Finite-volume formalism in the $2\xrightarrow{H_I+H_I}2$ transition: An application to the lattice QCD calculation of double beta decays

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We present the formalism for connecting a second-order electroweak $2\xrightarrow{H_I+H_I}2$ transition amplitudes in the finite volume (with two hadrons in the initial and final states) to the physical amplitudes in the infinite volume. Our study mainly focuses on the case where the low-lying intermediate state consists of two scattering hadrons. As a side product, we also reproduce the finite-volume formula for $2\xrightarrow{H_I}2$ transition, originally obtained by Briceño and Hansen [Phys. Rev. D **94**, 013008 (2016)]. With the available finite-volume formalism, we further discuss how to treat with the finite-volume problem in the double beta decays $nn \to ppee\bar{\nu}\bar{\nu}$ and $nn \to ppee$.

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I. INTRODUCTION

Lattice QCD provides a well-established nonperturbative approach to solve the quantum chromodynamics (QCD) theory of quarks and gluons. Using the high-performance supercomputers, the quarks and gluons are enclosed and simulated in a discretized, finite-volume lattice. Controlling the various systematic effects such as lattice discretization effects, finite-volume effects, and unphysical quark mass effects is required for the lattice QCD calculation to make the high-precision predication from first principles. On the other hand, in some cases, the study of the systematic effects is much more than the reduction of the uncertainty. It could lead to the new methodology to solve the interesting physics problems. For example, the study of the pion mass dependence from lattice QCD interplays with the chiral perturbation theory, yielding a deeper understanding of the chiral dynamics of QCD. Another example is the pioneering work on the finite-volume formalism by Lüscher [1–3]. It allows us to connect the discrete energy spectrum calculated from lattice QCD to the infinite-volume scattering phase and has played an important role in understanding the hadron spectra and hadron-hadron scattering.

The finite-volume formalism generically includes three topics.

(i) Finite-volume energy quantization relates the discrete energy in the finite volume to the scattering phase in the infinite volume. The best examples under well investigation are the pion-pion scattering in the isospin $I = 2$ [4–14], $I = 1$ (ρ resonance relevant) [15–30], and $I = 0$ (σ resonance and disconnected diagrams relevant) [31–37] channels. Due to the good signals provided by the pion-pion system, a lot of attentions are paid to these scattering channels in the past years. For more lattice calculations of scattering amplitudes, we refer to a recent review [38].

(ii) The Lellouch-Lüscher relation [39] connects the finite-volume matrix element with two hadrons in either the initial or final state to the physical matrix element in the infinite volume. Such examples include $0\xrightarrow{J}2$ decays, e.g., the timelike pion form factor [21,28,30,40], $1\xrightarrow{H_I}2$ and $1\xrightarrow{J}2$ decays[1]

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[1]By using the current J, it means that the transition goes through a quark bilinear vertex. By using H_I, it indicates an interaction term such as a four-quark operator, where the $1\xrightarrow{J}1$ subprocesses do not exist.

including $K \to \pi\pi$ [41–45] and $\pi\pi \to \pi\gamma^*$ transition [46–48], and $2\xrightarrow{J}2$ decays recently studied in Refs. [49,50].

(iii) A finite-volume formula for long-distance electroweak amplitudes [51–53] relates the bilocal matrix element in the finite volume to the physical one in the infinite volume. This formalism is first developed to solve the finite-volume problem for K_L-K_S mixing [54–57] and has been used for other second-order electroweak processes such as rare kaon decays [58–63]. Recently, the formalism is generalized in Ref. [64] to access more long-distance observables.

It is found by Ref. [53] that the above three finite-volume formulas can be derived in a uniform way in the framework of quantum field theory using the techniques of Kim *et al.* (KSS) [65].

In this work, we present the derivation of the finite-volume formula for long-distance electroweak amplitudes with two hadrons in both the initial and final states ($2\xrightarrow{H_I+H_I}2$). We consider the scattering process with two channels, which are mixed by the electroweak interaction. We label these two channels by α and β. The master formula is given as

$$\begin{aligned} &\frac{d(\phi+\delta_\alpha^{(0)})}{dE}\Delta E_\alpha + \Delta\delta_\alpha \\ &\quad = \frac{1}{4}\cot(\phi+\delta_\beta^{(0)})|\langle E,\mathrm{in},\beta|H_I|E,\mathrm{in},\alpha\rangle|^2, \\ &\text{at } E = E_\alpha^{(0)}, \end{aligned} \tag{1}$$

where $E_\alpha^{(0)}$ is discrete energy for the initial/final state without non-QCD correction. ΔE_α is the energy shift when turning on the second-order electroweak interaction, and it is equal to the $2\xrightarrow{H_I+H_I}2$ finite-volume matrix element calculated on the lattice. ϕ is a known, kinematic function, originally introduced by Lüscher in Eq. (6.12) of Ref. [2]. $\delta_\alpha^{(0)}$ is the strong scattering phase for the initial/final state, and $\delta_\beta^{(0)}$ is the scattering phase for the low-lying two-hadron intermediate state. Here, we consider the case that the lowest intermediate state consists of two interacting hadrons. $\Delta\delta_\alpha$ is the shift in the total scattering phase with the existence of the non-QCD interaction. It is equivalent to the infinite-volume $2\xrightarrow{H_I+H_I}2$ matrix element as we explain later. The derivation is performed using the perturbative approach proposed by Lellouch and Lüscher [39] together with the coupled-channel finite-volume energy quantization condition [66,67]. The derivation of Eq. (1) is performed under the assumption that there are no $1\xrightarrow{H_I}1$ subprocesses, which significantly simplifies the analysis. As a side product, we also obtain the finite-volume formula for the $2\xrightarrow{H_I}2$ transition with H_I carrying the vanishing momentum. For the more general cases of $2\xrightarrow{J}2$, we refer to Refs. [49,50].

We find that the KSS approach [65] treats the finite-volume problem in a thorough and fundamental way using the Poisson summation formula. Many new developments of the finite-volume formalism have made progress along the direction proposed by KSS. On the other hand, the approach invented by Lellouch and Lüscher [39] creates another possibility that one can obtain the finite-volume formalism in a relatively simple way as the finite-volume information is already incorporated inside Lüscher's quantization condition, and it is not necessary to investigate it again using Poisson summation formula.

The paper is organized as follows. In Sec. II, we discuss the discrete energy shift in the finite volume due to the existence of the $2\xrightarrow{H_I+H_I}2$ transition. In Sec. III, we discuss the infinite-volume scattering amplitude relevant for the $2\xrightarrow{H_I+H_I}2$ transition. In Sec. IV, the energy shift is related to the scattering amplitude using the coupled-channel quantization condition, and thus, the finite-volume formalism Eq. (1) is obtained. In Sec. V, we discuss the possible solutions to the finite-volume problems in the double beta decays.

II. $2\xrightarrow{H_I+H_I}2$ TRANSITION IN THE FINITE VOLUME

We consider the full Hamiltonian including both QCD and non-QCD interactions as

$$H^L = H_0^L + H_I^L, \tag{2}$$

where H_0^L stands for the pure strong interaction and H_I^L indicates the non-QCD ones, induced by, e.g., an electromagnetic or weak interaction. The superscript L reminds us that all the interactions are constrained by a finite volume. For simplicity, we assume that the $1\xrightarrow{H_I}1$ subprocesses do not exist.

When the interaction H_I is turned on, it is possible that two independent strong scattering (or bound) channels are mixed by the non-QCD interaction. For example, we can consider a transition process of $\alpha\xrightarrow{H_I}\beta\xrightarrow{H_I}\alpha$, where the two-particle state α mixes with β. To specify this character of the $2\xrightarrow{H_I+H_I}2$ transition, we assign two low-lying eigenstates of the Hamiltonian H_0^L as $|\alpha\rangle^L$ and $|\beta\rangle^L$, which satisfy the normalization conditions,

$${}^L\langle\alpha|H_0^L|\alpha\rangle^L = E_\alpha^{(0)}, \quad {}^L\langle\beta|H_0^L|\beta\rangle^L = E_\beta^{(0)}, \quad {}^L\langle\beta|H_0^L|\alpha\rangle^L = 0, \tag{3}$$

and $E_\alpha^{(0)}$ and $E_\beta^{(0)}$ are the corresponding energy eigenvalues. These two states are independent when turning off the non-QCD interactions but mix with each other when turning on these interactions. In the finite volume, the spectra of QCD Hamiltonian are discrete, and it allows for multiple nearly degenerate states. Here, we focus on only one of them and classify all the other states as $|n_\alpha\rangle^L$ and $|n_\beta\rangle^L$, where $|n_\alpha\rangle^L$ and $|n_\beta\rangle^L$ have the same quantum number as $|\alpha\rangle^L$ and $|\beta\rangle^L$, respectively. We introduce the projectors,

$$Q = \sum_{n=\alpha,\beta} |n\rangle^{LL}\langle n|, \qquad P = 1 - Q, \tag{4}$$

to construct a two-state subspace.

The eigenvalue equation for the full Hamiltonian is given by

$$(H_0^L + H_I^L)|n\rangle_I^L = E_n|n\rangle_I^L. \tag{5}$$

In the notation of the eigenstate $|n\rangle_I^L$, the subscript I is used to indicate the existence of the non-QCD interaction. Acting P and Q on the above equation, we have

$$\begin{aligned} H_0^L P|n\rangle_I^L + PH_I^L(Q+P)|n\rangle_I^L &= E_n P|n\rangle_I^L, \\ H_0^L Q|n\rangle_I^L + QH_I^L(Q+P)|n\rangle_I^L &= E_n Q|n\rangle_I^L. \end{aligned} \tag{6}$$

This results in

$$\begin{aligned} PH_I^L Q|n\rangle_I^L &= (E_n - H_0^L - PH_I^L P)P|n\rangle_I^L, \\ QH_I^L P|n\rangle_I^L &= (E_n - H_0^L - QH_I^L Q)Q|n\rangle_I^L. \end{aligned} \tag{7}$$

Inserting $P|n\rangle_I^L = P(E_n - H_0^L - PH_I^L P)^{-1}PH_I^L Q|n\rangle_I^L$ into the second line of Eq. (7), we have

$$\begin{aligned} &QH_I^L P(E_n - H_0^L - PH_I^L P)^{-1}PH_I^L Q|n\rangle_I^L \\ &\quad = (E_n - H_0^L - QH_I^L Q)Q|n\rangle_I^L. \end{aligned} \tag{8}$$

By neglecting the $O(H_I^3)$ terms, we obtain the equations,

$$(\tilde{H}_0 + \tilde{H}_I)Q|n\rangle_I^L = E_n Q|n\rangle_I^L, \tag{9}$$

with

$$\tilde{H}_0 = H_0^L + QH_I^L Q, \quad \tilde{H}_I = QH_I^L P(E_n - H_0^L)^{-1}PH_I^L Q. \tag{10}$$

The existence of the nonzero solutions for equations,

$$\begin{aligned} &{}^L\langle\alpha|\tilde{H}_0 + \tilde{H}_I|\alpha\rangle^{LL}\langle\alpha|\alpha\rangle_I^L + {}^L\langle\alpha|\tilde{H}_0 + \tilde{H}_I|\beta\rangle^{LL}\langle\beta|\alpha\rangle_I \\ &\quad = E_\alpha{}^L\langle\alpha|\alpha\rangle_I^L, \\ &{}^L\langle\beta|\tilde{H}_0 + \tilde{H}_I|\alpha\rangle^{LL}\langle\alpha|\alpha\rangle_I^L + {}^L\langle\beta|\tilde{H}_0 + \tilde{H}_I|\beta\rangle^{LL}\langle\beta|\alpha\rangle_I \\ &\quad = E_\alpha{}^L\langle\beta|\alpha\rangle_I^L, \end{aligned} \tag{11}$$

requires that the secular equation holds

$$\begin{vmatrix} {}^L\langle\alpha|\tilde{H}_0 + \tilde{H}_I|\alpha\rangle^L - E_\alpha & {}^L\langle\alpha|\tilde{H}_0 + \tilde{H}_I|\beta\rangle^L \\ {}^L\langle\beta|\tilde{H}_0 + \tilde{H}_I|\alpha\rangle^L & {}^L\langle\beta|\tilde{H}_0 + \tilde{H}_I|\beta\rangle^L - E_\alpha \end{vmatrix} = 0. \tag{12}$$

For the general case with $E_\alpha^{(0)} \neq E_\beta^{(0)}$, the solution of E_α is given by

$$\begin{aligned} E_\alpha &= E_\alpha^{(0)} + \Delta E_\alpha, \\ \Delta E_\alpha &= \frac{|{}^L\langle\beta|H_I^L|\alpha\rangle^L|^2}{E_\alpha^{(0)} - E_\beta^{(0)}} + \sum_{n_\beta\neq\beta} \frac{|{}^L\langle n_\beta|H_I^L|\alpha\rangle^L|^2}{E_\alpha^{(0)} - E_{n_\beta}^{(0)}}. \end{aligned} \tag{13}$$

The energy shift ΔE_α is exactly the finite-volume long-distance matrix element obtained from a lattice QCD calculation.

Here, we obtain Eq. (13) using the second-order degenerate perturbation theory. In fact, Eq. (13) is the standard result from the second-order perturbation theory, and we expect the derivation could be simpler using the common perturbation theory.

III. $2\xrightarrow{H_I+H_I}2$ TRANSITION IN THE INFINITE VOLUME

Now we consider the $2\xrightarrow{H_I+H_I}2$ transition in the infinite volume. For simplicity, we only discuss the case that the low-lying intermediate state is given by a two-particle scattering state or a one-particle bound state. For the former, the transition amplitude involves the input of a 2×2 scattering S matrix. For the latter, a single-channel S matrix is relevant.

A. Process of $2\xrightarrow{H_I}2\xrightarrow{H_I}2$

We first consider the scattering state by turning off the non-QCD interactions. In the infinite volume, we use $|E, \mathrm{in}, \alpha\rangle$ to describe the incoming scattering state and $\langle E, \mathrm{out}, \alpha|$ for the outgoing scattering state. The low-lying intermediate scattering state is described by $|E, \mathrm{in}, \beta\rangle$. For simplicity, here we only consider the S-wave scattering. The relevant normalization condition is assigned as

$$\langle E', \mathrm{in}, \beta|E, \mathrm{in}, \alpha\rangle = 2\pi\delta(E - E')\delta_{\alpha\beta}. \tag{14}$$

The scattering S matrix is defined as

$$\begin{pmatrix} \langle E', \text{out}, \alpha | E, \text{in}, \alpha \rangle & \langle E', \text{out}, \beta | E, \text{in}, \alpha \rangle \\ \langle E', \text{out}, \alpha | E, \text{in}, \beta \rangle & \langle E', \text{out}, \beta | E, \text{in}, \beta \rangle \end{pmatrix}$$
$$= 2\pi\delta(E - E')S,$$
$$S = \begin{pmatrix} e^{2i\delta_\alpha^{(0)}} & 0 \\ 0 & e^{2i\delta_\beta^{(0)}} \end{pmatrix}. \tag{15}$$

Without non-QCD interactions, there is no mixing between α and β states. Thus, S is a diagonal matrix with $\delta_\alpha^{(0)}$ and $\delta_\beta^{(0)}$ the scattering phases for pure strong interaction. We use $|\alpha'\rangle$ and $|\beta'\rangle$ to stand for the excited states, which have the same quantum number as $|E, \text{in}, \alpha\rangle$ and $|E, \text{in}, \beta\rangle$, respectively. We assume that the threshold energy E_{th} for these excited states are above the energy region we are interested in.

When turning on the non-QCD interactions, the scattering state for full Hamiltonian $H = H_0 + H_I$ is given by

$$|E, \text{in}, \alpha\rangle_I = |E, \text{in}, \alpha\rangle + G_E^{(+)} H_I |E, \text{in}, \alpha\rangle_I, \tag{16}$$

where

$$G_E^{(+)} = \frac{1}{E - H_0 + i\varepsilon} = \mathcal{PV}\frac{1}{E - H_0} - i\pi\delta(E - H_0) \tag{17}$$

is the standard Green's function. With non-QCD interactions, we parametrize the S matrix following Refs. [66,67],

$$S_I = \begin{pmatrix} ce^{2i\delta_\alpha} & ise^{i\delta_\alpha + i\delta_\beta} \\ ise^{i\delta_\alpha + i\delta_\beta} & ce^{2i\delta_\beta} \end{pmatrix}, \tag{18}$$

where the real values c and s satisfy the relation $c^2 + s^2 = 1$. This parametrization makes the derivation of the finite-volume formalism very straightforward. (In some other cases, e.g., in the $K \to \pi\pi$ decay where $I = 0$ and $I = 2$ $\pi\pi$ states mix due to the existence of electromagnetic interactions [68], it is simpler to use the parametrization proposed by Ref. [69].)

It is useful to relate the S matrix to the T matrix using the relation $S = 1 + iT$. After turning on the non-QCD interaction, the change of the T matrix is given by

$$\Delta T = -i\begin{pmatrix} ce^{2i\delta_\alpha} - e^{2i\delta_\alpha^{(0)}} & ise^{i\delta_\alpha + i\delta_\beta} \\ ise^{i\delta_\alpha + i\delta_\beta} & ce^{2i\delta_\beta} - e^{2i\delta_\beta^{(0)}} \end{pmatrix}. \tag{19}$$

On the other hand, the matrix of ΔT can be constructed using the scattering state through

$$\Delta T = -\begin{pmatrix} \langle E,\text{out},\alpha|H_I|E,\text{in},\alpha\rangle_I & \langle E,\text{out},\beta|H_I|E,\text{in},\alpha\rangle_I \\ \langle E,\text{out},\alpha|H_I|E,\text{in},\beta\rangle_I & \langle E,\text{out},\beta|H_I|E,\text{in},\beta\rangle_I \end{pmatrix}. \tag{20}$$

We can make the perturbative expansion of ΔT. Up to $O(H_I^2)$, we find

$$\Delta T = -\begin{pmatrix} e^{2i\delta_\alpha^{(0)}}(K_\alpha - i|J|^2/2) & e^{i\delta_\alpha^{(0)} + i\delta_\beta^{(0)}} J \\ e^{i\delta_\alpha^{(0)} + i\delta_\beta^{(0)}} J^* & e^{2i\delta_\beta^{(0)}}(K_\beta - i|J|^2/2) \end{pmatrix}, \tag{21}$$

where

$$K_\alpha = \mathcal{PV}\int \frac{dE'}{2\pi} \frac{|\langle E', \text{in}, \beta | H_I | E, \text{in}, \alpha\rangle|^2}{E - E'} + \sum\!\!\!\!\!\!\int_{\beta'} \frac{|\langle \beta' | H_I | E, \text{in}, \alpha\rangle|^2}{E - E_{\beta'}},$$
$$J = e^{i\delta_\beta^{(0)} - i\delta_\alpha^{(0)}} \langle E, \text{in}, \beta | H_I | E, \text{in}, \alpha\rangle. \tag{22}$$

Here, we have used the simplified symbol $\not{\Sigma}\!\!\!\!\int_{\beta'} \equiv \sum_{\beta'} \int_{E_{\text{th}}}^{\infty} \frac{dE_{\beta'}}{2\pi}$. Under the symmetry of the time reversal invariance, J is a real quantity. By exchanging α and β for K_α, one gets the expression for K_β.

Equating Eqs. (19) and (21), we obtain

$$s = -J, \qquad \Delta\delta_\alpha \equiv \delta_\alpha - \delta_\alpha^{(0)} = -\frac{K_\alpha}{2},$$
$$\Delta\delta_\beta \equiv \delta_\beta - \delta_\beta^{(0)} = -\frac{K_\beta}{2}. \tag{23}$$

B. Process of $2 \xrightarrow{H_I} 1 \xrightarrow{H_I} 2$

For the $2 \xrightarrow{H_I + H_I} 2$ process with a deeply bound intermediate state, the first example comes from $\pi\pi \to K \to \pi\pi$ in L. Lellouch and M. Lüscher's work [39]. Later, H. Meyer extended it to the case of $\pi\pi \to W \to \pi\pi$ [40], where a massive gauge boson W is introduced and annihilated with an auxiliary vector field to obtain a finite-volume formula for the timelike pion form factor. In Ref. [51], N. Christ used again the $\pi\pi \to K \to \pi\pi$ transition amplitude to obtain a finite-volume correction for the K_L-K_S mass difference. Here, we include the process of $2 \xrightarrow{H_I} 1 \xrightarrow{H_I} 2$ simply for the completeness of the discussion.

If β is a deeply bound state, it is not necessary to introduce a 2×2 S matrix. The correction to the T matrix due to the non-QCD interaction is given by

$$\Delta T = -\langle E, \text{out}, \alpha | H_I | E, \text{in}, \alpha\rangle_I. \tag{24}$$

Using Eq. (16) and inserting the $|\beta\rangle$ and $|\beta'\rangle$ states into ΔT, one can obtain

$$\Delta T = -e^{2i\delta_\alpha^{(0)}}\left(\frac{|\langle\beta|H_I|E,\mathrm{in},\alpha\rangle|^2}{E-E_\beta}\right.$$
$$\left.+\sum\!\!\!\!\!\!\!\int_{\beta'}\frac{|\langle\beta'|H_I|E,\mathrm{in},\alpha\rangle|^2}{E-E_{\beta'}}\right). \tag{25}$$

It results in

$$\Delta\delta_\alpha \equiv \delta_\alpha - \delta_\alpha^{(0)} = -\frac{\hat{K}_\alpha}{2},$$
$$\hat{K}_\alpha = \frac{|\langle\beta|H_I|E,\mathrm{in},\alpha\rangle|^2}{E-E_\beta} + \sum\!\!\!\!\!\!\!\int_{\beta'}\frac{|\langle\beta'|H_I|E,\mathrm{in},\alpha\rangle|^2}{E-E_{\beta'}}. \tag{26}$$

IV. FINITE-VOLUME FORMALISM

In this section, we present the finite-volume formalism, which connects the matrix elements that can be calculated in the finite volume using lattice QCD to the infinite-volume transition amplitudes.

We first discuss the $2\xrightarrow{H_I}2\xrightarrow{H_I}2$ transition. The coupled-channel finite-volume energy quantization condition has been first established by Refs. [66,67] in 2005 using quantum mechanics. Later, there have been a number of papers studying the generalization of Lüscher's quantization condition to multiple channels [69–73]. For example, in Ref. [69], the quantization condition is extended to quantum field theory using the KSS approach [65].

When turning on the non-QCD interaction, we adopt the quantization condition from Refs. [66,67],

$$(e^{-2i(\phi+\delta_\alpha)}-c)(e^{-2i(\phi+\delta_\beta)}-c)+s^2=0, \quad \text{at } E=E_\alpha, \tag{27}$$

where the angle ϕ is a known function of a discrete, finite-volume energy E [2]. [By multiplying a factor of $e^{2i\delta_\alpha+2i\delta_\beta}$, Eq. (27) can reproduce Eq. (34) in Ref. [66].] When turning off the non-QCD interaction, we have

$$e^{-2i(\phi+\delta_\alpha^{(0)})}-1=0, \quad \text{at } E=E_\alpha^{(0)}. \tag{28}$$

Comparing Eqs. (27) and (28) and using the relation $s^2=|\langle E,\mathrm{in},\beta|H_I|E,\mathrm{in},\alpha\rangle|^2$ given in Eq. (23), we obtain the master formula given in Eq. (1). We copy it here for the sake of an easier read,

$$\frac{d(\phi+\delta_\alpha^{(0)})}{dE}\Delta E_\alpha + \Delta\delta_\alpha$$
$$=\frac{1}{4}\cot(\phi+\delta_\beta^{(0)})|\langle E,\mathrm{in},\beta|H_I|E,\mathrm{in},\alpha\rangle|^2,$$
$$\text{at } E=E_\alpha^{(0)}, \tag{29}$$

where ΔE_α is the finite-volume matrix element defined in Eq. (13), and $\Delta\delta_\alpha$ is the infinite-volume matrix element defined in Eq. (23). It is not surprising that the finite-volume correction formula takes the form of Eq. (29) as the initial/final state receives a correction of the Lellouch-Lüscher factor $\frac{d(\phi+\delta_\alpha^{(0)})}{dE}$, and the intermediate state receives a correction of the factor $\cot(\phi+\delta_\beta^{(0)})$ as first obtained by Refs. [52,53]. It is known that the energy quantization condition can be used for a shallow bound state through the analytical continuation [74,75]. Thus, the master formula derived here can be extended from a scattering state to a shallow bound state.

In the limit of $E_\beta^{(0)}\to E_\alpha^{(0)}$, both ΔE_α and $\cot(\phi+\delta_\beta^{(0)})$ in Eq. (29) become singular. By equating the residue of the poles, we obtain

$$h'_\alpha|^L\langle\beta|H_I^L|\alpha\rangle^L|^2h'_\beta = \frac{1}{4}|\langle E,\mathrm{in},\beta|H_I|E,\mathrm{in},\alpha\rangle|^2,$$
$$\text{at } E=E_\alpha^{(0)} \quad \text{and} \quad E_\beta^{(0)}\to E_\alpha^{(0)}, \tag{30}$$

where $h_i=\phi+\delta_i^{(0)}$ and $h'_i=dh_i/dE$ for $i=\alpha,\beta$. We thus reproduce the finite-volume correction formula for the $2\xrightarrow{J}2$ transition matrix in the special case that there are no $1\xrightarrow{J}1$ subprocesses, which is originally studied by Ref. [49].

For the $2\xrightarrow{H_I}1\xrightarrow{H_I}2$ transition, the corresponding finite-volume formula is given by

$$\frac{d(\phi+\delta_\alpha^{(0)})}{dE}\Delta E_\alpha + \Delta\delta_\alpha = 0, \tag{31}$$

where ΔE_α is given by Eq. (13) and $\Delta\delta_\alpha$ is given by Eq. (26).

V. APPLICATION TO DOUBLE BETA DECAYS

The observation of neutrinoless double beta ($0\nu 2\beta$) decays would prove neutrinos as Majorana fermions and the lepton number violation in nature. As a result, the study of double beta decays attracts a lot of interests from both experimental and theoretical sides. Current knowledge of second-order weak-interaction nuclear matrix elements needs to be improved, as various nuclear models lead to discrepancies on the order of 100% [76]. A promising approach to improving the reliability of the theoretical predication is to combine the chiral effective field theory (χEFT) [77–84] with lattice QCD and then provide well-constrained few-body inputs to *ab initio* many-body calculations [76]. Efforts have been invested to calculate double beta decays in both pion [85–89] and nucleon [90,91] sector from lattice QCD.

We start the discussion of the finite-volume problem for the double beta decays in the pion sector, taking the

$\pi^-\pi^- \to \pi^- e\bar{\nu} \to ee$ and $\pi^- \to \pi^0 e\bar{\nu} \to \pi^+ ee$ as examples. If we only consider the hadronic particles, the former process is a $2 \xrightarrow{J} 1 \xrightarrow{J} 0$ transition, and the latter is a $1 \xrightarrow{J} 1 \xrightarrow{J} 1$ transition. However, one needs to pay attention to the finite-volume effects caused by the massless neutrino in the intermediate state. For the case of the $\pi^-\pi^- \to \pi^- e\bar{\nu} \to ee$ transition, there are two sources of power-law finite-volume effects [86]. One arises from the $\pi^-\pi^-$ initial state and is corrected by the inclusion of the Lellouch-Lüscher factor. The other one originates from the massless neutrino and is estimated as an $O(L^{-2})$ effect by using the $\mathrm{QED_L}$ technique. In the study of the $\pi^- \to \pi^0 e\bar{\nu} \to \pi^+ ee$ transition [88], a novel method called *infinite-volume reconstruction* [92] is used to treat the massless neutrino in the intermediate state. This method reduces the usual power-law finite-volume effect induced by the neutrino-pion loop to an exponentially suppressed effect. With the finite-volume corrections, Refs. [86,88] produce the lattice results for the double beta decay amplitudes, which are well consistent with the χEFT formula [79] and much more accurate than the estimates from the phenomenological study [93]. In an exploratory study [87], Detmold and Murphy make an attempt to use a massive neutrino for $\pi^- \to \pi^0 e\bar{\nu} \to \pi^+ ee$ and then study the neutrino mass dependence. (In a recent work [89], the authors use the massless neutrinos in their latest results, where a power-law finite-volume effect is a relevant issue.) We consider the massive neutrino a good solution to the finite-volume problem, particularly in $0\nu 2\beta$ decay $nn \to ppee$ as we will explain below. A similar idea to use the massive photon as an infrared regularization scheme for lattice QCQ + QED can be found in Ref. [94].

A. $2\nu 2\beta$ decay $nn \to ppee\bar{\nu}\bar{\nu}$

The pioneering lattice QCD calculation of $nn \to ppee\bar{\nu}\bar{\nu}$ has been performed by the NPLQCD Collaboration [90,91]. At the physical pion mass, it is well known that the $^1S_0$ state is a scattering state while the $^3S_1$ is a shallow bound state below the threshold and a scattering state above the threshold. In general, one can treat the shallow bound state as a two-body system. Note that the $nn \to ppee\bar{\nu}\bar{\nu}$ transition involves the $n \to pe\bar{\nu}$, a $1 \xrightarrow{J} 1$ subprocess. It limits the direct usage of the finite-volume formalism developed in this work.

In Fig. 1, we show three examples of two-particle loop diagrams relevant for the development of the finite-volume formalism. For diagram (a), since it does not involve the $1 \xrightarrow{J} 1$ subprocess, the finite-volume formalism is consistent with what we have derived. For diagram (b), Briceno and Hansen have proposed a beautiful solution to the three-propagator loop integral in Ref. [49]. For diagram (c), where a four-propagator loop is involved, the problem becomes much more challenging. First, the loop integral is more singular than that in diagrams (a) and (b). Second, it is unclear how to relate the finite-volume effects, which originates from the singularities of the four-propagator loop, to the on shell physical amplitudes. In Ref. [95], Davoudi and Kadam provide a finite-volume formalism based on pionless effective field theory, where a nonrelativistic kinematic is assumed. The relativistic field-theory solution for the finite-volume problem in the $nn \to ppee\bar{\nu}\bar{\nu}$ still remains a challenge.

For many interesting $2 \to 2 \to 2$ quantities, including the double beta decays, the Compton tensor of deuteron, the neutrino-nucleus scattering, and the laser spectroscopy of muonic deuterium, the $1 \xrightarrow{J} 1$ subprocess is unavoidable. In this sense, the formalism derived in this work may be considered as a first attempt for a simplified case, which includes diagram (a) only. The development of the complete finite-volume formalism, including especially the diagram (c), is still an open and important problem. It requires more investigations in the future.

B. $0\nu 2\beta$ decay $nn \to ppee$

The finite-volume problem for a $0\nu 2\beta$ decay $nn \to ppee$ is more complicated for two reasons. First, the neutrino, proton, and neutron in the low-lying intermediate states form a three-body system. Second, the massless neutrino enclosed in a finite-size box results in an additional power-law finite-volume effect. Although Ref. [92] developed the infinite-volume reconstruction method to eliminate the power-law finite-volume effects for the system with a massless photon and a stable hadron in the low-lying intermediate state, it is much harder to do this for a system with a massless neutrino and two hadrons in the intermediate state.

As pointed out by Ref. [81], a leading-order, short-range contribution needs to be introduced in the χEFT study of the $nn \to ppee$ decay. Such a short-range contribution breaks down Weinberg's power-counting scheme. New local operators need to be introduced in the effective

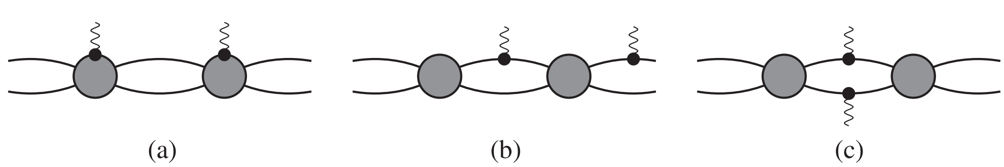

FIG. 1. The examples of two-particle loop diagrams relevant for the development of the finite-volume formalism.

action to account for this contribution. Our goal of the lattice calculation is to calculate the low energy constants for these new local operators. Fortunately, these low energy constants are irrelevant with the ultrasoft region where neutrino's energy is much smaller than the pion mass. Besides, the ultrasoft information from the $nn \to ppee$ decay is not very useful for the heavy-nuclei $0\nu 2\beta$ decay. In that case, the ultrasoft neutrino can feel the complete nucleus instead of just the nucleons. One would rely on the *ab initio* many-body theory to treat the nuclei properly.

We thus propose to introduce a nonzero mass for neutrino to remove the ultrasoft contribution. For simplicity, the neutrino mass can be chosen the same as the pion mass. Such a choice would unavoidably introduce the unphysical effects. However, as far as the lattice QCD calculation and the χEFT use the same unphysical neutrino mass, the low energy constants can be determined in a clean way. Compared to the other IR regulator, such as the $\mathrm{QED_L}$ technique, introducing the massive neutrino is relatively simple for χEFT. As far as the nonzero neutrino mass is introduced, at the threshold of dibaryon, the three particles in the intermediate state (two nucleons and one neutrino) cannot be on shell simultaneously. Thus, one can effectively treat the double beta decay as a $2 \xrightarrow{H_{\mathrm{eff}}} 2$ system with H_{eff} an effective Hamiltonian generated by two weak-interaction operators. The formula in Eq. (30) can be applied to this case.

VI. CONCLUSION

In this work, we derive the finite-volume formula, which connects a $2 \xrightarrow{H_I + H_I} 2$ transition amplitude in the finite volume to the physical amplitudes in the infinite volume. We discuss the cases with the low-lying intermediate state consisting of two scattering hadrons or a single stable hadron. Using the idea originally proposed by Lellouch and Lüscher, the derivation is simple and straightforward. As a side product, we reproduce the finite-volume formalism for the $2 \xrightarrow{J} 2$ transition previously obtained by Ref. [49].

We discuss the application of the finite-volume formula of the $2 \xrightarrow{H_I + H_I} 2$ transition to the lattice QCD calculation of the double beta decay. In the case of the $nn \to ppee$ decay, we propose to use the massive neutrino to avoid the complication of the finite-volume problem induced by the long-range massless neutrino.

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