Electric field of a charge in the vicinity of a higher dimensional black hole

David Garfinkle[*]

Department of Physics, Oakland University, Rochester, Michigan 48309, USA
and Leinweber Center for Theoretical Physics, Randall Laboratory of Physics,
University of Michigan, Ann Arbor, Michigan 48109-1120, USA

(Received 3 December 2020; accepted 14 January 2021; published 28 January 2021)

We find the electric field of a point charge in the presence of a higher-dimensional black hole. As the charge is lowered to the horizon, all higher multipole moments go to zero, and only the Coulomb field remains.

DOI: 10.1103/PhysRevD.103.024056

I. INTRODUCTION

A black hole has no hair. That is, the properties of a stationary black hole in four spacetime dimensions are entirely determined by its mass, spin, and charge [1–4]. When objects fall into a black hole, the black hole settles down to this simple, unique, stationary state. A nice illustration of this phenomenon is contained in the paper of Cohen and Wald [5], which calculated the electric field of a static point charge in the presence of a Schwarzschild black hole. While this paper contains a detailed expression for the electric field, its main result is that as the position of the charge approaches the event horizon all higher multipole moments of the electric field go to zero, and only the Coulomb field remains.

In more than four spacetime dimensions, there are many more exotic possibilities for black holes (for a review see Ref. [6]). Nonetheless, for static black holes the theorems of Refs. [1,2] generalize [7]. A static, vacuum, asymptotically flat black hole in $n+1$ spacetime dimensions is the Schwarzschild-Tangherlini black hole [8]. In the electrovac case, it is the charged generalization of the Schwarzschild-Tangherlini black hole.

Given the uniqueness result of Ref. [7] one would expect the result of Ref. [5] to generalize to higher dimensions. This issue was addressed by Fox [9], who considered the problem of a point charge in the presence of a Schwarzschild-Tangherlini black hole. The claimed result of Ref. [9] is that in contrast to the $3+1$-dimensional case, the higher multipoles do not go away as the charge is lowered to the horizon.

In this paper, we calculate the electric field of a point charge in the presence of a Schwarzschild-Tangherlini black hole. In contrast to Ref. [9] we find that the higher multipole moments vanish as the charge is lowered to the horizon, just as they did in Ref. [5]. The calculation of the electric field is given in Sec. II, with some of the details of the calculation provided in Sec. III. Conclusions are given in Sec. IV.

II. FIELD CALCULATION

The line element of the Schwarzschild-Tangherlini black hole in $n+1$ spacetime dimensions takes the form

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \gamma_{AB} dx^A dx^B). \quad (1)$$

Here the quantity in parentheses is the line element of the $n-1$-dimensional sphere, with γ_{AB} being the metric of the $n-2$-dimensional sphere. The reason for writing the metric in this way is that we will choose the position of the charge to be the z axis, and will thus consider functions depending only on r and θ. The quantity f is given by

$$f = 1 - \frac{2M}{r^{n-2}}. \quad (2)$$

For the most part, our treatment will be a straightforward generalization of the treatment in Ref. [5], with one exception: we will begin by choosing a different set of coordinates. The reason for this is that the t coordinate is singular on the horizon. Therefore imposing smoothness conditions on tensor fields using the coordinate system of Eq. (1) must involve careful calculation of the behavior of invariant quantities. In contrast, given a smooth coordinate system, all that is needed is to check that the coordinate components of the relevant tensor fields are smooth functions of the coordinates. We will choose ingoing Eddington coordinates [10] (sometimes called Eddington-Finkelstein coordinates [11]) given by

$$dv = dt + f^{-1} dr. \quad (3)$$

This puts the line element of Eq. (1) in the form

[*]garfinkl@oakland.edu

$$ds^2 = -f dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\gamma_{AB}dx^A dx^B). \quad (4)$$

In terms of metric components we have

$$g_{vv} = -f, \qquad g_{vr} = g_{rv} = 1, \qquad g_{rr} = 0,$$
$$g_{\theta\theta} = r^2, \qquad \sqrt{g} = r^{n-1}(\sin\theta)^{n-2}\sqrt{\gamma}. \quad (5)$$

The static Killing vector, ξ^a has component $\xi^v = 1$ with all other components vanishing.

For the electrostatic field of a point charge on the z axis, the only nonzero components of the electromagnetic field tensor F^{ab} are $F^{vr} = -F^{rv}$ and $F^{v\theta} = -F^{\theta v}$ where these components are functions of only r and θ. From the Maxwell equation $\nabla_{[a}F_{bc]} = 0$ we obtain

$$0 = \partial_v F_{r\theta} + \partial_r F_{\theta v} + \partial_\theta F_{vr} \quad (6)$$

which using Eq. (5) becomes

$$0 = \partial_r(-fr^2F^{\theta v}) + \partial_\theta F^{rv}. \quad (7)$$

Therefore there is a scalar ψ for which

$$F^{rv} = \partial_r\psi, \qquad F^{\theta v} = f^{-1}r^{-2}\partial_\theta\psi. \quad (8)$$

From the second Maxwell equation

$$-4\pi j^\beta = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}F^{\alpha\beta}) \quad (9)$$

and Eq. (8) we find

$$\partial_r\partial_r\psi + \frac{n-1}{r}\partial_r\psi + r^{-2}f^{-1}(\partial_\theta\partial_\theta\psi + (n-2)\cot\theta\partial_\theta\psi) = -4\pi j^v. \quad (10)$$

We consider a point charge e located on the z axis at $r = b$. Away from the charge, we look for solutions of Eq. (10) by the method of separation of variables. That is, we seek a solution of the form $\psi = A(r)B(u)$ where $u = \cos\theta$. We then find that Eq. (10) gives

$$\frac{d^2A}{dr^2} + \frac{n-1}{r}\frac{dA}{dr} - \frac{K}{r^2 f}A = 0, \quad (11)$$

$$(1-u^2)\frac{d^2B}{du^2} + (1-n)u\frac{dB}{du} + KB = 0, \quad (12)$$

where K is the separation constant of the equation. The solutions of the second of these equations are the Gegenbauer polynomials $C_\ell^\alpha(u)$. Here, ℓ is the order of the polynomial and $\alpha = (n-2)/2$. The separation constant is $K = \ell(\ell + n - 2)$. For $n = 3$ the Gegenbauer polynomials are just the usual Legendre polynomials. The Gegenbauer polynomials are orthogonal with weight function $(1-u^2)^{(n-3)/2}$ and satisfy the normalization

$$\int_{-1}^{1}(C_\ell^\alpha)^2(1-u^2)^{(n-3)/2}du = \frac{\pi 2^{1-2\alpha}\Gamma(\ell+2\alpha)}{\ell!(\ell+\alpha)(\Gamma(\alpha))^2}. \quad (13)$$

We will use the symbol Q_ℓ^α to denote the somewhat complicated looking normalization constant on the right-hand side of Eq. (13).

With the known value of the separation constant, Eq. (11) then becomes

$$\frac{d^2A_\ell}{dr^2} + \frac{n-1}{r}\frac{dA_\ell}{dr} - \frac{\ell(\ell+n-2)}{r^2 f}A_\ell = 0. \quad (14)$$

For each ℓ we must find separate solutions of Eq. (14): one for $r < b$ and one for $r > b$. The solution must be continuous at $r = b$, and we will compute the discontinuity in dA/dr using Eq. (10).

We will treat the $\ell = 0$ case separately. Here $B = 1$ and

$$\frac{d^2A_0}{dr^2} + \frac{n-1}{r}\frac{dA_0}{dr} = 0. \quad (15)$$

It then follows that $F^{v\theta} = 0$ and

$$F^{vr} = c_0 r^{1-n} \quad (16)$$

where the constant c_0 must be chosen separately for $r < b$ and $r > b$. Since the black hole has no charge, we must choose $c_0 = 0$ for the $r < b$ solution. Since the charge as calculated from the field at large distances must equal e, it follows from Eq. (10) that for the $r > b$ solution

$$c_0 = -\frac{4\pi e}{\mathcal{A}_{n-1}}. \quad (17)$$

Here $\mathcal{A}_{n-1}$ is the area of the $n-1$ sphere and is given explicitly by

$$\mathcal{A}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \quad (18)$$

Thus, we find that the $\ell = 0$ part of the electromagnetic field is given by

$$F^{vr} = 0 \quad \text{for } r < b, \qquad F^{vr} = -\frac{4\pi e}{\mathcal{A}_{n-1}}r^{1-n} \quad \text{for } r > b. \quad (19)$$

Now we consider the $\ell > 0$ part of the electromagnetic field. Since $f \to 1$ at large r it follows that the solutions of Eq. (14) behave like r^ℓ and $r^{-(\ell+n-2)}$ at large r. For $r > b$ we must choose the solution that goes to zero at large distances. Denote this solution $g_\ell(r)$ with its normalization chosen so that $g_\ell(r) = r^{-(\ell+n-2)}$ at large distances. Since f vanishes on the horizon, it follows from Eq. (8) that in order to have a smooth electromagnetic field, the solution of Eq. (14) must vanish on the horizon. Denote by $h_\ell(r)$ this solution, with the normalization chosen so that $h_\ell(r) = r^\ell$ at large distances. Then the $\ell > 0$ part of ψ takes the form

$$\psi_{\ell>0} = \sum_{\ell>0} c_\ell g_\ell(b) h_\ell(r) C^\alpha_\ell(\cos\theta) \quad r < b,$$
$$\psi_{\ell>0} = \sum_{\ell>0} c_\ell h_\ell(b) g_\ell(r) C^\alpha_\ell(\cos\theta) \quad r > b, \tag{20}$$

for some set of constants c_ℓ.

Now for each $\ell > 0$ multiply Eq. (10) by $\sqrt{g} C^\alpha_\ell(\cos\theta)$ and integrate over all angular variables to obtain

$$-4\pi e C^\alpha_\ell(1)\delta(r-b) = \mathcal{A}_{n-2} Q^\alpha_\ell \left[\frac{d}{dr}\left(r^{n-1}\frac{dA_\ell}{dr}\right) - \ell(\ell+n-2) r^{n-3} f^{-1} A_\ell\right]. \tag{21}$$

Integrating Eq. (21) from $b-\epsilon$ to $b+\epsilon$ we obtain

$$-4\pi e C^\alpha_\ell(1) = \mathcal{A}_{n-2} Q^\alpha_\ell c_\ell b^{n-1} W(h_\ell, g_\ell). \tag{22}$$

Here the Wronskian of two solutions $W(u_1, u_2)$ is defined to be $W \equiv u_1 u_2' - u_2 u_1'$ and it is to be evaluated at $r = b$. However, since the differential equation that the solutions satisfy is Eq. (14) we obtain

$$\frac{dW}{dr} = -\frac{n-1}{r} W \tag{23}$$

and therefore there is a constant k for which $W = k r^{1-n}$. But with our chosen normalization for g_ℓ and h_ℓ we find that at large distances $W = -(2\ell + n - 2) r^{1-n}$ and therefore that the constant k is equal to $-(2\ell + n - 2)$. Using this result in Eq. (22) we obtain

$$-4\pi e C^\alpha_\ell(1) = -(2\ell + n - 2)\mathcal{A}_{n-2} Q^\alpha_\ell c_\ell \tag{24}$$

and therefore

$$c_\ell = \frac{4\pi e C^\alpha_\ell(1)}{(2\ell + n - 2)\mathcal{A}_{n-2} Q^\alpha_\ell}. \tag{25}$$

[Note that for the case $n = 3$ the expression of Eq. (25) becomes $c_\ell = e$.] Using Eq. (25) in Eq. (20) we find that the $\ell > 0$ part of ψ is given by

$$\psi_{\ell>0} = \frac{4\pi e}{\mathcal{A}_{n-2}} \sum_{\ell>0} \frac{C^\alpha_\ell(1)}{(2\ell+n-2) Q^\alpha_\ell} g_\ell(b) h_\ell(r) C^\alpha_\ell(\cos\theta) \quad r < b, \tag{26}$$

$$\psi_{\ell>0} = \frac{4\pi e}{\mathcal{A}_{n-2}} \sum_{\ell>0} \frac{C^\alpha_\ell(1)}{(2\ell+n-2) Q^\alpha_\ell} h_\ell(b) g_\ell(r) C^\alpha_\ell(\cos\theta) \quad r > b. \tag{27}$$

To obtain explicit expressions for $\psi_{\ell>0}$ we need explicit expressions for $g_\ell(r)$ and $h_\ell(r)$. However, there is already enough information in Eq. (27) to work out the fate of the higher multipole field as the charge is lowered to the horizon. Since $h_\ell(r)$ vanishes on the horizon, it follows that $h_\ell(b)$ goes to zero as the charge is lowered to the horizon. Therefore in this limit the right-hand side of Eq. (27) vanishes. Thus all higher multipole parts of the field vanish and only the Coulomb field of Eq. (19) remains.

III. SOLUTIONS OF THE RADIAL EQUATION

We now turn to the problem of obtaining explicit expressions for $g_\ell(r)$ and $h_\ell(r)$. Since g_ℓ behaves like $r^{-(\ell+n-2)}$ near infinity, we define $\tilde{A}_\ell$ by $\tilde{A}_\ell \equiv r^{\ell+n-2} A_\ell$ and find that Eq. (14) takes the form

$$\frac{d^2\tilde{A}_\ell}{dr^2} - \frac{2\ell + n - 3}{r}\frac{d\tilde{A}_\ell}{dr} + \frac{\ell(\ell+n-2)}{r^2}(1 - f^{-1})\tilde{A}_\ell = 0. \tag{28}$$

Defining the coordinate $\rho \equiv 1 - f$ we find that Eq. (28) takes the form

$$\rho(\rho-1)\frac{d^2\tilde{A}_\ell}{d\rho^2} + (\rho-1)(2s+2)\frac{d\tilde{A}_\ell}{d\rho} + s(s+1)\tilde{A}_\ell = 0 \tag{29}$$

where the quantity s is defined by

$$s \equiv \frac{\ell}{n-2}. \tag{30}$$

Note that $r \to \infty$ corresponds to $\rho = 0$ and the horizon is at $\rho = 1$. Thus we are interested in solutions to Eq. (28) on the interval (0,1). Furthermore, g_ℓ is the solution that vanishes at $\rho = 0$ and h_ℓ is the solution that vanishes at $\rho = 1$.

Equation (29) has the form of the hypergeometric differential equation. Recall [12] that the hypergeometric differential equation for a function $y(x)$ has three parameters (a_1, a_2, a_3) and takes the form

$$x(x-1)\frac{d^2y}{dx^2} + [(a_1 + a_2 + 1)x - a_3]\frac{dy}{dx} + a_1 a_2 y = 0. \tag{31}$$

Furthermore, the solution to the hypergeometric equation that is regular at $x = 0$ is the hypergeometric function $F(a_1, a_2, a_3, x)$. Comparing Eq. (29) to Eq. (31) we find that the values of the parameters are

$$a_1 = s, \qquad a_2 = 1 + s, \qquad a_3 = 2 + 2s. \tag{32}$$

It then follows that g_ℓ is given by

$$g_\ell = r^{-(\ell+n-2)} F(s, 1+s, 2+2s, \rho). \tag{33}$$

Since $F(a_1, a_2, a_3, 0) = 1$, it follows that Eq. (33) has the normalization for g_ℓ that we chose in the previous section.

We could attempt to find h_ℓ by using a linear combination of the singular solution and the nonsingular solution of Eq. (29). However, it turns out to be both easier and more straightforward to use f as a variable instead of ρ and to build in the property that h_ℓ needs to vanish at the horizon: from Eq. (29) we obtain

$$f(f-1)\frac{d^2}{df^2}(f^{-1}\tilde{A}_\ell) + [(2s+4)f-2]\frac{d}{df}(f^{-1}\tilde{A}_\ell) + (s+1)(s+2)f^{-1}\tilde{A}_\ell = 0. \quad (34)$$

Equation (34) is also the hypergeometric equation, but now with the parameters

$$a_1 = 1+s, \qquad a_2 = 2+s, \qquad a_3 = 2. \quad (35)$$

Taking the nonsingular solution, we then find that h_ℓ is given by

$$h_\ell = k_\ell r^{-(\ell+n-2)} f F(1+s, 2+s, 2, f). \quad (36)$$

Here k_ℓ is a normalization constant to be chosen to satisfy the normalization condition chosen in the previous section.

IV. CONCLUSION

We have found that all the higher multipole moments vanish as the charge is lowered to the horizon. What then went wrong in the analysis of Ref. [9] to yield the opposite conclusion? Simply put, the treatment of Ref. [9] chooses solutions of Maxwell's equations that are singular on the horizon, with that choice being obscured by the coordinate systems used. The method of Ref. [9] uses the t coordinate throughout, and uses the ρ coordinate to analyze all solutions of the radial equation, which makes for a very complicated analysis at the horizon. Using the Eddington coordinate v, one can immediately see from Eq. (8) that the higher multipole part of ψ must vanish at the horizon. But in any coordinate system, one can calculate invariant quantities and demand that they be nonsingular. From Eqs. (5) and (8) it follows that the electromagnetic invariant $F^{ab}F_{ab}$ is given by

$$F^{ab}F_{ab} = -2[(\partial_r\psi)^2 + f^{-1}r^{-4}(\partial_\theta\psi)^2]. \quad (37)$$

Therefore, from an examination of this invariant one can conclude that the nonmonopole part of ψ must vanish on the horizon. The treatment of Ref. [9] fails to impose this condition and is therefore not treating the correct electromagnetic field.

In contrast, we impose smoothness on the horizon and find that everything proceeds as a straightforward generalization of Ref. [5] with the same conclusion: all higher multipoles vanish as the charge approaches the horizon. There may be many cases in which higher-dimensional black holes lead to exotic, unexpected behavior, but this is not one of them.

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. PHY-1806219.

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