Reply to "Comment on 'Coulomb-nuclear interference effects in proton-proton scattering: A simple new eikonal approach'"

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We clarify the connection between the results of Petrov in the preceding Comment and those in our previous paper on Coulomb-nuclear interference in proton-proton scattering in the region of very small momentum transfers. We calculate the correction to the approximation we used in that region, but not elsewhere, and show that it agrees with corresponding terms in Petrov's expression. The corrections are negligible in either form, and do not affect our previous results.

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In the preceding Comment [1], Petrov points out that an approximation used at small momentum transfers in our earlier paper on Coulomb-nuclear interference in proton-proton scattering [2] would, if taken as exact, imply that the ratio of the real to the imaginary part of the nuclear scattering amplitude $f_N(s,q^2)$ must be independent of the momentum transfer q^2. This is in apparent conflict with the results presented in [2] where the ratio $\mathrm{Re}f_N(s,q^2)/\mathrm{Im}f_N(s,q^2)$ was found to decrease rapidly as q^2 increases from 0.

We show here that this is actually not a problem. The approximation was used only in the region of very small q^2 where the corrections are negligible and do not affect the q^2 dependence of the ratio. In particular, they do not affect the determination of the ratio $\rho(s)=\mathrm{Re}f_N(s,0)/\mathrm{Im}f_N(s,0)$ using Coulomb-nuclear interference effects in the scattering. We derive the corrections at small q^2, connect the results to those of Petrov, and show why they are negligible in either form.

The strong-interaction or nuclear part of the pp scattering amplitude including the effects of the Coulomb interaction and electromagnetic form-factor corrections is given in the additive eikonal model in Eq. (27) of [2] as an integral over impact parameters b as

$$f_{N,c}(s,q^2) = i\int_0^\infty db\, b e^{2i\delta_c(b,s)+2i\delta_c^{FF}(b,s)}(1-e^{2i\delta_N(b,s)})J_0(qb). \quad (1)$$

Here $\delta_N=\delta_{N,R}+i\delta_{N,I}$ is the complex nuclear phase shift, $\delta_c(b,s)$ is the Coulomb phase shift, and $\delta_c^{FF}(b,s)$ is the phase shift associated with the form-factor corrections. In particular, $\delta_c(b,s)=\alpha(\log p(W)b+\gamma)$ where $p(W)$ is the center-of-mass momentum of either particle at total center-of-mass energy W, and γ is Euler's constant. The form-factor phase shift $\delta_c^{FF}(b,s)$ is given by a sum of hyperbolic Bessel functions with an overall factor of α for the standard dipole form factor $F_Q(q^2)=\mu^4/(q^2+\mu^2)^2$ ([2], Sec. II C).

It was shown in [2] by direct numerical calculation using the successful eikonal model in [3] that the ratio $|f_{N,c}(s,q^2)/f_N(s,q^2)|$ of the magnitudes of the corrected amplitude to the pure nuclear amplitude

$$f_N(s,q^2) = i\int_0^\infty db\, b(1-e^{2i\delta_N(b,s)})J_0(qb) \quad (2)$$

was equal 1 to better than a part per thousand in the region $q^2 \lesssim 0.15$ GeV^2 at 13 000 GeV, and to higher q^2 at lower energies, covering the regions important for the determination of $\rho(s)$ from Coulomb-nuclear interference effects. See Fig. 1 in [2]. This correction is significantly smaller than the experimental uncertainties in the most accurate data available at present, and does not affect the determination of ρ from 10 to 13 000 GeV or somewhat above. The ratio differs significantly from unity and must be taken into account in calculations at larger q^2, again as seen in the numerical results in Fig. 1 in [2].

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Since our focus was on the interference effects at small q^2, this very small correction to the ratio of magnitudes was dropped in [2] as noted preceding Eq. (29), and $f_{N,c}(s,q^2)$ was taken as

$$f_{N,c}(s,q^2) \approx e^{\Delta\Phi(s,q^2)} f_N(s,q^2) \tag{3}$$

in the remainder of the analysis, where $\Delta\Phi$ is the difference between the phases of $f_{N,c}$ and f_N,

$$\begin{aligned}\Delta\Phi_N(s,q^2) &= \arg f_{N,c}(s,q^2) - \arg f_N(s,q^2) \\ &= \arg\left(f_{N,c}(s,q^2)/f_N(s,q^2)\right).\end{aligned} \tag{4}$$

The fact that the magnitude correction was dropped on the right-hand side of Eq. (3) in our small-q^2 analysis is the origin of the apparent discrepancy between the results in [2] and those of Cahn [4] and of Kundrát and Lokajiček [5] noted by Petrov [1]. It must, of course, be included at large q^2.

It is straightforward to obtain the correction to the magnitude of $f_{N,c}(s,q^2)$ relative to $f_N(s,q^2)$ at small q^2. We note first that we can write the Coulomb phase factor in Eq. (1) as $\exp(2i\alpha(\ln pb_0+\gamma))\cdot\exp(2i\alpha\ln(b/b_0)$ where b_0, introduced for dimensional reasons, is constant. The first factor in this expression is independent of the impact parameter b so factors out of the integral, and has magnitude 1 in $|f_{N,c}(s,q^2)|^2$. We will therefore use the modified, momentum-independent phase $\delta'_c(b,s) = \alpha\log(b/b_{\mathrm{peak}})$, choosing $b_0 = b_{\mathrm{peak}}$ as the value of b at the peak in the impact-parameter distribution of $|(1-e^{2i\delta_N(b,s)})|^2$, essentially the peak in the distribution for the imaginary part of f_N. A good estimate is $b_{\mathrm{peak}} \approx \sqrt{\sigma_{\mathrm{tot}}/4\pi}$ for the reasonably symmetric peaked distributions found in the eikonal model, with σ_{tot} the total cross section [3]. The exact result for $|f_{N,c}|^2$ is, of course, independent of b_{peak}. Its use here gives a convenient way of estimating the corrections since $\delta'_c(b_{\mathrm{peak}},s)=0$ and $\delta_c^{FF}(b_{\mathrm{peak}},s)$ is also small.

With this specification, and $\delta' = \delta'_c + \delta_c^{FF}$, an expansion of the ratio $|f_{N,c}/f_N|$ in inverse powers of $|f_N|$ gives

$$|f_{N,c}(s,q^2)| = |f_N(s,q^2)|\left(1 + \frac{\mathrm{Re}f_N(s,q^2)\mathrm{Re}\Delta f_{N,c}(s,q^2) + \mathrm{Im}f_N(s,q^2)\mathrm{Im}\Delta f_{N,c}(s,q^2)}{|f_N(s,q^2)|^2} + \cdots\right), \tag{5}$$

where

$$\Delta f_{N,c}(s,q^2) = -i\int_0^\infty db\, b(1-e^{2i\delta'(b,s)})(1-e^{2i\delta_N(b,s)})J_0(qb) \tag{6}$$

$$\approx -\int_0^\infty db\, b 2\delta'(b,s)(1-e^{2i\delta_N(b,s)})J_0(qb) + \cdots. \tag{7}$$

The expansion in Eq. (7) is justified because $2\delta'$ is small, of order α, and the remaining factor in the integrand is compact in b, a major advantage of the impact-parameter description of the scattering amplitude as used in [2]. This gives the corrections to leading order in α as

$$\mathrm{Re}\Delta f_{N,c}(s,q^2) = -\int_0^\infty db\, b 2\delta'(b,s)(1-\cos 2\delta_{N,R}e^{-2\delta_{N,I}})J_0(qb), \tag{8}$$

$$\mathrm{Im}\Delta f_{N,c}(s,q^2) = \int_0^\infty db\, b 2\delta'(b,s)\sin 2\delta_{N,R}e^{-2\delta_{N,I}}J_0(qb). \tag{9}$$

The factors in the integrals for $\mathrm{Re}\Delta f_{N,c}$ and $\mathrm{Im}\Delta f_{N,c}$ other than $2\delta'(b,s)$ are just the integrands for $\mathrm{Im}f_N$ and $\mathrm{Re}f_N$, respectively. Those integrands vanish at $b=0$ and large b, with smooth peaks near $b=b_{\mathrm{peak}}$ for $\mathrm{Im}f_N$ at $q^2=0$, and slightly beyond for $\mathrm{Re}f_N$.[6] If the phase shift $\delta'(b,s)$ were constant over the peak region, it would factor approximately out of the integrals and we would have $\mathrm{Re}\Delta f_{N,c} \approx -2\delta'\mathrm{Im}f_N$ and $\mathrm{Im}\Delta f_{N,c} \approx 2\delta'\mathrm{Re}f_N$ and the correction to $|f_N|$ would vanish. Because of the variation in $\delta'(b,s)$ over the peak regions and the small difference in the locations of the peaks in the integrands, the cancellation is only approximate.

Given the structure of the correction in Eq. (5), it is clear that it is at most of order $\alpha\rho(s) \sim 10^{-3}$. It is smaller in the exact calculations. This estimate extends over the region in q^2 in which the variation of the Bessel function $J_0(qb) = 1-\frac{1}{4}(qb)^2+\cdots$ can be ignored, $q^2 \lesssim 4/b_{\mathrm{peak}}^2$, e.g., $q^2 \lesssim 0.12\ \mathrm{GeV}^2$ at 13 000 GeV. This is in agreement with the exact results in Fig. 1 of [2], where the magnitude of the correction begins to increase significantly beyond that point.

The expression in Eq. (6) is the starting point for the treatment of the Coulomb effects by Cahn [4] in his Eq. (15), with the form-factor effects included later in Eq. (28). This treatment was later sharpened by Kundrát and Lokajiček [5]. Cahn uses the Fourier convolution theorem to write the integral as the convolution of the Fourier transforms of the two factors multiplying the Bessel function in Eq. (6). To obtain convergence of the transform of the factor $(1 - e^{2i\delta'})$, he first replaces the Coulomb phase shift by that for a massive photon, $\alpha/q^2 \to \alpha/(q^2 + \lambda^2)$, and then rearranges the terms in the full expression into a form in which he can take the limit $\lambda \to 0$, with the final result given in his Eq. (30). This gives

$$\Delta f_{N,c}(s, q^2) = -\frac{i}{\pi}\int d^2k \frac{2\alpha}{(\mathbf{k}-\mathbf{q})^2} F_Q^2((\mathbf{k}-\mathbf{q})^2)[f_N(s,k^2) - f_N(s,q^2)] \tag{10}$$

with our notation and normalization.

In this approach, the expression for the fractional change in $|f_{N,c}|^2$ is given by

$$\delta|f_{N,c}| = \frac{1}{\pi}\frac{1}{|f_N(s,q^2)|^2}\int d^2k \frac{2\alpha}{(\mathbf{k}-\mathbf{q})^2} F_Q^2((\mathbf{k}-\mathbf{q})^2)[\mathrm{Re} f_N(s,q^2)\mathrm{Im} f_N(s,k^2) - \mathrm{Im} f_N(s,q^2)\mathrm{Re} f_N(s,k^2)]. \tag{11}$$

This expression has the same structure as Eq. (6), is nonsingular, and of order $\alpha\rho(s)$. It would only vanish identically for the ratio $\mathrm{Re} f_N(s,k^2)/\mathrm{Im} f_N(s,k^2)$ constant and equal to $\mathrm{Re} f_N(s,q^2)/\mathrm{Im} f_N(s,q^2)$. This would require a constant nuclear phase as observed by Petrov [1]. While correct, this remark is not relevant to the treatment of Coulomb-nuclear interference in [2], where the correction was only omitted in a region in which it was shown to be negligibly small, and not elsewhere.

In the eikonal model in [3] and other models which respect the constraints on the phase of f_N imposed by unitarity and analyticity [7], $\mathrm{Re} f_N$ actually decreases at small k^2 (or q^2) substantially more rapidly than $\mathrm{Im} f_N$ as k^2 (q^2) increases, so the cancellation in Eq. (11) is not complete. The result is consistent with that obtained above, where the shift of the impact-parameter distribution for $\mathrm{Re} f_N$ toward larger b than that for $\mathrm{Im} f_N$ leads to the more rapid decrease of $\mathrm{Re} f_N$ through the earlier onset of the effects of the Bessel function in Eq. (2).

The results in Eqs. (5) and (11) are equivalent, and the corrections to $f_{N,c}$ in Eq. (3) are very small, e.g., about 6×10^{-4} at $q^2 = 0$ for either at energies from 100 to 13 000 GeV, and of similar sizes throughout the region $q^2 \lesssim 4/b^2_{\mathrm{peak}}$ as shown in [2]. The ratio of magnitudes $|f_{N,c}(s,q^2)/f_N(s,q^2)|$ is therefore 1 to high accuracy throughout the region where the Coulomb-nuclear interference effects are significant. We would emphasize, however, that the phase of the nuclear amplitude is not constant in this region as might be suggested by Petrov's argument, but actually decreases rapidly as shown in the exact calculations. This decrease, by ~60%–70% from $q^2 = 0$ to 0.15 GeV^2 for energies above about 500 GeV, results from a diffraction zero in $\mathrm{Re} f_N$ at a significantly smaller q^2 than that in $\mathrm{Im} f_N$ (see Fig. 8 in [2]).

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