Cosmological perturbations in the interacting dark sector: Mapping fields and fluids

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There is no unique way to describe the dark energy–dark matter interaction, as we have little information about the nature and dynamics of the dark sector. Hence, in many of the phenomenological dark matter fluid interaction models in the literature, the interaction strength Q_ν in the dark sector is introduced by hand. Demanding that the interaction strength Q_ν in the dark sector must have a field theory description, we obtain a unique form of the interaction strength. We show the equivalence between the fields and fluids for the $f(R,\chi)$ model where f is an arbitrary, smooth function of R and a classical scalar field χ, which represents dark matter. Up to first order in perturbations, we show that the one-to-one mapping between the *classical* field theory description and the phenomenological fluid description of interacting dark energy and dark matter exists *only* for this unique form of interaction. We then classify the interacting dark energy models considered in the literature into two categories based on the field-theoretic description. We introduce a novel autonomous system and its stability analysis for the general interacting dark sector. We show that the dark-energy-dominated epoch occurs earlier than the noninteracting systems for a specific scalar field potential and a range of coupling strengths.

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I. INTRODUCTION

Dark matter dominates the mass of galaxies, and dark energy forms the majority of our Universe's energy density [1]. However, we have little information about the properties of these two components that dominate the energy content of the Universe today [2]. The only information we have about the two components is that (i) dark energy contributes a negative pressure to the energy budget, and (ii) dark matter has negligible (possibly zero) pressure [1]. The above properties are based on gravitational interactions. More importantly, we do not know how they interact with each other and baryons/photons.

In the early Universe, due to the tight coupling of baryons and photons, the baryons participate in the acoustic oscillations of the photons, and also cause Silk damping [3]. Near recombination, the baryons decouple from the photons, and photons propagate freely. Solar eclipse measurements rule out dark matter interactions with photons. Local gravity measurements rule out dark energy interactions with baryons [4]. However, the current observations cannot constrain (or rule out) the interaction strength between dark matter and dark energy. Interestingly, the dark matter–dark energy interaction provides a mechanism to alleviate the coincidence problem (see, for instance, Ref. [5]). Besides this, it was recently shown that the dark matter–dark energy interaction can reconcile the tensions in the Hubble constant H_0 [6].

Naturally, there has been a surge in constructing dark energy–dark matter models [7–32]. Phenomenologically, in all of these models the interaction is proposed between the fluid terms in the dark sector. More specifically, dark matter (DM) and dark energy (DE) do not individually satisfy the conservation equations; however, the combined sector satisfies the energy conservation equation [5], i.e.,

$$\nabla^\mu T^{(\mathrm{DE,DM})}_{\mu\nu} = Q^{(\mathrm{DE,DM})}_{\nu} \quad (1)$$

such that

$$Q^{(\mathrm{DE})}_{\nu} + Q^{(\mathrm{DM})}_{\nu} = 0, \quad (2)$$

where Q determines the interaction strength between dark matter and dark energy. Since the gravitational effects on dark matter and dark energy are opposite, even a small interaction can impact the cosmological evolution [5]. Since we have little information about the dark sector, in many of these models the interaction strength Q_ν in the dark sector is put in by hand.

However, it is unclear whether these broad classes of phenomenological models can be obtained from a field theory action. More specifically, can the above interaction strength Q_ν in the dark sector be derived systematically from a field theory action? Attempts have been made in the

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literature to obtain the interaction strength from the field-theoretic action [30]. The correspondence between the fluid description and field-theoretic description is established *only* for the background cosmology and not for the perturbations. The analysis of cosmological perturbations is essential to provide a complete understanding of these models and, more importantly, to determine if the perturbations are stable in the presence of the interaction Q_ν.

In this work, we show the equivalence up to first order in the perturbations of the $f(R,\chi)$ model, where f is an arbitrary, smooth function of R and the classical scalar field χ which represents dark matter. More specifically, under conformal transformations, we show that $f(R,\chi)$ is equivalent to a model with two coupled scalar fields. The coupling between the classical scalar fields, which gives rise to the dark energy–dark matter interaction (1), can be represented by the evolution equations of the dark energy (represented by a scalar field) and dark matter (represented by a fluid). We show that the interaction between the dark sectors can be rewritten in terms of the trace of the energy-momentum tensor of the dark matter fluid and a coupling function depending on the dark energy field. We then look at several interacting dark sector models proposed in the literature and identify those that are compatible with the field theory action proposed here.

We define a set of dimensionless variables and construct an autonomous system that completely describes the dark energy–dark matter interaction and background evolution. We analyze the fixed points of the system and show that the system has a stable attractor solution, corresponding to the late-time accelerated expansion of the Universe. To our knowledge, this is the first time such an approach has been used to study a general class of interacting dark sector models. We consider a specific dark energy–dark matter interaction model and study the background evolution. We show that for a range of (both positive and negative) coupling strengths, the dark-energy-dominated epoch occurs earlier with an interacting dark sector than in the noninteracting dark sector.

In this work we use the natural units where $c = 1$, $\kappa^2 = 8\pi G$, and the metric signature $(-,+,+,+)$. Greek letters denote the four-dimensional space-time coordinates and latin letters denote the three-dimensional spatial coordinates. Overbarred quantities [like $\bar{\rho}(t)$, $\bar{P}(t)$] are evaluated for the Friedmann-Robertson-Walker (FRW) background, and a dot represents a derivative with respect to cosmic time t. Unless otherwise specified, the subscript "$,\phi$" denotes a derivative with respect to ϕ, the subscript "$,\chi$" denotes a derivative with respect to χ, and the subscript m denotes dark matter.

II. DARK SECTOR INTERACTION FROM A FIELD THEORY ACTION

In the field theory description of the interacting dark energy–dark matter models, the coupling between the dark sector components is represented by a coupling term, which is an arbitrary function of the dark energy scalar field. It can be shown that modified gravity models such as $f(\tilde{R},\tilde{\chi})$ gravity can lead to such models [33]. Consider the following action in the Jordan frame:

$$S_J = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}f(\tilde{R},\tilde{\chi}) - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\chi}\tilde{\nabla}_\nu\tilde{\chi} - V(\tilde{\chi})\right], \quad (3)$$

where $f(\tilde{R},\tilde{\chi})$ is an arbitrary, smooth function of the Ricci scalar and scalar field $\tilde{\chi}$, and $V(\chi)$ is the self-interaction potential of the scalar field $\tilde{\chi}$. Under the conformal transformation

$$g_{\mu\nu} = \Omega^2\tilde{g}_{\mu\nu}, \quad \text{where } \Omega^2 = F(\tilde{R},\tilde{\chi}) \equiv \frac{\partial f(\tilde{R},\tilde{\chi})}{\partial\tilde{R}} \quad (4)$$

and a field redefinition, the action in the Einstein frame takes the following form:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - U(\phi) - \frac{1}{2}e^{2\alpha(\phi)}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - e^{4\alpha(\phi)}V(\chi)\right), \quad (5)$$

where

$$U = \frac{F\tilde{R} - f}{2\kappa^2F^2}.$$

This action has also been considered in the context of a multifield inflationary scenario (see, for instance, Ref. [34]). Recently, the same action was also considered in Ref. [30]. However, to our knowledge, we have not seen an explicit calculation that shows the derivation of the above action in the Einstein frame. Appendix A contains the details of the transformations in field space used to derive the above action.

From the above action (5), the field equations for χ and ϕ are, respectively,

$$-\nabla^\mu\nabla_\mu\chi - 2\alpha_{,\phi}(\phi)\nabla_\mu\phi\nabla^\mu\chi + e^{2\alpha(\phi)}V_{,\chi}(\chi) = 0, \quad (6)$$

$$-\nabla^\mu\nabla_\mu\phi + 4e^{4\alpha}\alpha_{,\phi}(\phi)V(\chi) + e^{2\alpha}\alpha_{,\phi}(\phi)\nabla^\mu\chi\nabla_\mu\chi + U_{,\phi}(\phi) = 0, \quad (7)$$

where the notations such as $V_{,\chi}$ and $U_{,\phi}$ denote $\partial V/\partial\chi$ and $\partial U/\partial\phi$. The variation of the action (5) with respect to the metric $g_{\mu\nu}$ gives the Einstein equation

$$G_{\mu\nu} = \kappa^2T_{\mu\nu}, \quad (8)$$

where the stress-tensor is given by

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\phi\nabla_\sigma\phi - g_{\mu\nu}U(\phi) + e^{2\alpha}\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}e^{2\alpha}g_{\mu\nu}\nabla^\sigma\chi\nabla_\sigma\chi - e^{4\alpha}g_{\mu\nu}V(\chi). \tag{9}$$

In the field-theoretic description, the two field equations (6)–(7) and the Einstein equation (8) completely describe the system.

Since dark matter and dark energy constitute up to 95% of the energy content of the Universe today, it is a good approximation to assume that the total energy-momentum tensor of the Universe is given by Eq. (9). Demanding the local conservation of the energy-momentum tensor leads to

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu T^{(\phi)}_{\mu\nu} + \nabla^\mu T^{(\chi)}_{\mu\nu} = 0, \tag{10}$$

where $T^{(\phi)}_{\mu\nu}$ and $T^{(\chi)}_{\mu\nu}$ refer to the stress tensors corresponding to the scalar fields ϕ and χ, respectively. Due to the interaction between the two fields ϕ and χ, there is no unique way to write the stress tensor corresponding to the scalar fields, and the conservation of the energy-momentum tensor of the individual components is violated. Following Eqs. (1), (6), and (7), the interaction between the two scalar fields can be described as

$$-\nabla^\mu T^{(\phi)}_{\mu\nu} = Q^{(\mathrm{F})}_\nu = \nabla^\mu T^{(\chi)}_{\mu\nu}, \tag{11}$$

where

$$T^{(\chi)}_{\mu\nu} = e^{2\alpha(\phi)}\left(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\chi\nabla_\sigma\chi - e^{2\alpha(\phi)}g_{\mu\nu}V(\chi)\right), \tag{12}$$

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\phi\nabla_\sigma\phi - g_{\mu\nu}U(\phi), \tag{13}$$

$$Q^{(\mathrm{F})}_\nu = \nabla^\mu T^{(\chi)}_{\mu\nu} = -e^{2\alpha(\phi)}\alpha_{,\phi}(\phi)\nabla_\nu\phi[\nabla^\sigma\chi\nabla_\sigma\chi + 4e^{2\alpha(\phi)}V(\chi)]. \tag{14}$$

It is important to note that, starting from Eq. (3), we can obtain interaction strength $Q^{(\mathrm{F})}$ in terms of ϕ and $V(\chi)$. We can equally rewrite $Q^{(\mathrm{F})}$ in terms of $U(\phi)$. While this field theory description may be considered a fundamental description of the system, the fluid description turns out to be more useful to analyze the cosmological observations. In that regard, the most common description of the interacting dark sector is in terms of dark matter fluid.

A. Fluid description of the interacting dark sector

In the fluid description, it is often convenient to consider the dark matter to be a fluid. For this purpose, we replace the dark matter scalar field and related quantities by the corresponding energy density ρ_m and pressure p_m of the dark matter fluid [30]:

$$p_m = -\frac{1}{2}e^{2\alpha}[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi + e^{2\alpha}V(\chi)],$$
$$\rho_m = -\frac{1}{2}e^{2\alpha}[g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - e^{2\alpha}V(\chi)]. \tag{15}$$

The four-velocity u_μ of the dark matter fluid is given by

$$u_\mu = -[-g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi]^{-\frac{1}{2}}\nabla_\mu\chi. \tag{16}$$

In this description, the Einstein equation can be rewritten in terms of the dark energy scalar field and dark matter fluid:

$$G_{\mu\nu} = \kappa^2\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\phi\nabla_\sigma\phi - g_{\mu\nu}V(\phi) + p_m g_{\mu\nu} + (\rho_m + p_m)u_\mu u_\nu\right], \tag{17}$$

where the energy-momentum tensor for the dark matter fluid is given by

$$T^{(m)\mu}_\nu = p_m g_{\mu\nu} + (\rho_m + p_m)u_\mu u_\nu, \tag{18}$$

and the interaction term can be rewritten as

$$\begin{aligned} Q^{(\mathrm{F})}_\nu &= \nabla_\mu T^{(m)\mu}_\nu \\ &= -e^{2\alpha(\phi)}\alpha_{,\phi}(\phi)\nabla_\nu\phi[\nabla^\sigma\chi\nabla_\sigma\chi + 4e^{2\alpha(\phi)}V(\chi)] \\ &= -\alpha_{,\phi}(\phi)\nabla_\nu\phi(\rho_m - 3p_m). \end{aligned} \tag{19}$$

Identifying $T^{(m)} = T^{(m)\mu}_\mu = -(\rho_m - 3p_m)$, we get

$$Q^{(\mathrm{F})}_\nu = T^{(m)}\nabla_\nu\alpha(\phi). \tag{20}$$

Thus, we see that in the fluid description of interacting dark matter the interaction term is proportional to the trace of the energy-momentum tensor of the dark matter and the coupling α. It is important to note that, starting from the Jordan frame action (3), the form of the interaction term $Q^{(\mathrm{F})}_\nu$ is *uniquely* written in terms of the dark energy scalar field and dark matter fluid.

This has to be contrasted with the dark matter interaction fluid models in the literature [7–23,25–32], where Q_ν can take any form. In the next section we show that a one-to-one correspondence between the fields and the fluids is only true if the interaction term is given by $Q^{(\mathrm{F})}_\nu$ in Eq. (20).

III. COSMOLOGICAL EVOLUTION WITH DARK ENERGY–DARK MATTER INTERACTION

To study the cosmological evolution with an interacting dark sector, we consider the spatially flat FRW metric with first-order scalar perturbations in synchronous gauge[1] [3]:

$$g_{00} = -1, \qquad g_{0i} = 0,$$
$$g_{ij} = a^2\left[(1+A)\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j}\right], \tag{21}$$

where $a \equiv a(t)$ is the scale factor with the Hubble parameter given by $H = \dot{a}/a$, and $A \equiv A(t,x,y,z)$ and $B \equiv B(t,x,y,z)$ are scalar perturbations. At the linear order, the scalar, vector, and tensor perturbations decouple and can be treated separately. Since the scalar perturbations couple to the energy density $(\delta\rho)$ and pressure (δP) leading to the growing inhomogeneities, we only consider scalar perturbations.

The scalar fields ϕ and χ, dark matter fluid energy density (ρ_m), dark matter fluid pressure (p_m), and interaction strength (Q_ν) can be split into background and perturbed parts as

$$\phi = \bar{\phi} + \delta\phi, \qquad \chi = \bar{\chi} + \delta\chi, \qquad \rho_m = \bar{\rho}_m + \delta\rho_m,$$
$$p_m = \bar{p}_m + \delta p_m, \qquad Q_\nu = \bar{Q}_\nu + \delta Q_\nu. \tag{22}$$

In the literature, in the fluid description the dark matter is usually assumed to be pressureless dust, i.e., $\bar{p}_m = \delta p_m = 0$. In this work, we *do not* make this assumption for the dark matter fluid, i.e., $\bar{p}_m \neq 0$ and $\delta p_m \neq 0$. However, all of our calculations are valid in the special case of pressureless dust.

The components of the dark matter fluid four-velocity can be written as

$$u_\mu = \bar{u}_\mu + \delta u_\mu, \qquad \bar{u}_0 = -1, \qquad \delta u_0 = 0,$$
$$\bar{u}_i = 0, \qquad \delta u_i = \frac{\partial \delta u^s}{\partial x^i}, \qquad \delta u^s = -\frac{\delta\chi}{\dot{\bar{\chi}}}. \tag{23}$$

In the following subsections, we present the evolution equations for the background and the first-order perturbations.

A. Correspondence between fields and fluids in the FRW background

In the fluid description, the Friedmann equations for the interacting dark sector are given by [5]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\left(\bar{\rho}_m + \frac{\dot{\bar{\phi}}^2}{2} + U(\bar{\phi})\right),$$
$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{\kappa^2}{3}\left(\bar{p}_m + \frac{\dot{\bar{\phi}}^2}{2} - U(\bar{\phi})\right). \tag{24}$$

[1]For the evolution equations in Newtonian gauge, see Appendix B.

From Eq. (1), the conservation equations for the dark energy field and dark matter fluid in the FRW background are given by

$$\ddot{\bar{\phi}}\,\dot{\bar{\phi}} + 3H\dot{\bar{\phi}}^2 + U_{,\phi}(\bar{\phi})\dot{\bar{\phi}} = \bar{Q},$$
$$\dot{\bar{\rho}}_m + 3H(\bar{\rho}_m + \bar{p}_m) = -\bar{Q}. \tag{25}$$

In the phenomenological description of the dark matter fluid interaction, there is no unique form of $\bar{Q}$. Several authors have considered many different forms of $\bar{Q}$ in the literature (see, for instance, Refs. [7–23,25–32]). However, as discussed in Sec. II A, starting from the Jordan frame action (3) the interaction term $Q_\nu^{(\mathrm{F})}$ in Eq. (20) is written uniquely in terms of the dark energy scalar field and dark matter fluid. In this case, the background interaction term is given by

$$\bar{Q}^{(\mathrm{F})} = -\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}(\bar{\rho}_m - 3\bar{p}_m). \tag{26}$$

We now show that the above equations are consistent with the field theory description *only* for this form of the interaction term $\bar{Q}^{(\mathrm{F})}$. Using the definitions of p_m and ρ_m in Eq. (15), the evolution equations for the scalar fields ϕ and χ are given by

$$\ddot{\bar{\chi}} + 3H\dot{\bar{\chi}} + e^{2\alpha}V_{,\chi}(\bar{\chi}) + 2\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}\dot{\bar{\chi}} = 0,$$
$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + U_{,\phi}(\bar{\phi}) + 4e^{4\alpha}\alpha_{,\phi}(\bar{\phi})V(\bar{\chi}) - e^{2\alpha}\alpha_{,\phi}(\bar{\phi})\dot{\bar{\chi}}^2 = 0. \tag{27}$$

The background interaction term in the field theory picture can also be obtained by a direct substitution of the variables:

$$\bar{Q}^{(\mathrm{F})} = \alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}e^{2\alpha(\bar{\phi})}[\dot{\bar{\chi}}^2 - 4e^{2\alpha}V(\bar{\chi})]. \tag{28}$$

Similarly, the Friedmann equations in the field theory description can be obtained by substituting $\bar{\rho}_m$ and $\bar{p}_m$ with the corresponding field theory variables. From the above analysis, it is clear that there is a one-to-one correspondence between the fluids and fields *only* for the interaction term $\bar{Q}^{(\mathrm{F})}$. For any other form of the interaction term, the correspondence may not exist. In Sec. IV, we classify various models used in the literature based on this correspondence.

B. Correspondence between fields and fluids in first-order perturbations

In the fluid description, the first-order scalar perturbations in synchronous gauge satisfy the following equations [3]:

$$\dot{A} = \kappa^2[(\bar{p}_m + \bar{\rho}_m)\delta u^s - \dot{\bar{\phi}}\delta\phi], \tag{29}$$

$$\ddot{B} + 3H\dot{B} - \frac{A}{a^2} = 0, \tag{30}$$

$$\frac{3}{2}\ddot{A} + \nabla^2\left[\frac{1}{2}\ddot{B} + H\dot{B}\right] + 3H\dot{A} = \frac{\kappa^2}{2}[-\delta\rho_m - 3\delta p_m - 4\dot{\bar{\phi}}\,\dot{\delta\phi} + 2U_{,\phi}(\bar{\phi})\delta\phi], \tag{31}$$

$$-\frac{1}{2}\ddot{A} + \frac{1}{2a^2}\nabla^2 A - 3H\dot{A} - \frac{1}{2}H\nabla^2\dot{B} = \frac{\kappa^2}{2}[-\delta\rho_m + \delta p_m - 2U_{,\phi}(\bar{\phi})\delta\phi]. \tag{32}$$

From Eq. (1), the conservation equations for the dark energy field and dark matter fluid in the first-order perturbations are given by

$$\dot{\delta\rho_m} + 3H(\delta p_m + \delta\rho_m) + (\bar{p}_m + \bar{\rho}_m)\left[\frac{\nabla^2\delta u^s}{a^2} + \frac{3}{2}\dot{A} + \frac{\nabla^2\dot{B}}{2}\right] = -\delta Q, \tag{33}$$

$$\dot{\bar{\phi}}\left(\ddot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} + U_{,\phi\phi}(\bar{\phi})\delta\phi\right) + \dot{\delta\phi}(\ddot{\bar{\phi}} + 6H\dot{\bar{\phi}} + U_{,\phi}(\bar{\phi})) + \frac{\dot{\bar{\phi}}^2}{2}(\nabla^2\dot{B} + 3\dot{A}) = \delta Q. \tag{34}$$

The above equations are generic equations for the coupled dark matter fluid and dark energy field with an arbitrary interaction term δQ. As mentioned earlier, there is no unique form of δQ in the phenomenological description of the dark matter fluid interaction. Several authors have considered many different forms of δQ in the literature (see, for instance, Refs. [7–32]). However, as discussed in Sec. II A, starting from the Jordan frame action (3) the interaction term $Q^{(\mathrm{F})}_\nu$ in Eq. (20) is uniquely written in terms of the dark energy scalar field and dark matter fluid. In this case, the perturbed interaction term is given by

$$\delta Q^{(\mathrm{F})} = -(\delta\rho_m - 3\delta p_m)\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}} - (\bar{\rho}_m - 3\bar{p}_m)[\alpha_{,\phi\phi}(\bar{\phi})\dot{\bar{\phi}}\delta\phi + \alpha_{,\phi}(\bar{\phi})\dot{\delta\phi}]. \tag{35}$$

Like in the previous subsection, we now show that the above equations are consistent with the field theory description *only* for this form of interaction $Q^{(\mathrm{F})}$. Substituting ρ_m, p_m, $\delta\rho_m$, and δp_m from Eq. (15), the perturbed equations of motion for ϕ and χ are, respectively,

$$\ddot{\delta\chi} - \frac{\nabla^2\delta\chi}{a^2} + e^{2\alpha}V_{,\chi\chi}(\bar{\chi})\delta\chi + \frac{\dot{\bar{\chi}}}{2}(\nabla^2\dot{B} + 3\dot{A}) + 3H\dot{\delta\chi} + 2\alpha_{,\phi}(\bar{\phi})(\dot{\bar{\phi}}\,\dot{\delta\chi} + \dot{\bar{\chi}}\,\dot{\delta\phi}) + 2\delta\phi[\dot{\bar{\phi}}\dot{\bar{\chi}}\,\alpha_{,\phi\phi}(\bar{\phi}) + e^{2\alpha}\alpha_{,\phi}(\bar{\phi})V_{,\chi}(\bar{\chi})] = 0, \tag{36}$$

$$\ddot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} + U_{,\phi\phi}(\bar{\phi})\delta\phi + \frac{\dot{\bar{\phi}}}{2}(\nabla^2\dot{B} + 3\dot{A}) + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})[2e^{2\alpha}V_{,\chi}(\bar{\chi})\delta\chi - \dot{\bar{\chi}}\,\dot{\delta\chi}] + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})^2\delta\phi[8e^{2\alpha}V(\bar{\chi}) - \dot{\bar{\chi}}^2] + e^{2\alpha}\alpha_{,\phi\phi}(\bar{\phi})\delta\phi[4e^{2\alpha}V(\bar{\chi}) - \dot{\bar{\chi}}^2] = 0. \tag{37}$$

The above perturbed field equations are identical to the equations obtained from Eqs. (6) and (7), respectively. The perturbed interaction term in the field theory picture can also be obtained by a direct substitution of the variables:

$$\delta Q^{(\mathrm{F})} = 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}[\dot{\bar{\chi}}\,\dot{\delta\chi} - 2e^{2\alpha}V_{,\chi}(\bar{\chi})\delta\chi] + e^{2\alpha}\alpha_{,\phi\phi}(\bar{\phi})\dot{\bar{\phi}}\delta\phi[\dot{\bar{\chi}}^2 - 4V(\bar{\chi})] + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})^2\dot{\bar{\phi}}\delta\phi[\dot{\bar{\chi}}^2 - 8e^{2\alpha}V(\bar{\chi})] + e^{2\alpha}\alpha_{,\phi}(\bar{\phi})\dot{\delta\phi}[\dot{\bar{\chi}}^2 - 4e^{2\alpha}V(\bar{\chi})]. \tag{38}$$

We would like to stress the following points regarding the above results. First, there is no unique form of δQ in the phenomenological description of the dark matter fluid interaction. However, demanding a one-to-one correspondence between the field and fluid pictures leads to a unique interaction term $Q^{(\mathrm{F})}_\nu$. Second, we see that apart from the convenience of relating the variable to cosmological observables, the evolution equations in the fluid description are simpler than those in the fluid theory description, which simplifies the numerical analysis of the model. Third, while the form of the interaction term is unique, it still contains unknown functions like $\alpha(\phi)$, χ, and $V(\chi)$. In the next section, we will use this correspondence to clarify the phenomenological dark matter fluid interaction models in the literature [7–31].

IV. INTERACTING DARK ENERGY MODELS IN THE LITERATURE

Since we have little information about the nature and dynamics of the dark sector, there is no unique way of describing the interaction between dark energy and dark matter. So far, the interaction strength Q_ν has been described by phenomenological models, with model parameters constrained by cosmological observations [5]. In many of the models, the interaction strength Q_ν in the dark sector is constructed using the energy densities of dark energy and dark matter and other dynamic quantities appearing in the model. However, it is not clear whether the models can be written from a field-theoretic action.

In this work, starting from the Jordan frame action (3), we showed that the interaction term $Q_\nu^{(F)}$ is unique. We showed that this interaction provides a one-to-one mapping between the field and fluid descriptions of the dark matter sector. Armed with this, in this section we classify the interacting dark energy models considered in the literature into two categories based on the field-theoretic description. The table below identifies the models that can (or cannot) be described by the field theory approach considered in this work. The list is not exhaustive but gives a good representation of the various models discussed in the literature.

Interacting DE-DM model	DE-DM Interaction $\nabla^\mu T_{\mu\nu}^{(\mathrm{DE,DM})} = Q_\nu^{(\mathrm{DE,DM})}$	Is $Q_\nu \propto Q_\nu^{(\mathrm{F})}$?
Amendola—1999 [7]	$\dot\rho_m + 3H\rho_m = -C\rho_m\dot\phi$	Yes
Amendola—1999 [8]	$\dot\rho_m + 3H\rho_m = -C\rho_m\dot\phi$	Yes
Billyard & Coley—1999 [9]	$\dot\phi(\ddot\phi + 3H\dot\phi + kV) = \frac{(4-3\gamma)}{2\sqrt{\omega+\frac{3}{2}}}\dot\phi\mu$	Yes
Olivares *et al.*—2005 [10]	$\frac{d\rho_c}{dt} + 3H\rho_c = 3Hc^2(\rho_c + \rho_x)$	No
Amendola *et al.*—2006 [11]	$\dot\rho_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} - \delta(a)H\rho_{\mathrm{DM}} = 0$	No
Olivares *et al.*—2007 [12]	$\dot\rho_c + 3H\rho_c = 3Hc^2(\rho_x + \rho_c)$	No
Boehmer *et al.*—2008 [13]	$\dot\rho_c + 3H\rho_c = -\sqrt{2/3}\kappa\beta\rho_c\dot\varphi$	Yes
	$\dot\rho_c + 3H\rho_c = -\alpha H\rho_c$	No
Caldera-Cabral *et al.*—2008 [14]	$\dot\rho_c = -3H\rho_c + 3H(\alpha_x\rho_x + \alpha_c\rho_c)$	No
	$\dot\rho_c = -3H\rho_c + 3(\Gamma_x\rho_x + \Gamma_c\rho_c)$	No
He & Wang—2008 [15]	$\dot\rho_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} - \delta H\rho_{\mathrm{DM}} = 0$	No
	$\dot\rho_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} - \delta H(\rho_{\mathrm{DM}} + \rho_{\mathrm{DE}}) = 0$	No
Pettorino & Baccigalupi—2008 [16]	$\phi'' + 2\mathcal{H}\phi' + a^2U_{,\phi} = a^2C_c\rho_c$	Yes
Quartin *et al.*—2008 [17]	$\frac{\mathrm{d}\rho_c}{\mathrm{d}N} + 3\rho_c = 3\lambda_x\rho_x + \lambda_c\rho_c$	No
Boehmer *et al.*—2009 [18]	$\dot\rho_c = -3H\rho_c - \frac{\alpha}{M_0}\rho_\varphi^2$	No
	$\dot\rho_c = -3H\rho_c - \frac{\beta}{M_0}\rho_c^2$	No
	$\dot\rho_c = -3H\rho_c - \frac{\gamma}{M_0}\rho_\varphi\rho_c$	No
Beyer *et al.*—2010 [19]	$\ddot\varphi + 3H\dot\varphi - \alpha M^3 e^{-\alpha\varphi/M} = \frac{\beta}{M}\rho_\chi$	Yes
Lopez Honorez *et al.*—2010 [20]	$\dot\rho_{dm} + 3H\rho_{dm} = \beta(\phi)\rho_{dm}\dot\phi$	Yes
Avelino & Silva—2012 [21]	$\dot\rho_m + 3H\rho_m = \alpha H a^\beta\rho_w$	No
Pan *et al.*—2012 [22]	$\dot\rho_m + 3H\rho_m = 3\lambda_m H\rho_m + 3\lambda_d H\rho_d$	No
Salvatelli *et al.*—2013 [23]	$\dot\rho_{dm} + 3\mathcal{H}\rho_{dm} = \xi\mathcal{H}\rho_{de}$	No
Chimento *et al.*—2013 [24]	$\rho'_{\mathrm{m}} + \gamma_{\mathrm{m}}\rho_{\mathrm{m}} = -\alpha\rho'\rho$	No
Amendola *et al.*—2014 [25]	$\dot\rho_\alpha + 3H\rho_\alpha = -\kappa\sum_i C_{i\alpha}\dot\phi_i\rho_\alpha$	Yes
Marra—2015 [26]	$\dot\rho_m + 3H\rho_m = \nu\delta_m^n\rho_m\dot\phi/M_{\mathrm{Pl}}$	No
Bernardi & Landim—2016 [27]	$\dot\rho_m + 3H\rho_m = Q(\rho_\phi + \rho_m)\dot\phi$	No
	$\dot\rho_m + 3H\rho_m = Q\rho_\phi\dot\phi$	No
Pan & Sharov—2016 [28]	$\dot\rho_{dm} + 3\mathcal{H}\rho_{dm} = 3\lambda_m H\rho_{dm} + 3\lambda_d H\rho_d$	No
Bruck & Mifsud—2017 [29] [a]	$\nabla^\mu T_{\mu\nu}^{\mathrm{DM}} = Q\nabla_\nu\phi$	Yes
	$Q = \frac{C_{,\phi}}{2C}T_{\mathrm{DM}} + \frac{D_{,\phi}}{2C}T_{\mathrm{DM}}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu[\frac{D}{C}T_{\mathrm{DM}}^{\mu\nu}\nabla_\nu\phi]$	if $D = 0$
Gonzalez & Trodden—2018 [30]	$\dot\rho_\chi + 3H\rho_\chi = \alpha'\dot\phi\rho_\chi$	Yes
Barros *et al.*—2018 [31]	$\dot\rho_c + 3H\rho_c = -\kappa\beta\dot\phi\rho.$	Yes
Landim—2019 [32]	$\ddot\phi + 3H\dot\phi + V'(\phi) = -Q\rho_m$	Yes

[a]Violates the causality condition $[D(\phi) > 0]$ for the disformal transformations [35].

V. DETAILED ANALYSIS OF BACKGROUND EVOLUTION

This section considers the class of interacting models with a one-to-one mapping between the fields and fluids and shows that one can exactly solve the evolution equation of the dark matter fluid. We perform a detailed analysis of the background evolution of the dark sector in the fluid picture.

The background interaction term (20) in the fluid picture can be rewritten as

$$\bar{Q}^{(\mathrm{F})} = -\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}(\bar{\rho}_m - 3\bar{p}_m) = -\dot{\alpha}(\bar{\phi})(\bar{\rho}_m - 3\bar{p}_m). \quad (39)$$

Assuming a matter fluid with a constant equation of state $\omega_m = \bar{p}_m/\bar{\rho}_m$, we can rewrite the continuity equation for matter fluid as

$$\dot{\bar{\rho}}_m + 3H\bar{\rho}_m(1+\omega_m) = \dot{\alpha}(\bar{\phi})\bar{\rho}_m(1-3\omega_m). \quad (40)$$

Solving this, we get

$$\bar{\rho}_m = \bar{\rho}_{m_0} a^{-3(1+\omega_m)} e^{[\alpha(\bar{\phi})-\alpha_0](1-3\omega_m)}, \quad (41)$$

where $\bar{\rho}_{m_0}$ and $\bar{\phi}_0$ are the current values of $\bar{\rho}_m$ and $\bar{\phi}$, and $\alpha_0 = \alpha(\bar{\phi}_0)$. The evolution of ϕ is determined by the equation of motion of ϕ. As expected, setting $\alpha(\phi) = 0$ results in the evolution of ρ_m in the ΛCDM model.

A. Autonomous system of interacting dark energy–dark matter model

To study the cosmological evolution in the interacting dark sector, we write the equations in dimensionless variables and describe them as an autonomous system of equations [2,36]. To our knowledge, this is the first time such an approach has been used to study a general class of interacting dark sector models.

To study and analyze a general class of dark sector interaction models, we define the following dimensionless variables:

$$x = \sqrt{\frac{C_1}{6}}\frac{\dot{\phi}}{HM_{\mathrm{Pl}}}, \qquad y = \sqrt{\frac{C_1}{3}}\frac{\sqrt{U}}{HM_{\mathrm{Pl}}}, \quad (42)$$

$$\lambda = -\frac{M_{\mathrm{Pl}}}{\sqrt{C_1}}\frac{U_{,\phi}}{U}, \qquad \Gamma = \frac{UU_{,\phi\phi}}{U_{,\phi}^2}, \quad (43)$$

$$\alpha = \alpha(\phi), \qquad \beta = -\frac{M_{\mathrm{Pl}}}{\sqrt{C_1}}\frac{\alpha_{,\phi}}{\alpha}, \qquad \gamma = \frac{\alpha\alpha_{,\phi\phi}}{\alpha_{,\phi}^2}, \quad (44)$$

where a dot represents a derivative with respect to time, and C_1 is a constant. (C_1 is defined in Appendix C.) Here α, β, and γ describe a general interaction function and its properties. These variables can be used to study a large class of interacting dark energy–dark matter models. The following equations give the autonomous system of the interacting dark-sector model:

$$x' + \frac{3}{2}x\left(1 - x^2 + y^2 - \frac{\Omega_r}{3}\right) - \sqrt{\frac{3}{2}}\left(\lambda y^2 + \frac{q}{x}\right) = 0, \quad (45)$$

$$y' + \frac{3}{2}y\left(\sqrt{\frac{2}{3}}\lambda x - x^2 + y^2 - \frac{\Omega_r}{3} - 1\right) = 0, \quad (46)$$

$$\Omega_m' + \Omega_m(3y^2 - 3x^2 - \Omega_r) + \sqrt{6}q = 0, \quad (47)$$

$$\Omega_r' + \Omega_r(1 - 3x^2 + 3y^2 - \Omega_r) = 0, \quad (48)$$

$$\lambda' + \sqrt{6}\lambda^2 x(\Gamma - 1) = 0, \quad (49)$$

$$\beta' + \sqrt{6}\beta^2 x(\gamma - 1) = 0, \quad (50)$$

$$\alpha' + \sqrt{6}\alpha\beta x = 0, \quad (51)$$

and the energy constraint is given by

$$x^2 + y^2 + \Omega_m + \Omega_r - 1 = 0. \quad (52)$$

Here a prime denotes a derivative with respect to the number of e-foldings $N \equiv \ln(a)$. For the pressureless matter fluid, the scaled interaction term (q) is defined as

$$q \equiv \alpha\beta x\Omega_m = -\frac{\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}\bar{\rho}_m}{3\sqrt{6}H^3M_{\mathrm{Pl}}^2} = \frac{\bar{Q}}{3\sqrt{6}H^3M_{\mathrm{Pl}}^2}. \quad (53)$$

Note that various cosmological parameters can be expressed in terms of these variable as

$$\Omega_\phi = x^2 + y^2, \qquad \omega_\phi = \frac{x^2 - y^2}{x^2 + y^2}, \quad (54)$$

$$\rho_i = 3H^2M_{\mathrm{Pl}}^2\Omega_i, \quad \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(x^2 - y^2 + \frac{\Omega_r}{3} + 1\right). \quad (55)$$

B. Stability analysis of the autonomous system

To get further insight into the background evolution of the Universe with the interacting dark sector, we look at the fixed points of the autonomous system introduced in Sec. V A. We consider two cases.

1. **Case (i)**: Models with constant λ (exponential scalar field potential) and a linear interaction function (constant $\alpha\beta$).
2. **Case (ii)**: Models with general scalar field potential ($\lambda \neq \mathrm{const}$) and a general coupling function ($\alpha\beta \neq \mathrm{const}$). To our knowledge, a general stability analysis for this case has not been done before.

TABLE I. Fixed points of the autonomous system with a given λ and linear coupling function.

Fixed point	x^*	y^*	Ω_r^*	Ω_ϕ^*	ϵ^*
1a	-1	0	0	1	3
1b	0	0	1	0	2
1c	1	0	0	1	3
1d	$\frac{1}{\sqrt{6}\alpha\beta}$	0	$1-\frac{1}{2\alpha^2\beta^2}$	$\frac{1}{6\alpha^2\beta^2}$	2
1e	$\sqrt{\frac{2}{3}}\alpha\beta$	0	0	$\frac{2}{3}\alpha^2\beta^2$	$\frac{3}{2}+\alpha^2\beta^2$
1f	$-\sqrt{\frac{3}{2}}\frac{1}{\alpha\beta-\lambda}$	$\sqrt{\frac{3}{2}+\alpha^2\beta^2-\alpha\beta\lambda}\frac{1}{\alpha\beta-\lambda}$	0	$\frac{\alpha^2\beta^2-\alpha\beta\lambda+3}{(\lambda-\alpha\beta)^2}$	$-\frac{3\lambda}{2(\alpha\beta-\lambda)}$
1g	$\sqrt{\frac{2}{3}}\frac{2}{\lambda}$	$\frac{2}{\sqrt{3}\lambda}$	$1-\frac{4}{\lambda^2}$	$\frac{4}{\lambda^2}$	2
1h	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	1	$\frac{\lambda^2}{2}$

Table I contains the list of fixed points of the autonomous system for case (i).

As mentioned above, the fixed points are for the case of an exponential scalar field potential and a linear interaction function. The fixed points can be considered as instantaneous fixed points for other potentials and interaction functions. For constant λ and $\alpha\beta$, it has been shown that a sequence of radiation-dominated era, matter-dominated era, and an accelerated attractor (like, 1b → 1e → 1h) is cosmologically viable for a range of parameter values [2,37].

Table II contains the list of fixed points of the autonomous system for case (ii), which is for a system with an evolving λ and a general coupling function $\alpha(\phi)$ (for which λ and $\alpha\beta$ are not necessarily constants).

In some cases, the fixed-point conditions are satisfied for any physically realizable values of the parameters. These are represented as empty cells in the table. We can classify the fixed points in Table II into radiation-dominated, matter-dominated, and late-time accelerated phase fixed points.

1. Radiation-dominated phase: We have two radiation-dominated fixed points: “2a” and “2g.” Looking at the eigenvalues of the Jacobian matrix of the system, we see that both of them are saddle points.
2. Matter-dominated phase: The matter-dominated era can be realized by “2b,” “2c,” and “2f,” and all of them are saddle points.

TABLE II. Fixed points of the autonomous system with varying λ and a general coupling function α.

Fixed point	x^*	y^*	Ω_r^*	Ω_ϕ^*	Ω_m^*	λ^*	α^*	β^*	ϵ^*
2a	0	0	1	0	0	…	…	…	2
2b	0	0	0	0	1	…	0	…	$\frac{3}{2}$
2c	0	0	0	0	1	…	…	0	$\frac{3}{2}$
2d	0	1	0	1	0	0	…	…	0
2e	-1	0	0	1	0	0	…	0	3
2f	0	0	0	0	1	0	…	0	$\frac{3}{2}$
2g	0	0	1	0	0	0	…	0	2
2h	0	1	0	1	0	0	…	0	0
2i	1	0	0	1	0	0	…	0	3

3. Accelerated phase: Looking at the values of ϵ^*, we see that dark-energy-dominated accelerated expansion can be realized by “2d” and “2h.” Both of these fixed points are attractors.

From the above analysis, it is clear that the interacting dark sector model can lead to a radiation-dominated era followed by a matter-dominated era, followed by an accelerated phase (e.g., 2a → 2c → 2d). The attractor behavior of the accelerated fixed point ensures that the late Universe stays in the accelerated phase, leading to the de Sitter Universe, which is indicated by $\epsilon^* = 0 \Rightarrow H^* = \text{const}$.

C. Dark energy–dark matter interaction: A specific example

In the previous subsections we have shown that the interacting dark-sector model can be expressed as an autonomous system. However, the analysis of an arbitrary model is not possible. Here, we consider a quintessence dark energy model [38] with a linear interaction function:

$$C_1 = \frac{M_{\rm Pl}^2}{2}, \qquad U(\phi) = \frac{8\pi M_{\rm Pl}^2 \kappa}{2}\frac{1}{\phi}, \qquad \alpha(\phi) = \frac{C}{\sqrt{2}}\phi, \tag{56}$$

where κ and the coupling strength C are constants. In the rest of this section, we consider the background evolution for two different scenarios: $C \geq 0$ and $C \leq 0$. For both scenarios, we solve the above set of equations numerically in the redshift range $1500 < z < 0$, and the evolution of various cosmological parameters are plotted with respect to the number of e-foldings (N).

For the background evolution, we choose the following initial conditions and parameter values:

$$x_i = 1.5\times10^{-5}, \qquad y_i = 2.5\times10^{-5},$$
$$\Omega_{r_i} = 0.4, \qquad \Omega_{m_i} = 1 - x_i^2 - y_i^2 - \Omega_{r_i},$$
$$\lambda_i = 0.6, \qquad \alpha_i = C/\lambda_i, \qquad \beta_i = -\lambda_i,$$
$$\Gamma = 2, \qquad \gamma = 0.$$

The initial values are chosen so that the evolution is consistent with the observed values of the cosmological parameters. It is important to note that a range of initial conditions will lead to the accelerated expansion of the Universe with a dark-energy-dominated phase. These specific initial conditions are chosen as representative values for the background evolution. The values of Γ and γ are fixed by the choice of the dark energy scalar field potential U and the coupling function α, respectively.

1. Scenario I: $C \geq 0$

Figure 1 contains the plots of the evolution of the scaled interaction term q [defined in Eq. (53)] and slow-roll parameter ϵ [defined in Eq. (55)]. Here we see that the interaction term takes both positive and negative values during the evolution, and the strength of the interaction decreases in the late Universe. All of the cases result in the late-time accelerated expansion ($\epsilon < 1$). The interacting dark-sector model leads to an early dark-energy-dominated phase compared to the noninteracting dark sector.

In the previous subsection, we showed that the interacting dark-sector model has a stable attractor solution, which corresponds to the late-time accelerated Universe. To demonstrate this point, we fix the variables other than x and y to be constants with the values of corresponding functions at $N = 7$. As we see in the left panel of Fig. 2, this is a reasonable assumption since the relevant variables are nearly constant for $N > 3$. From the right panel of Fig. 2, we see that a large range of the x and y parameters lead to a dark-energy-dominated attractor. It has to be noted that this is a rough representation of the phase-space evolution since the other parameters in the system are slowly varying. For simplicity, we have kept them constant while plotting the phase- space diagram. Hence, the attractor in the phase-space diagram is an instantaneous attractor. It is important to note that various potentials, including the one we have considered here, have been shown to have a dark-energy-dominated attractor in the noninteracting scenario [37,39]. Figure 3 contains the plots of the evolution of the energy density parameters for dark matter and dark energy. The figure shows that different coupling strengths lead to a dark-energy-dominated Universe, and the dark-energy-dominated phase starts earlier as compared to the noninteracting scenario. To investigate further, in Fig. 4 we plot

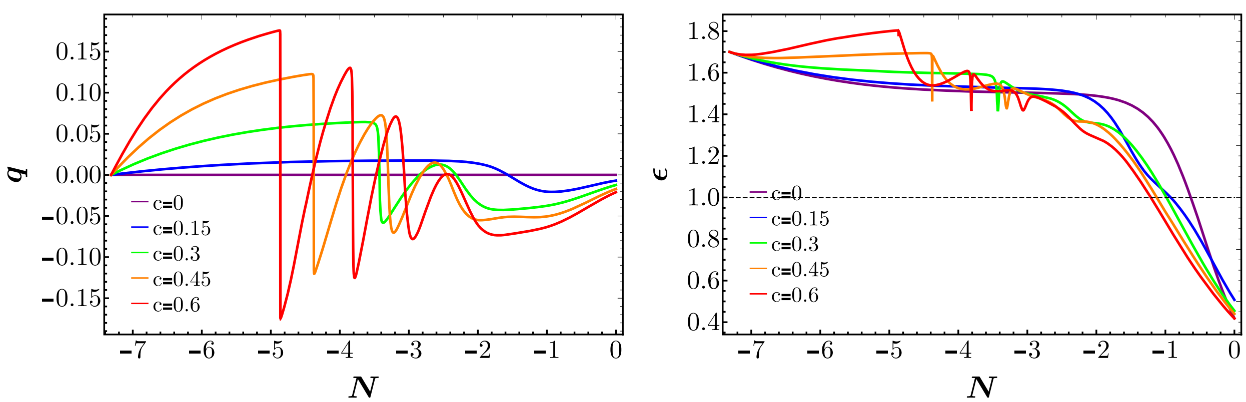

FIG. 1. Left panel: evolution of the interaction term $q \equiv \alpha\beta x \Omega_m$ as a function of N. Right panel: slow-roll parameter ϵ as a function of N.

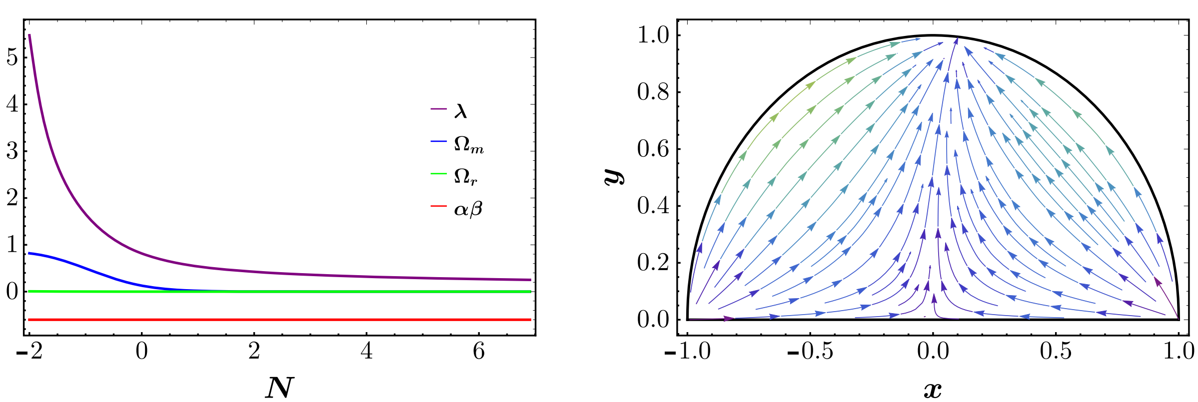

FIG. 2. Left panel: evolution of various parameters in the future ($N > 0$). Right panel: x-y phase space with a dark-energy-dominated attractor point for $C = 0.6$.

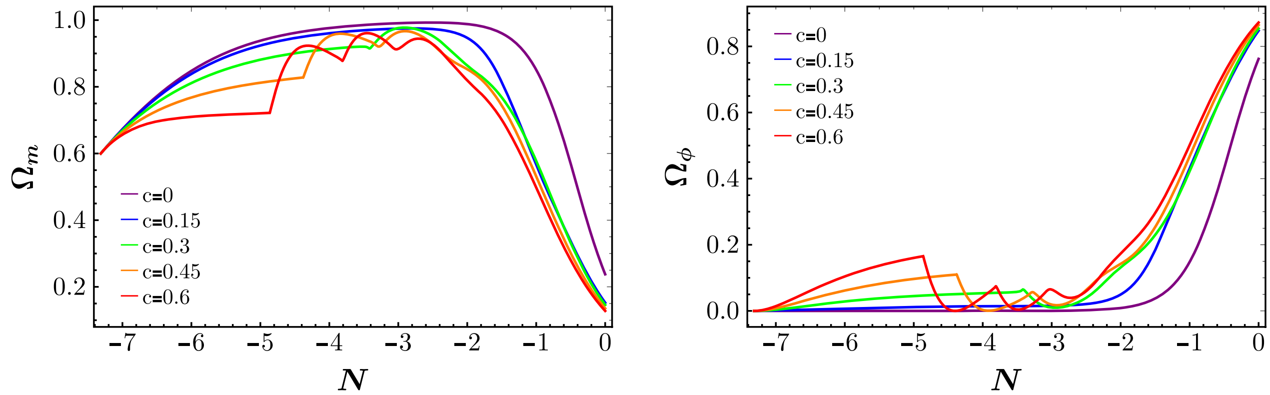

FIG. 3. Evolution of energy density parameters as functions of N. Left panel: dark matter. Right panel: dark energy.

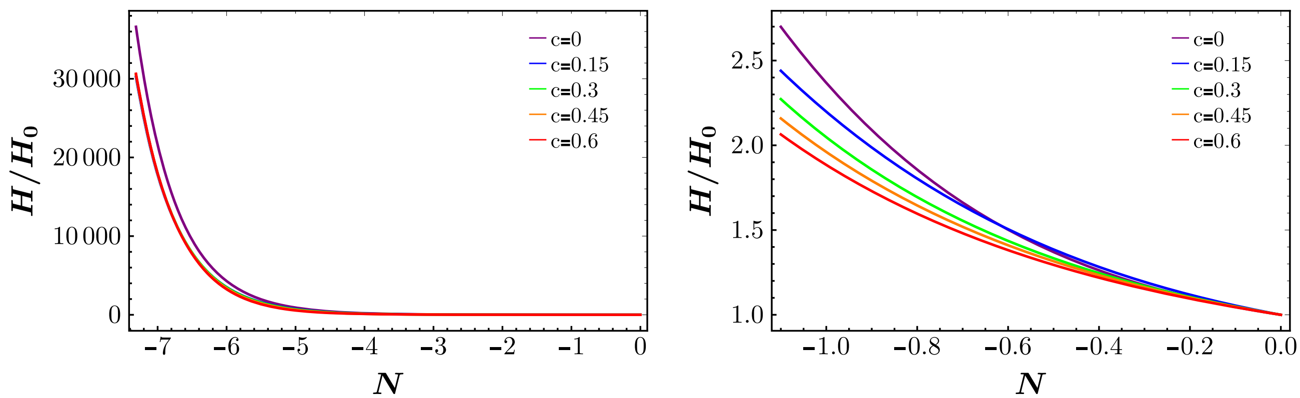

FIG. 4. Evolution of the Hubble parameter $h = H/H_0$ as a function of N.

the evolution of the scaled Hubble parameter $h = H/H_0$. The plot in the right panel (for redshift range $2 < z < 0$) is the zoomed-in version of the plot in the left panel (for redshift range $1500 < z < 0$). From the plots, it is clear that while the dark energy dominates early, the scaled Hubble parameter is smaller in the interacting dark-sector models than in the noninteracting dark-sector model.

2. Scenario II: $C \leq 0$

In this scenario, the initial value of the coupling function is non-negative. Figure 5 contains the plots of the evolution of the scaled interaction term q [defined in Eq. (53)] and slow-roll parameter ϵ [defined in Eq. (55)]. As in the earlier scenario, the evolution with $C < 0$ leads to accelerated expansion, and the interaction function stays positive

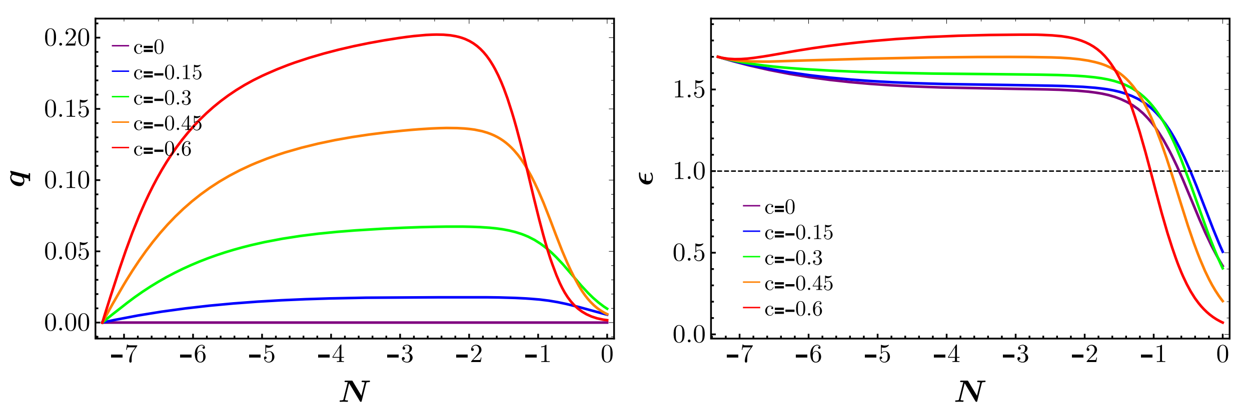

FIG. 5. Left panel: evolution of the interaction term $q \equiv \alpha\beta x \Omega_m$ as a function of N. Right panel: slow-roll parameter ϵ as a function of N

during the evolution. However, the late-time evolution of these parameters is different from that of the $C \geq 0$ scenario.

Like in the $C \geq 0$ scenario, Fig. 6 contains the evolution of the system in the x-y phase plane. This scenario also leads to a stable dark-energy-dominated attractor. Figure 7 contains the evolution of Ω_ϕ and Ω_m. Like in the earlier scenario, the plots show the dark-energy-dominated phase in the late Universe. However, the late-time evolution of these parameters is different from the $C \geq 0$ scenario. Figure 8 also shows a similar trend in the evolution of the Hubble parameter.

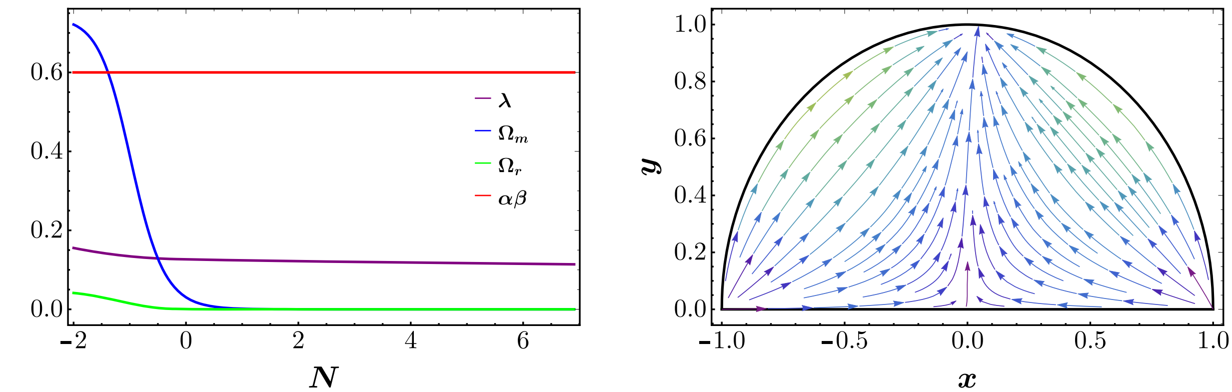

FIG. 6. Left panel: evolution of various parameters in the future ($N > 0$). Right panel: x-y phase space with a dark-energy-dominated attractor point for $C = -0.6$.

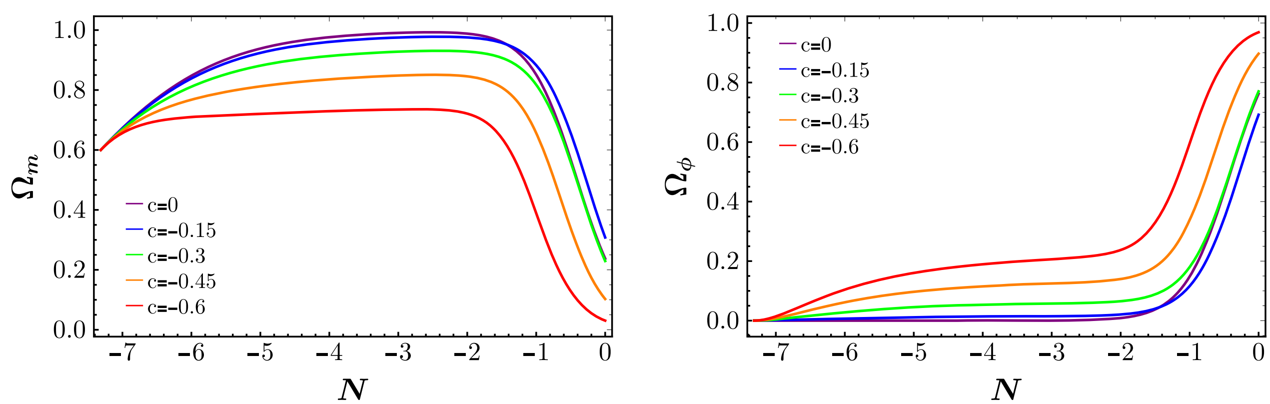

FIG. 7. Evolution of the energy density parameters as functions of N. Left panel: dark matter. Right panel: dark energy.

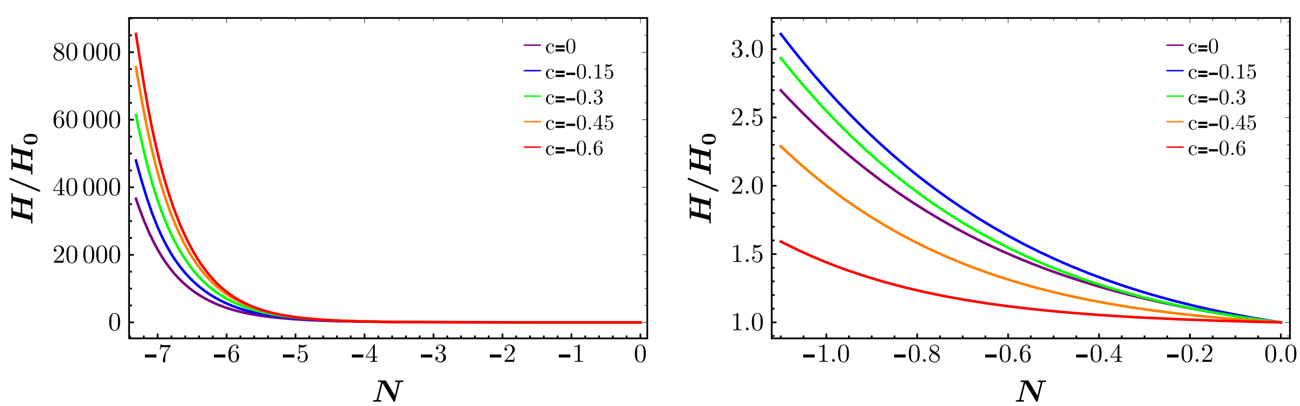

FIG. 8. Evolution of the Hubble parameter $h = H/H_0$ as a function of N.

V. CONCLUSIONS AND DISCUSSIONS

In this work, we have constructed the dark energy–dark matter interaction from a classical field theory action. This action is obtained from the $f(\tilde{R},\tilde{\chi})$ action using a conformal transformation and redefinition of the scalar fields. While the total energy-momentum tensor is conserved due to the interaction, the energy-momentum of the individual components in the dark sector is not satisfied. This leads to a unique interaction term $Q_\nu^{(F)}$.

While the field theory description helps us to obtain the interaction from the action principle, the fluid description turns out to be more useful for analyzing cosmological observations. In that regard, the most common description of the interacting dark sector is in terms of a dark matter fluid. However, in the phenomenological description of the dark matter fluid interaction, there is no unique form of Q_ν. In many of the models in the literature the interaction strength Q_ν in the dark sector is introduced by hand. We have systematically shown that the one-to-one correspondence between the fluids and fields is possible only if the interaction term is given by $Q_\nu^{(F)}$. In this specific case, the equations in the field theory description can be obtained from the fluid equations by a simple substitution of the variables.

We classified the interacting dark energy models considered in the literature into two categories based on the field-theoretic description. While many of the models have a field-theoretic description, many of the dark matter fluid interaction models do not have a field-theoretic description like that used in this work. The field-theoretic description used in this work is the simplest one possible. It may be possible that by considering a generalized action, like the Horndeski Lagrangian, some of these models may have a field-theoretic description [40]. This needs further investigation.

We defined a set of dimensionless variables and constructed a novel autonomous system that describes the evolution of a general quintessence dark energy interacting with dark matter. Studying the fixed points of the autonomous system, we showed that the interacting dark sector model has a stable attractor solution that describes the late-time accelerated Universe. As an example, we considered the model with $U(\phi) \propto 1/\phi$ and $\alpha(\phi) \propto \phi$. We have shown that a stable, dark-energy-dominated solution exists for this model for a range of coupling strengths (both positive and negative).

While the form of the interaction term ($Q_\nu^{(F)}$) is unique, it still contains unknown functions like $\alpha(\phi)$, χ, and $V(\chi)$. These can be constrained using particle physics models [41]. The immediate question that arises is whether one can use some other tools to further constrain the suitable dark matter–dark energy model from the observations. This is currently under investigation.

ACKNOWLEDGMENTS

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APPENDIX A: FIELD-THEORETIC FORMULATION OF THE DARK ENERGY–DARK MATTER INTERACTION

Consider the following Jordan frame action:

$$S_J = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}f(\tilde{R},\tilde{\chi}) - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\chi}\tilde{\nabla}_\nu\tilde{\chi} - V(\tilde{\chi})\right], \tag{A1}$$

where $f(\tilde{R},\tilde{\chi})$ is an arbitrary, smooth function of the Ricci scalar ($\tilde{R}$) defined in the four-dimensional metric $\tilde{g}_{\mu\nu}$, and the scalar field $\tilde{\chi}$. Under conformal transformation and redefining the scalar fields, one can bring it to the Einstein frame with two interacting scalar fields [34].

To keep calculations tractable, we assume the following form for $f(\tilde{R},\tilde{\chi})$:

$$f(\tilde{R},\tilde{\chi}) = h(\tilde{\chi})f(\tilde{R}). \tag{A2}$$

The above action can be rewritten as

$$S = \int d^4x\sqrt{-\tilde{g}}\left[h(\tilde{\chi})\left(\frac{F\tilde{R}}{2\kappa^2} - U\right) - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi} - V(\tilde{\chi})\right], \tag{A3}$$

where

$$F\frac{\partial f}{\partial\tilde{R}} \quad \text{and} \quad \tilde{U} = \frac{F\tilde{R} - f}{2\kappa^2}.$$

Under the conformal transformation

$$\hat{g}_{\mu\nu} = F\tilde{g}_{\mu\nu}, \tag{A4}$$

the above action (A3) becomes

$$S = \int d^4x\sqrt{-\hat{g}}\left[h(\tilde{\chi})\frac{\hat{R}}{2\kappa^2} - h(\tilde{\chi})\hat{U} + h(\tilde{\chi})\sqrt{\frac{3}{2\kappa^2}}\hat{\Box}\psi - \frac{h(\tilde{\chi})}{2}\hat{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{e^{-\sqrt{\frac{2\kappa^2}{3}}\psi}}{}\hat{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi} - \hat{V}(\tilde{\chi})\right], \tag{A5}$$

where

$$\psi = \sqrt{\frac{3}{2\kappa^2}}\ln F, \qquad \hat{U} = \frac{\tilde{U}}{F^2}, \qquad \hat{V} = \frac{V}{F^2}. \tag{A6}$$

Introducing one more conformal transformation,

$$g_{\mu\nu} = h(\tilde{\chi})\hat{g}_{\mu\nu}, \tag{A7}$$

the above action can be rewritten as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \left[\frac{1}{2he^{\sqrt{\frac{2\kappa^2}{3}}\psi}} + \frac{3h^2_{,\tilde{\chi}}}{4\kappa^2 h^2}\right] g^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi} - \frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \sqrt{\frac{3}{2\kappa^2}}\frac{h_{,\tilde{\chi}}}{h}g^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\psi - \hat{W}\right], \tag{A8}$$

where

$$\hat{W} = \frac{FR - f}{\kappa h F^2} + \frac{V}{h^2 F^2}. \tag{A9}$$

The above action in the Einstein frame neatly separates into the Ricci scalar and the scalar fields. However, the scalar fields are not in a canonical form. Since the metric $g_{\mu\nu}$ appears in all of the kinetic parts of the scalar fields, the field-space line element can be written as

$$d\ell^2 = \left[\frac{1}{he^{\sqrt{\frac{2\kappa^2}{3}}\psi}} + \frac{3h^2_{,\tilde{\chi}}}{2\kappa^2 h^2}\right] d\tilde{\chi}^2 + 2\sqrt{\frac{3}{2\kappa^2}}\frac{h_{,\tilde{\chi}}}{h}d\tilde{\chi}d\psi + d\psi^2. \tag{A10}$$

It has to be noted that it is impossible to bring the above line element to Euclidean form by a redefinition of the fields. Thus, the field-space line element can be written in many different ways, leading to a different interaction between the two scalar fields. We list two cases below.

1. One of the simplest options is to redefine the fields as [33]

$$\sqrt{\frac{3}{2\kappa^2}}\ln h + \psi = \phi, \qquad \tilde{\chi} = \chi. \tag{A11}$$

Then, the field-space line element (A10) reduces to

$$d\ell^2 = \frac{1}{he^{\sqrt{\frac{2\kappa^2}{3}}\psi}} d\tilde{\chi}^2 + \left[d\left(\sqrt{\frac{3}{2\kappa^2}}lnh + \psi\right)\right]^2 = e^{-\sqrt{\frac{2\kappa^2}{3}}\phi}d\chi^2 + d\phi^2. \tag{A12}$$

Under this field redefinition, the Einstein frame action (A8) is given by

$$S_E = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - U(\phi) - \frac{1}{2}e^{-\sqrt{\frac{2\kappa^2}{3}}\phi}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - e^{-2\sqrt{\frac{2\kappa^2}{3}}\phi}V(\chi)\right]. \tag{A13}$$

2. Let us now consider the following redefinition of the fields:

$$e^{2\alpha(\phi)}\left(\frac{\partial\chi}{\partial\psi}\right)^2 + \left(\frac{\partial\phi}{\partial\psi}\right)^2 = 1,$$
$$e^{2\alpha(\phi)}\frac{\partial\chi}{\partial\tilde{\chi}}\frac{\partial\chi}{\partial\psi} + \frac{\partial\phi}{\partial\tilde{\chi}}\frac{\partial\phi}{\partial\psi} = \sqrt{\frac{3}{2\kappa^2}}\frac{h_{,\tilde{chi}}}{h},$$
$$e^{2\alpha(\phi)}\left(\frac{\partial\chi}{\partial\tilde{\chi}}\right)^2 + \left(\frac{\partial\phi}{\partial\tilde{\chi}}\right)^2 = \frac{1}{he^{\sqrt{\frac{2\kappa^2}{3}}\psi}} + \frac{3}{2\kappa^2}\frac{h^2_{,\tilde{\chi}}}{h^2}, \tag{A14}$$

where $\chi \equiv \chi(\tilde{\chi}, \psi)$, $\phi \equiv \phi(\tilde{\chi}, \psi)$, and $\alpha(\phi)$ is an arbitrary function of ϕ. Under this redefinition, the field space line element (A10) reduces to

$$ds^2 = e^{2\alpha(\phi)}d\chi^2 + d\phi^2. \tag{A15}$$

Thus, the Einstein frame action takes the form

$$S_E = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - U(\phi) - \frac{1}{2}e^{2\alpha(\phi)}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - e^{4\alpha(\phi)}V(\chi)\right) \tag{A16}$$

and is identical to the action (5) in Sec. II. This action describes two interacting scalar fields with an arbitrary coupling represented by the function $\alpha(\phi)$.

APPENDIX B: COSMOLOGICAL EVOLUTION IN NEWTONIAN GAUGE

Consider the spatially flat FRW metric with first-order scalar perturbations in Newtonian gauge [3]:

$$g_{00} = -(1+2\Phi), \quad g_{0i} = 0, \quad g_{ij} = a^2(1-2\Psi)\delta_{ij}, \tag{B1}$$

where $a \equiv a(t)$ is the scale factor with the Hubble parameter given by $H = \dot{a}/a$, and $\Phi \equiv \Phi(t, x, y, z)$ and $\Psi \equiv \Psi(t, x, y, z)$ are scalar perturbations.

The scalar fields ϕ and χ, the dark matter fluid energy density (ρ_m), dark matter fluid pressure (p_m), and the interaction strength (Q_ν) can be split into background and perturbed parts as

$$\phi = \bar{\phi} + \delta\phi, \qquad \chi = \bar{\chi} + \delta\chi, \qquad \rho_m = \bar{\rho}_m + \delta\rho_m,$$
$$p_m = \bar{p}_m + \delta p_m, \qquad Q_\nu = \bar{Q}_\nu + \delta Q_\nu, \tag{B2}$$

$$u_\mu = \bar{u}_\mu + \delta u_\mu, \qquad \bar{u}_0 = -1, \qquad \delta u_0 = -\Phi,$$
$$\bar{u}_i = 0, \qquad \delta u_i = \frac{\partial\delta u^s}{\partial x^i}, \qquad \delta u^s = -\frac{\delta\chi}{\dot{\bar{\chi}}}. \tag{B3}$$

In the following subsection, we present the evolution equations for the first-order perturbations in Newtonian gauge.

1. Correspondence between fields and fluids in first-order perturbations

In the fluid description, the first-order scalar perturbations, in Newtonian gauge, satisfy the following equations [3]:

$$\Psi - \Phi = 0, \tag{B4}$$

$$\dot{\Psi} + H\Phi = \frac{\kappa^2}{2}[\dot{\bar{\phi}}\delta\phi - (\bar{\rho}_m + \bar{p}_m)\delta u^s], \tag{B5}$$

$$3H\dot{\Psi} - \frac{\nabla^2\Psi}{a^2} + 3H^2\Phi = -\frac{\kappa^2}{2}(\delta\rho_m + \dot{\delta\phi}\,\dot{\bar{\phi}} - \Phi\dot{\bar{\phi}}^2 + U_{,\phi}(\bar{\phi})\delta\phi), \tag{B6}$$

$$3\ddot{\Psi} + \frac{\nabla^2\Phi}{a^2} + 6\Phi(H^2 + \dot{H}) + 3H(2\dot{\Psi} + \dot{\Phi}) = \frac{\kappa^2}{2}(\delta\rho_m + 3\delta p_m + 4\dot{\delta\phi}\dot{\bar{\phi}} - 4\Phi\dot{\bar{\phi}}^2 - 2U_{,\phi}(\bar{\phi})\delta\phi). \tag{B7}$$

From Eq. (1), the conservation equations for the dark energy field and dark matter fluid in the first-order perturbations are given by

$$\dot{\delta\rho_m} + 3H(\delta p_m + \delta\rho_m) + (\bar{p}_m + \bar{\rho}_m)\left[\frac{\nabla^2\delta u^s}{a^2} - 3\dot{\Psi}\right] = -\delta Q, \tag{B8}$$

$$\dot{\bar{\phi}}\left(\ddot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} - 2\Phi\ddot{\bar{\phi}} + U_{,\phi\phi}(\bar{\phi})\delta\phi\right) + \dot{\delta\phi}(\ddot{\bar{\phi}} + 6H\dot{\bar{\phi}} + U_{,\phi}(\bar{\phi})) - \frac{\dot{\bar{\phi}}^2}{2}(3\dot{\Psi} + \dot{\Phi} + 6H\Phi) = \delta Q. \tag{B9}$$

The above equations are generic equations for the coupled dark matter fluid and dark energy field with an arbitrary interaction term δQ. As discussed in Sec. II A, starting from the Jordan frame action (3), the interaction term $Q_\nu^{(\mathrm{F})}$ in Eq. (20) is uniquely written in terms of the dark energy scalar field and dark matter fluid. In this case, the perturbed interaction term is given by

$$\delta Q^{(\mathrm{F})} = -(\delta\rho_m - 3\delta p_m)\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}} - (\bar{\rho}_m - 3\bar{p}_m)[\alpha_{,\phi\phi}(\bar{\phi})\dot{\bar{\phi}}\delta\phi + \alpha_{,\phi}(\bar{\phi})\dot{\delta\phi}]. \tag{B10}$$

We now show that the above equations are consistent with the field theory description *only* for this form of interaction $Q^{(\mathrm{F})}$. Substituting ρ_m, p_m, $\delta\rho_m$, and δp_m from Eq. (15), the perturbed equations of motion for ϕ and χ are, respectively,

$$\ddot{\delta\chi} - \frac{\nabla^2\delta\chi}{a^2} + e^{2\alpha}V_{,\chi\chi}(\bar{\chi})\delta\chi - \dot{\bar{\chi}}(3\dot{\Psi} + \dot{\Phi}) + 2e^{2\alpha}V_{,\chi}(\bar{\chi})\Phi + 3H\dot{\delta\chi} + 2\alpha_{,\phi}(\bar{\phi})(\dot{\bar{\phi}}\,\dot{\delta\chi} + \dot{\bar{\chi}}\,\dot{\delta\phi}) + 2\delta\phi[\dot{\bar{\phi}}\,\dot{\bar{\chi}}\,\alpha_{,\phi\phi}(\bar{\phi}) + e^{2\alpha}\alpha_{,\phi}(\bar{\phi})V_{,\chi}(\bar{\chi})] = 0, \tag{B11}$$

$$\ddot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} + 3H\dot{\delta\phi} + U_{,\phi\phi}(\bar{\phi})\delta\phi - \dot{\bar{\phi}}(3\dot{\Psi} + \dot{\Phi}) + 2U_{,\phi}(\bar{\phi})\Phi + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})[2e^{2\alpha}V_{,\chi}(\bar{\chi})\delta\chi - \dot{\bar{\chi}}\,\dot{\delta\chi}] + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})^2\delta\phi[8e^{2\alpha}V(\bar{\chi}) - \dot{\bar{\chi}}^2] + e^{2\alpha}\alpha_{,\phi\phi}(\bar{\phi})\delta\phi[4e^{2\alpha}V(\bar{\chi}) - \dot{\bar{\chi}}^2] = 0. \tag{B12}$$

The above perturbed field equations are identical to the equations obtained from Eqs. (6) and (7), respectively. The perturbed interaction term in the field theory picture can also be obtained by a direct substitution of the variables:

$$\delta Q^{(\mathrm{F})} = 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})\dot{\bar{\phi}}[\dot{\bar{\chi}}\,\dot{\delta\chi} - 2e^{2\alpha}V_{,\chi}(\bar{\chi})\delta\chi - \dot{\bar{\chi}}^2\Phi] + e^{2\alpha}\alpha_{,\phi\phi}(\bar{\phi})\dot{\bar{\phi}}\delta\phi[\dot{\bar{\chi}}^2 - 4e^{2\alpha}V(\bar{\chi})] + 2e^{2\alpha}\alpha_{,\phi}(\bar{\phi})^2\dot{\bar{\phi}}\delta\phi[\dot{\bar{\chi}}^2 - 8e^{2\alpha}V(\bar{\chi})] + e^{2\alpha}\alpha_{,\phi}(\bar{\phi})\dot{\delta\phi}[\dot{\bar{\chi}}^2 - 4e^{2\alpha}V(\bar{\chi})]. \tag{B13}$$

APPENDIX C: BACKGROUND EVOLUTION IN A GENERAL INTERACTING DARK ENERGY–DARK MATTER MODEL

In various quintessence models considered in the literature, the scalar field's dimensions differ depending on the nature of the potential, especially in the case of the power-law potentials [38]. To include those scenarios, we rewrite the action for the interacting dark sector as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - C_1\left(\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + U(\phi)\right) - C_2\left(\frac{1}{2}e^{2\alpha(\phi)}\nabla^\mu\chi\nabla_\mu\chi + e^{4\alpha(\phi)}V(\chi)\right)\right], \tag{C1}$$

where C_1 and C_2 are constants. Then, the background energy density and pressure of the dark matter are defined by

$$\rho_m = C_2 e^{2\alpha(\phi)}\left(\frac{\dot{\chi}^2}{2} + e^{2\alpha(\phi)}V(\chi)\right), \tag{C2}$$

$$p_m = C_2 e^{2\alpha(\phi)}\left(\frac{\dot{\chi}^2}{2} - e^{2\alpha(\phi)}V(\chi)\right). \tag{C3}$$

Then, the energy conservation equations become

$$C_1(\ddot{\phi} + 3H\dot{\phi} + U_\phi(\phi))\dot{\phi} = Q, \tag{C4}$$

$$\dot{\rho_m} + 3H(\rho_m + p_m) = -Q, \quad \text{(C5)}$$

where the interaction term Q is given by

$$Q = C_2\alpha_\phi(\phi)\dot{\phi}(e^{2\alpha(\phi)}\dot{\chi}^2 - 4e^{\alpha(\phi)}V(\chi)) = -\alpha_\phi(\phi)\dot{\phi}(\rho_m - 3p_m). \quad \text{(C6)}$$

The Friedmann equations are given by

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left[\rho_m + C_1\left(\frac{\dot{\phi}^2}{2} + U(\phi)\right)\right], \quad \text{(C7)}$$

$$2\dot{H} = -\frac{1}{M_{\rm Pl}^2}(\rho_m + p_m + C_1\dot{\phi}^2). \quad \text{(C8)}$$

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