Remarks on the analysis of the reaction $e^+e^- \to \Sigma^0\bar{\Sigma}^0$

Göran Fäldt*
Department of Physics and Astronomy, Uppsala University, Box 516, S-751 20 Uppsala, Sweden

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We investigate roads for evaluating model-independent cross-section-distribution functions for the sequential-hyperon decay $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$ and its corresponding antihyperon decay. The Σ^0 and $\bar{\Sigma}^0$ hyperons are produced in the reaction $e^+e^- \to J/\psi \to \bar{\Sigma}^0\Sigma^0$. Cross-section-distribution functions are calculated using the folding technique, but a comparison with results using the helicity technique is also made.

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I. INTRODUCTION

The BESIII experiment [1] is exploring new venues into hyperon physics, based on e^+e^- annihilation into hyperon-antihyperon pairs. In a recent paper [2], we investigated in some detail the reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$ and its associated decay chains $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$ and $\bar{\Sigma}^0 \to \bar{\Lambda}\gamma$; $\bar{\Lambda} \to \bar{p}\pi^+$. By measuring this process in the vicinity of the J/ψ-vector-charmonium state, one gains information on the strong baryon-antibaryon-decay process of the J/ψ-vector-charmonium state and also, it offers a model-independent way of measuring weak-decay-asymmetry parameters, that in turn could probe CP symmetry [3].

The diagram for the basic reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$ is graphed in Fig. 1. Its structure is governed by two vertices. The strength of the lepton-vertex function is determined by a single parameter, the electromagnetic-fine-structure constant α_e, but two complex form factors $G_M^\psi(s)$ and $G_E^\psi(s)$ are needed for the baryonic-vertex function. However, we shall not work with the form factors themselves but with certain combinations thereof: the strength of form factors $D_\psi(s)$; the ratio of form-factor magnitudes $\eta_\psi(s)$; and the relative phase of form factors $\Delta\Phi_\psi(s)$. These form-factor combinations are defined in Appendix A.

The theoretical description of the annihilation reaction of Fig. 1 can be found in Ref. [4]. Accurate experimental results for the form-factor parameters η_ψ and $\Delta\Phi_\psi$ and the weak-interaction parameters $\alpha_\Lambda(\alpha_{\bar{\Lambda}})$ for the J/ψ annihilation process are all reported in Ref. [3]. In addition, the graph can be generalized to include hyperons that decay sequentially.

Our analysis of the cross-section-distribution function for the annihilation reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$, followed by its subsequent hyperon decays, starts from the master formula of Ref. [2], and which is reproduced in the following section. The purpose of our investigation is to find out which coordinate choice would be most convenient when evaluating the master formula, and at the same time being able to compare our result to those of others.

II. MASTER FORMULA

In several previous publications we studied e^+e^- annihilation into hyperon pairs $Y\bar{Y}$ and the subsequent decays of those pairs. Photon as well as charmonium induced annihilaton was considered. In the present investigation we limit ourselves to the hyperon-decay chain $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$, and its corresponding antihyperon-decay chain $\bar{\Sigma}^0 \to \bar{\Lambda}\gamma$; $\bar{\Lambda} \to \bar{p}\pi^+$, again when simultaneously occurring in the reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$.

In Ref. [2] it was shown that the cross-section-distribution function for a J/ψ induced joint production and subsequent decay of a $\Sigma^0\bar{\Sigma}^0$ pair can be summarized in the master formula

$$d\sigma = \frac{d\sigma}{d\Omega_{\Sigma^0}}(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) \times \left[\frac{\mathcal{W}(\boldsymbol{\xi})}{\mathcal{R}}\right] d\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p}). \qquad (2.1)$$

As can be seen the master formula involves three factors, describing the annihilation of a lepton pair into a hyperon pair, the folded product of spin densities $\mathcal{W}(\boldsymbol{\xi})$ representing hyperon production and decay, and the phase space element of sequential hyperon decays. Each event is specified by a

*goran.faldt@physics.uu.se

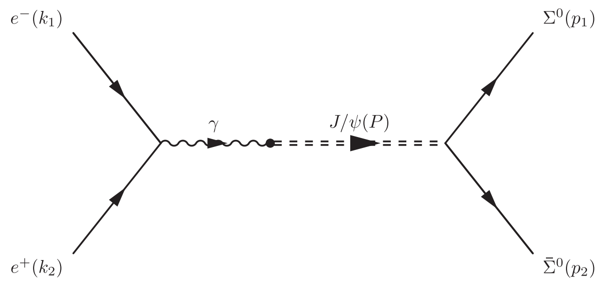

FIG. 1. Graph describing the psionic annihilation reaction $e^+e^- \to J/\psi \to \bar{\Sigma}^0\Sigma^0$. The same reaction can also proceed hadronicly via other vector-charmonium states such as ψ' or $\psi(2S)$, or electromagnetically via photons.

nine-dimensional vector $\boldsymbol{\xi} = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$, with θ the scattering angle in the $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ subprocess.

Following Refs. [4,2] we write the cross-section-distribution function for the J/ψ induced annihilation reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$ as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\Sigma^0}}(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) = \frac{p}{k}\frac{\alpha_\psi \alpha_g}{(s - m_\psi^2)^2 + m_\psi^2 \Gamma(m_\psi)} D_\psi(s)\mathcal{R}, \tag{2.2}$$

where the strength function $D_\psi(s)$ is defined in Appendix A, and the structure function $\mathcal{R}$ in Appendix B. The electromagnetic-coupling constant α_ψ is determined by the electromagnetic-decay width $\Gamma(J/\psi \to e^+e^-)$, and the hadronic-coupling constant α_g similarly by the hadronic-decay width $\Gamma(J/\psi \to \Sigma^0\bar{\Sigma}^0)$, as illustrated in Fig. 2.

The differential-spin-distribution function $\mathcal{W}(\boldsymbol{\xi})$ of Eq. (2.1) is obtained by *folding* a product of five spin densities,

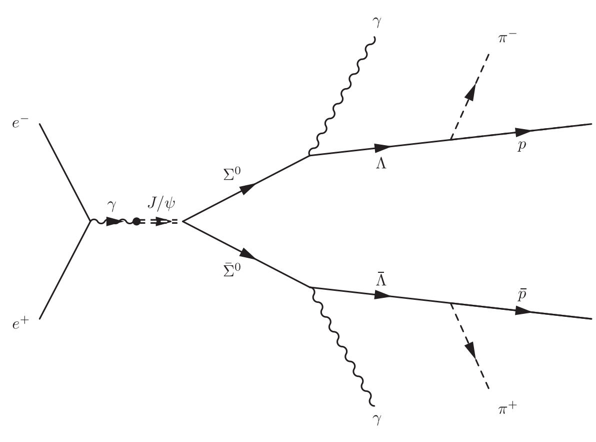

FIG. 2. Graph describing the reaction $e^+e^- \to \bar{\Sigma}^0\Sigma^0$, and the subsequent decays, $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$ and $\bar{\Sigma}^0 \to \bar{\Lambda}\gamma$; $\bar{\Lambda} \to \bar{p}\pi^+$. The reaction graphed can, in addition to photons, be mediated by vector charmonia, such as J/ψ, ψ' and $\psi(2S)$. Solid lines refer to baryons, dashed to mesons, and wavy to photons.

$$\mathcal{W}(\boldsymbol{\xi}) = \langle S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}) G(\mathbf{n}_{\Sigma^0}, \mathbf{n}_\Lambda) G(\mathbf{n}_\Lambda, \mathbf{n}_p) \times G(\mathbf{n}_{\bar{\Sigma}^0}, \mathbf{n}_{\bar{\Lambda}}) G(\mathbf{n}_{\bar{\Lambda}}, \mathbf{n}_{\bar{p}}) \rangle_{\mathbf{n}}, \tag{2.3}$$

in accordance with the prescription of Ref. [5] and of Eq. (5.1). The folding operation $\langle \ldots \rangle_{\mathbf{n}}$ applies to each of the six hadron spin vectors, $\mathbf{n}_{\Sigma^0}, \ldots, \mathbf{n}_{\bar{p}}$.

The function $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ represents the spin-density distribution for the $\Sigma^0\bar{\Sigma}^0$ hyperon pair. This function also depends on the unit vectors $\mathbf{l}_{\Sigma^0}$ and $\mathbf{l}_{\bar{\Sigma}^0}$, which are unit vectors in the directions of motion of the Σ^0 and $\bar{\Sigma}^0$ hyperons in the center-of-momentum (c.m.) frame of the event. The four remaining spin-density-distribution functions $G(\mathbf{n}_{Y_1}, \mathbf{n}_{Y_2})$ represent spin-density distributions for the hyperon decays $\Sigma^0 \to \Lambda\gamma$; or $\Lambda \to p\pi^-$, or their antihyperon counterparts.

The spin-decay-distribution functions $G(\mathbf{n}_{Y_1}, \mathbf{n}_{Y_2})$ are normalized to unity, which means their spin independent terms are unity. However, for convenience the spin-density-distribution function $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ is normalized to $\mathcal{R}$.

The phase-space factor, $\mathrm{d}\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p})$ of the master equation, describes the normalized phase-space element for the sequential decays of the two baryons Σ^0 and $\bar{\Sigma}^0$,

$$\mathrm{d}\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p}) = \mathrm{d}\Omega_{\Sigma^0} \cdot \frac{\Gamma(\Sigma^0 \to \Lambda\gamma)}{\Gamma(\Sigma^0 \to all)} \frac{\mathrm{d}\Omega_\Lambda}{4\pi} \cdot \frac{\Gamma(\Lambda \to p\pi^-)}{\Gamma(\Lambda \to all)} \frac{\mathrm{d}\Omega_p}{4\pi} \cdot \frac{\Gamma(\bar{\Sigma}^0 \to \bar{\Lambda}\gamma)}{\Gamma(\bar{\Sigma}^0 \to all)} \frac{\mathrm{d}\Omega_{\bar{\Lambda}}}{4\pi} \cdot \frac{\Gamma(\bar{\Lambda} \to \bar{p}\pi^-)}{\Gamma(\bar{\Lambda} \to all)} \frac{\mathrm{d}\Omega_{\bar{p}}}{4\pi}. \tag{2.4}$$

The widths are defined in the usual way. For $\Gamma(\Sigma^0 \to \Lambda\gamma)$ this means forming an average over the Σ^0 spin directions, and summing over the Λ and γ spin directions. However, since the $\Sigma^0 \to \Lambda\gamma$ decay rate is 100% we also have $\Gamma(\Sigma^0 \to \Lambda\gamma) = \Gamma(\Sigma^0 \to all)$.

The angles Ω_Λ define the direction of motion of the Λ hyperon in the Σ^0 rest system, the angles Ω_p the direction of motion of the p baryon in the Λ rest system, and so on.

III. e^+e^- ANNIHILATION INTO $\Sigma^0\bar{\Sigma}^0$ PAIRS

The cross-section-distribution function for e^+e^- annihilation into a $\Sigma^0\bar{\Sigma}^0$ pair appears in two places in the master formula of Eq. (2.1). The unpolarized-cross-section-distribution function is a prefactor in the master formula, and the hyperon-spin-density-distribution function enters as a factor in the spin-density-distribution function of Eq. (2.3).

The cross-section distribution for polarized-final-state hyperons was derived in Refs. [2,4] as

$$\frac{d\sigma}{d\Omega_{\Sigma^0}}(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) = \frac{p}{4k}\frac{\alpha_\psi \alpha_g D_\psi(s)}{(s-m_\psi^2)^2 + m_\psi^2\Gamma(m_\psi)} S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}), \quad (3.1)$$

where $D_\psi(s)$ is the strength function of Eq. (A1), $\mathbf{n}_{\Sigma^0}$ and $\mathbf{n}_{\bar{\Sigma}^0}$ the spin vectors of the Σ^0 and $\bar{\Sigma}^0$ hyperons, and $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ the spin-density-distribution function for the final-state hyperons. This spin-density-distribution function is normalized so that its spin-independent part equals $\mathcal{R}$, with

$$\mathcal{R} = 1 + \eta_\psi \cos^2\theta, \quad (3.2)$$

according to Eq. (B1). Consequently, summing over the final-state-hyperon polarizations gives the unpolarized cross-section-distribution function

$$\frac{d\sigma}{d\Omega_{\Sigma^0}}(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) = \frac{p}{k}\frac{\alpha_\psi \alpha_g}{(s-m_\psi^2)^2 + m_\psi^2\Gamma(m_\psi)} D_\psi(s)\mathcal{R}. \quad (3.3)$$

The branching rate for the decay channel $J/\psi \to \Sigma^0\bar{\Sigma}^0$ is $(1.07 \pm 0.08) \times 10^{-3}$, and for the channel $J/\psi \to \Lambda\bar{\Lambda}$ it is $(1.89 \pm 0.09) \times 10^{-3}$ [6].

For a spin-one-half baryon of four-momentum $\mathbf{p}$, the four-vector spin $s(p)$ is related to the three-vector-spin direction $\mathbf{n}$, the spin in the rest system, by

$$s(\mathbf{p}, \mathbf{n}) = \frac{n_\parallel}{M}(|\mathbf{p}|, E\hat{\mathbf{p}}) + (0, \mathbf{n}_\perp). \quad (3.4)$$

Longitudinal and transverse directions of vectors are relative to the $\hat{\mathbf{p}}$ direction.

In the global c.m. system kinematics simplifies. There, three-momenta $\mathbf{p}$ and $\mathbf{k}$ are defined such that

$$\mathbf{p}_{\Sigma^0} = -\mathbf{p}_{\bar{\Sigma}^0} = \mathbf{p}, \quad (3.5)$$

$$\mathbf{k}_{e^+} = -\mathbf{k}_{e^-} = \mathbf{k}, \quad (3.6)$$

and the scattering angle θ such that $\cos\theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}$. For the Σ^0 and $\bar{\Sigma}^0$ unit vectors $\mathbf{l}_{\Sigma^0}$ and $\mathbf{l}_{\bar{\Sigma}^0}$, we have $\mathbf{l}_{\Sigma^0} = -\mathbf{l}_{\bar{\Sigma}^0} = \hat{\mathbf{p}}$.

The spin-density-distribution function $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ is a sum of seven mutually orthogonal contributions [7],

$$\begin{aligned} S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}) = {} & \mathcal{R} + \mathcal{S}\mathbf{N}\cdot\mathbf{n}_{\Sigma^0} + \mathcal{S}\mathbf{N}\cdot\mathbf{n}_{\bar{\Sigma}^0} + \mathcal{T}_1 \mathbf{n}_{\Sigma^0}\cdot\hat{\mathbf{p}}\mathbf{n}_{\bar{\Sigma}^0}\cdot\hat{\mathbf{p}} \\ & + \mathcal{T}_2 \mathbf{n}_{\Sigma^0\perp}\cdot\mathbf{n}_{\bar{\Sigma}^0\perp} + \mathcal{T}_3 \mathbf{n}_{\Sigma^0\perp}\cdot\hat{\mathbf{k}}\mathbf{n}_{\bar{\Sigma}^0\perp}\cdot\hat{\mathbf{k}}/\sin^2\theta \\ & + \mathcal{T}_4(\mathbf{n}_{\Sigma^0}\cdot\hat{\mathbf{p}}\mathbf{n}_{\bar{\Sigma}^0\perp}\cdot\hat{\mathbf{k}} + \mathbf{n}_{\bar{\Sigma}^0}\cdot\hat{\mathbf{p}}\mathbf{n}_{\Sigma^0\perp}\cdot\hat{\mathbf{k}})/\sin\theta, \end{aligned} \quad (3.7)$$

where $\mathbf{N}$ is normal to the scattering plane,

$$\mathbf{N} = \frac{1}{\sin\theta}\hat{\mathbf{p}} \times \hat{\mathbf{k}}. \quad (3.8)$$

The six structure functions $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ of Eq. (3.7) depend on the scattering angle θ, the ratio function $\eta_\psi(s)$, and the phase function $\Delta\Phi_\psi(s)$. For their definitions we refer to Appendix B, but be careful, our original definitions were slightly different [7].

IV. ASSORTED SPIN DENSITIES

To be able to calculate the differential-distribution function of Eq. (2.3) we need in addition to the spin-density-distribution function for the $\Sigma^0\bar{\Sigma}^0$ final-state pair, the spin-density-distribution functions for the decays $\Sigma^0 \to \Lambda\gamma$ and $\Lambda \to p\pi^-$, and their antiparticle conjugate decays.

Weak decays of spin-one-half baryons, such as $\Lambda \to p\pi^-$, involve both S- and P-wave amplitudes, and the spin-density-decay distribution is commonly parametrized by three parameters, denoted $\alpha\beta\gamma$, and which fulfill a relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \quad (4.1)$$

Details of this description can be found in Refs. [8] or [2].

The spin-density-distribution function $G(\mathbf{n}_\Lambda, \mathbf{n}_p)$, describing the decay $\Lambda \to p\pi^-$, is a scalar, which we choose to evaluate in the rest system of the Λ hyperon, to get

$$G(\mathbf{n}_\Lambda, \mathbf{n}_p) = 1 + \alpha_\Lambda \mathbf{n}_\Lambda\cdot\mathbf{l}_p + \alpha_\Lambda \mathbf{n}_p\cdot\mathbf{l}_p + \mathbf{n}_\Lambda\cdot\mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p), \quad (4.2)$$

with the vector-valued function $\mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p)$ defined as

$$\mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p) = \gamma_\Lambda \mathbf{n}_p + [(1-\gamma_\Lambda)\mathbf{n}_p\cdot\mathbf{l}_p]\mathbf{l}_p + \beta_\Lambda \mathbf{n}_p \times \mathbf{l}_p. \quad (4.3)$$

Here, $\mathbf{n}_\Lambda$ and $\mathbf{n}_p$ are the spin vectors of the Λ hyperon and the p baryon, and $\mathbf{l}_p$ a unit vector in the direction of motion of the proton in the rest system of the Λ hyperon. The Λ indices remind us the parameters refer to a Λ decay. An important aspect of the spin-density-distribution function is its normalization. The spin-independent term is unity.

The spin-density-distribution function $G(\mathbf{n}_{\bar{\Lambda}}, \mathbf{n}_{\bar{p}})$ for the antiparticle-conjugate decay $\bar{\Lambda} \to \bar{p}\pi^+$ has exactly the same functional structure as $G(\mathbf{n}_\Lambda, \mathbf{n}_p)$, but the decay parameters take other numerical values. For *CP* conserving interactions the asymmetry parameters of the Λ-hyperon decay are related to those of the $\bar{\Lambda}$-hyperon decay by [9,10]

$$\alpha_\Lambda = -\alpha_{\bar{\Lambda}}, \qquad \beta_\Lambda = -\beta_{\bar{\Lambda}}, \qquad \gamma_\Lambda = \gamma_{\bar{\Lambda}}. \quad (4.4)$$

Numerical values for the weak-interaction parameters are given in Ref. [6]: for the decay $\Lambda \to p\pi^-$ we have $[\alpha = 0.732 \pm 0.014]$, and $[\phi = \arctan(\beta/\gamma) = -6.5 \pm 3.5]$; and for the decay $\bar{\Lambda} \to \bar{p}\pi^+$, we have $[\alpha = -0.758 \pm 0.010 \pm 0.007]$.

Next, we turn to the electromagnetic M1 transition $\Sigma^0 \to \Lambda\gamma$. It is caused by a transition-magnetic moment, of strength

$$\mu_{\Sigma\Lambda} = eF_2(0)/(m_\Sigma + m_\Lambda). \tag{4.5}$$

The normalized-spin-density-distribution function for a $\Sigma^0 \to \Lambda\gamma$ transition to a final state of fixed photon helicity λ_γ is, according to Ref. [2],

$$G_\gamma(\mathbf{n}_{\Sigma^0}, \mathbf{n}_\Lambda; \lambda_\gamma) = 1 - \mathbf{n}_{\Sigma^0} \cdot \mathbf{l}_\gamma \mathbf{l}_\gamma \cdot \mathbf{n}_\Lambda + \lambda_\gamma(\mathbf{n}_{\Sigma^0} \cdot \mathbf{l}_\gamma - \mathbf{n}_\Lambda \cdot \mathbf{l}_\gamma), \tag{4.6}$$

where $\mathbf{l}_\gamma$ is a unit vector in the direction of motion of the photon, and $\mathbf{l}_\Lambda = -\mathbf{l}_\gamma$ a unit vector in the direction of motion of the Λ hyperon, both in the rest system of the Σ^0 baryon. The photon helicities λ_γ take on the values ± 1.

We notice that when both hadron spins are parallel or antiparallel to the photon momentum, then the decay probability vanishes, a property of angular-momentum conservation.

Summing, in Eq. (4.6), the contributions from the two photon-helicity states gives the normalized-spin-density-distribution function

$$G(\mathbf{n}_{\Sigma^0}, \mathbf{n}_\Lambda) = 1 - \mathbf{n}_{\Sigma^0} \cdot \mathbf{l}_\gamma \mathbf{l}_\gamma \cdot \mathbf{n}_\Lambda. \tag{4.7}$$

The normalized-spin-density-distribution function for the conjugate transition, $\bar{\Sigma}^0 \to \bar{\Lambda}\gamma$, is obtained by replacing, in expression (4.7), the particle spin vectors $\mathbf{n}_{\Sigma^0}$ and $\mathbf{n}_\Lambda$ by the antiparticle-spin vectors $\mathbf{n}_{\bar{\Sigma}^0}$ and $\mathbf{n}_{\bar{\Lambda}}$.

V. SEQUENTIAL DECAY OF HYPERONS

A factor of our master formula for hyperon production and decay, Eq. (2.1), is the differential-spin-distribution function $\mathcal{W}(\boldsymbol{\xi})$ of Eq. (2.3), which is obtained by folding a product of five spin densities. The folding prescription is especially adapted to spin one-half baryons. A folding operation implies forming an average over intermediate-spin directions $\mathbf{n}$ according to the prescription of Ref. [5],

$$\langle 1 \rangle_\mathbf{n} = 1, \qquad \langle \mathbf{n} \rangle_\mathbf{n} = 0, \qquad \langle \mathbf{n} \cdot \mathbf{k} \mathbf{n} \cdot \mathbf{l} \rangle_\mathbf{n} = \mathbf{k} \cdot \mathbf{l}. \tag{5.1}$$

The spin-density distribution $W(\mathbf{n}_{\Sigma^0}, \mathbf{n}_p)$ for the decay chain $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$ is obtained by folding the product of the spin density distributions in the decay chain. We obtain

$$W(\mathbf{n}_{\Sigma^0}, \mathbf{n}_p) = \langle G(\mathbf{n}_{\Sigma^0}, \mathbf{n}_\Lambda) G(\mathbf{n}_\Lambda, \mathbf{n}_p) \rangle_{\mathbf{n}_\Lambda}, \tag{5.2}$$

where the two spin-density-distribution functions on the right-hand side are defined in Eqs. (4.7) and (4.2). Performing the folding operation gives

$$W(\mathbf{n}_{\Sigma^0}, \mathbf{n}_p) = U_{\Sigma^0} + \mathbf{n}_{\Sigma^0} \cdot \mathbf{V}_{\Sigma^0}, \tag{5.3}$$

$$U_{\Sigma^0} = 1 + \alpha_\Lambda \mathbf{n}_p \cdot \mathbf{l}_p, \tag{5.4}$$

$$\mathbf{V}_{\Sigma^0} = -\mathbf{l}_\gamma[\alpha_\Lambda \mathbf{l}_\gamma \cdot \mathbf{l}_p + \mathbf{n}_p \cdot \mathbf{L}_\Lambda(\mathbf{l}_\gamma, -\mathbf{l}_p)], \tag{5.5}$$

and the same for $W(\mathbf{n}_{\bar{\Sigma}^0}, \mathbf{n}_{\bar{p}})$.

VI. PRODUCTION AND DECAY OF $\Sigma^0\bar{\Sigma}^0$ PAIRS

Now, we come to our final task: production and decay of $\Sigma^0\bar{\Sigma}^0$ pairs. The starting point is the reaction $e^+e^- \to \Sigma^0\bar{\Sigma}^0$, the spin-density-distribution function of which was calculated in Sec. III, and named $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$. The spin-density-distribution function $W(\mathbf{n}_{\Sigma^0}, \mathbf{n}_p)$ which represents the decay chain $\Sigma^0 \to \Lambda\gamma$; $\Lambda \to p\pi^-$ was calculated in Sec. V, and so for the antichain-decay function $W(\mathbf{n}_{\bar{\Sigma}^0}, \mathbf{n}_{\bar{p}})$.

The final-state-angular distributions are obtained by folding the spin distributions for production and decay, according to prescription (5.1). Invoking Eq. (3.7) for the production step and Eq. (5.3) and its antidistribution for the decay steps, we get the differential-spin-density-distribution function

$$\begin{aligned}
\mathcal{W}(\boldsymbol{\xi}) &= \langle S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}) W(\mathbf{n}_{\Sigma^0}, \mathbf{n}_p) W(\mathbf{n}_{\bar{\Sigma}^0}, \mathbf{n}_{\bar{p}}) \rangle_{\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}} \\
&= \mathcal{R} U_{\Sigma^0} U_{\bar{\Sigma}^0} + \mathcal{S} U_{\bar{\Sigma}^0} \mathbf{N} \cdot \mathbf{V}_{\Sigma^0} + \mathcal{S} U_{\Sigma^0} \mathbf{N} \cdot \mathbf{V}_{\bar{\Sigma}^0} \\
&\quad + \mathcal{T}_1 \mathbf{V}_{\Sigma^0} \cdot \hat{\mathbf{p}} \mathbf{V}_{\bar{\Sigma}^0} \cdot \hat{\mathbf{p}} + \mathcal{T}_2 \mathbf{V}_{\Sigma^0 \perp} \cdot \mathbf{V}_{\bar{\Sigma}^0 \perp} \\
&\quad + \mathcal{T}_3 \mathbf{V}_{\Sigma^0 \perp} \cdot \hat{\mathbf{k}} \mathbf{V}_{\bar{\Sigma}^0 \perp} \cdot \hat{\mathbf{k}} / \sin^2\theta \\
&\quad + \mathcal{T}_4 (\mathbf{V}_{\Sigma^0} \cdot \hat{\mathbf{p}} \mathbf{V}_{\bar{\Sigma}^0 \perp} \cdot \hat{\mathbf{k}} + \mathbf{V}_{\bar{\Sigma}^0} \cdot \hat{\mathbf{p}} \mathbf{V}_{\Sigma^0 \perp} \cdot \hat{\mathbf{k}}) / \sin\theta.
\end{aligned} \tag{6.1}$$

The functions U_{Σ^0} and $\mathbf{V}_{\Sigma^0}$ are defined in Sec. B, and

$$U_{\Sigma^0} = 1 + \alpha_\Lambda \mathbf{n}_p \cdot \mathbf{l}_p, \tag{6.2}$$

$$\mathbf{V}_{\Sigma^0} = -\mathbf{l}_\gamma[\alpha_\Lambda \mathbf{l}_\gamma \cdot \mathbf{l}_p + \mathbf{n}_p \cdot \mathbf{L}_\Lambda(\mathbf{l}_\gamma, -\mathbf{l}_p)]. \tag{6.3}$$

We observe that U_{Σ^0} depends on the weak interaction parameter α_Λ, whereas $\mathbf{V}_{\Sigma^0}$ in addition depends on the parameters β_Λ and γ_Λ through the vector function $\mathbf{L}_\Lambda$, of Eq. (4.3).

The angular distributions of Eq. (6.1), which are the most general ones, still depend on the spin vectors $\mathbf{n}_p$ and $\mathbf{n}_{\bar{p}}$. In case we are satisfied with considering their averages, then the variables U and $\mathbf{V}$ simplify,

$$U_{\Sigma^0} = 1, \qquad \mathbf{V}_{\Sigma^0} = -\alpha_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_p \mathbf{l}_\Lambda,$$
$$U_{\bar{\Sigma}^0} = 1, \qquad \mathbf{V}_{\bar{\Sigma}^0} = -\alpha_{\bar{\Lambda}} \mathbf{l}_{\bar{\Lambda}} \cdot \mathbf{l}_{\bar{p}} \mathbf{l}_{\bar{\Lambda}}. \qquad (6.4)$$

When $U_{\Sigma^0} = U_{\bar{\Sigma}^0} = 1$ the effect of the folding is to make the replacements $\mathbf{n}_{\Sigma^0} \to \mathbf{V}_{\Sigma^0}$ and $\mathbf{n}_{\bar{\Sigma}^0} \to \mathbf{V}_{\bar{\Sigma}^0}$ in the spin-density function $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ of Eq. (3.7). We notice that the U and $\mathbf{V}$ variables now are independent of the weak-asymmetry parameters β_Λ and γ_Λ.

Inserting the expressions of Eq. (6.4) into the spin-density function of Eq. (6.1), we get

$$\begin{aligned}\mathcal{W}(\boldsymbol{\xi}) = \mathcal{R} &- \alpha_\Lambda \mathcal{S} \mathbf{N} \cdot \mathbf{l}_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_p - \alpha_{\bar{\Lambda}} \mathcal{S} \mathbf{N} \cdot \mathbf{l}_{\bar{\Lambda}} \mathbf{l}_{\bar{\Lambda}} \cdot \mathbf{l}_{\bar{p}} \\ &+ \alpha_\Lambda \alpha_{\bar{\Lambda}} \mathbf{l}_\Lambda \cdot \mathbf{l}_p \mathbf{l}_{\bar{\Lambda}} \cdot \mathbf{l}_{\bar{p}} [\mathcal{T}_1 \mathbf{l}_\Lambda \cdot \hat{\mathbf{p}} \mathbf{l}_{\bar{\Lambda}} \cdot \hat{\mathbf{p}} \\ &+ \mathcal{T}_2 \mathbf{l}_{\Lambda\perp} \cdot \mathbf{l}_{\bar{\Lambda}\perp} + \mathcal{T}_3 \mathbf{l}_{\Lambda\perp} \cdot \hat{\mathbf{k}} \mathbf{l}_{\bar{\Lambda}\perp} \cdot \hat{\mathbf{k}} / \sin^2\theta \\ &+ \mathcal{T}_4 (\mathbf{l}_\Lambda \cdot \hat{\mathbf{p}} \mathbf{l}_{\bar{\Lambda}\perp} \cdot \hat{\mathbf{k}} + \mathbf{l}_{\bar{\Lambda}} \cdot \hat{\mathbf{p}} \mathbf{l}_{\Lambda\perp} \cdot \hat{\mathbf{k}}) / \sin\theta]. \qquad (6.5)\end{aligned}$$

Thus, this is the angular distribution obtained when folding the product of spin densities for production and decay. These results were previously reported in Ref. [2].

VII. DIFFERENTIAL-SPIN DISTRIBUTIONS

A closer inspection of the differential-spin-density-distribution function of Eq. (6.5) shows that the weak-interaction parameters α_Λ and $\alpha_{\bar{\Lambda}}$ always come in the combinations $\alpha_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_p$ or $\alpha_{\bar{\Lambda}} \mathbf{l}_{\bar{\Lambda}} \cdot \mathbf{l}_{\bar{p}}$. Therefore, it is convenient to define the following functions:

$$\lambda_\Lambda(\theta_{\Lambda p}) = \alpha_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_p = \alpha_\Lambda \cos(\theta_{\Lambda p}), \qquad (7.1)$$

$$\lambda_{\bar{\Lambda}}(\theta_{\bar{\Lambda}\bar{p}}) = \alpha_{\bar{\Lambda}} \mathbf{l}_{\bar{\Lambda}} \cdot \mathbf{l}_{\bar{p}} = \alpha_{\bar{\Lambda}} \cos(\theta_{\bar{\Lambda}\bar{p}}). \qquad (7.2)$$

Then, the differential-spin-density-distribution function of Eq. (6.5) can be rewritten as

$$\begin{aligned}\mathcal{W}(\boldsymbol{\xi}) = \mathcal{R} &- [\lambda_\Lambda Q_\Lambda + \lambda_{\bar{\Lambda}} Q_{\bar{\Lambda}}] \mathcal{S} \\ &+ \lambda_\Lambda \lambda_{\bar{\Lambda}} [Q_1 \mathcal{T}_1 + Q_2 \mathcal{T}_2 + Q_3 \mathcal{T}_3 + Q_4 \mathcal{T}_4], \qquad (7.3)\end{aligned}$$

with the argument $\boldsymbol{\xi}$ a nine-dimensional vector $\boldsymbol{\xi} = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$ representing the scattering angle and four directional-unit vectors of particle motion.

The six structure functions $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are functions of the scattering angle θ and the ratio of form factors η_ψ. The six kinematic Q functions are functions of $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$. Their dependencies on the unit vectors $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$ reside solely in the functions λ_Λ and $\lambda_{\bar{\Lambda}}$ of Eqs. (7.1) and (7.2).

The analytic expressions for the six functions $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ are obtained by comparing Eqs. (6.5) and (7.3),

$$\begin{aligned} Q_\Lambda &= \mathbf{N} \cdot \mathbf{l}_\Lambda, \\ Q_{\bar{\Lambda}} &= \mathbf{N} \cdot \mathbf{l}_{\bar{\Lambda}}, \\ Q_1 &= \mathbf{l}_\Lambda \cdot \hat{\mathbf{p}} \mathbf{l}_{\bar{\Lambda}} \cdot \hat{\mathbf{p}}, \\ Q_2 &= \mathbf{l}_{\Lambda\perp} \cdot \mathbf{l}_{\bar{\Lambda}\perp}, \\ Q_3 &= \mathbf{l}_{\Lambda\perp} \cdot \hat{\mathbf{k}} \mathbf{l}_{\bar{\Lambda}\perp} \cdot \hat{\mathbf{k}} / \sin^2\theta, \\ Q_4 &= [\mathbf{l}_\Lambda \cdot \hat{\mathbf{p}} \mathbf{l}_{\bar{\Lambda}\perp} \cdot \hat{\mathbf{k}} + \mathbf{l}_{\bar{\Lambda}} \cdot \hat{\mathbf{p}} \mathbf{l}_{\Lambda\perp} \cdot \hat{\mathbf{k}}] / \sin\theta. \qquad (7.4)\end{aligned}$$

Here, longitudinal and transverse components of vectors are defined relative to $\hat{\mathbf{p}}$, the direction of motion of the Σ^0 hyperon.

The differential-spin-density distribution of Eq. (7.3), and the angular functions above, depend on a number of unit vectors; $\hat{\mathbf{p}}$ and $-\hat{\mathbf{p}}$ are unit vectors along the directions of motion of the Σ^0 and the $\bar{\Sigma}^0$ in the c.m. system; $\hat{\mathbf{k}}$ and $-\hat{\mathbf{k}}$ are unit vectors along the directions of motion of the incident electron and positron in the c.m. system; $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$ are unit vectors along the directions of motion of the Λ and $\bar{\Lambda}$ in the rest systems of the Σ^0 and the $\bar{\Sigma}^0$; and $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$ are unit vectors along the directions of motion of the p and the $\bar{p}$ in the rest frames of the Λ and the $\bar{\Lambda}$.

VIII. SPIN POLARIZATIONS

As an application we shall now investigate what can be learned by concentrating our attention to the proton leg. The spin-density-distribution function $\mathcal{W}(\mathbf{n}_p, \mathbf{n}_{\bar{p}})$ for final-state-spin vectors $\mathbf{n}_p$ and $\mathbf{n}_{\bar{p}}$ is described by Eq. (6.1). We start by averaging over the final-state antiproton-spin directions $\mathbf{n}_{\bar{p}}$, to get

$$\begin{aligned}\mathcal{W}(\mathbf{n}_p) &= \langle \mathcal{W}(\mathbf{n}_p, \mathbf{n}_{\bar{p}}) \rangle_{\mathbf{n}_{\bar{p}}} \\ &= (X_a + X_b) + a X_a + b X_b, \qquad (8.1)\end{aligned}$$

where the functions X_a and X_b are defined as

$$X_a = \mathcal{R} - \lambda_{\bar{\Lambda}} Q_{\bar{\Lambda}} \mathcal{S}, \qquad (8.2)$$

$$X_b = -\lambda_\Lambda Q_\Lambda \mathcal{S} + \lambda_\Lambda \lambda_{\bar{\Lambda}} [Q_1 \mathcal{T}_1 + Q_2 \mathcal{T}_2 + Q_3 \mathcal{T}_3 + Q_4 \mathcal{T}_4], \qquad (8.3)$$

and the functions a and b as

$$a = \alpha_\Lambda \mathbf{n}_p \cdot \mathbf{l}_p, \qquad (8.4)$$

$$b = -\mathbf{n}_p \cdot \mathbf{L}_\Lambda(-\mathbf{l}_\Lambda, -\mathbf{l}_p). \qquad (8.5)$$

Since the vector-valued function $\mathbf{L}_\Lambda(-\mathbf{l}_\Lambda, -\mathbf{l}_p)$ is defined in Eq. (4.3), and $X = X_a + X_b$ equals $\mathcal{W}(\boldsymbol{\xi})$ of Eq. (7.3), it follows that in this particular case the final-state-proton polarization $\mathbf{P}_p$ becomes

$$\mathbf{P}_p = (\alpha_\Lambda X_a \mathbf{l}_p - X_b \mathbf{L}_\Lambda(-\mathbf{l}_\Lambda, -\mathbf{l}_p))/X. \quad (8.6)$$

Hence, apart from the Λ weak-interaction parameters, the final-state-proton-polarization vector is built on the vectors $\mathbf{l}_p$ and $\mathbf{L}_\Lambda$.

It is instructive to compare this result with the spin-density-distribution function $G(\mathbf{n}_\Lambda, \mathbf{n}_p)$ of Eq. (4.2) describing the decay $\Lambda \to p\pi^-$. For a Λ hyperon of initial-state poarization $\mathbf{n}_\Lambda = \mathbf{P}_\Lambda$, the spin-density-distribution function reads

$$G(\mathbf{P}_\Lambda, \mathbf{n}_p) = 1 + \alpha_\Lambda \mathbf{P}_\Lambda \cdot \mathbf{l}_p + \alpha_\Lambda \mathbf{n}_p \cdot \mathbf{l}_p + \mathbf{P}_\Lambda \cdot \mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p), \quad (8.7)$$

and implies a proton polarization

$$\mathbf{P}_p = (\alpha_\Lambda \mathbf{l}_p - \mathbf{L}_\Lambda(-\mathbf{P}_\Lambda, -\mathbf{l}_p))/(1 + \alpha_\Lambda \mathbf{P}_\Lambda \cdot \mathbf{l}_p). \quad (8.8)$$

We immediately notice the similarity between the final-state-proton polarizations of Eqs. (8.6) and (8.8).

However, it should be remembered that the spin polarization $\mathbf{P}_p$ of Eq. (8.6) is only one of many possible.

IX. GLOBAL ANGULAR FUNCTIONS

The differential-spin-density distribution (6.5) is a function of several unit vectors. In order to handle them we need a common coordinate system, which we call global and define as follows. The scattering plane of the reaction $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ is spanned by the unit vectors $\hat{\mathbf{p}} = \mathbf{l}_{\Sigma^0}$ and $\hat{\mathbf{k}} = \mathbf{l}_e$, as measured in the c.m. system. The scattering plane makes up the xz plane, with the y axis along the normal to this plane. We choose a right-handed coordinate system with basis vectors

$$\mathbf{e}_z = \hat{\mathbf{p}},$$
$$\mathbf{e}_y = \frac{1}{\sin\theta}(\hat{\mathbf{p}} \times \hat{\mathbf{k}}),$$
$$\mathbf{e}_x = \frac{1}{\sin\theta}(\hat{\mathbf{p}} \times \hat{\mathbf{k}}) \times \hat{\mathbf{p}}, \quad (9.1)$$

and where the initial-state-lepton momentum is decomposed as

$$\hat{\mathbf{k}} = \sin\theta \mathbf{e}_x + \cos\theta \mathbf{e}_z. \quad (9.2)$$

The reason we call this coordinate system global is that we use it whenever studying a subprocess of the e^+e^- annihilation.

In spherical xyz coordinates the unit vectors $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$ associated with the directions of motion of the Λ and $\bar{\Lambda}$ hyperons are

$$\mathbf{l}_\Lambda = (\cos\phi_\Lambda \sin\theta_\Lambda, \sin\phi_\Lambda \sin\theta_\Lambda, \cos\theta_\Lambda),$$
$$\mathbf{l}_{\bar{\Lambda}} = (\cos\phi_{\bar{\Lambda}} \sin\theta_{\bar{\Lambda}}, \sin\phi_{\bar{\Lambda}} \sin\theta_{\bar{\Lambda}}, \cos\theta_{\bar{\Lambda}}). \quad (9.3)$$

However, in order to make our formulas more transparent we introduce the notations $\mathbf{l}_\Lambda = \mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{l}_{\bar{\Lambda}} = \mathbf{F} = (F_x, F_y, F_z)$. In this Cartesian notation, the expressions for kinematic functions $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ of Eq. (7.4) are

$$Q_\Lambda = E_y, \qquad Q_{\bar{\Lambda}} = F_y,$$
$$Q_1 = E_z F_z, \qquad Q_2 = E_x F_x + E_y F_y,$$
$$Q_3 = E_x F_x, \qquad Q_4 = E_x F_z + E_z F_x. \quad (9.4)$$

Inserting them into Eq. (7.3), the differential-spin-density-distribution function becomes

$$\begin{aligned}\mathcal{W}(\boldsymbol{\xi}(\Omega)) &= 1 + \eta_\psi \cos^2\theta - \sqrt{1-\eta_\psi^2}\sin(\Delta\Phi_\psi)\sin\theta\cos\theta[\lambda_\Lambda E_y + \lambda_{\bar{\Lambda}} F_y] \\ &+ \lambda_\Lambda \lambda_{\bar{\Lambda}}[(1+\eta_\psi)E_z F_z + \sin^2\theta(E_x F_x - E_z F_z - \eta_\psi E_y F_y) \\ &+ \sqrt{1-\eta_\psi{}^2}\cos(\Delta\Phi_\psi)\sin\theta\cos\theta(E_x F_z + E_z F_x)]. \quad (9.5)\end{aligned}$$

Now, the phase-space-angular variables are hidden inside the $\mathbf{E}(\theta_\Lambda, \phi_\Lambda)$ and $\mathbf{F}(\theta_{\bar{\Lambda}}, \phi_{\bar{\Lambda}})$ functions.

The differential-spin-density-distribution function $\mathcal{W}(\boldsymbol{\xi})$ of Eq. (9.5) involves two parameters related to the $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ reaction that can be determined by data: the ratio of form factors η_ψ, and the relative phase of form factors $\Delta\Phi_\psi$. In addition, the distribution function $\mathcal{W}(\boldsymbol{\xi})$ depends on the weak-asymmetry parameters α_Λ and $\alpha_{\bar{\Lambda}}$ of the two Lambda-hyperon decays. The dependencies on the weak-asymmetry parameters β and γ drop out, when final-state-proton and antiproton spins are unidentified.

An important conclusion to be drawn from the differential distribution of Eq. (9.5) is that when the phase $\Delta\Phi_\psi$ is small, the parameters α_Λ and $\alpha_{\bar{\Lambda}}$ are strongly correlated and therefore difficult to separate. In order to contribute to the experimental precision value of α_Λ and $\alpha_{\bar{\Lambda}}$ a nonzero value of $\Delta\Phi_\psi$ is required.

X. GLOBAL2 ANGULAR FUNCTIONS

In the global2-coordinate system, the scattering plane of the reaction $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ is still spanned by the unit vectors $\hat{\mathbf{p}} = \mathbf{l}_{\Sigma^0}$ and $\hat{\mathbf{k}} = \mathbf{l}_e$, as measured in the c.m. system, and with scattering angle $\cos\theta = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$. Again, the scattering plane makes up the $x'z'$ plane, and the y' axis is normal to this plane. In the $x'y'z'$ coordinate system we choose a right-handed set of basis vectors

$$\mathbf{e}'_z = \hat{\mathbf{k}},$$
$$\mathbf{e}'_y = \frac{1}{\sin\theta}(\hat{\mathbf{k}} \times \hat{\mathbf{p}}),$$
$$\mathbf{e}'_x = \frac{1}{\sin\theta}(\hat{\mathbf{k}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{k}}. \tag{10.1}$$

We observe that by this definition the xyz and $x'y'z'$ coordinate bases are related by an interchange of the $\hat{\mathbf{p}}$ and $\hat{\mathbf{k}}$ momenta. Moreover, in global2 coordinates the final-state-hyperon momentum can be decomposed as

$$\hat{\mathbf{p}} = \sin\theta\,\mathbf{e}'_x + \cos\theta\,\mathbf{e}'_z, \tag{10.2}$$

with $\mathbf{N} = -\mathbf{e}'_y$ normal to the scattering plane, for $\mathbf{N}$ defined in Eq. (3.8).

In spherical $x'y'z'$ coordinates the unit vectors $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$ associated with the directions of motion of the Λ and $\bar{\Lambda}$ hyperons are

$$\mathbf{l}_\Lambda = (\cos\phi'_\Lambda \sin\theta'_\Lambda, \sin\phi'_\Lambda \sin\theta'_\Lambda, \cos\theta'_\Lambda),$$
$$\mathbf{l}_{\bar{\Lambda}} = (\cos\phi'_{\bar{\Lambda}} \sin\theta'_{\bar{\Lambda}}, \sin\phi'_{\bar{\Lambda}} \sin\theta'_{\bar{\Lambda}}, \cos\theta'_{\bar{\Lambda}}), \tag{10.3}$$

and similarly for the unit vectors $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$. More generally, we use the prime notation for vectors in global2 coordinates, $\mathbf{l}_\Lambda = \mathbf{E}' = (E'_x, E'_y, E'_z)$ and $\mathbf{l}_{\bar{\Lambda}} = \mathbf{F}' = (F'_x, F'_y, F'_z)$.

In order to be able to determine the spin-density-distribution function in terms of the angles of Eqs. (10.3), first we need to determine the angular dependencies of the six kinematic functions $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ of Eq. (7.4). In principle, this is straightforward but it turns out to be more involved than for the global case, since some of the $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ functions will depend on the scattering angle θ.

The basis vectors of Eqs. (10.1) and (9.1) are related by

$$\mathbf{e}_x = -\cos\theta\,\mathbf{e}'_x + \sin\theta\,\mathbf{e}'_z,$$
$$\mathbf{e}_y = -\mathbf{e}'_y,$$
$$\mathbf{e}_z = \sin\theta\,\mathbf{e}'_x + \cos\theta\,\mathbf{e}'_z. \tag{10.4}$$

From this relation one obtains a corresponding relation for the xyz components F_k, and the $x'y'z'$ components F'_k, of the directional unit vector $\mathbf{l}_{\bar{\Lambda}} = \mathbf{F}$ associated with the $\bar{\Lambda}$ hyperon,

$$F_x = -\cos\theta F'_x + \sin\theta F'_z,$$
$$F_y = -F'_y,$$
$$F_z = \sin\theta F'_x + \cos\theta F'_z, \tag{10.5}$$

and the same for the Λ hyperon case.

The new set of the six $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ functions of Eq. (7.4) is obtained by replacing global-vector components by global2-vector components, which give

$$Q_\Lambda = -E'_y,$$
$$Q_{\bar{\Lambda}} = -F'_y,$$
$$Q_1 = (\sin\theta E'_x + \cos\theta E'_z)(\sin\theta F'_x + \cos\theta F'_z),$$
$$Q_2 = Q_3 + E'_y F'_y,$$
$$Q_3 = (-\cos\theta E'_x + \sin\theta E'_z)(-\cos\theta F'_x + \sin\theta F'_z),$$
$$Q_4 = (-\cos\theta E'_x + \sin\theta E'_z)(\sin\theta F'_x + \cos\theta F'_z)$$
$$+ (\sin\theta E'_x + \cos\theta E'_z)(-\cos\theta F'_x + \sin\theta F'_z). \tag{10.6}$$

This global2 set of functions has a decidedly more complex dependence on the scattering angle θ than the global set of Eq. (9.4), which is independent of the scattering angle.

The differential-distribution function as defined in Eq. (7.3) now takes the form

$$\mathcal{W}(\boldsymbol{\xi}(\Omega'))$$
$$= 1 + \eta_\psi \cos^2\theta + \sqrt{1-\eta_\psi^2}\sin(\Delta\Phi_\psi)\sin\theta\cos\theta[\lambda_\Lambda E'_y + \lambda_{\bar{\Lambda}} F'_y]$$
$$+ \lambda_\Lambda\lambda_{\bar{\Lambda}}[(1+\eta_\psi)Q_1 + \sin^2\theta((Q_3 - Q_1) + \eta_\psi(Q_3 - Q_2))$$
$$+ \sqrt{1-\eta_\psi{}^2}\cos(\Delta\Phi_\psi)\sin\theta\cos\theta Q_4], \tag{10.7}$$

with the functions $Q(\Omega')$ of Eq. (10.6).

This ends our involvement with $x'y'z'$ global2 coordinates, since the xyz coordinates seem considerably easier to work with.

XI. CROSS-SECTION DISTRIBUTIONS

The differential-cross-section-distribution function $\mathcal{W}(\boldsymbol{\xi}(\Omega))$ of Eq. (7.3) is a Cartesian scalar. Its argument $\boldsymbol{\xi}$ is a nine-dimensional vector $\boldsymbol{\xi} = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$, which represents the scattering angle and four directional-unit vectors of particle motion. In view of the findings of the previous sections, we propose evaluating this cross-section-distribution function in the global xyz coordinate system of Eq. (9.1), and so for each event.

The desired expression for the cross-section-distribution function $\mathcal{W}(\boldsymbol{\xi}(\Omega))$ is in our global coordinates already known, and displayed in Eq. (9.5), where the symbol Ω refers to spherical angles, $\Omega = (\theta, \phi)$, in the xyz coordinate system.

It might be remembered we introduced the notation $\mathbf{E} = \mathbf{l}_\Lambda$ and $\mathbf{F} = \mathbf{l}_{\bar{\Lambda}}$, with Cartesian components as defined in Eq. (9.3). A unit vector such as $\mathbf{l}_\Lambda$, which is a unit vector in the direction of motion of the Λ hyperon in the rest system of the Σ^0 hyperon, can be expressed in either Cartesian xyz or spherical-angular variables,

$$\mathbf{l}_\Lambda = (l_{\Lambda x}, l_{\Lambda y}, l_{\Lambda z}) = (\cos\phi_\Lambda \sin\theta_\Lambda, \sin\phi_\Lambda \sin\theta_\Lambda, \cos\theta_\Lambda). \tag{11.1}$$

The decomposition into spherical coordinates needs to be known since in our treatment the phase-space element $d\Omega_\Lambda$ is expressed in terms of spherical-angular variables.

It was already noticed in Sec. VII that the angular variables Ω_p and $\Omega_{\bar{p}}$ only appear in the multiplicative parameters $\lambda_\Lambda(\theta_{\Lambda p})$ and $\lambda_{\bar{\Lambda}}(\theta_{\bar{\Lambda}\bar{p}})$ of Eqs. (7.1) and (7.2). Averaging these parameters over Ω_p or $\Omega_{\bar{p}}$ give a vanishing result, as e.g.,

$$\int \frac{d\Omega_p}{4\pi} \alpha_\Lambda \cos(\theta_{\Lambda p}) = 0. \tag{11.2}$$

The phase-space element $d\Phi(\boldsymbol{\xi})$ of Eq. (2.4), associated with the spin-density-distribution function $\mathcal{W}(\boldsymbol{\xi})$, is nine-dimensional as the dimensionality of the $\boldsymbol{\xi}$ vector. The corresponding nine-dimensional-cross-section distribution is that of the master formula Eq. (2.1). Certainly, it should be possible to determine the weak-interaction parameters $\alpha_\Lambda(\alpha_{\bar{\Lambda}})$, and the amplitude parameters $\eta_\psi(s)$ and $\Delta\Phi_\psi(s)$, from this distribution.

Lower dimensional cross-section distributions may contain as much information as the nine-dimensional one. To investigate this claim let us integrate over the antihyperon angles $\Omega_{\bar{p}}$ and $\Omega_{\bar{\Sigma}^0}$. The result is a five-dimensional cross-section-distribution function

$$\int \frac{d\Omega_{\bar{p}}}{4\pi} \frac{d\Omega_{\bar{\Sigma}^0}}{4\pi} \mathcal{W}(\boldsymbol{\xi}) = \mathcal{W}(\boldsymbol{\xi}'), \tag{11.3}$$

with $\boldsymbol{\xi}' = (\theta, \Omega_\Lambda, \Omega_p)$, and

$$\begin{aligned} \mathcal{W}(\boldsymbol{\xi}') &= \mathcal{R} - \lambda_\Lambda Q_\Lambda \mathcal{S}, \\ &= \mathcal{R}(\theta, \eta_\psi) - \alpha_\Lambda \mathbf{N} \cdot \mathbf{l}_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_p \mathcal{S}(\theta, \Delta\Phi_\psi). \end{aligned} \tag{11.4}$$

Thus, we realize the five-dimensional cross-section-distribution function contains as much information as the nine-dimensional one.

A further reduction of phase-space into a three-dimensional space can be obtained by integrating over the hyperon angles Ω_Λ, giving

$$\int \frac{d\Omega_\Lambda}{4\pi} \mathcal{W}(\boldsymbol{\xi}') = \mathcal{W}(\boldsymbol{\xi}''), \tag{11.5}$$

with $\boldsymbol{\xi}'' = (\theta, \Omega_p)$, and

$$\begin{aligned} \mathcal{W}(\boldsymbol{\xi}'') &= \mathcal{R} - \lambda_\Lambda Q_\Lambda \mathcal{S}, \\ &= \mathcal{R}(\theta, \eta_\psi) - \frac{1}{3} \alpha_\Lambda \mathbf{N} \cdot \mathbf{l}_p \mathcal{S}(\theta, \Delta\Phi_\psi). \end{aligned} \tag{11.6}$$

Since $\mathbf{N} = \mathbf{e}_y$ it follows that $\mathbf{N} \cdot \mathbf{l}_p = l_{py} = \sin\phi_p \sin\theta_p$. Again we are forced to conclude that the three-dimensional phase-space harbors as much information as the nine-dimensional one.

We end our investigation with a remark on polarization. If we integrate the cross-section-distribution function over the antiparticle leg, which will then be the polarizations of the Σ^0 and Λ baryons? After the integration, we get the Σ^0 polarization from Eq. (3.7)

$$\mathbf{P}_{\Sigma^0} = \mathcal{S}/\mathcal{R}\mathbf{N}, \tag{11.7}$$

and, similarly, the Λ polarization can be picked out from Eq. (6.5)

$$\mathbf{P}_\Lambda = -\mathcal{S}/\mathcal{R}\mathbf{N} \cdot \mathbf{l}_\Lambda \mathbf{l}_\Lambda. \tag{11.8}$$

Thus, the polarization of the Σ^0 is directed along the normal to the scattering plane, and the polarization of the Λ directed along its own momentum.

XII. SUMMARY

This is a study of joint production and simultaneous sequential decay of $\Sigma^0\bar{\Sigma}^0$ pairs produced in e^+e^- annihilation. It starts from a master formula which is a product of three factors, describing: the annihilation of a lepton pair into a hyperon pair, the spin-density distribution $\mathcal{W}(\boldsymbol{\xi})$ representing the spin dependence in hyperon production and decay, and the phase-space element in sequential hyperon decay. Each measured event is specified by a nine-dimensional vector $\boldsymbol{\xi} = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$, with θ the scattering angle in the $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ subprocess.

The dynamics of the process is described by four unit-three vectors $\mathbf{l}_p$, $\mathbf{l}_\Lambda$, $\mathbf{l}_{\bar{p}}$, $\mathbf{l}_{\bar{\Lambda}}$, directed along the directions of motion of the final state baryons $(\Omega_p, \Omega_\Lambda, \Omega_{\bar{p}}, \Omega_{\bar{\Lambda}})$. We have arranged so that the spin-density-distribution function can be written as

$$\begin{aligned} \mathcal{W}(\boldsymbol{\xi}) = \mathcal{R} &- [\lambda_\Lambda Q_\Lambda + \lambda_{\bar{\Lambda}} Q_{\bar{\Lambda}}]\mathcal{S} \\ &+ \lambda_\Lambda \lambda_{\bar{\Lambda}} [Q_1 \mathcal{T}_1 + Q_2 \mathcal{T}_2 + Q_3 \mathcal{T}_3 + Q_4 \mathcal{T}_4]. \end{aligned} \tag{12.1}$$

Here, the six functions $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are functions of the scattering angle θ and the ratio of form factors η_ψ, whereas the six functions Q are functions of $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$, and of $\hat{\mathbf{p}} = \mathbf{l}_{\Sigma^0}$ and $\hat{\mathbf{k}} = \mathbf{l}_e$. The unit vectors $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$ only enter the weak-asymmetry functions λ_Λ and $\lambda_{\bar{\Lambda}}$ of Eqs. (7.1) and (7.2).

It remains to connect the four kinematic unit vectors to measured quantities. To this end we imbed Cartesian-coordinate systems in our events. Then, with the Lambda hyperon as an example,

$$\mathbf{l}_\Lambda = (l_{\Lambda x}, l_{\Lambda y}, l_{\Lambda z}) = (\cos\phi_\Lambda \sin\theta_\Lambda, \sin\phi_\Lambda \sin\theta_\Lambda, \cos\theta_\Lambda). \tag{12.2}$$

Our preferred coordinate system is named global and has the xz plane as scattering plane, and $\hat{\mathbf{p}}$ along the z direction.

In global coordinates the building blocks of the spin-density-distribution function $\mathcal{W}(\boldsymbol{\xi})$ in Eq. (12.1) have the simple structure mentioned above. In particular, the six Q functions are independent of the scattering angle θ.

An alternative to global coordinates is helicity-like coordinates, when the $x'z'$ plane is the scattering plane, and $\mathbf{k}$ directed along the z' axis. Several of the Q functions now depend on the scattering angle θ in a complex way, even though the two coordinate systems are related by a rotation.

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APPENDIX A: BARYON FORM FACTORS

The diagram in Fig. 1 describes the annihilation reaction $e^-(k_1)e^+(k_2) \to Y(p_1)\bar{Y}(p_2)$ and involves two vertex functions: one of them leptonic, the other one baryonic. The strength of the lepton-vertex function is determined by the fine-structure constant α_e, but two complex form factors $G_M^\psi(s)$ and $G_E^\psi(s)$ are needed for a proper parametrization of the baryonic vertex function, as of Ref. [4]. The values of these form factors vary with energy, $s = (p_1 + p_2)^2$.

The strength of the baryon form factors is measured by the function $D_\psi(s)$,

$$D_\psi(s) = s|G_M^\psi|^2 + 4M^2|G_E^\psi|^2, \tag{A1}$$

with the M-variable representing the hyperon mass. The ratio of form factors is measured by $\eta_\psi(s)$,

$$\eta_\psi(s) = \frac{s|G_M^\psi|^2 - 4M^2|G_E^\psi|^2}{s|G_M^\psi|^2 + 4M^2|G_E^\psi|^2}, \tag{A2}$$

with $\eta_\psi(s)$ satisfying $-1 \le \eta_\psi(s) \le 1$. The relative phase of form factors is measured by $\Delta\Phi_\psi(s)$,

$$\frac{G_E^\psi}{G_M^\psi} = e^{i\Delta\Phi_\psi(s)}\left|\frac{G_E^\psi}{G_M^\psi}\right|. \tag{A3}$$

A model involving both strong and electromagnetic amplitudes, and simultaneously describing the J/ψ decays into baryon-antibaryon pairs, $J/\psi \to Y\bar{Y}$, is investigated in Ref. [11]. The model parameters are determined by fitting to available experimental data. For the parameters we need, those of the decay $J/\psi \to \Sigma^0\bar{\Sigma}^0$, experimental data exist [12]. In particular, $[\eta_\psi = -0.467 \pm 0.014]$ and $[\Delta\Phi_\psi = 0.092 \pm 0.030]$.

APPENDIX B: STRUCTURE FUNCTIONS

The six structure functions $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ of Eq. (3.7) depend on the scattering angle θ, the ratio function $\eta_\psi(s)$, and the phase function $\Delta\Phi_\psi(s)$. To be specific [4,7],

$$\mathcal{R} = 1 + \eta_\psi \cos^2\theta, \tag{B1}$$

$$\mathcal{S} = \sqrt{1-\eta_\psi^2}\,\sin\theta\cos\theta\sin(\Delta\Phi_\psi), \tag{B2}$$

$$\mathcal{T}_1 = \eta_\psi + \cos^2\theta, \tag{B3}$$

$$\mathcal{T}_2 = -\eta_\psi \sin^2\theta, \tag{B4}$$

$$\mathcal{T}_3 = (1+\eta_\psi)\sin^2\theta, \tag{B5}$$

$$\mathcal{T}_4 = \sqrt{1-\eta_\psi^2}\,\sin\theta\cos\theta\cos(\Delta\Phi_\psi). \tag{B6}$$

The parameters η_ψ and $\Delta\Phi_\psi$ are defined in Eqs. (A2) and (A3). The function $\mathcal{T}_3$ of Eq. (B5) differs from the corresponding function $\mathcal{T}_3$ of Ref. [2] by the $\sin^2\theta$ factor. Similarly, the function $\mathcal{T}_4$ of Eq. (B6) differs from the corresponding function $\mathcal{T}_4$ of Ref. [2] by the $\sin\theta$ factor.

APPENDIX C: INTRODUCING ANGULAR VARIABLES

The angular functions $Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}})$ of Eq. (7.4) and the λ parameters of Eqs. (7.1) and (7.2) are expressed in terms of unit vectors such as $\mathbf{l}_p$ and $\mathbf{l}_\Lambda$, which are not directly measurable but which must be calculated. We suggest the following approach.

For each event we imbed the particle momenta in its c.m. system and with coordinate axes as defined in Eq. (9.1). For the Σ^0 hyperon the components of the momentum are, by definition,

$$\hat{\mathbf{p}}_{\Sigma^0} = (0, 0, 1). \tag{C1}$$

Then, let us consider the proton and the hyperon of the final state, with momenta $\mathbf{p}_p$ and $\mathbf{p}_\Lambda$ in the c.m. system. In the rest system of the Lambda hyperon $\mathbf{L}_p$ denotes the proton momentum, which is given by the expression

$$\mathbf{L}_p = \mathbf{p}_p + B_{\Lambda p}\mathbf{p}_\Lambda, \tag{C2}$$

$$B_{\Lambda p} = \frac{1}{m_\Lambda}\left[\frac{1}{E_\Lambda + m_\Lambda}\mathbf{p}_\Lambda \cdot \mathbf{p}_p - E_\Lambda\right]. \tag{C3}$$

Now, the length of the vector $\mathbf{L}_p$ is well known, being the momentum in the hyperon decay $\Lambda \to \pi N$, and therefore

$$|\mathbf{L}_p| = \frac{1}{2m_\Lambda}[(m_\Lambda^2 + m_\pi^2 - m_N^2)^2 - 4m_\Lambda^2 m_\pi^2]^{1/2}. \tag{C4}$$

Hence, the unit vector $\mathbf{l}_p$ appearing in our equations should be

$$\mathbf{l}_p = \mathbf{L}_p/|\mathbf{L}_p|, \tag{C5}$$

$$= (\cos\phi_p \sin\theta_p, \sin\phi_p \sin\theta_p, \cos\theta_p). \tag{C6}$$

Also, the equation for $\mathbf{l}_\Lambda$ in the decay $\Sigma^0 \to \Lambda\gamma$ is easily written down, as are the corresponding equations for the antiparticles, $\bar{p}$ and $\bar{\Lambda}$.

APPENDIX D: HELICITY APPROACH

Working within the helicity formalism, Adlarson and Kupsc [13,14] have derived an expression, $\mathcal{W}_{\rm H}(\mathbf{E},\mathbf{F})$, for the spin-density-distribution function describing reaction, $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0 \to \Lambda\gamma\bar{\Lambda}\gamma \to p\pi^-\gamma\bar{p}\pi^+\gamma$. Subsequently, this function has been employed by Heikkilä [12] for a study of hyperon-decay parameters.

The corresponding spin-density-distribution function in the global formalism is $\mathcal{W}_{\rm G}(\mathbf{E},\mathbf{F})$ of Eq. (9.5).

The two density-distribution functions can be unified into a single equation,

$$\begin{aligned}\mathcal{W}_{\rm G,H}(\mathbf{E},\mathbf{F}) = 1 + \eta_\psi\cos^2\theta + \sqrt{1-\eta_\psi^2}\sin(\Delta\Phi_\psi)\sin\theta\cos\theta[\pm\lambda_\Lambda E_y - \lambda_{\bar{\Lambda}}F_y]\\ - \lambda_\Lambda\lambda_{\bar{\Lambda}}[\pm(1+\eta_\psi)E_zF_z - \sin^2\theta(E_xF_x \pm E_zF_z \pm \eta_\psi E_yF_y)\\ - \sqrt{1-{\eta_\psi}^2}\cos(\Delta\Phi_\psi)\sin\theta\cos\theta(\pm E_zF_x - E_xF_z)],\end{aligned} \tag{D1}$$

with the $\pm$ symbol being $+$ for $\mathcal{W}_{\rm H}$ and $-$ for $\mathcal{W}_{\rm G}$, and the following shorthand notations being understood:

$$\begin{aligned}\mathbf{E}(\theta_\Lambda,\phi_\Lambda) &= (E_x,E_y,E_z)\\ &= (\cos\phi_\Lambda\sin\theta_\Lambda, \sin\phi_\Lambda\sin\theta_\Lambda, \cos\theta_\Lambda),\end{aligned} \tag{D2}$$

$$\begin{aligned}\mathbf{F}(\theta_{\bar{\Lambda}},\phi_{\bar{\Lambda}}) &= (F_x,F_y,F_z)\\ &= (\cos\phi_{\bar{\Lambda}}\sin\theta_{\bar{\Lambda}}, \sin\phi_{\bar{\Lambda}}\sin\theta_{\bar{\Lambda}}, \cos\theta_{\bar{\Lambda}}).\end{aligned} \tag{D3}$$

The λ factors of Eq. (D1) are originally defined in Eqs. (7.1) and (7.2),

$$\lambda_\Lambda = \alpha_\Lambda\cos\theta_{\Lambda p} = \alpha_\Lambda\mathbf{l}_\Lambda\cdot\mathbf{l}_p \tag{D4}$$

$$\lambda_{\bar{\Lambda}} = \alpha_{\bar{\Lambda}}\cos\theta_{\bar{\Lambda}\bar{p}} = \alpha_{\bar{\Lambda}}\mathbf{l}_{\bar{\Lambda}}\cdot\mathbf{l}_{\bar{p}}, \tag{D5}$$

where the angles $\theta_{\Lambda p}$ and $\theta_{\bar{\Lambda}\bar{p}}$ are the hyperon-helicity angles.

In *global coordinates* the components of the unit vectors $\mathbf{E}(\theta_\Lambda,\phi_\Lambda) \equiv \mathbf{l}_\Lambda(\theta_\Lambda,\phi_\Lambda)$ and $\mathbf{F}(\theta_{\bar{\Lambda}},\phi_{\bar{\Lambda}}) \equiv \mathbf{l}_{\bar{\Lambda}}(\theta_{\bar{\Lambda}},\phi_{\bar{\Lambda}})$ of Eqs. (D2) and (D3), as well as the unit vectors $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$, are defined relative to the Cartesian base $(\hat{x},\hat{y},\hat{z}) \equiv (\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ of Eq. (9.1). Inspection shows that in these coordinates the differential-spin-density-distribution functions $\mathcal{W}_{\rm G}(\mathbf{E},\mathbf{F})$ and $\mathcal{W}(\mathbf{E}(\theta_\Lambda,\phi_\Lambda),\mathbf{F}(\theta_{\bar{\Lambda}},\phi_{\bar{\Lambda}}))$ of Eq. (9.5) are one and the same function, identically,

$$\mathcal{W}_{\rm G}(\mathbf{E},\mathbf{F}) \equiv \mathcal{W}(\mathbf{E}(\theta_\Lambda,\phi_\Lambda),\mathbf{F}(\theta_{\bar{\Lambda}},\phi_{\bar{\Lambda}})). \tag{D6}$$

Working in *helicity coordinates* implies working in several coordinate systems in parallel. In the present application it is conventional to consider two different coordinate systems.

The first one is for particles, such as the Λ and their decays, and spanned by basis vectors $(\hat{x}_1,\hat{y}_1,\hat{z}_1) = (-\mathbf{e}_x,-\mathbf{e}_y,\mathbf{e}_z)$. To emphasize the coordinate system, components are marked by a ‘prime,’ such that $\mathbf{E}'(\theta,\phi) = (E'_x,E'_y,E'_z) = (E_x(\Omega'_E),E_y(\Omega'_E),E_z(\Omega'_E))$, and with spherical coordinates $\Omega'_E = (\theta'_E,\phi'_E)$.

The second coordinate system is for antiparticles, such as anti-Lambda and their decays, and spanned by basis vectors $(\hat{x}_2,\hat{y}_2,\hat{z}_2) = (-\mathbf{e}_x,\mathbf{e}_y,-\mathbf{e}_z)$. Vector components are marked by a ‘bis,’ such that $\mathbf{F}''(\theta,\phi) = (F''_x,F''_y,F''_z) = (F_x(\Omega''_F),F_y(\Omega''_F),F_z(\Omega''_F))$. Relations among basis vectors lead to relations among components, e.g., $E_x = -E'_x = -E''_x$.

Let us then return to $\mathcal{W}_{\rm H}$ of Eq. (D1).

The spin-density function $\mathcal{W}_{\rm H}$, which is the one encountered in the helicity-based work, differs from the spin-density function $\mathcal{W}_{\rm G}$, encountered in the global-based work, through some signs. They can be absorbed into the $\mathbf{E}$ and $\mathbf{F}$ functions, yielding new functions ${}^\eta\mathbf{E}$ and ${}^\eta\mathbf{F}$ that satisfy, by definition,

$$\mathcal{W}_{\rm H}(\mathbf{E}',\mathbf{F}'') = \mathcal{W}_{\rm G}({}^\eta\mathbf{E},{}^\eta\mathbf{F}). \tag{D7}$$

From Eq. (D1) we deduce that a redefinition with this demand can be achieved in two different ways:

$$\begin{aligned}({}^\eta E_x,{}^\eta E_y,{}^\eta E_z) &= (-E'_x,-E'_y,E'_z),\\ ({}^\eta F_x,{}^\eta F_y,{}^\eta F_z) &= (-F''_x,F''_y,-F''_z); \quad \text{(A)}\end{aligned}$$

and

$$\begin{aligned}({}^\eta E_x,{}^\eta E_y,{}^\eta E_z) &= (E'_x,-E'_y,-E'_z),\\ ({}^\eta F_x,{}^\eta F_y,{}^\eta F_z) &= (F''_x,F''_y,F''_z). \quad \text{(B)}\end{aligned}$$

We first check case (A).

By construction ${}^\eta\mathbf{E}$ is a vector with components as given, in the base $\hat{x}_1\hat{y}_1\hat{z}_1$. Comparing the basis vectors of this coordinate with those of the global-coordinate system $\hat{x}\,\hat{y}\,\hat{z}$ we notice the minus signs match. Therefore, we have the relation $(-E'_x, -E'_y, E'_z) = (E_x, E_y, E_z)$; in short ${}^{\eta}\mathbf{E} = \mathbf{E}$. This represents a rotation by π around the $\hat{z}$ axis.

Repeating this analysis for the $\mathbf{F}$ vector we get ${}^{\eta}\mathbf{F} = \mathbf{F}$. Now, the rotation is around the $\hat{z}$ axis. This all means that,

$$\mathcal{W}_{\mathrm{H}}(\mathbf{E}', \mathbf{F}'') = \mathcal{W}_{\mathrm{G}}({}^{\eta}\mathbf{E}, {}^{\eta}\mathbf{F}) = \mathcal{W}_{\mathrm{G}}(\mathbf{E}, \mathbf{F}), \quad \text{(D8)}$$

which in turn means that in the initial spin-density functional $\mathcal{W}_{\mathrm{H}}$ the vector $\mathbf{E}'$ is a vector in the base $\hat{x}_1\hat{y}_1\hat{z}_1$, and $\mathbf{F}''$ a vector in the base $\hat{x}_2\hat{y}_2\hat{z}_2$. In the global spin-density functional $\mathcal{W}_{\mathrm{G}}$, the vectors $\mathbf{E}$ and $\mathbf{F}$ are vectors in the base $\hat{x}\,\hat{y}\,\hat{z}$. Equation (D8) is the connection we set out to prove.

Next, we turn to case (B).

We already have ${}^{\eta}\mathbf{F} = \mathbf{F}''$, which is a vector in the base $\hat{x}_2\hat{y}_2\hat{z}_2$. Concerning the ${}^{\eta}\mathbf{E}$ we notice it involves a rotation by π around the $\hat{x}$ axis, or equivalently, two successive rotations around the $\hat{y}$ and $\hat{z}$ axes.

Concerning the ${}^{\eta}\mathbf{E}$ vector we notice the signs enter in such a way that ${}^{\eta}\mathbf{E} = \mathbf{E}$. Thus, we obtain a relation,

$$\mathcal{W}_{\mathrm{H}}(\mathbf{E}', \mathbf{F}'') = \mathcal{W}_{\mathrm{G}}(\mathbf{E}, \mathbf{F}''), \quad \text{(D9)}$$

between the helicity $\mathcal{W}_{\mathrm{H}}$ and the global $\mathcal{W}_{\mathrm{G}}$ spin-density functionals. It follows that case (B) is not the relation we were looking for, but it was fun anyway.

The relations of Eqs. (D8) and (D9) can alternatively be formulated as relations among spherical-angular coordinates. The calculation is elementary and therefore we only give results. Hence, for case (A) the relations between the spherical angles of the Λ-directional-unit-vector $\mathbf{E}$ and the spherical angles of the $\bar{\Lambda}$-directional-unit-vector $\mathbf{F}$ as required by the helicity (H) and global (G) calculations are

$$\begin{cases} \theta_{\Lambda H} = \theta_{\Lambda G} \\ \phi_{\Lambda H} = \pi + \phi_{\Lambda G} \end{cases} \qquad \begin{cases} \theta_{\bar{\Lambda} H} = \pi - \theta_{\bar{\Lambda} G} \\ \phi_{\bar{\Lambda} H} = \pi - \phi_{\bar{\Lambda} G} \end{cases}.$$

The notation should be obvious. For case (B) the corresponding relations read

$$\begin{cases} \theta_{\Lambda H} = \theta_{\Lambda G} \\ \phi_{\Lambda H} = \pi + \phi_{\Lambda G} \end{cases} \qquad \begin{cases} \theta_{\bar{\Lambda} H} = \theta_{\bar{\Lambda} G} \\ \phi_{\bar{\Lambda} H} = \phi_{\bar{\Lambda} G} \end{cases}.$$

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