

Cosmic censorship hypothesis and entropy bound on black holes in a canonical ensemble

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This paper argues that the weak cosmic censorship hypothesis implies that the Schwarzschild black hole has maximal entropy in all stationary black holes of fixing temperature, or equivalently, to store a same amount of information the Schwarzschild black hole has highest temperature. It then gives the independent mathematical proofs for 4-dimensional general static black holes and stationary-axisymmetric black holes which have “ t - ϕ ” reflection isometry. This result does not only provide a new universal bound between temperature and entropy of black holes but also offers us new evidence to support the weak cosmic censorship hypothesis.

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I. INTRODUCTION

In general relativity, the Penrose's and Hawking's theorems about singularity show that spacetime singularity will be inevitable if matters satisfy a few very general conditions [1,2]. To keep the predicability in general relativity, it has been conjectured for a long time that spacetime singularity arising from gravitational collapse of physically “reasonable” matter must be shrouded by an event horizon, which is called “weak cosmic censorship” hypothesis [3,4]. Despite its clear significance, however, a complete proof is still open [5–7]. In the absence of a complete proof, theoretical tests of cosmic censorship are of significant value.

Well-studied, the theoretical test was proposed by Penrose in the early 1970s. It considers an asymptotically flat solution of Einstein equations with matter satisfying the dominant energy condition. Then if a Cauchy slice of this solution contains an outer-trapped 2-surface S of area $A(S)$, and if M is the Arnowitt-Deser-Misner (ADM) mass of the data on the slice, the inequality $A(S) \leq 16\pi M^2$ must be true if the “weak cosmic censorship” is true. Though the proof in general case is still open, people have proven it in a large class of cases [8–10]. Particularly, if the initial dataset just forms a stationary black hole, we have

$$A_H \leq 16\pi M^2. \quad (1)$$

Here A_H is just the area of event horizon. The inequality (1) has been also generalized into asymptotically anti-de Sitter (AdS) black holes [11,12]. Recently, a modified version by

taking quantum effects into account was also discussed in Refs. [13,14].

We have known that the black hole is not only a mechanical system but also a thermodynamical system, of which the entropy is given by Bekenstein-Hawking entropy $S = A_H/4$ and temperature is given by Hawking temperature $T_H = \kappa/(2\pi)$, where κ is the surface gravity of event horizon. It is clear that the Penrose inequality can be regarded as an entropy bound. As total energy is fixed, this is an entropy bound in microcanonical ensemble. Once we reconsider the inequality (1) from the thermodynamics, it is natural to ask the question: what will happen if we consider the black hole in the canonical ensemble? One natural expectation is that the Schwarzschild black hole may also have maximal entropy in canonical ensemble, i.e.,

$$S \leq \frac{1}{16\pi T_H^2} \quad \text{i.e.,} \quad A_H \leq \frac{1}{4\pi T_H^2}. \quad (2)$$

This is not a trivial corollary of bound (1), as the physics of black holes may be not equivalent in different ensembles. For example, under certain circumstances, the partition function obtained by using the path-integral approach turned out to depend on the boundary conditions [15–17]. Thus, we cannot use the inequality (1) to directly argue that inequality (2) must be true in canonical ensemble.

The inequality (2) was first conjectured by Ref. [18] according to the computation of Hawking temperature in static spherically symmetric black holes. However, to our knowledge, no any progress was achieved beyond the static spherically symmetric case up to now. This paper will make a first step toward the proof of inequality (2) in general case. Particularly, we will argue that the bound (2) is a necessary condition of “weak cosmic censorship.” Then we

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will prove that: in 4-dimensional Einstein’s gravity theory, for a static black hole or a stationary-axisymmetric black hole which has the “ t - ϕ ” reflection isometry [19], if (i) the weak energy condition is satisfied and (ii) the horizon has topology $S^2 \times \mathbb{R}$, then inequality (2) is always true. The requirement (ii) is redundant in an asymptotically flat black hole if we use dominant energy condition to replace (i) [20–23]. Note that the temperature (i.e., surface gravity) is constant automatically in the case considered here [24].

II. A PHYSICAL HEURISTIC ARGUMENT

Let us first argue that, similar to the Penrose inequality (1), the bound (2) is also implied by “weak cosmic censorship.” Roughly speaking, the weak cosmic censorship states that the singularity originating from a gravitational collapse should be hidden by the event horizon. To connect this dynamics process with inequality (2), we consider the following “thought experiment” shown by Fig. 1. First, we assume that there is a stationary initial black hole with temperature T_H and horizon area A_H . Then we immersed it into a big thermal source at its boundary and the thermal source also has a fixed temperature T_H . Though the temperatures of the thermal source and black hole are same, the black hole and thermal source may have different chemical potential. The black hole then will evolve by various isothermal processes and can exchange energy, particles, charges and so on with the thermal source.

It needs to note that, the vacuum black hole in asymptotically flat spacetime is unstable due to negative heat capacity. Thus, here we assume that exterior of black hole is full of classical matters which offer positive heat capacity during the evolution of the black hole and the dynamics of these isothermal processes is dominated by classical physics. The “weak cosmic censorship” guarantees that an asymptotically flat spacetime with regular initial conditions will be strongly asymptotically predictable [22]. Then the null energy condition implies that the area of the event horizon will not decrease during these processes [25], i.e.,

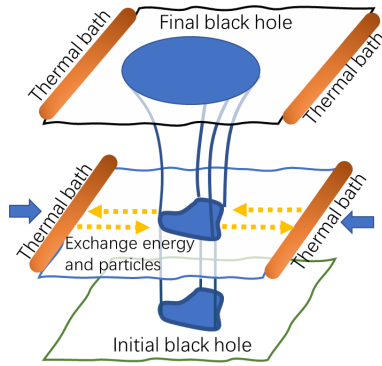


FIG. 1. A schematic diagram of the “thought experiment” which shows how the inequality (2) is relevant to weak cosmic censorship.

$$A_H \leq A_{H,f}. \tag{3}$$

Here $A_{H,f}$ is the area of event horizon in final black hole. In physics, it is reasonable to expect the black hole will settle down to a Kerr-Newman black hole by referring “no-hair” theorem of black hole. For the final Kerr-Newman black hole, let us assume that M is the mass, Ma is the angular momentum and Q is the total charge. We then have the following relationships

$$A_{H,f} = 4\pi(r_h^2 + a^2), \quad T_H = \frac{1}{2\pi} \frac{r_h - M}{r_h^2 + a^2}, \tag{4}$$

with $r_h = M + \sqrt{M^2 - a^2 - Q^2}$. We can verify that

$$T_H A_{H,f} = 2r_h - 2M \leq r_h \leq \sqrt{\frac{A_{H,f}}{4\pi}}. \tag{5}$$

Combine Eqs. (3) and (5) and we will obtain desired inequality (2).

The reader will have noticed that the above argument makes a lot of global assumptions about the resulting space-times, and our current understanding is much too poor to be able to settle those one way or another. The “no-hair” theorem can also be broken in some physical acceptable situations [26,27]. It is clear this heuristic cannot be treated as a valid proof. In the following we will first give the proof for the general static case and then discuss it in the stationary-axisymmetric case.

III. COORDINATES GAUGE IN BONDI-SACHS FORMALISM

In order to prove inequality (2), we will use the Bondi-Sachs formalism, which foliates the spacetime by a series of null surfaces [28–30] and can be used for arbitrary spacetime. Here we first briefly explain about how to build such formalism when the black hole is static or stationary axisymmetric with “ t - ϕ ” reflect isometry. In this paper, the Greek indexes $\{\mu, \nu, \dots\}$ runs from 0 to 3 and the capital Latin indexes $\{A, B, \dots\}$ run from 2 to 3.

In the static case or stationary axisymmetric case with “ t - ϕ ” reflect isometry, there is a Killing vector ξ^μ which is both tangent and normal to event horizon H . The Killing vector ξ^μ will generate a 1-parameter group of diffeomorphisms Φ_u , i.e., $\forall p$ in spacetime $\Phi_0(p) = p$ and the curve $\{\Phi_u(p) | u \in \mathbb{R}\}$ gives us an orbit of ξ^μ . Assume that H is the event horizon (a 3-dimensional null surface) and S_{r_h} is its one spacelike cross section. As H has topology of $S^2 \times \mathbb{R}$, S_{r_h} is a topology sphere. The outward light rays of S_{r_h} form a 3-dimensional null surface W_0 , which is labeled by $u = 0$. See Fig. 2(a). Using map Φ_u we can obtain a series of equal- u surfaces $W_u := \Phi_u(W_0)$. We can prove that W_u are all null by using the fact that ξ^μ is a Killing

vector. These null surfaces are labeled by $u = \text{constant}$ and so we have $g^{\mu\nu}(\partial u)_\mu(\partial u)_\nu = 0$.

Let us now explain how to build $\{r, x^A\}$ coordinates. In the null surface W_0 , the r -coordinate is chosen to satisfy equation

$$(\partial/\partial r)^\mu = \psi g^{\mu\nu}(\partial u)_\nu \quad (6)$$

with an arbitrary u -independent positive function ψ . The r -coordinate then is just the integral curve of Eq. (6). We can adjust the zero point so that the r -coordinate of S_{r_h} satisfies $r|_{S_{r_h}} = r_h$. The value of r_h will be determined later. We denote equal- r surface to be S_r and introduce 2-dimensional coordinates $\{x^A\}$ in S_r . The map Φ_u can bring $\{r, x^A\}$ from W_0 into all other null surfaces and so we obtain the coordinates $\{u, r, x^A\}$ for any point outside event horizon. As the result, we have $\xi^\mu = (\partial/\partial u)^\mu$. See Fig. 2(b) for a schematic explanation.

From Eq. (6) we see that $g_{rr} = g_{\mu\nu}(\partial/\partial r)^\mu(\partial/\partial r)^\nu = 0$ and $g_{rA} = g_{\mu\nu}(\partial/\partial x^A)^\mu(\partial/\partial r)^\nu = \psi \partial u / \partial x^A$. As the coordinate lines of x^A lie in an equal- u surface, we find $g_{rA} = \psi \partial u / \partial x^A = 0$. Thus, the metric in coordinates $\{u, r, x^A\}$ has the following form

$$ds^2 = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du). \quad (7)$$

At event horizon H we have $V|_H = 0$. As the spacetime is asymptotically flat, we then fix the boundary conditions $\beta|_{r \rightarrow \infty} = 0$, and

$$\frac{V}{r} \Big|_{r \rightarrow \infty} = 1, \quad h_{AB} \Big|_{r \rightarrow \infty} dx^A dx^B = d\hat{s}^2, \quad (8)$$

Here $d\hat{s}^2$ is the metric of unit sphere. In the asymptotic inertial frame, we require $r^2 U^A \rightarrow 0$ at the null infinity. It is also possible to choose a rotational frame with constant angular velocity, then we have $U^A = \text{constant}$. The ψ in

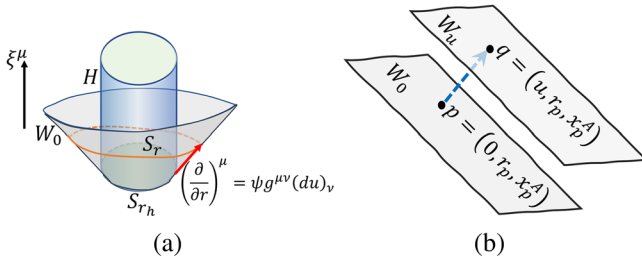


FIG. 2. (a) The schematic diagram about the event horizon H , spacelike cross section S_{r_h} , null hypersurface W_0 and equal- r surface S_r . (b) For a point $q = \Phi_u(p) \in W_u$, we define its coordinate to be (u, r_p, x_p^A) , where r_p and x_p^A are the r and x^A coordinates of p .

Eq. (6) is still free and we can fix this gauge freedom by a requirement

$$\partial_r h = 0. \quad (9)$$

As the result, we have $\sqrt{h} d^2x = \sqrt{h}|_{r \rightarrow \infty} d^2x = d\Omega$ and $d\Omega$ is the surface element of unit sphere. Then the area of event horizon has a simple formula

$$A_H = \int_{\text{fix } u, r=r_h} r^2 \sqrt{h} d^2x = 4\pi r_h^2 \quad (10)$$

and we find $r_h = \sqrt{A_H/(4\pi)}$. The surface gravity of event horizon is given by

$$\kappa^2 = -\frac{1}{2} (\nabla_\mu \xi_\nu) \nabla^\mu \xi^\nu \Big|_H = -\frac{1}{8} g^{\sigma\mu} g^{\tau\nu} (d\xi)_{\sigma\tau} (d\xi)_{\mu\nu} \Big|_H. \quad (11)$$

At event horizon κ is a constant [20–23]. Strictly speaking, the metric (7) with gauge condition (9) may cover only a neighborhood of the event horizon. Here we assume that they have no coordinates singularity in the whole spacetime outside the horizon.

A proposition will be useful in our proofs: for any null vector r^μ which satisfies $\xi_\mu r^\mu < 0$, if weak energy condition is satisfied, then $T_{\mu\nu} \xi^\mu r^\nu \Big|_H \geq 0$. Here $T_{\mu\nu}$ is the energy momentum tensor. The proof contains two steps. At the first step we use Raychaudhuri's equation at H

$$\xi^\mu \partial_\mu \Theta = \omega^{\mu\nu} \omega_{\mu\nu} - \kappa \Theta - \frac{\Theta^2}{2} - \sigma^{\mu\nu} \sigma_{\mu\nu} - R_{\mu\nu} \xi^\mu \xi^\nu. \quad (12)$$

where Θ , κ , $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ are the expansion, the surface gravity, the shear, and the rotation of ξ^μ , respectively. $R_{\mu\nu}$ is the Ricci tensor. As ξ^μ is a hypersurface-orthogonal null Killing vector at H , we find $\omega_{\mu\nu} = \sigma_{\mu\nu} = \Theta = 0$. Then Einstein's equation and Eq. (12) imply

$$T_{\mu\nu} \xi^\mu \xi^\nu \Big|_H = R_{\mu\nu} \xi^\mu \xi^\nu \Big|_H = 0. \quad (13)$$

In the second step, we take a vector $v^\mu = \xi^\mu + s r^\mu$. One can verify $v^\mu v_\mu \Big|_H = 2s \xi^\mu r_\mu$, so $v^\mu \Big|_H$ is timelike for all $s > 0$. The direct computation shows

$$T_{\mu\nu} v^\mu v^\nu \Big|_H = 2s T_{\mu\nu} \xi^\mu r^\nu \Big|_H + s^2 T_{\mu\nu} r^\mu r^\nu \Big|_H. \quad (14)$$

The weak energy condition requires $T_{\mu\nu} r^\mu r^\nu \geq 0$ and $T_{\mu\nu} v^\mu v^\nu \Big|_H \geq 0$ for all $s > 0$. Then $T_{\mu\nu} \xi^\mu r^\nu \Big|_H \geq 0$ follows.

IV. PROOF FOR STATIC CASE

We first focus on static case. We then choose the Killing vector $\xi^\mu = (\partial/\partial u)^\mu$ to be the one which stands for the static symmetry. So all components of metric is

independent of u and the reflection $u \rightarrow -u$ is a symmetry. Consider the induced metric of equal- r surface,

$$d\tilde{s}_r^2 = -\frac{V}{r}e^{2\beta}du^2 + r^2h_{AB}(dx^A - U^A du)(dx^B - U^B du). \quad (15)$$

The $\xi^\mu = (\partial/\partial u)^\mu$ lies in this subspacetime, so the reflection symmetry implies $U^A = 0$. Applying the metric (7) with $U^A = 0$, we find that Eq. (11) reduces into

$$\kappa^2 = -e^{-4\beta}g_{uu}(\partial_r\sqrt{-g_{uu}})^2|_H. \quad (16)$$

Using the fact $A_H = 4\pi r_h^2$, we finally obtain

$$\kappa\sqrt{A_H/4\pi} = \frac{1}{2}\partial_r V|_H. \quad (17)$$

Now let us apply Einstein's equation. Einstein's equation shows the following two relevant equations [31,32] (See Appendix A)

$$\partial_r\beta = \frac{r}{16}h^{AC}h^{BD}(\partial_r h_{AB})(\partial_r h_{CD}) + 2\pi r T_{rr} \quad (18)$$

and

$$e^{-2\beta}\partial_r V = \frac{{}^{(2)}R}{2} - \mathfrak{D}^2\beta - (\mathfrak{D}\beta)^2 - 8\pi r^2 e^{-2\beta}T_{ur} + 4\pi r V e^{-2\beta}T_{rr}. \quad (19)$$

Here ${}^{(2)}R$ and \mathfrak{D}_A stand for the scalar curvature and covariant derivative operator of h_{AB} .

Let us now prove $\partial_r\beta \geq 0$ and $\beta \leq 0$. As T_{rr} is a “null-null” component of energy momentum tensor, the weak energy condition insures $T_{rr} \geq 0$. We note that, for any surface of fixed r and u , the $\mathcal{X}_{AB} := \partial_r h_{AB}$ is a tensor of the 2-dimensional space spanned by the coordinates $\{x^A\}$, i.e., \mathcal{X}_{AB} will be transformed as a tensor under coordinates transformation $x^A \rightarrow \tilde{x}^A = \tilde{x}^A(x)$ (note that \tilde{x}^A does not depend on u and r). Then it is clear that $h^{AC}h^{BD}(\partial_r h_{AB})(\partial_r h_{CD})$ is invariant under such coordinates transformation. By using this invariance, we can always find a suitable coordinates transformation locally so that the inverse induced metric h^{AB} has a diagonal form with two positive eigenvalues $\{\lambda^A\}$. In these special coordinates, the components of inverse induced metric become $\lambda^A\delta^{AB}$ (no summation) and we assume that the components of $\partial_r h_{AB}$ become $\tilde{\mathcal{X}}_{AB}$. Then we have

$$\begin{aligned} h^{AC}h^{BD}(\partial_r h_{AB})(\partial_r h_{CD}) &= \sum_{A,B,C,D} \lambda^A\lambda^B\delta^{AC}\delta^{BD}\tilde{\mathcal{X}}_{AB}\tilde{\mathcal{X}}_{CD} \\ &= \sum_{A,B} \lambda^A\lambda^B(\tilde{\mathcal{X}}_{AB})^2 \geq 0. \end{aligned} \quad (20)$$

This proves $\partial_r\beta \geq 0$ and so we find $\beta \leq \beta(\infty) = 0$.

Take $r^\mu = (\partial/\partial r)^\mu$ and we will find that weak energy condition insures $T_{ur}|_H = T_{\mu\nu}\xi^\mu r^\nu|_H \geq 0$. Then at horizon we have

$$e^{-2\beta}\partial_r V|_H \leq \frac{{}^{(2)}R}{2} - \mathfrak{D}^2\beta. \quad (21)$$

Using the fact that $\partial_r V|_H \geq 0$, $\beta \leq 0$ and Eq. (17), we find $e^{-2\beta}\partial_r V|_H \geq \partial_r V|_H = 2\kappa\sqrt{A_H/4\pi}$ and so

$$\int_H \frac{\partial_r V}{e^{2\beta}}\sqrt{hd^2x} \geq 2\kappa\sqrt{\frac{A_H}{4\pi}} \int_H \sqrt{hd^2x} = 2\kappa\sqrt{4\pi A_H}. \quad (22)$$

Here we have used the fact that κ is constant at event horizon. Taking Eq. (21) into Eq. (22), we finally obtain

$$2\kappa\sqrt{4\pi A_H} \leq \int_{r=r_h} \left[\frac{{}^{(2)}R}{2} - \mathfrak{D}^2\beta \right] \sqrt{hd^2x} = 4\pi. \quad (23)$$

Here we have neglected the totally divergent term and used the Gauss-Bonnet theorem. Rewrite the surface gravity in terms of Hawking temperature $T_H = \kappa/(2\pi)$ and we will obtain the desired inequality (2).

V. PROOF OF STATIONARY-AXISYMMETRIC CASE WITH “ t - ϕ ” REFLECT ISOMETRY

In the stationary axisymmetric black hole, there are two commutative Killing vectors $t^\mu = (\partial/\partial t)^\mu$ (with $t^\mu t_\mu = -1$ at infinity) and $\Psi^\mu = (\partial/\partial\phi)^\mu$, which present the time translation symmetry and rotational symmetry respectively. In addition, the Killing vector Ψ^μ is tangent to event horizon H (but t^μ may not be). In the study of stationary axisymmetric black holes, “ t - ϕ ” reflected isometry is a usual assumption, which covers most of the physically interesting cases [19,33]. By this assumption, there is a constant Ω_H such that Killing vector $\xi^\mu = t^\mu + \Omega_H\Psi^\mu$ is both tangent and normal to event horizon. Here Ω_H is a constant and stands for the angular velocity of event horizon. The Hawking temperature then is given by the Killing vector ξ^μ rather than Killing vector t^μ [33].

At horizon, we choose S_{r_h} such that $(\partial/\partial\phi)^\mu$ lies on the surface S_{r_h} . Because of rotational symmetry, we can require the function ψ in Eq. (6) to satisfy $\Psi^\mu\partial_\mu\psi = 0$. As the result we can prove that the orbit of Ψ^μ , i.e., the coordinate line of ϕ , will always lie on a S_r . See Appendix B for a proof. We choose coordinate u by requiring $(\partial/\partial u)^\mu = \xi^\mu$ and choose coordinates $\{x^2 = \theta, x^3 = \phi\}$ for S_r . At null infinity, this

corresponds to a *rotational frame* rather than an internal frame.

Though U^A is not zero in general, it still satisfies a few properties. First, as the Killing vector ξ^μ is orthogonal to the event horizon, we then have $\xi_\mu(\partial/\partial\phi)^\mu|_H = g_{\mu\nu}(\partial/\partial\phi)^\mu(\partial/\partial u)^\nu|_H = 0$. This leads to $U^\phi|_H = 0$. On the other hand, consider the induced metric of fixing r , i.e., metric (15). The vector fields $(\partial/\partial u)^\mu$ and $(\partial/\partial\phi)^\mu$ are both tangent to equal- r hypersurface, so vector t^μ is also tangent to equal- r hypersurface. Thus, the orbits of two Killing vectors t^μ and $(\partial/\partial\phi)^\mu$ both lay in this timelike hypersurface. Then the “ t - ϕ ” reflection isometry requires metric (15) is invariant under the transformation $\{t \rightarrow -t, \phi \rightarrow -\phi\}$. According to the relationship between u , t , and ϕ , this requires that metric (15) is invariant under the transformation $\{u \rightarrow -u, \phi \rightarrow -\phi\}$. This leads to $U^\theta = 0$. To conclude, U^A satisfies the following two properties in our coordinates gauge,

$$U^\phi|_H = 0, \quad U^\theta = 0. \quad (24)$$

The formula of surface gravity is still given by Eq. (11). By using metric (7) and Eq. (24), we finally find that the κ is still given by Eq. (17).

Now we apply Einstein’s equation. The equation of β is still given by Eq. (18) but Eq. (19) is replaced by [31,32] (See Appendix A)

$$\begin{aligned} e^{-2\beta}\partial_r V &= \frac{({}^2R)}{2} - \mathfrak{D}^2\beta - (\mathfrak{D}\beta)^2 + \mathfrak{D}_A \left[\frac{e^{-2\beta}}{2r^2} \partial_r(r^4 U^A) \right] \\ &\quad - \partial_r(r^4 U^A) \mathfrak{D}_A \left(\frac{e^{-2\beta}}{2r^2} \right) - \frac{8\pi r^2}{e^{2\beta}} T_{ur} \\ &\quad - 8\pi r^2 U^A e^{-2\beta} T_{rA} + 4\pi r V e^{-2\beta} T_{rr} \\ &\quad - \frac{r^4}{4} e^{-4\beta} h_{AB} (\partial_r U^A) (\partial_r U^B). \end{aligned} \quad (25)$$

Here all variables are independent of $\{u, \phi\}$. Note that $U^\theta = 0$ leads to $[\partial_r(r^4 U^A)] \mathfrak{D}_A (e^{-2\beta}/r^2) = 0$, so Eq. (25) shows

$$\left. \frac{\partial_r V}{e^{2\beta}} \right|_H \leq \frac{({}^2R)}{2} - \mathfrak{D}^2\beta + \mathfrak{D}_A \left[\frac{e^{-2\beta}}{2r} \partial_r(r^4 U^A) \right]. \quad (26)$$

Here we also used the fact that $U^A|_H = 0$, which results from Eq. (24). Similar to Eq. (23), after integrating Eq. (26), we can still obtain $2\kappa\sqrt{4\pi A_H} \leq 4\pi$ and so the bound (2) follows.

VI. CONCLUSION AND DISCUSSION

To conclude, this paper proposes a new entropy bound (2) for black holes in canonical ensemble and shows the Schwarzschild black hole has the maximal entropy. This is a parallel version Penrose inequality in the canonical

ensemble. We argue that, in a certain circumstance, the bound (2) will be a necessary condition of “weak cosmic censorship.” We then prove it in the 4-dimensional general static case and stationary-axisymmetric case with the “ t - ϕ ” reflection isometry. The bound (2) also has an inverse interpretation: to store the same amount of information, the Schwarzschild black hole will have the highest temperature.

It is interesting to study the generalization of bound (2) in the higher dimensional case, where we assume that the horizon has topology $S^{d-1} \times R$. The bound (2) should become

$$A_H \leq \left(\frac{d-2}{4\pi T_H} \right)^{d-1} \Omega_{d-1} \quad (27)$$

Here Ω_{d-1} is the surface area of $(d-1)$ -dimensional unit sphere. To derive bound (27) one sufficient condition is

$$\Omega_{d-1}^{-1} \int_{S_{r_h}} ({}^{d-1}R) \leq (d-1)(d-2). \quad (28)$$

Here $({}^{d-1}R)$ is the scalar curvature of metric h_{AB} . In the case $d=3$, this follows from Gauss-Bonnet theorem. For the case $d \geq 4$, the situation is less clear. Inequality (28) will be true if the horizon has $(d-1)$ -dimensional spherical symmetry. If the metric does not have this symmetry however we have, without further information, little control of the integrand. Note that the bound (27) may still be true even if Eq. (28) is broken.

This paper focuses on Einstein gravity theory. It is interesting to study the generalizations in other gravity theories in the future, such as coupling with a dilation field or adding higher order curvature terms. In our proof, we apply a special coordinates gauge to simplify the discussion. However, outside the event horizon, we cannot prove that such coordinates system exists globally in all cases. This leaves an issue for further study. It is also worthy of studying how to prove bound (2) by a coordinate-independent method.

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APPENDIX A: HYPERSURFACE EQUATIONS IN BONDI-SACHS FORMALISM

In this appendix, we give a few mathematical formulas in the Bondi-Sachs formalism. In the main text, we focus on the static spacetime or axisymmetric stationary spacetime with t - ϕ reflection isometry. In fact, we can build local coordinates $\{u, r, x^A\}$ for arbitrary spacetime so that the metric has the following form [28–30]

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2h_{AB}(dx^A - U^A du)(dx^B - U^B du), \quad (\text{A1})$$

and satisfies the gauge $\partial_r h = 0$. Here all components may depend on $\{u, r, x^A\}$. The corresponding nonzero components of the inverse metric are

$$g^{ur} = -e^{-2\beta}, \quad g^{rr} = \frac{Ve^{-2\beta}}{r},$$

$$g^{rA} = -U^A e^{-2\beta}, \quad g^{AB} = \frac{h_{AB}}{r^2}. \quad (\text{A2})$$

The two relative equations in the main text come from the null hypersurface constraint equations. Following Ref. [31] and taking $\lambda = 1$ (or following Ref. [32]), we will have

$$\partial_r \beta = \frac{r}{16} h^{AC} h^{BD} (\partial_r h_{AB})(\partial_r h_{CD}) + 2\pi r T_{rr} \quad (\text{A3})$$

and

$$e^{-2\beta} \partial_r V = \frac{{}^{(2)}R}{2} - \mathfrak{D}^2 \beta - (\mathfrak{D}\beta)^2 - \frac{r^4}{4} e^{-4\beta} h_{AB} (\partial_r U^A)(\partial_r U^B) + \frac{e^{-2\beta}}{2r^2} \mathfrak{D}_A [\partial_r (r^4 U^A)] + 4\pi(r^2 T - h^{AB} T_{AB}). \quad (\text{A4})$$

Here T_{AB} is the projection of energy momentum tensor on the subspace spanned by $\{x^A\}$ and $T = g^{\mu\nu} T_{\mu\nu}$. Note here we use signature $(-, +, +, +)$ but Ref. [31] used the signature $(+, -, -, -)$. The convention on the definition of Riemannian curvature is also different from Ref. [31]. In the convention of Ref. [31], the unit sphere has curvature ${}^{(2)}R = -2$, however, in this paper the unit sphere has curvature ${}^{(2)}R = 2$.

To obtain the equations used in the main text, we first note the facts

$$\frac{e^{-2\beta}}{2r^2} \mathfrak{D}_A [\partial_r (r^4 U^A)] = \mathfrak{D}_A \left[\frac{e^{-2\beta}}{2r^2} \partial_r (r^4 U^A) \right] - \partial_r (r^4 U^A) \mathfrak{D}_A \left(\frac{e^{-2\beta}}{2r^2} \right) \quad (\text{A5})$$

and

$$T = g^{\mu\nu} T_{\mu\nu} + 2g^{ur} T_{ur} + 2g^{\mu A} T_{\mu A} + 2g^{rA} T_{rA} + g^{rr} T_{rr} + r^{-2} h^{AB} T_{AB} \quad (\text{A6})$$

Using Eq. (A2), we find Eq. (A6) becomes

$$T = -2e^{-2\beta} T_{ur} - 2U^A e^{-2\beta} T_{rA} + \frac{Ve^{-2\beta}}{r} T_{rr} + r^{-2} h^{AB} T_{AB}. \quad (\text{A7})$$

Then we obtain

$$e^{-2\beta} \partial_r V = \frac{{}^{(2)}R}{2} - \mathfrak{D}^2 \beta + \mathfrak{D}_A \left[\frac{e^{-2\beta}}{2r^2} \partial_r (r^4 U^A) \right] - (\mathfrak{D}\beta)^2 - \frac{1}{4} r^4 e^{-4\beta} h_{AB} (\partial_r U^A)(\partial_r U^B) + 4\pi r V e^{-2\beta} T_{rr} - \partial_r (r^4 U^A) \mathfrak{D}_A \left(\frac{e^{-2\beta}}{2r^2} \right) - 8\pi U^A e^{-2\beta} r^2 T_{rA} - 8\pi r^2 e^{-2\beta} T_{ur}. \quad (\text{A8})$$

This gives us Eqs. (19) and (25) in the main text.

APPENDIX B: A PROOF ON $(\partial/\partial\phi)^\mu$ LYING ON A S_r

In this appendix, we will prove following statement for stationary axisymmetric case: if (1) one orbit of vector field Ψ^μ lies on the surface S_{r_h} and (2) the function ψ in Eq. (6) satisfies $\Psi^\mu \partial_\mu \psi = 0$, then any orbit of vector field Ψ^μ must lie on a S_r .

The proof contains two parts. At the first part, we prove that the orbit of Ψ^μ must lie on a null hypersurface W_u , i.e., Ψ^μ is tangent to null hypersurface W_u . To do that, we only need to prove Ψ^μ is tangent to null hypersurface W_0 , i.e., $\Psi^\mu (du)_\mu|_{W_0} = 0$. Because Ψ^μ is tangent to S_{r_h} and $S_{r_h} \subset W_0$, we have

$$\Psi^\mu (du)_\mu|_{S_{r_h}} = \Psi^\mu (du)_\mu|_{r=r_h} = 0. \quad (\text{B1})$$

Define $r^\mu = (\partial/\partial r)^\mu$. We consider the Lie derivative of $\Psi^\mu (du)_\mu$ with respect to r^μ . The direct computation shows

$$\mathcal{L}_r (\Psi^\mu (du)_\mu) = (du)_\mu \mathcal{L}_r \Psi^\mu + \Psi^\mu \mathcal{L}_r (du)_\mu = (du)_\mu \mathcal{L}_r \Psi^\mu + \Psi^\mu (d\mathcal{L}_r u)_\mu. \quad (\text{B2})$$

Here we use the formula $d\mathcal{L}_r \omega = \mathcal{L}_r d\omega$ for a p -form field ω . As

$$\mathcal{L}_r u = r^\mu (du)_\mu = \psi g^{\mu\nu} (du)_\mu (du)_\nu = 0.$$

we see

$$\mathcal{L}_r (\Psi^\mu (du)_\mu) = (du)_\mu \mathcal{L}_r \Psi^\mu = -(du)_\mu \mathcal{L}_\Psi r^\mu. \quad (\text{B3})$$

Using the fact that Ψ^μ is a Killing vector and the definition $r^\mu = \psi g^{\mu\nu} (du)_\nu$, we find

$$\begin{aligned}
 (du)_\mu \mathcal{L}_\Psi r^\mu &= (du)_\mu \mathcal{L}_\Psi [\psi g^{\mu\nu} (du)_\nu] \\
 &= \frac{1}{2} \mathcal{L}_\Psi [(du)_\mu \psi g^{\mu\nu} (du)_\nu] = 0. \quad (\text{B4})
 \end{aligned}$$

Here we have used the fact $\mathcal{L}_\Psi(\psi g^{\mu\nu}) = 0$. This shows that $\Psi^\mu (du)_\mu$ is constant along the vector r^μ and so $\Psi^\mu (du)_\mu = \Psi^\mu (du)_\mu|_{r=r_h} = 0$. Thus, we find Ψ^μ is tangent to W_0 and all orbits of Ψ^μ lie on the surface W_0 .

In order to prove that the orbit of Ψ^μ lies on a S_r , we now only need to prove Ψ^μ is tangent to S_r , i.e., $\Psi^\mu (dr)_\mu = 0$. The method is similar. We first note

$$\Psi^\mu (dr)_\mu|_{S_{r_h}} = 0. \quad (\text{B5})$$

Then we consider the Lie derivative with respect to r^μ

$$\begin{aligned}
 \mathcal{L}_r[\Psi^\mu (dr)_\mu] &= \Psi^\mu (d\mathcal{L}_r r)_\mu - (dr)_\mu \mathcal{L}_\Psi r^\mu \\
 &= -(dr)_\mu \mathcal{L}_\Psi r^\mu \\
 &= -(dr)_\mu \psi g^{\mu\nu} \mathcal{L}_\Psi (du)_\nu \\
 &= -(dr)_\mu \psi g^{\mu\nu} (d\mathcal{L}_\Psi u)_\nu = 0. \quad (\text{B6})
 \end{aligned}$$

In the third line, it uses the facts $r^\mu = \psi g^{\mu\nu} (du)_\nu$ and $\mathcal{L}_\Psi(\psi g^{\mu\nu}) = 0$. In the last line, it uses the fact $\mathcal{L}_\Psi u = 0$ as Ψ^μ is tangent to the equal- u hypersurface. Equation (B6) shows that $\Psi^\mu (dr)_\mu$ is independent of coordinate r . Combining this result with Eq. (B5), we find $\Psi^\mu (dr)_\mu = 0$ on the whole W_0 . Thus, we prove that any orbit of Ψ^μ must lie on a surface S_r . Because of this reason, we can choose the orbit of Ψ^μ as one coordinate of S_r .

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