \langle \Lambda, u_k|$, with u_k a coset element at τ_k and Haar measure given in Eq. (13), to which we may insert at each partition point. The Wilson loop becomes

$$\mathcal{W} = \int \mathcal{D}\mu_C \exp \left\{ \frac{ig}{2} \oint_C (m^a A^a dx + \omega(x)) \right\}, \quad (16)$$

with $\mathcal{D}\mu$ corresponding to the product of the invariant Haar measure per coset element and $m^a = \langle \Lambda | u(x) T^a u^{-1}(x) | \Lambda \rangle$ and $\omega(x) = \langle \Lambda | u(x) du^{-1}(x) | \Lambda \rangle$. Let us notice that the argument of the exponential has been brought down to Abelian quantities allowing us to apply Stokes' theorem [hence coined the non-Abelian Stokes theorem as demonstrated for $SU(N)$ in [23,24]] and express \mathcal{W} over the surface bounded by C ,

$$\mathcal{W} = \oint \mathcal{D}\mu \exp \left\{ \frac{ig}{2} \int_0^T d\tau \text{tr} \left[\sigma_3 \left(u A \dot{x} u^{-1} + \frac{i}{g} u \dot{u}^{-1} \right) \right] \right\}. \quad (17)$$

Having demonstrated the Wilson loop may be represented in $SL(2, \mathbb{C})$ as a path integral over a complex isospin, let us now show how periodic solutions to Wong's equations in Euclidean spacetime are non-Abelian worldline instantons. In evaluating the trace we adopt coherent states, which are equipped with properties that will prove useful to explore more exotic field configurations. Coherent states, in terms of the language of group theory, are embedded in a topologically nontrivial geometrical space facilitating the perfect machinery to probe instanton and other nontrivial configurations.

B. Wong's equations

Wong's equations follow as classical equations of motion of the worldline action. Let us however first arrange terms in the action to more suitably elicit their connection to worldline instantons; to accomplish this we first take the Schwinger proper time integral, T , as was done in [16,17]. First, using the coherent state represented Wilson loop in Eq. (17), we have for the effective action, Eq. (3), after the substitution, $\tau \rightarrow T\tau'$

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \oint \mathcal{D}x \mathcal{D}\mu e^{-S}, \quad (18)$$

$$S = \int_0^1 d\tau \left\{ m^2 T + \frac{\dot{x}^2}{4T} - \frac{ig}{2} \text{tr} \left[\sigma_3 \left(u A \dot{x} u^{-1} - \frac{i}{g} u \dot{u}^{-1} \right) \right] \right\}. \quad (19)$$

Let us evaluate the proper time integral through steepest descents—or rather the Laplace method. We expand $T = T_n + K \exp(i\alpha_n)$ for stationary points, T_n , about $K \in [0, \infty)$, with phase $\alpha_n = \pi/2 - 1/2 \arg(f''(T_n))$, for $f(T) = -m^2 T - \frac{1}{4T} \int_0^1 d\tau \dot{x}^2$. We also have that $f''(T) = -\frac{1}{2T^3} \int_0^1 d\tau \dot{x}^2$. However, in what follows we will confine our attention to the case of only real and positive T_n , and hence the phase factor $\exp(i\alpha_n) = 1$. We find for the stationary points

$$T_n^2 = \frac{1}{4m^2} \int_0^1 d\tau \dot{x}^2, \quad (20)$$

whose n minima can be had alongside steepest descents in coordinate and isospin space. The proper time integral becomes

$$\int_0^\infty \frac{dT}{T} e^{-m^2 T - \frac{1}{4T} \int_0^1 d\tau \dot{x}^2} \approx \sum_n \sqrt{\frac{\pi T_n}{4m^2}} e^{-2m^2 T_n}. \quad (21)$$

Importantly, so that steepest descents be valid, a large mass and or weak fields—characterized by $\sqrt{\int_0^1 d\tau \dot{x}^2}$ [16,17]—must be assumed, i.e., $m^2 T_n \gg 1$. The criteria is safely met for a low probability of pair production occurrence. Furthermore, while we solve Wong's equations for complex trajectories, we will find that only real periodic paths will contribute to pair production for the fields examined. Last, the Schwinger effect is dominated by an exponential suppression, therefore we ignore prefactor terms. Let us then turn our attention to the worldline action, Eq. (19), for a given stationary point in proper time, Eq. (20),

$$S_n = m \sqrt{\int_0^1 d\tau \dot{x}^2} - \frac{ig}{2} \int_0^1 d\tau \text{tr} \left[\sigma_3 \left(u A \dot{x} u^{-1} - \frac{i}{g} u \dot{u}^{-1} \right) \right]. \quad (22)$$

Classical solutions of the above action lead to Wong's equations, and with the Euclidean periodicity criteria, also lead to worldline instantons. Let us begin with the gauge element, u , leading to Wong's equation describing the precession of isospin.

One may straightforwardly apply the Euler-Lagrange equations to the action in the gauge element through an introduction of an infinitesimal angle to the gauge element as outlined in [21]. Let us assume the gauge element can be parametrized by the independent set of variables given by Θ ; then we have for $\vec{\sigma}_i = (\vec{\sigma}, i\vec{\sigma})$,

$$u(\Theta(\beta)) := e^{i\beta \vec{\sigma}_i} u(\Theta), \quad (23)$$

$$\frac{\partial}{\partial \beta_i} u(\Theta(\beta)) \Big|_{\beta=0} = \frac{\partial \Theta_j}{\partial \beta_i} \frac{\partial}{\partial \Theta_j} u(\Theta(\beta)) \Big|_{\beta=0}, \quad (24)$$

where we define $M_{ij} := \frac{\partial \Theta_j}{\partial \beta_i} \Big|_{\beta=0}$. And we find

$$\frac{\partial}{\partial \Theta_j} u(\Theta) = M_{ij}^{-1} \frac{i}{2} \vec{\sigma}_i u(\Theta). \quad (25)$$

M must have a nonzero determinant owing to the linear independence of $\vec{\sigma}$. Then, using the fact that $\dot{u}^{-1} = -u^{-1} u \dot{u}^{-1}$ and likewise $\frac{\partial}{\partial \Theta_i} u^{-1} = -u^{-1} \left(\frac{\partial}{\partial \Theta_i} u \right) u^{-1}$, we find for parameter, Θ_i ,

$$\frac{\delta S}{\delta \Theta_i} = \frac{i}{2} M_{ji}^{-1} \text{tr}[u^{-1} \bar{\sigma}_j u \{ [I, igA\dot{x}] + [I, u^{-1}\dot{u}] \}]. \quad (26)$$

Here we have introduced the isospin variable,

$$I = \frac{1}{2} u^{-1} \sigma_3 u. \quad (27)$$

Then using that $\dot{I} = [I, u^{-1}\dot{u}]$, we can find for the isospin portion of Wong's equations as

$$\dot{I} = [igA\dot{x}, I]. \quad (28)$$

Let us digress on some properties of the isospin. It is conserved as $\text{tr}I^2 = (1/2)I_i I_i = 1/2$, where $I = (I_i/2)\sigma_i$ and $I_i = \text{tr}[\sigma_i I]$. We can see therefore that the isospin performs a precessional motion [19] despite its complex nature.

One may also write the isospin equation, Eq. (28), as $[(1/2)\sigma_3, uigA\dot{x}u^{-1} + \dot{u}u^{-1}] = 0$, and hence we can see that a solution to the isospin equation of motion is analogous to finding a gauge transformation of the Hamiltonian $igA\dot{x}$ such that the transformed Hamiltonian becomes diagonal. Alternatively, a solution to the isospin equation of motion furnishes such a gauge. An exact solution can be had in principal such that the off diagonal pieces of the geometric phase $\dot{u}u^{-1}$ cancel with those of the $uigA\dot{x}u^{-1}$ term. We, however, exploit an adiabatic approximation giving way to the familiar Berry's phase [52] to arrive at a solution of Eq. (28).

Wong's other equation is the non-Abelian equivalent of the Lorentz force equation. This can be found straightforwardly by minimizing the worldline action, Eq. (22), with respect to the coordinate x , making use of the isospin equation, Eq. (28), and the fact that \dot{x}^2 is a constant owing to the antisymmetric field strength tensor. We find for the Lorentz force portion of Wong's equations as

$$\ddot{x}_\mu = -\frac{ig|\dot{x}|}{m} \text{tr}[IG_{\mu\nu}] \dot{x}_\nu, \quad (29)$$

$|\dot{x}| := \sqrt{\dot{x}^2}$. Equations (28) and (29) make up Wong's equations. Their solutions in Euclidean spacetime about periodic boundary conditions are worldline instantons predicting a particle antiparticle tunneling from the QFT vacuum. Here, we focus on the dominant exponential suppression and confine our attention to the classical worldline action, whose form is

$$S_n = m|\dot{x}| - \int_0^1 d\tau \text{tr} \left[I igA\dot{x} + \frac{1}{2} \sigma_3 \dot{u}u^{-1} \right], \quad (30)$$

for instanton number, n , governed by Eqs. (28) and (29). Let us next look at the simplest case of non-Abelian fields that posses nonzero Chern-Pontryagin density; these are

homogeneous parallel fields. This step is instructive in that we can check the validity of the extension of the worldline instanton method to non-Abelian fields. But moreover, a key difference between the electric and magnetic field applicability to the WI method is revealed, which is beneficial to expose in a comparatively simple setting. The non-Abelian WI method devised above is new, and let us remark that one may also arrive at our results starting from an auxiliary field approach [31,32] to a complexification of the group to $SU(2)^C$.

C. Homogeneous Abelian-like parallel fields

Let us examine homogeneous parallel fields in the \hat{x}_3 direction, which are in an Abelian-like representation:

$$G_{12} = B\sigma_3, \quad G_{34} = iE\sigma_3, \quad (31)$$

$$A_2 = Bx_1\sigma_3, \quad A_4 = iEx_3\sigma_3. \quad (32)$$

These fields correspond to $SU(2)$ parallel fields, $\vec{B} = B\hat{x}_3$ and $\vec{E} = E\hat{x}_3$, in Minkowski spacetime; here they are in $SL(2, \mathbb{C})$. For Abelian-like (here proportional to σ_3), non-Abelian fields the isospin, according to Eq. (28), takes a trivial solution. Note that for fields not proportional to σ_3 , the Schwinger effect characteristics differ markedly [53]. Since $[\sigma_3, A_\mu] = 0$ we may select a proptime independent gauge; let us take $u_{-,+} = i\sigma_{1,3}$ and $u_{+,-}^{-1} = -i\sigma_{1,3}$. Then for $I_\pm^H = \frac{1}{2} u_\pm^{-1} \sigma_3 u_\pm$, we find two independent solutions for the isospin as

$$I_\pm^H = \pm \frac{1}{2} \sigma_3. \quad (33)$$

Variables of solutions in a homogeneous parallel field background are affixed with superscript, H , to contrast later solutions in a BPST instanton background, I , and a complex equivalent, CI .

The Lorentz force equation, Eq. (29) reduces to

$$\ddot{x}_{\pm 1}^H = \mp \frac{ig|\dot{x}|}{m} B \dot{x}_{\pm 2}^H, \quad \ddot{x}_{\pm 2}^H = \pm \frac{ig|\dot{x}|}{m} B \dot{x}_{\pm 1}^H, \quad (34)$$

$$\ddot{x}_{\pm 3}^H = \pm \frac{g|\dot{x}|}{m} E \dot{x}_{\pm 4}^H, \quad \ddot{x}_{\pm 4}^H = \mp \frac{g|\dot{x}|}{m} E \dot{x}_{\pm 3}^H. \quad (35)$$

As we are looking for only solutions that yield real and positive $|\dot{x}|$ we examine only solutions corresponding to the electric field. One can readily see that the coordinates associated with the magnetic field will be periodic about an imaginary argument for a hyperbolic sinusoidal function, whereas the electric field periodicity is governed by a real argument in a sinusoidal function. No simultaneous solutions for both the electric and magnetic parts exist. Physically, this stems from the fact that a sole magnetic field cannot elicit Schwinger pair production.

Therefore one may take for the magnetic field coordinates, x_1^H and x_2^H , a trivial constant value. Whereas for the electric field coordinates, we find for the non-Abelian WIs the same as in the Abelian case [16,17]:

$$\begin{aligned} x_{+3}^H(\tau) &= x_{-4}^H(\tau) = \frac{m}{gE} \sin\left(\frac{g|\dot{x}|E}{m}\tau\right), \\ x_{+4}^H(\tau) &= x_{-3}^H(\tau) = \frac{m}{gE} \cos\left(\frac{g|\dot{x}|E}{m}\tau\right), \end{aligned} \quad (36)$$

with two distinct WIs circling either clockwise or counter-clockwise according to the isospin, Eq. (33). And to satisfy periodic boundary conditions we must have similarly

$$|\dot{x}| = \frac{2n\pi m}{gE} \quad \forall n \in \mathbb{Z}^+. \quad (37)$$

The WIs indicate an exponential suppression of Schwinger pair production according to Eq. (30) of

$$S_{n\pm} = \frac{\pi n m^2}{gE}, \quad (38)$$

which is the same for either I_{\pm} . The exponential suppression follows the exact solution, Eq. (A5), as is calculated in the Appendix. Having illustrated the simplest case of non-Abelian pair production via the worldline instanton method, let us address the case of pair production in a YMI background; we will find as expected pair production is absent.

IV. PAIR PRODUCTION (OR LACK THEREOF) IN A BPST INSTANTON BACKGROUND

To provide a convenient point of comparison for later discussions, let us write out the BPST instanton (YMI) field, A^I , in the regular gauge,

$$A_{\mu}^I(x) = \frac{i}{g} \frac{x^2}{x^2 + R^2} G(\hat{x})^{\dagger} \partial_{\mu} G(\hat{x}), \quad (39)$$

where R denotes the size of the instanton, but has no effect on the topological winding number. The anti-YMI, $A^{\bar{I}}$, can be found with the replacement $G \rightleftharpoons G^{\dagger}$. We use the conventions of [54]. Then for the gauge element $G(\hat{x}) = \hat{x}_{\mu} \bar{\sigma}_{\mu}$, where $\sigma_{\mu} := (i\vec{\sigma}, 1)$ and $\bar{\sigma}_{\mu} := (-i\vec{\sigma}, 1)$, the gauges read

$$A_{\mu}^I(x) = \frac{1}{g} \frac{2}{x^2 + R^2} \sigma_{\mu\nu} x_{\nu}, \quad A_{\mu}^{\bar{I}}(x) = \frac{1}{g} \frac{2}{x^2 + R^2} \bar{\sigma}_{\mu\nu} x_{\nu}, \quad (40)$$

where $\sigma_{\mu\nu} := \frac{1}{4i} [\sigma_{\mu} \bar{\sigma}_{\nu} - \sigma_{\nu} \bar{\sigma}_{\mu}]$ and $\bar{\sigma}_{\mu\nu} := \frac{1}{4i} [\bar{\sigma}_{\mu} \sigma_{\nu} - \bar{\sigma}_{\nu} \sigma_{\mu}]$. And the field strength tensors are of course

$$G_{\mu\nu}^I = -\frac{4R^2}{g(x^2 + R^2)^2} \sigma_{\mu\nu}, \quad G_{\mu\nu}^{\bar{I}} = -\frac{4R^2}{g(x^2 + R^2)^2} \bar{\sigma}_{\mu\nu}. \quad (41)$$

For the calculations that follow it is convenient to use a matrix form for Lorentz indices; contractions are understood and $x^T = (x_1, x_2, x_3, x_4)$. Let us show this for the 't Hooft symbols. For $\sigma_{\mu\nu} = \frac{1}{2} \eta_{a\mu\nu} \sigma_a$ and $\bar{\sigma}_{\mu\nu} = \frac{1}{2} \bar{\eta}_{a\mu\nu} \sigma_a$, we have for the symbols, $\eta_{a\mu\nu} = \varepsilon_{a\mu\nu 4} + \delta_{a\mu} \delta_{\nu 4} - \delta_{a\nu} \delta_{\mu 4}$ and $\bar{\eta}_{a\mu\nu} = \varepsilon_{a\mu\nu 4} - \delta_{a\mu} \delta_{\nu 4} + \delta_{a\nu} \delta_{\mu 4}$. The 't Hooft symbols can be shown to satisfy several relationships. They are anti-symmetric: $\eta_i^T = -\eta_i$ and $\bar{\eta}_i^T = -\bar{\eta}_i$. And also, since the symbols transform under $SO(4) = SU(2) \otimes SU(2)$, we can find the following:

$$\eta_i \eta_j = -[\delta_{ij} + \varepsilon_{ijk} \eta_k], \quad \bar{\eta}_i \bar{\eta}_j = -[\delta_{ij} + \varepsilon_{ijk} \bar{\eta}_k]. \quad (42)$$

Let us go ahead and express Wong's equations in the above form for the YMI, Eq. (39); they are

$$\dot{I}_c^I = -\frac{2}{x^{I2} + R^2} \varepsilon_{abc} \dot{x}^{IT} \eta_a x^I I_b^I, \quad (43)$$

$$\dot{x}^I = \frac{i|\dot{x}^I|}{m} \frac{2R^2}{(x^{I2} + R^2)^2} \eta \cdot I^I \dot{x}^I. \quad (44)$$

Before explicit computation, we can see that as was the case for the magnetic fields in the previous section, see Sec. III C and Eq. (31), the Lorentz force equation has a real field strength argument, in contrast to the electric field. Moreover, eigenvalues of $\text{tr}[I^I G_{\mu\nu}^I]$ are all real, and hence only project magnetic parts. We can explore this more deeply with the aid of a large instanton limit and through fixing the isospin in the direction of the gauge. Let us also point out that Wong's equation in the YMI have been studied in [55]; however, our WI approach as well as calculation technique are new.

A. Adiabatic theorem and the large instanton

Let us first evaluate the isospin equation of motion. Consider for fictitious Hamiltonian, $H = igA\dot{x}$, the isospin equation of motion provided by $[\frac{1}{2}\sigma_3, uHu^{-1} + iu^{-1}] = 0$; see also Eq. (28). One can immediately see a solution is provided by the selection of a gauge element, u , such that H takes a diagonal form, and that off diagonal parts of the geometric phase, iu^{-1} , may be ignored. This is an adiabatic theorem leading to Berry's phase(s) [52], one for each monopole singularity governing level crossing in H . We will demonstrate shortly that the adiabatic approximation is equivalent to (complex) circular solutions for the WIs, as we calculated in Sec. III C, and the importance of which we highlighted in Sec. II. Moreover, there, for the homogeneous fields circular solutions to Wong's equations were found, and homogeneous fields were also found to be the limiting form in the large instanton limit, Eq. (1). Thus the

adiabatic theorem and large instanton limits go hand in hand. We take for the gauge element, u^I , such that

$$u^I A^I \dot{x}^I (u^I)^{-1} = \sqrt{A^{Ia} \dot{x}^I A^{Ia} \dot{x}^I} \sigma_3, \quad (45)$$

and likewise for $A^{\bar{I}}$. Let us just treat the YMI from this point, and report on the anti-YMI below. One can see that the above corresponds to

$$I_a^I = \frac{\dot{x}^{IT} \eta_a x^I}{\sqrt{\dot{x}^{IT} \eta_a x^I \dot{x}^{IT} \eta_a x^I}}. \quad (46)$$

We can see the isospin is fixed in the direction of the gauge field, whose magnitude is always unity since the isospin is an element of the coset $SU(2)/U(1)$ and is tracing out a point on the surface of the unit sphere. One can see an adiabatic approximation entails isospin also to be independent of proper time, since according to Eq. (43), $\dot{I}^I = 0$. Then consider Eq. (46), since η_a are all antisymmetric tensors we can see that to satisfy $\dot{I}^I = 0$ we must have that

$$x^I \propto \dot{x}^I. \quad (47)$$

We also take the large instanton limit $R^2 \gg x^2$; then one can see the field strength tensor takes on the following form (cf. Sec. II)

$$\dot{x}^I \approx \frac{2i|\dot{x}^I|}{mR^2} \eta \cdot I^I \dot{x}^I. \quad (48)$$

It proves convenient to evaluate the above using projection operators of the field strength tensor, as one might do for the Abelian Lorentz force equation [56]. However, due to the self-duality of the YMI, we find a compact form for the operators, only linear in $\text{tr}[I^I G^I]$; they are

$$P_-^I := \frac{1}{2}(1 + i\eta \cdot I^I), \quad P_+^I := \frac{1}{2}(1 - i\eta \cdot I^I). \quad (49)$$

Apart from a Lorentz index assignment there is no real identification for electric and magnetic projection operators; they both project out magnetic field components. Some useful properties of the projection operators include idempotency, a completeness, and an orthogonality between unlike projectors:

$$P_{\pm}^{I2} = P_{\pm}^I, \quad P_-^I + P_+^I = 1, \quad P_{\pm}^I P_{\mp}^I = 0. \quad (50)$$

Last, the projection operators project their respective eigenvalues of the field strength tensor,

$$\eta \cdot I P_-^I = -i P_-^I, \quad \eta \cdot I P_+^I = i P_+^I. \quad (51)$$

One can use the projection operators to decouple the isospin from the Lorentz force equation.

Let us use the projection operators to separate the coordinates into two parts

$$x_+^I := P_+^I x^I, \quad x_-^I := P_-^I x^I. \quad (52)$$

Note also that the projection operators under interchange of the Lorentz indices satisfy the following relationship: $P_{I\pm}^T = P_{I\mp}$. Hence, we would have $x_{\pm}^{IT} = x^{IT} P_{\mp}^I$. Furthermore, under the adiabatic approximation we found the isospin becomes proper time independent. Therefore, using Eq. (51), we find the Lorentz force equation decouples as

$$\ddot{x}_+^I = -\frac{2|\dot{x}^I|}{mR^2} \dot{x}_+^I, \quad \ddot{x}_-^I = \frac{2|\dot{x}^I|}{mR^2} \dot{x}_-^I. \quad (53)$$

One can readily find the solutions to the above as

$$x_+^I(\tau) = \frac{mR^2}{2|\dot{x}^I|} \left[1 - \exp\left(-\frac{2|\dot{x}^I|}{mR^2} \tau\right) \right] \dot{x}_+^I(0) + x_+^I(0), \quad (54)$$

$$x_-^I(\tau) = \frac{mR^2}{2|\dot{x}^I|} \left[\exp\left(\frac{2|\dot{x}^I|}{mR^2} \tau\right) - 1 \right] \dot{x}_-^I(0) + x_-^I(0), \quad (55)$$

which can be combined giving one

$$x^I(\tau) = \frac{mR^2}{2|\dot{x}^I|} \left[\sinh\left(\frac{2|\dot{x}^I|}{mR^2} \tau\right) + \cosh\left(\frac{2|\dot{x}^I|}{mR^2} \tau\right) i\eta \cdot I^I \right] \dot{x}^I(0), \quad (56)$$

with I^I given by Eq. (46). We have also applied the constraint given in Eq. (47); this gives $x^I(0) = i \frac{mR^2}{2|\dot{x}^I|} \eta \cdot I^I \dot{x}^I(0)$.

As anticipated, one can readily see that to satisfy the periodicity requirement, $x^I(0) = x^I(1)$, one must have

$$|\dot{x}^I| = imR^2 \pi n \quad \forall n \in \mathcal{Z}^+, \quad (57)$$

in contradiction to the requirement of a real stationary point, Eq. (20), and hence real worldline action, Eq. (30). And thus, there can be no pair production. The YMI acts like a magnetic field, as we saw in Sec. III C, which does not give rise to the Schwinger effect. Also, evaluation of the anti-YMI would result in Eq. (57) as well.

Let us look at the worldline action. Using Eq. (56), one can find that $\int_0^1 d\tau \text{tr}[I^I i g A^I \dot{x}^I] = (1/2)m|\dot{x}^I|$. Also, the Berry's phase term in Eq. (30), given by $\dot{u}^I (u^I)^{-1}$ may also only introduce a trivial factor of $4\pi i$ into the worldline action. Thus the exponential suppression goes as $(1/2)im^2 R^2 \pi n$, cf. Eq. (38). If one were to have real electric fields, one would expect a similar but real quantity. Then, let us explore just such a scenario.

V. PAIR PRODUCTION IN A COMPLEX BPST INSTANTON BACKGROUND

Above we demonstrated no pair production could occur in a BPST instanton (YMI), and then (as alluded to before) one may ask for what topologically nontrivial background fields could one see pair production. In the homogeneous parallel field case, Sec. III C, a real Minkowski electric field was needed for the vacuum instability to be present. Furthermore, in Sec. II it was shown in a large instanton limit the YMI resembled in Minkowski spacetime non-Abelian homogeneous imaginary electric fields and real magnetic fields. An intuitive extension of the YMI that might furnish pair production then would be to seek field configurations in which both real electric and magnetic non-Abelian fields are present. We construct such a background field here.

Let us however point out that Minkowski Yang-Mills solutions with topology do exist [57–59], whose effect on the anomaly have been studied [60]. But are, however, analytically cumbersome for our purposes. The importance of parallel real fields in Minkowski space for Yang-Mills tunneling is stressed in [59]. Let us also note that our gauge construction is similar in objective as the one demonstrated in [61], and indeed we find here too that the Yang-Mills equations of motion are not satisfied. Finally, we remark that solutions to the Yang-Mills equations of motion in Minkowski spacetime need not have integer Chern-Simons number [58].

A. A complex BPST instanton

The desired background field can be had from a simple identification: whereas a BPST instanton (YMI) represents a solution with winding number difference in Euclidean time, to wit, $x_4 \rightarrow -\infty$ to $x_4 \rightarrow \infty$, the desired background field stipulates a winding number difference in Minkowski real time for $x^0 \rightarrow -\infty$ to $x^0 \rightarrow \infty$ such that

$$\begin{aligned} \Delta N_{\text{Mink}} &= N_{\text{Mink}}(x^0 \rightarrow \infty) - N_{\text{Mink}}(x^0 \rightarrow -\infty) \\ &= \int dx^0 d^3x \frac{g^2}{16\pi^2} \text{tr}[G_{\text{Mink}} \tilde{G}_{\text{Mink}}]. \end{aligned} \quad (58)$$

Let us define the field under a Minkowski metric, $g_{\mu\nu} = \text{diag}(-, +, +, +)$, with map

$$\mathcal{G}_{\text{Mink}}(x_0, \vec{x}) = \frac{1}{\rho}(x_0 - i\vec{x} \cdot \vec{\sigma}) \in \text{SU}(2), \quad (59)$$

$$\mathcal{G}_{\text{Mink}}^{-1}(x_0, \vec{x}) = \frac{1}{\rho}(x_0 + i\vec{x} \cdot \vec{\sigma}), \quad (60)$$

with $\rho = \sqrt{x_\mu x^\mu + 2x_0^2}$. $\mathcal{G}_{\text{Mink}}$ has the topology of S^3 , however unlike the map in the YMI, Eq. (39), $\mathcal{G}_{\text{Mink}}$ cannot

map from a vacuum S^3 ; as we will demonstrate our fields are not classical Yang-Mills solutions. Also, ρ , is not a Minkowski four-vector of unit length. Last, one may indeed envision $\mathcal{G}_{\text{Mink}}$ as living in a Minkowski four-dimensional cylinder with end caps in real time at $N_{\text{Mink}}(x^0 \rightarrow \pm\infty)$, and disappearing at spatial infinity.

Using the above one can construct the following gauge connection with $\Delta N_{\text{Mink}} = 1$,

$$A_{\text{Mink}\mu}^{CI}(x_0, \vec{x}) = \frac{i}{g} f(x_0, \vec{x}) \mathcal{G}_{\text{Mink}}^{-1}(x_0, \vec{x}) \partial_\mu \mathcal{G}_{\text{Mink}}(x_0, \vec{x}). \quad (61)$$

Here f is a function such that $f(0) = 0$ and $f(\rho \rightarrow \infty) = 1$. The above we can see is in precise analogy to the YMI with $x_4 \rightarrow x_0$, and thus analogous arguments hold here as well. Notably Eq. (61) interpolates winding numbers at asymptotic real times. The field is also localized in real time and space.

WIs have an intuitive periodic structure in Euclidean time, therefore let us examine the above field in a Euclidean metric. This provides the additional benefit of contrast with the YMI. In a Euclidean metric sense, we refer to the following solutions as complex (anti-)Yang-Mills instantons (CYMI). For Wick rotation, $x_0 = ix_4$, we have

$$\mathcal{G}(x) = \frac{1}{\rho}(ix_4 - i\vec{x} \cdot \vec{\sigma}) = \hat{x}_\mu \bar{\sigma}_\mu^C \in \text{SL}(2, \mathbb{C}), \quad (62)$$

$$\mathcal{G}^{-1}(x) = \frac{1}{\rho}(ix_4 + i\vec{x} \cdot \vec{\sigma}) = \hat{x}_\mu \sigma_\mu^C, \quad (63)$$

with now $\rho = \sqrt{x^2 - 2x_4^2}$, and also we have $\sigma_\mu^C := (i\vec{\sigma}, i)$ and $\bar{\sigma}_\mu^C := (-i\vec{\sigma}, i)$. Then for $A_\mu^{CI}(x) = (i/g)f(\rho)\mathcal{G}(\hat{x})^{-1}\partial_\mu\mathcal{G}(\hat{x})$, one can find

$$\begin{aligned} G_{\mu\nu}^{CI}(x) &= \frac{2}{g\rho^2} \left\{ \left[\partial_\mu f + \frac{2}{\rho}(f^2 - f)\partial_\mu \rho \right] \sigma_{\nu\sigma}^C x_\sigma \right. \\ &\quad \left. - \left[\partial_\nu f + \frac{2}{\rho}(f^2 - f)\partial_\nu \rho \right] \sigma_{\mu\sigma}^C x_\sigma \right\} \\ &\quad + \frac{4}{g\rho^2} (f^2 - f) \sigma_{\mu\nu}^C, \end{aligned} \quad (64)$$

where we have defined

$$\sigma_{\mu\nu}^C := \frac{1}{4i} [\sigma_\mu^C \bar{\sigma}_\nu^C - \sigma_\nu^C \bar{\sigma}_\mu^C], \quad \bar{\sigma}_{\mu\nu}^C := \frac{1}{4i} [\bar{\sigma}_\mu^C \sigma_\nu^C - \bar{\sigma}_\nu^C \sigma_\mu^C]. \quad (65)$$

Then in analogy to the YMI we seek a solutions such that $\partial_\mu f + \frac{2}{\rho}(f^2 - f)\partial_\mu \rho = 0$; this can be found as $f = \rho^2/(\rho^2 + R^2)$, and hence we have for the CYMI and anti-CYMI,

$$A_\mu^{CI}(x) = \frac{2}{g\rho^2 + R^2} \sigma_{\mu\nu}^C x_\nu, \quad G_{\mu\nu}^{CI}(x) = -\frac{4}{g(\rho^2 + R^2)^2} \sigma_{\mu\nu}^C, \quad (66)$$

$$A_\mu^{C\bar{I}}(x) = \frac{2}{g\rho^2 + R^2} \bar{\sigma}_{\mu\nu}^C x_\nu, \quad G_{\mu\nu}^{C\bar{I}}(x) = -\frac{4}{g(\rho^2 + R^2)^2} \bar{\sigma}_{\mu\nu}^C. \quad (67)$$

$G^{C\bar{I}}$ describes a field configuration in which $\Delta N_{\text{Mink}} = -1$.

One can readily demonstrate the sought electric and magnetic field decomposition of the CYMI is

$$E_i^{CI} = -\frac{2}{g(\rho^2 + R^2)^2} R^2 \sigma_i, \quad E_i^{C\bar{I}} = \frac{2}{g(\rho^2 + R^2)^2} R^2 \sigma_i, \quad (68)$$

$$B_k^{CI} = B_k^{C\bar{I}} = -\frac{2}{g(\rho^2 + R^2)^2} R^2 \sigma_k. \quad (69)$$

One can clearly see that the CYMI corresponds to ‘‘parallel’’ fields, whereas the anti-CYMI corresponds to ‘‘anti-parallel’’ fields in Minkowski spacetime (imaginary electric fields in Euclidean spacetime). This property we will show gives rise to pair production. Let us remark though that real Minkowski electric fields are not a sufficient requirement for pair production, a simple Abelian counterexample is provided through plane waves. Before solving Wong’s equations, let us discuss some basic properties of the fields.

The CYMI are not self-dual, but one can show the totally antisymmetric field strength tensor is related to an imaginary anti-CYMI. To show this let us introduce ‘t Hooft symbols,

$$\eta_{a\mu\nu}^C = \varepsilon_{a\mu\nu 4} + i\delta_{a\mu}\delta_{\nu 4} - i\delta_{a\nu}\delta_{\mu 4}, \quad (70)$$

$$\bar{\eta}_{a\mu\nu}^C = \varepsilon_{a\mu\nu 4} - i\delta_{a\mu}\delta_{\nu 4} + i\delta_{a\nu}\delta_{\mu 4}, \quad (71)$$

for $\sigma_{\mu\nu}^C = \frac{1}{2}\eta_{a\mu\nu}^C \sigma_a$ and $\bar{\sigma}_{\mu\nu}^C = \frac{1}{2}\bar{\eta}_{a\mu\nu}^C \sigma_a$. Then we can find the following identities:

$$\bar{\eta}_{a\mu\nu}^C = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\eta_{a\alpha\beta}^C = i\bar{\eta}_{a\mu\nu}^C, \quad (72)$$

$$\tilde{\eta}_{a\mu\nu}^C = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\bar{\eta}_{a\alpha\beta}^C = -i\eta_{a\mu\nu}^C. \quad (73)$$

To further explore the properties, let us introduce in matrix form the ‘t Hooft symbols as combinatory rotation and translation (imaginary rotation) generators of $\text{SO}(4)$. Namely we have that

$$K_{a\mu\nu} = i\delta_{a\mu}\delta_{\nu 4} - i\delta_{a\nu}\delta_{\mu 4}, \quad L_{a\mu\nu} = \varepsilon_{a\mu\nu 4}. \quad (74)$$

And we can write for the ‘t Hooft symbols $\eta_i^C = L_i + K_i$ and $\bar{\eta}_i^C = L_i - K_i$. We have that $K_i^T = -K_i$ and $L_i^T = -L_i$. The generators satisfy the following algebra:

$$[K_i, K_j] = \varepsilon_{ijk}L_k, \quad [L_i, L_j] = -\varepsilon_{ijk}L_k, \quad (75)$$

$$[L_i, K_j] = -\varepsilon_{ijk}K_k. \quad (76)$$

One may also show by introducing the tensor, $\Delta_{\mu\nu} := \delta_{\mu\nu} - 2\delta_{\mu 4}\delta_{\nu 4}$, the following relation holds: $\{L_i, K_j\} = \varepsilon_{ijk}K_k\Delta$. Note, we also have that $\{L_i, L_j\} = \{K_i, K_j\}$ for $i \neq j$. Last, one can determine that

$$\{\eta_i^C, \bar{\eta}_j^C\} = 2\varepsilon_{ijk}\Delta K_k - 2\delta_{ij}. \quad (77)$$

Using the generators, Eq. (74), one may directly find that not only do the CYMI have finite energies but also that they vanish,

$$\text{tr}G_{\mu\nu}^{CI}G_{\nu\mu}^{CI} = 0, \quad \text{tr}G_{\mu\nu}^{C\bar{I}}G_{\nu\mu}^{C\bar{I}} = 0. \quad (78)$$

One may also, using Eq. (77), confirm that we have

$$\text{tr}[G_{\mu\nu}^{CI}\tilde{G}_{\nu\mu}^{CI}] = -\frac{4 \cdot 24i}{g^2} \frac{R^4}{(\rho^2 + R^2)^4}, \quad (79)$$

and then in a Euclidean picture,

$$\frac{g^2}{16\pi^2} \int d^4x \text{tr}G_{\mu\nu}^{CI}\tilde{G}_{\nu\mu}^{CI} = -1, \quad (80)$$

in agreement with the Minowski definition, Eq. (58).

Likewise we have that $\frac{g^2}{16\pi^2} \int d^4x \text{tr}G_{\mu\nu}^{C\bar{I}}\tilde{G}_{\nu\mu}^{C\bar{I}} = 1$.

The CYMI, however, does not solve the Yang-Mills equation of motion. In fact one can find using the above identities that

$$[\mathcal{D}_\mu^{CI}, G_{\mu\nu}^{CI}] = \frac{8}{g(\rho^2 + R^2)^3} \{x^T \Delta(L + K) \cdot \sigma - x^T K \cdot \sigma\}_\nu. \quad (81)$$

However the Bianchi identity holds as it should, $[\mathcal{D}_\mu^{CI}, \tilde{G}_{\mu\nu}^{CI}] = 0$.

Wong’s equations in a CYMI written in a Lorentz index matrix representation read,

$$j_c^{CI} = -\frac{2}{\rho^2 + R^2} \varepsilon_{abc} \dot{x}^{cIT} \eta_a^C x^{CI} I_b^{CI}, \quad (82)$$

$$\dot{x}^{cCI} = \frac{i|\dot{x}^{cCI}|}{m} \frac{2R^2}{(\rho^2 + R^2)^2} \eta^C \cdot I^{CI} \dot{x}^{cCI}, \quad (83)$$

and we can evaluate them similar to as was accomplished in Sec. IV A, namely through the usage of a large parameter limit for the CYMI coupled with the adiabatic theorem in isospin.

B. Adiabatic theorem and the large complex instanton

Employing the adiabatic theorem one can determine a gauge element, $u^{CI} \in \text{SL}(2, \mathbb{C})$, such that the isospin equation of motion, Eq. (82), is satisfied and

$$u^{CI} A^{CI} \dot{x}^{CI} (u^{CI})^{-1} = \sqrt{A^{CIa} \dot{x}^{CI} A^{CIa} \dot{x}^{CI}} \sigma_3; \quad (84)$$

cf. Eq. (45). This gives us

$$I_a^{CI} = \frac{\dot{x}^{CIT} \eta_a^C x^{CI}}{\sqrt{\dot{x}^{CIT} \eta_a^C x^{CI} \dot{x}^{CIT} \eta_a^C x^{CI}}}. \quad (85)$$

The isospin for the CYMI is an element of the coset $SL(2, \mathbb{C})/SU(2)$. Again we have proptime independence, $\dot{I}^{CI} = 0$. And that we may apply the adiabatic theorem and in turn ignore off diagonal parts to Berry's phase, owing to the antisymmetric tensors, η_a^C , one must have that $x^{CI} \propto \dot{x}^{CI}$.

For the evaluation of the Lorentz force equation we examine the large complex instanton limit such that $R^2 \gg \rho^2$. And we have that

$$\dot{x}^{CI} \approx \frac{2i|\dot{x}^{CI}|}{mR^2} \eta^C \cdot I^{CI} \dot{x}^{CI}, \quad (86)$$

which we evaluate using electric and magnetic projection operators:

$$P_B^{CI} := \frac{1}{2}[1 - (\eta^C \cdot I^{CI})^2], \quad P_E^{CI} := \frac{1}{2}[1 + (\eta^C \cdot I^{CI})^2], \quad (87)$$

such that

$$(\eta^C \cdot I^{CI})^2 P_B^{CI} = -P_B^{CI}, \quad (\eta^C \cdot I^{CI})^2 P_E^{CI} = P_E^{CI}, \quad (88)$$

with the properties of idempotency, $(P_{B,E}^{CI})^2 = P_{B,E}^{CI}$, and also that $P_E^{CI} + P_B^{CI} = 1$ and $P_E^{CI} P_B^{CI} = 0$. We may use the projection operators to decouple the Lorentz force equation, and moreover the coordinates into electric and magnetic degrees of freedom,

$$x_E^{CI} = P_E^{CI} x^{CI}, \quad x_B^{CI} = P_B^{CI} x^{CI}. \quad (89)$$

The Lorentz force equation may be readily solved using projection operators; see [56] for details. Alternatively, one may simply find an $SO(4)$ transformation such that the coordinates in Eq. (86) are parallel in an arbitrary direction, which also projects the electric and magnetic parts, and leads to the exact same result. Solutions of the decoupled Lorentz force yield

$$\dot{x}_E^{CI}(\tau) = \left[\cos\left(\frac{2|\dot{x}^{CI}|}{mR^2}\tau\right) + i\eta^C \cdot I^{CI} \sin\left(\frac{2|\dot{x}^{CI}|}{mR^2}\tau\right) \right] \dot{x}_E^{CI}(0), \quad (90)$$

$$\begin{aligned} \dot{x}_B^{CI}(\tau) = & \left[\cosh\left(\frac{2|\dot{x}^{CI}|}{mR^2}\tau\right) \right. \\ & \left. + i\eta^C \cdot I^{CI} \sinh\left(\frac{2|\dot{x}^{CI}|}{mR^2}\tau\right) \right] \dot{x}_B^{CI}(0). \end{aligned} \quad (91)$$

The situation here is analogous to the one encountered for homogeneous fields in Sec. III C. Namely, we cannot both satisfy both equations given the periodic boundary conditions; to maintain a real stationary point and hence real $|\dot{x}^{CI}|$ we take the magnetic components trivial, i.e., $x_B^{CI} = \text{constant}$. The electric components, given the circular instanton constraint through $x^{CI} \propto \dot{x}^{CI}$, can be found as

$$x_E^{CI}(\tau) = \frac{imR^2}{2|\dot{x}^{CI}|} \bar{\eta}^C \cdot I^{CI} \dot{x}_E^{CI}(\tau). \quad (92)$$

Then the WIs are solely determined from the electric fields. We can immediately write down the WI periodic criteria as

$$|\dot{x}^{CI}| = n\pi m R^2 \quad \forall n \in \mathbb{Z}^+, \quad (93)$$

as anticipated, cf. Eq. (57). We do have pair production for the CYMI, the topological fields decay by a vacuum instability.

The Schwinger effect exponential suppression is $(1/2)n\pi m^2 R^2$, understood from the worldline action, Eq. (30). One can calculate using Eqs. (90) and (92) the contribution to the worldline action of $\int_0^1 d\tau \text{tr}[I^{CI} ig A^{CI} \dot{x}^{CI}] = (m/2)|\dot{x}^{CI}|$. The Berry's phase term, here too, only introduces a trivial factor of $4\pi i$.

VI. CONCLUSIONS

Schwinger pair production has been analyzed in the topological BPST instanton (YMI), and due to the Hermiticity of its construction, no vacuum decay via the Schwinger effect was found as anticipated. However, as an anomaly cancellation is found for Abelian homogeneous fields, (one which is revived through the Schwinger mechanism), it is likewise anticipated that a non-Abelian field configuration with Chern-Pontryagin density should be present and decay via the Schwinger effect. We construct such a field that is gauge invariant in $SL(2, \mathbb{C})$, in a Euclidean metric, and $SU(2)$, in a Minkowski metric. The field resembles parallel or antiparallel fields in Minkowski spacetime. To accomplish calculations, we extended the WI method to non-Abelian fields.

The WI method is important for the study of Schwinger pair production in inhomogeneous fields. Thus (apart from the study of the Schwinger effect in YMI/CYMI), our twofold scope included the development of the WI method for a generically complex $SL(2, \mathbb{C})$ background field. To arrive at Wong's equations—the non-Abelian equivalent of the Lorentz force equation—we made use of the coherent state method. There, color degrees of freedom were summed over in a Haar measure extended for the non-compact group.

The WI method in non-Abelian systems may prove useful for not only the fields discussed in this work but also for color-glass condensate [62] backgrounds, as are thought

present in the early stages of heavy-ion collisions. Furthermore, we expect the non-Abelian WI method to be essential in the development of the worldline formalism to handle a variety of topological field theories through the construction of bulk or boundary G/H type coset theories, such as the Wess-Zumino-Witten theory, conformal sigma models, and Chern-Simons theories [33,63]. Last, a realization of large N $SU(N)$ Yang-Mills and nonlinear sigma models through coherent states has been reported in [24], presenting us with an array of chiral models which would be worthwhile to revisit under the light of the WI method introduced here.

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APPENDIX: EXACT NONPERSISTENCE IN ABELIAN-LIKE HOMOGENEOUS PARALLEL FIELDS

Here we evaluate the contributions to Schwinger pair production coming from the effective action exactly in non-Abelian homogeneous parallel fields given by Eq. (31). This is accomplished by summing over the eigenvalues of the worldline Hamiltonian. Let us begin by writing the effective action, Eq. (2), in Schwinger proper time as $\Gamma[A] = \text{tr} \int_0^\infty \frac{dT}{T} \exp[-(-D^2 + m^2)T]$. The homogeneous fields are given in Eq. (32). It proves useful to first take the color trace, summing over both colors. This is permissible as the Hamiltonian is already diagonal in color. The effective action becomes

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int d^4x \langle x | [e^{D_+^2 T} + e^{D_-^2 T}] | x \rangle, \quad (\text{A1})$$

$$D_\pm^2 = \partial_1^2 + \partial_3^2 + (\partial_2 \mp igBx_1)^2 + (\partial_4 \mp i^2 gEx_3)^2. \quad (\text{A2})$$

Let us define a Euclidean Fourier transform such that $p \langle x | p \rangle = -i\partial/\partial x \langle x | p \rangle$, $\langle x_2 | p_2 \rangle \sim \exp(ip_2 x_2)$, $\int \frac{dp}{2\pi} |p\rangle \langle p| = 1$, and $\langle p | p' \rangle = 2\pi \delta(p - p')$. Upon insertion of complete sets of states, one may then find the eigenvalues of Eq. (A2) as $E_{\pm n, m} = \mp 2(n + \frac{1}{2})gB \mp 2(m + \frac{1}{2})igE$, for $n, m \in [0, \infty)$. Upon summing over the eigenvalues one finds for the effective action

$$\Gamma[A] = \frac{i}{2} \int_0^\infty \frac{dT}{T} \int dx_{2,4} \int \frac{dp_{2,4}}{(2\pi)^2} \frac{\exp(-m^2 T)}{\sinh(gBT) \sin(gET)}, \quad (\text{A3})$$

where the contribution from D_-^2 is the same as that of D_+^2 . The coordinate and momenta integrals are divergent; this is due to our selection of homogeneous fields which are unbounded. We may normalize the action by considering a closed box with Landau modes such that $\int dx_2 = L$ and $\int dp_2 = gBL$, and likewise for Euclidean time for general length L . Hence

$$\Gamma[A] = \frac{2ig^2 EBL^4}{(4\pi)^2} \int_0^\infty \frac{dT}{T} \frac{\exp(-m^2 T)}{\sinh(gBT) \sin(gET)}. \quad (\text{A4})$$

We can find the contribution which pertains to Schwinger pair production by noting that $\text{Im}\Gamma_{\text{Mink}}[A] = \text{Re}\Gamma_{\text{Euclidean}}[A]$. This can be found through the poles on the real positive axis in the sine function. Taking the residues one can find the exact to one-loop pair production nonpersistence as

$$\text{Re}\Gamma[A] = \frac{g^2 EBL^4}{8\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{\pi n m^2}{gE}} \sinh^{-1} \left(\frac{n\pi B}{E} \right). \quad (\text{A5})$$

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