New rotating black holes in nonlinear Maxwell $f(\mathcal{R})$ gravity

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We investigate static and rotating charged spherically symmetric solutions in the framework of $f(\mathcal{R})$ gravity, allowing additionally the electromagnetic sector to depart from linearity. Applying a convenient, dual description for the electromagnetic Lagrangian, and using as an example the square-root $f(\mathcal{R})$ correction, we solve analytically the involved field equations. The obtained solutions belong to two branches, one that contains the Kerr-Newman solution of general relativity as a particular limit, and one that arises purely from the gravitational modification with no general relativity limit. The novel black hole solution has a true central singularity which is hidden behind a horizon; however, for particular parameter regions the horizon disappears and the singularity becomes a naked one. Furthermore, we investigate the thermodynamical properties of the solutions, such as the temperature, energy, entropy, heat capacity, and Gibbs free energy. We extract the entropy and quasilocal energy positivity conditions, we show that negative-temperature, ultracold, black holes are possible, and we show that the obtained solutions are thermodynamically stable for suitable model parameter regions.

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I. INTRODUCTION

The detection of gravitational waves from the binary black hole and binary neutron star mergers by the LIGO-VIRGO collaboration [1–3] opened the new era of multimessenger astronomy. In this novel window to investigate the universe the central role is played by spherically symmetric compact objects and black holes. Since their properties are determined by the underlying gravitational theory recently there has been an increased interest in studying such solutions in various extensions of general relativity (GR).

The simplest modification of GR arises by generalizing the action through arbitrary functions of the Ricci scalar, resulting in $f(\mathcal{R})$ gravity [4,5]. Nevertheless, one can build more complicated constructions using higher-order corrections, such as the Gauss-Bonnet term G and its functions [6–9], Lovelock combinations [10,11], Weyl combinations [12], higher spatial derivatives as in Hořava-Lifshitz gravity [13], etc. On the other hand, one can be based in the teleparallel formulation of gravity, and construct its modifications such as in $f(T)$ gravity [14–16], in $f(T, T_G)$ gravity [17], etc. Hence, in all these classes of modified gravity one can extract the spherically symmetric black hole solutions and study their properties [18–57].

In general, the obtained spherically symmetric solutions can be classified either in branches which are extensions of the corresponding GR solutions, coinciding exactly with them in a particular limit, or to novel branches that appear purely from the gravitational modification and do not possess a GR limit. In both cases, the obtained black holes and compact objects present new properties which may be potentially detectable in the gravitational waves arising from mergers. Thus, studying the properties of spherically symmetric solutions in various modified gravities is crucial in order to put the new observational tool of multimessenger astronomy to work.

It is the aim of the present study to derive new charged black hole solutions in the context of $f(\mathcal{R})$ gravity, allowing additionally for possible nonlinearities in the Maxwell sector. The plan of the manuscript is as follows: In Sec. II, we present a convenient way to handle the possible electrodynamic nonlinearities. In Sec. III we extract static and rotating spherically symmetric black hole solutions and in Sec. IV we calculate all the thermodynamical quantities such as the entropy, Hawking temperature, heat capacity, and Gibbs free energy, analyzing additionally the stability of the solutions. Finally, Sec. V is devoted to discussion and conclusions.

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II. DUAL REPRESENTATION OF NONLINEAR ELECTRODYNAMICS

In this section we present a new way for the description of nonlinear electrodynamics, which is valid independently of the specific electromagnetic Lagrangian and which allows for an easy handling concerning the derivation of field equations. We start with a general gauge-invariant electromagnetic Lagrangian of the form $\mathcal{L}(\mathcal{F})$, where $\mathcal{F}=\frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$ is the usual antisymmetric Faraday tensor defined as $\mathcal{F}_{\alpha\beta}=2V_{[\alpha,\beta]}$, with V_μ the gauge potential 1-form and where square brackets denote symmetrization [58]. As usual, linear, Maxwell electrodynamics is obtained for $\mathcal{L}(\mathcal{F})=4\mathcal{F}$.

For the purposes of this work we consider a dual representation, introducing the auxiliary field $\mathcal{P}_{\alpha\beta}$, which has been proven convenient if one desires to embed electromagnetism in the framework of general relativity [59,60]. In particular, we impose the Legendre transformation

$$\mathbb{H}=2\mathcal{F}\mathcal{L}_\mathcal{F}-\mathcal{L}, \tag{1}$$

with $\mathcal{L}_\mathcal{F}\equiv\frac{\partial\mathcal{L}}{\partial\mathcal{F}}$. Defining

$$\mathcal{P}_{\mu\nu}=\mathcal{L}_\mathcal{F}\mathcal{F}_{\mu\nu}, \tag{2}$$

we immediately find that $\mathbb{H}$ is an arbitrary function of the invariant

$$\mathcal{P}=\frac{1}{4}\mathcal{P}_{\alpha\beta}\mathcal{P}^{\alpha\beta}=\mathcal{L}_\mathcal{F}^2\mathcal{F}. \tag{3}$$

Using (1), the Lagrangian of nonlinear electrodynamics can be represented in terms of $\mathcal{P}$ as

$$\mathcal{L}=2\mathcal{P}\mathbb{H}_\mathcal{P}-\mathbb{H}, \tag{4}$$

while

$$\mathcal{F}_{\mu\nu}=\mathbb{H}_\mathcal{P}\mathcal{P}_{\mu\nu}, \tag{5}$$

with $\mathbb{H}_\mathcal{P}=\frac{\partial\mathbb{H}}{\partial\mathcal{P}}$.

The field equations thus acquire the form [59]

$$\partial_\nu(\sqrt{-g}\mathcal{P}^{\mu\nu})=0, \tag{6}$$

and the corresponding energy-momentum tensor is given as

$$\mathfrak{T}^{nlem}{}_\mu^\nu\equiv 2(\mathbb{H}_\mathcal{P}\mathcal{P}_{\mu\alpha}\mathcal{P}^{\nu\alpha}-\delta_\mu^\nu[2\mathcal{P}\mathbb{H}_\mathcal{P}-\mathbb{H}]). \tag{7}$$

We mention that in general (7) has a nonvanishing trace

$$\mathfrak{T}^{nlem}=8(\mathbb{H}-\mathbb{H}_\mathcal{P}\mathcal{P}), \tag{8}$$

which becomes zero only in the case of the linear theory. Finally, the electric and magnetic fields in spherical coordinates can be calculated as [59,60]

$$\begin{aligned} E&=\int\mathcal{F}_{tr}dr=\int\mathbb{H}_\mathcal{P}\mathcal{P}_{tr}dr,\\ B_r&=\int\mathcal{F}_{r\phi}d\phi=\int\mathbb{H}_\mathcal{P}\mathcal{P}_{r\phi}d\phi,\\ B_\theta&=\int\mathcal{F}_{\theta r}dr=\int\mathbb{H}_\mathcal{P}\mathcal{P}_{\theta r}dr,\\ B_\phi&=\int\mathcal{F}_{\phi r}dr=\int\mathbb{H}_\mathcal{P}\mathcal{P}_{\phi r}dr. \end{aligned} \tag{9}$$

III. STATIC AND ROTATING BLACK HOLE SOLUTIONS IN NONLINEAR MAXWELL $f(\mathcal{R})$ GRAVITY

In this section we consider nonlinear electrodynamics in a gravitational background governed by $f(\mathcal{R})$ gravity, and we extract charged black hole solutions. The total action is written as [61]

$$S_t=\frac{1}{2\kappa}\int\sqrt{-g}f(\mathcal{R})d^4x+\int\sqrt{-g}\mathcal{L}(\mathcal{F})d^4x, \tag{10}$$

where $\sqrt{-g}$ is the determinant of the metric $g_{\mu\nu}$ and κ is the gravitational constant (from now on we set $\kappa=1$ and all quantities are measured in these units). Performing variation with respect to the metric leads to the gravitational field equations [62,63]:

$$\begin{aligned}\xi_{\mu\nu}&=\mathcal{R}_{\mu\nu}f_\mathcal{R}-\frac{1}{2}g_{\mu\nu}f(\mathcal{R})-2g_{\mu\nu}\Lambda+g_{\mu\nu}\Box f_\mathcal{R}\\ &\quad-\nabla_\mu\nabla_\nu f_\mathcal{R}-8\pi\mathfrak{T}^{nlem}{}_{\mu\nu}\equiv 0,\end{aligned} \tag{11}$$

where $\Box$ is the D'Alembertian operator defined as $\Box=\nabla_\alpha\nabla^\alpha$, $\nabla_\alpha V^\beta$ is the covariant derivative of the vector V^β, $f_\mathcal{R}\equiv\frac{df(\mathcal{R})}{d\mathcal{R}}$, and the electromagnetic energy-momentum tensor $\mathfrak{T}^{nlem}{}_{\mu\nu}$ is given by (7). Additionally, taking the trace of (11) gives

$$\xi=\mathcal{R}f_\mathcal{R}-2f(\mathcal{R})-8\Lambda+3\Box f_\mathcal{R}-\mathfrak{T}^{nlem}, \tag{12}$$

with $\mathfrak{T}^{nlem}$ given by (8).

A. Static solutions

In order to extract black hole solutions we consider a spherically symmetric metric of the form

$$ds^2=H(r)dt^2-\frac{dr^2}{H(r)}-r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{13}$$

Thus, the corresponding Ricci scalar becomes

$$\mathcal{R} = \frac{2 - r^2 H'' - 4rH' - 2H}{r^2}, \tag{14}$$

where from now on primes denote derivatives with respect to r. Concerning the electromagnetic potential 1-form we consider the general ansatz [64]

$$V \coloneqq q(r)dt + n(\phi)dr + s(r)d\phi, \tag{15}$$

with $q(r)$,$s(r)$,$n(\phi)$ three free functions reproducing the electric and magnetic charges in the vector potential where $\mathcal{P} = dV$ and $V = V_\nu dx^\nu$.

In the following, without loss of generality, and just to provide an example of the method at hand, we focus on the square-root $f(\mathcal{R})$ correction to general relativity, where $f(\mathcal{R}) = \mathcal{R} - 2\alpha\sqrt{\mathcal{R}}$ [65,66]. Inserting the metric (13) and the potential (15) into the general field equations (11), (12), (6) we obtain the following nonvanishing field equations:

$$\begin{aligned}
\xi_t{}^t =& \frac{1}{4r^6\sqrt{R^5}}\{2Rr^6HH'''+3r^6HH'''^2+r^3H'''[r^2H''(12H-rH')-4r^2H'^2-2r(31H-1)H'+48H(1-H)]\\
&+2r^6H''^3+4r^4H''^2(6rH'+15H-4)+2r^2H''[57r^2H'^2+14rH'(57H-5)+4(5+4H-3H^2)]\\
&+200r^3H'^3+4r^2H'^2(96H-85)+8rH'(H-1)(27H-23)+32(H-1)^2(2H-1)\}\\
&-\frac{2r\sin^2\theta q'q''(2r^2\mathbb{H}-rH'+1-H)}{2r^3\sin^2\theta q'q''+(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]}\\
&-\frac{(s'-n_\phi)\{2rHs''(1-rH'-H-\mathbb{H}r^2)-(s'-n_\phi)[r^2H'^2-r(1+H+2r^2\mathbb{H})]+2H(H-1-r^3\mathbb{H}'-r^2\mathbb{H})\}}{2r^2[2rHs''(s'-n_\phi)+2r^3\sin^2\theta q'q''+(s'-n_\phi)^2(rH'-2H)]}=0,
\end{aligned} \tag{16}$$

$$\xi_t{}^\phi = \frac{4rH\mathbb{H}'q'(s'-n_\phi)}{2r^3\sin^2\theta q'q''+(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]} = 0, \tag{17}$$

$$\begin{aligned}
\xi_r{}^r =& \frac{1}{4r^4\sqrt{R^3}}\{4r^2\sqrt{R^3}(2r^2\mathbb{H}-rH'-H+1)+\alpha[r^3H'''(rH'+4H)-2r^4H''^2+4r^2H''[(3+H)-4rH']\\
&-50r^2H'^2+4rH'(15-17H)-16(1+2H^2-3H)]\}=0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\xi_\theta{}^\theta =& \frac{1}{4r^6\sqrt{R^5}\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}}\\
&\times\{r^5\sqrt{R^5}\{rH''\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}-\frac{1}{2}rH'^2(s'-n_\phi)^2-2r^4\mathbb{H}'q'^2\sin^2\theta\\
&-[q'q''r^3\sin^2\theta-2rHs''(s'-n_\phi)](2r\mathbb{H}-H')-(H+r^2\mathbb{H})[(s'-n_\phi)^2-2rH(\phi'^2-2s'n_\phi+s'^2)]\}\\
&-\alpha\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}\\
&\times\{2r^6RHH'''-3r^6HH'''^2+2r^3H'''[r^2H''(rH'-7H)+4r^2H'^2+2rH'(14H-1)-22H(1-H)]\\
&-4r^4[r^2H''^3+H''^2(18H+9rH'-5)]-4r^2H''[33r^2H'^2+rH'(27H-34)+10H+8-18H^2]\\
&-208r^3H'^3-4r^2H'^2(81H-74)+8r(15H-16)(1-H)+16(1-4H+5H^2-2H^3)\}\}\equiv 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\xi_\phi{}^\phi =& \frac{1}{4r^6\sqrt{R^5}\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}}\\
&\times\{2r^5\sqrt{R^5}\{rH''\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}-[rH'^2/2-(H+r^2\mathbb{H})](s'-n_\phi)^2\\
&-r[q'q''r^2\sin^2\theta+2Hs''(s'-n_\phi)](2r\mathbb{H}-H')-2r[H\mathbb{H}(\phi'^2-2s'n_\phi+s'^2)-r^3\mathbb{H}'q'^2\sin^2\theta]\}\\
&+\alpha\{2r^3\sin^2\theta q'q''-(s'-n_\phi)[2Hs''+(s'-n_\phi)(rH'-2H)]\}\\
&\times\{2r^6RHH''''+3r^6HH'''^2-2r^3H'''[r^2H''(rH'-7H)+4r^2H'^2+2rH'(14H-1)-22H(1-H)]\\
&+4r^4[r^2H''^3+H''^2(18H+9rH'-5)]+4r^2H''[33r^2H'^2+rH'(27H-34)+10H+8-18H^2]\\
&+208r^3H'^3+4r^2H'^2(81H-74)-8rH'(15H-16)(1-H)-16(1-4H+5H^2-2H^3)\}\}\equiv 0,
\end{aligned} \tag{20}$$

$$\xi_\phi{}^t = \frac{4r^3\mathbb{H}'\sin^2\theta q'(s'-n_\phi)}{2rHs''(s'-n_\phi) - 2r^3\sin^2\theta q'q'' + (s'-n_\phi)^2(rH'-2H)}. \tag{21}$$

Finally, the trace equation (12) becomes

$$\begin{aligned}\xi = {} & \frac{1}{2r^6\sqrt{R^5}\{2r^3\sin^2\theta q'q'' - (s'-n_\phi)[2Hs'' + (s'-n_\phi)(rH'-2H)]\}} \\ & \times \{r^4\sqrt{R^5}\{r^2H''\{2r^3\sin^2\theta q'q'' - (s'-n_\phi)[2rHs'' + (s'-n_\phi)(rH'-2H)]\} + 4r^5\mathbb{H}'q'^2\sin^2\theta \\ & - r(1+4r^2\mathbb{H} - 2rH' - H)[2q'q''r^2\sin^2\theta - 2Hs''(s'-n_\phi)] - [2r^2H'^2 - r(3H+4r^2\mathbb{H}+1)](s'-n_\phi)^2 \\ & - 2H[\phi'^2 - 2s'n_\phi + s'^2](1+2r^3\mathbb{H}' - H + 4r^2H)\} \\ & + \alpha\{2r^3\sin^2\theta q'q'' - (s'-n_\phi)[2Hs'' + (s'-n_\phi)(rH'-2H)]\} \\ & \times \{2r^6RHH'''' + 3r^6HH'''^2 - 2r^3H'''[r^2H''(rH'-6H) + 4r^2H'^2 + 2rH'(16H-1) - 24H(1-H)] \\ & + 4r^6H''^3 + 2r^4H''^2(3-17H-5rH') - 4r^2H''[41r^2H'^2 - 2rH'(16H'-23) + 4(H+3-4H^2)] \\ & - 136r^3H'^3 + 2r^2H'^2(106-117H) + 8rH'(15H-13)(1-H) - 16(2H-1)(1-H)^2\}\} \equiv 0. \end{aligned} \tag{22}$$

From Eqs. (17) and (21) we acquire

$$s(r) = c_4 r, \qquad n(\phi) = c_4\phi. \tag{23}$$

Inserting (23) into Eqs. (19) and (20) it is easy to show that two of the other equations coincide, namely $\xi_\theta{}^\theta = \xi_\phi{}^\phi$. Therefore, the system of differential equations (16), (18), (19), and (20) reduces to three differential equations of three unknowns, $H(r)$, $\mathbb{H}$ and, $q(r)$, which can be solved to give the following analytical solutions:

$$\begin{aligned} -H(r) &= \frac{c}{2} + \frac{c_1}{r} + \frac{c_2}{r^2}, \qquad q(r) = \frac{c_3}{r}, \\ \mathbb{H}(r) &= \frac{\alpha[3c_2 + 4(c-1)r]r^2 + \sqrt{2-c}[(c-2)r^2 - 2c_1]}{4\sqrt{2-c}r^4}, \end{aligned} \tag{24}$$

$$\begin{aligned} -H(r) &= \frac{c}{2} - \frac{1}{3\alpha r} - \frac{1}{3\alpha r^2}, \qquad q(r) = \frac{c_3}{r}, \\ \mathbb{H}(r) &= \frac{\alpha[-3 + 12(c-1)r]r^2 + \sqrt{2-c}[3\alpha(c-2)r^2 + 2]}{12\sqrt{2-c}r^4}. \end{aligned} \tag{25}$$

We stress that we adjust the constants c_1 and c_2 so that solutions (24) and (25) satisfy the trace Eq. (22), too, and hence the whole solution structure is consistent. Additionally, concerning the parameter c we deduce that it must be non-negative in order to maintain the metric signature and also the value 0 is excluded too in order to obtain asymptotic flat spacetime at $r \to \infty$. If $\alpha = 0$ then solution (24) holds for any $0 < c$ while solution (25) does not exist; nevertheless if $\alpha \neq 0$ then we should restrict c to $0 < c < 2$ in both solutions in order to acquire real $\mathbb{H}(r)$.

Concerning the function $\mathcal{P}$ we find $\mathcal{P}(r) = \frac{c_3^2}{2r^4}$. Hence, knowing $\mathbb{H}(r)$ we can find that

$$\begin{aligned} \mathbb{H}(\mathcal{P}) = {} & \frac{\mathcal{P}}{4c_3^2}\{-4c_1 + \sqrt{2}(c-2)c_3\mathcal{P}^{-1/2} \\ & + \alpha c_3[\mathcal{P}(1-c/2)]^{-1/2}[3c_2 + 2^{7/4}(c-1)\sqrt{c_3}\mathcal{P}^{-1/4}]\}. \end{aligned} \tag{26}$$

Therefore, knowing from (1) that $\mathbb{H} = 2\mathcal{F}\mathcal{L}_\mathcal{F} - \mathcal{L}$ and from (3) that $\mathcal{P} = \mathcal{L}_\mathcal{F}^2\mathcal{F}$, we can rewrite (26) as

$$\begin{aligned} 2\mathcal{F}\mathcal{L}_\mathcal{F} - \mathcal{L} = {} & \frac{\mathcal{L}_\mathcal{F}^2\mathcal{F}}{4c_3^2}\{-4c_1 + \sqrt{2}(c-2)c_3\mathcal{L}_\mathcal{F}^{-1}\mathcal{F}^{-1/2} \\ & + \alpha c_3\mathcal{L}_\mathcal{F}^{-1}[\mathcal{F}(1-c/2)]^{-1/2} \\ & \times [3c_2 + 2^{7/4}(c-1)\sqrt{c_3}\mathcal{L}_\mathcal{F}^{-1/2}\mathcal{F}^{-1/4}]\}, \end{aligned} \tag{27}$$

which is a differential equation for $\mathcal{L}(\mathcal{F})$.

According to the value of c the solution of the above differential equation will give a corresponding correction to the standard electromagnetic Lagrangian. Since for our novel solution we have $0 < c < 2$, in the rest of the work we focus on the case $c = 1$, since this leads to the simple solution (but still a novel solution comparing to general relativity)

$$\mathcal{L}(\mathcal{F}) = \frac{c_3^2}{c_1}\mathcal{F} + \frac{c_3}{\sqrt{c_1}}\sqrt{\mathcal{F}}c_4 + c_1c_4^2. \tag{28}$$

As one can see, the first term is standard linear electromagnetism, while the second term is the nonlinear correction of a square-root form. In the general c case the correction terms take more complicated forms.

Let us analyze the properties of the obtained spherically symmetric solutions (24), (25). These can be rewritten in the standard form as

$$ds_1{}^2 = -\left(\frac{c}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \frac{dr^2}{(\frac{c}{2} - \frac{2M}{r} + \frac{q^2}{r^2})} + r^2 d\Omega^2, \quad \text{where } M = -\frac{c_1}{2}, \qquad q = \sqrt{c_2}, \tag{29}$$

for (24), and

$$ds_2{}^2 = -\left(\frac{c}{2} - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \frac{dr^2}{(\frac{c}{2} - \frac{2M}{r} + \frac{q^2}{r^2})} + r^2 d\Omega^2, \quad \text{where } M = \frac{1}{6\alpha}, \qquad q = \frac{1}{\sqrt{6\alpha}}, \tag{30}$$

for (25). The first solution branch includes the GR solution in the limit $\alpha \to 0$ and $c \to 2$ (solution (24) is the generalization of those obtained in [67] in the static and noncharged case; see also [37,68–70]). On the other hand, the second branch exists only in the case $\alpha \neq 0$, and thus it is a novel solution that arises from the $f(\mathcal{R})$ gravitational modification as well as from the electrodynamic nonlinearity. Hence, the two solutions, although looking similar, they are fundamentally different, and the fact that the mass of solution (30) depends only on $1/\alpha$ is a reflection of the novelty of the solution (such a connection between the gravitational modification parameters with the black hole quantities, in specific exact solutions, is known to be the case in many modified gravity theories). In this work we are interested in solution (30), i.e., (25), exactly because it is a novel one with no general relativity limit.

In order to investigate the horizons and singularities of the above solutions we calculate the Kretschmann, the Ricci tensor square, and the Ricci invariants. For (24) we find

$$\begin{aligned}\mathcal{R}^{\mu\nu\lambda\rho}\mathcal{R}_{\mu\nu\lambda\rho} &= r^{-8}\{8c_1(7c_1 - r^2) + 4c_1 r(12c_2 + cr) \\ &\quad + cr^4(c-4) + 4c_2 r^2(3c_2 - 2r) + 4r^3(r + c_2 c)\}, \\ \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} &= \frac{8c_1(c_1 + r^2) + cr^2(cr^2 - 4c_1) + 4r^4(1-c)}{18\alpha^2 r^8}, \\ \mathcal{R} &= \frac{2-c}{r^2},\end{aligned} \tag{31}$$

while for (25) we acquire

$$\begin{aligned}-\mathcal{R}^{\mu\nu\lambda\rho}\mathcal{R}_{\mu\nu\lambda\rho} &= (9\alpha^2 r^8)^{-1}\{56 + 9r^4\alpha^2[c-2]^2 \\ &\quad - 12\alpha r^3(c-2) - 12r^2[\alpha(c-2) - 1] + 48r\}, \\ \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} &= \frac{9r^4\alpha^2(c-2)^2 + 12\alpha r^2(c-2) + 8}{18\alpha^2 r^8}, \\ \mathcal{R} &= \frac{2-c}{r^2}.\end{aligned} \tag{32}$$

Expressions (31), (32) reveal that the spherically symmetric solutions exhibit a true singularity at $r = 0$. Although in the GR case this singularity is always hidden by a horizon, when the $f(\mathcal{R})$ correction is switched on this is not always the case; namely, a naked singularity may appear. This issue will be investigated in the next section.

Lastly, concerning the electric and magnetic charges, expressions (9) give

$$\begin{aligned}E &= \frac{[(c-2)r^2 - 8m]\sqrt{2-c} + r^2\alpha[(c-1)r + 3q^2]}{4c_3 r\sqrt{2-c}}, \\ B_\phi &= \frac{c_4 r([2(c-2)r^2 + 16m]\sqrt{2-c} + 3r^2\alpha[(c-1) + 2q^2])}{24c_3{}^2\sqrt{2-c}}, \\ B_r &= \frac{c_4\phi([(c-2)r^2 + 8m]\sqrt{2-c} + r^2\alpha[2(c-1) + 3q^2])}{4c_3{}^2\sqrt{2-c}},\end{aligned} \tag{33}$$

while $B_\theta = 0$. Equation (33) shows in a clear way that when the constant $c_4 = 0$ we have no magnetic fields; namely, the magnetic fields are related to the integration constant c_4.

B. Rotating solutions

In this subsection we derive rotating solutions that satisfy the field equations (11), (12), and (6). In order to achieve this we apply the following transformation [71,72]:

$$\begin{aligned}\bar{\phi} &= \Xi\phi + \omega t, \\ \bar{t} &= \Xi t + \omega\phi,\end{aligned} \tag{34}$$

with ω being the rotation parameter and $\Xi = \sqrt{1+\omega^2}$. Applying (34) to the metric (13) we obtain

$$\begin{aligned}ds^2 &= [\Xi^2 H(r) - \omega^2 r^2 \sin^2\theta]dt^2 - \frac{dr^2}{H(r)} - r^2 d\theta^2 \\ &\quad - [\Xi^2 r^2 \sin^2\theta - \omega^2 H]d\phi^2 + 2\omega\Xi[H - r^2\sin^2\theta]dt d\phi,\end{aligned} \tag{35}$$

where $H(r)$ is given by the previously extracted static solutions (24), (25), and $0 \leq r < \infty$, $-\infty < t < \infty$, $0 \leq \phi < 2\pi$. We mention that the static configuration (13) can be recovered as a special case of the above general metric, if the rotation parameter ω is set to zero. Hence, for the general gauge potential (15) we acquire the form

$$\bar{V} = [\Xi q(r) + \omega s(r)]d\bar{t} + n(\phi)dr + [\omega q(r) + \Xi s(r)]d\bar{\phi}. \tag{36}$$

Note here that although the transformation (34) leaves the local properties of spacetime unaltered, it does change them globally as has been shown in [71], since it mixes compact and noncompact coordinates. Thus, the two metrics (13) and (35) can be locally mapped into each other but not globally [71,72].

To conclude, we have succeeded in deriving new rotating charged black hole solutions in $f(\mathcal{R})$ gravity, using as a specific example the case $f(\mathcal{R}) = \mathcal{R} - 2\alpha\sqrt{\mathcal{R}}$. Similarly to the static case, these belong to two branches, one that contains the Kerr-Newman metric, namely the rotating charged black hole solution of general relativity, as a particular limit [the one arising inserting (24) into (35)] and one that arises purely from the gravitational modification and does not recover the general relativity solution [the one arising inserting (25) into (35)]. Concerning the singularity properties, as is clear from (35), these will be the same with the static solution (13). Therefore, at $r = 0$ we obtain a true singularity, and close to $r = 0$ the behavior of the invariants is $(K, R_{\mu\nu}R^{\mu\nu}) \sim r^{-8}$ and $(R) \sim r^{-2}$.

IV. THERMODYNAMICS

In this section we focus on the investigation of the thermodynamic properties of the obtained black hole solutions. Since solution (24) contains the general relativity result, in the following analysis we focus on the novel solution (25) that arises solely from the gravitational modification [64,66,73].

We start by introducing the Hawking temperature as [74–77]

$$T_h = \frac{H'(r_h)}{4\pi}, \tag{37}$$

where the event horizon is located at $r = r_h$ which represents the largest positive root of $H(r_h) = 0$ that satisfies $H'(r_h) \neq 0$. The Bekenstein-Hawking entropy of $f(\mathcal{R})$ gravitational theory is given by [78,79]

$$S(r_h) = \frac{1}{4} A f_R(r_h), \tag{38}$$

with A being the area of the event horizon. Additionally, the quasilocal energy in $f(\mathcal{R})$ gravity is defined as [78,79]

$$E(r_h) = \frac{1}{4}\int dr_h [r_h{}^2\{f(R(r_h)) - R(r_h)f_R(r_h)\} + 2f_R(r_h)]. \tag{39}$$

Finally, we can express the black hole mass as a function of the horizon r_h and the charge q, which for the case (25), (30) becomes

$$M_h = \frac{r_h}{2}\left[\frac{c}{2} + \frac{q^2}{r_h{}^2}\right]. \tag{40}$$

The relation between the metric function $H(r)$ and the radial coordinate r is presented in Fig. 1, which shows the possible horizons of the solution. Note that since, in this work for simplicity, we are using natural units; in order to be closer to physical cases we should have taken much larger values of M and q, and then the radial distance would take much larger values too while α would take much smaller values. However, since in mathematical terms the physical properties of the solutions do not depend on the scale, and in order to avoid graphs with very large/small numbers, we prefer to remain in these representative numbers of order one since they are adequate in order to provide the physical features of the solution.

Moreover, the relation between M_h and the horizon radius is depicted in Fig. 2. As one can see, there is a limiting horizon radius after which there is no horizon and the black hole singularity will be a naked singularity. This "degenerate horizon" value r_{dg} can thus be calculated by the condition $\frac{\partial M_h}{\partial r_h} = 0$, which for solution (25), (30) yields

$$r_{dg} = \frac{\sqrt{2}q}{\sqrt{c}}.$$

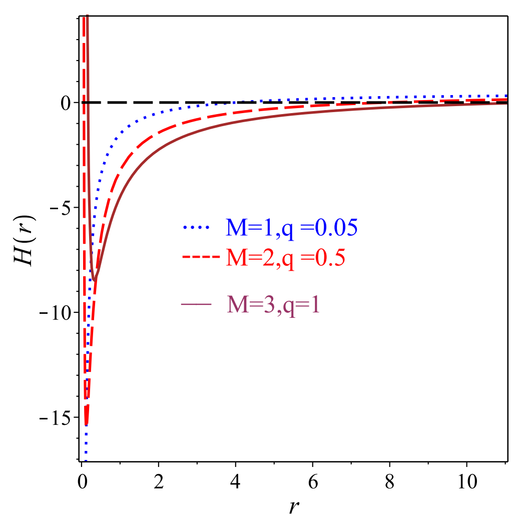

FIG. 1. The metric function $H(r)$ vs the radial coordinate r, for solution (25) with $c = 1$, for various mass and charge choices. The black hole horizon is determined by the condition $H(r) = 0$. The "degenerate horizon" r_{dg} marks the limiting value after which there is no horizon and the central singularity becomes a naked one (see text).

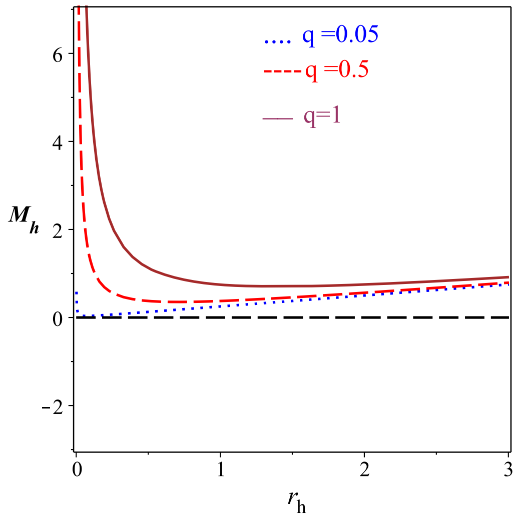

FIG. 2. The black hole mass as a function of the horizon r_h, for solution (25) with $c = 1$, for various charge choices.

Hence, as we can see, the cosmic censorship theorem can be violated in nonlinear Maxwell $f(\mathcal{R})$ gravity.

Concerning the Hawking temperature (37), calculating it using the black hole solution (25) we find

$$T_h = \frac{c r_h^2 - 2q^2}{8\pi r_h^3}. \tag{41}$$

We mention that T_h does not depend directly on the gravitational modification parameter α (although it indirectly does since the latter affects the horizon). In Fig. 3 we depict the temperature behavior as a function of the horizon. As we observe, for suitable parameter values this can be negative, and, according to (41), this happens when $c < \frac{2q^2}{r_h^2}$. This implies a formation of an ultracold black hole [80,81], which reveals the capabilities of the scenario at hand.

As a next step, using expression (38) the entropy of the black hole (25) is calculated as

$$S_h = \frac{\pi r_h^2(\sqrt{2-c} - \alpha r_h)}{\sqrt{2-c}}. \tag{42}$$

In Fig. 4 we show the behavior of the entropy as a function of the horizon. Hence, by imposing the entropy positivity condition we obtain

$$\alpha < \frac{\sqrt{c-2}}{r_h}.$$

This is one of the main results of the present work, and shows that the gravitational correction of $f(\mathcal{R})$ gravity must be suitably small in order to avoid nonphysical black hole properties (see also the discussion in [82–87] for the entropy negativity in various theories of modified gravity).

Similarly, using expression (39) we find the quasilocal energy of the black hole (25) as

$$E_h = \frac{r_h(4\sqrt{2-c} + 4 r_h \alpha - 3 r_h c \alpha)}{8\sqrt{2-c}}. \tag{43}$$

From (43) we conclude that in order to have a positive value of the quasilocal energy we must have

$$c > \frac{4}{3} \quad \text{and} \quad \alpha < \frac{4\sqrt{2-c}}{r_h(3c-4)} \tag{44}$$

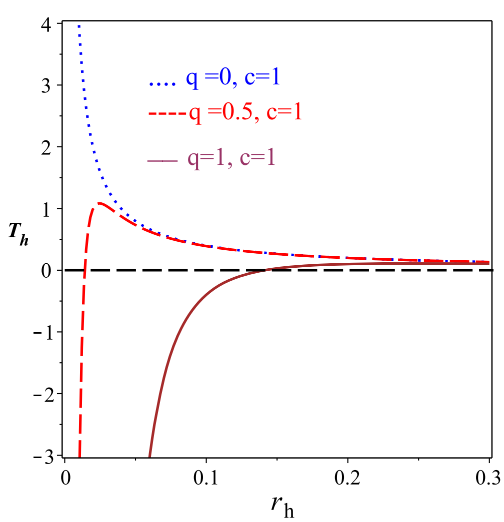

FIG. 3. The black hole temperature (41) as a function of the horizon r_h, for solution (25), for solution (25) with $c = 1$, for various charge choices.

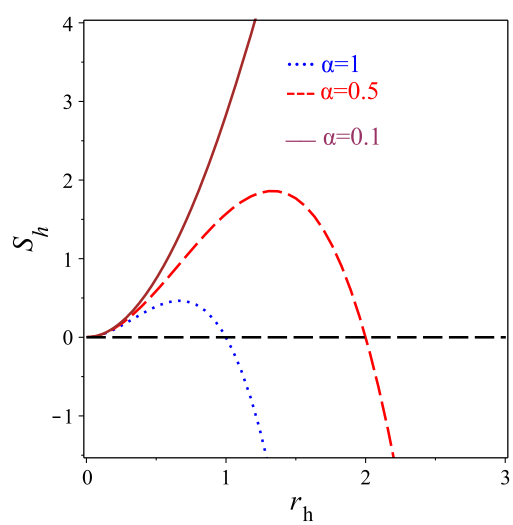

FIG. 4. The black hole entropy (42) as a function of the horizon r_h, for solution (25), for various choices of the gravitational modification parameter α.

or

$$c < \frac{4}{3} \quad \text{and} \quad \alpha > \frac{4\sqrt{2-c}}{r_h(3c-4)}. \tag{45}$$

We continue by examining the black hole thermodynamical stability. As it is known, in order to analyze it one has to examine the sign of its heat capacity C_h, given as [88–90]:

$$C_h = \frac{dE_h}{dT_h} = \frac{\partial M_h}{\partial r_h}\left(\frac{\partial T_h}{\partial r_h}\right)^{-1}, \tag{46}$$

where E_h is the energy. If the heat capacity $C_h > 0$ then the black hole is thermodynamically stable; i.e., a black hole with a negative heat capacity is thermally unstable. Concerning the heat capacity of the black hole solution (25), using Eq. (46) we acquire

$$C_h = \frac{2\pi r_h{}^2(2q^2 - c r_h{}^2)}{c r_h{}^2 - 6q^2}. \tag{47}$$

We mention that C_h does not depend directly on the gravitational modification parameter α, but only indirectly through the effect of α on the horizon. This expression implies that in order to obtain a positive heat capacity we must have

$$q > \pm 0.5 r\sqrt{c}. \tag{48}$$

In Fig. 5 we depict C_h as a function of the horizon, where we observe that if q satisfies the above inequality then stability is obtained. We mention here that a negative heat capacity is associated with a negative temperature, which corresponds to $r_h < r_{dg}$. At $r_h = r_{dg}$ both the temperature and the heat capacity are exactly zero on the black hole horizon. When $r_h > r_{dg}$, both temperature and heat capacity are positive and the solution is in thermal equilibrium. Indeed, the thermodynamical stability of charged black holes has been widely studied in various modified gravity theories, e.g., the thermodynamics of Bardeen (regular) black holes [91], of Schwarzschild-AdS solutions in two vacuum scales case [92], of solutions in noncommutative geometry [93–96], etc.

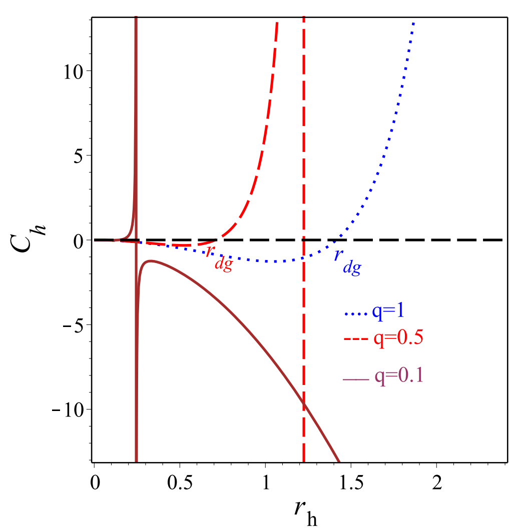

FIG. 5. The heat capacity (47) as a function of the horizon r_h, for solution (25) with $c = 1$, for various charge choices.

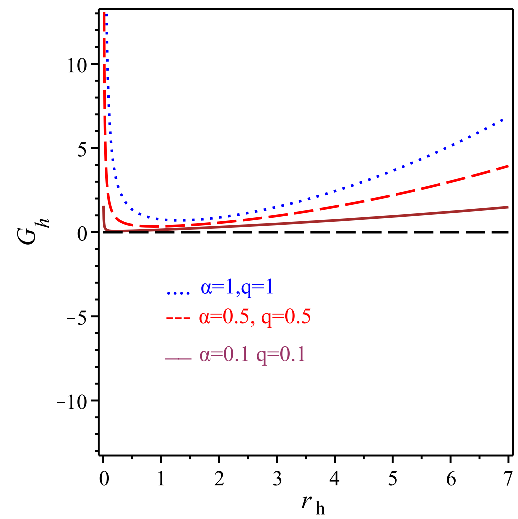

FIG. 6. The black hole Gibbs free energy (50) as a function of the horizon r_h, for solution (25) with $c = 1$, for various choices of charge and gravitational modification.

Finally, let us make some comments on the Gibbs free energy, namely the free energy in the grand canonical ensemble, defined as [79,97]

$$G(r_h) = E(r_h) - T(r_h)S(r_h). \tag{49}$$

Inserting (41), (42), and (43) into (49) we find

$$G_h = \frac{(6q^2 + c r_h^2)\sqrt{2-c} + \alpha r_h(r_h^2 - 2q^2)}{8 r_h\sqrt{2-c}}. \tag{50}$$

The behavior of the Gibb's energy of the black holes (25) is presented in Fig. 6 for particular values of the model parameters. As we can see it is always positive when $\alpha > 0$ which implies that it is more globally stable.

V. DISCUSSION AND CONCLUSION

The radical advance in multimessenger astronomy opens the possibility to test general relativity and investigate modified gravity by the gravitational and electromagnetic waves profile that arise from mergers of spherically symmetric objects, such as black holes and neutron stars. Hence, it is crucial to study such object's properties in various theories of modified gravity in the presence of the Maxwell sector.

In this work we investigated static and rotating charged spherically symmetric solution in the framework of $f(\mathcal{R})$ gravity, allowing additionally the electromagnetic sector to depart from linearity. Applying a convenient, dual description for the electromagnetic Lagrangian, and using as an example the square-root $f(\mathcal{R})$ correction, we were able to solve analytically the involved field equations. The obtained solutions belong to two branches: One that contains the Kerr-Newman metric, namely the rotating charged black hole solution of general relativity, as a particular limit and one that arises purely from the gravitational modification and does not recover the general relativity solution. Moreover, we have shown that the two components of the magnetic fields, of the nonlinear electrodynamics, are connected by a constant which if it is vanished we acquire a charged black hole with electric field only [66].

Analyzing the novel black hole solution that does not have a general relativity limit we found that it has a true central singularity which is hidden behind a horizon; however, for particular parameter regions the horizon disappears and the singularity becomes a naked one; i.e., we obtain a violation of the cosmic censorship theorem.

Furthermore, we investigated the thermodynamical properties of the solutions, such as the temperature, energy, entropy, heat capacity, and Gibbs free energy. We extracted the conditions on the gravitational modification parameter in order to obtain entropy and quasilocal energy positivity. Concerning temperature, we showed that it can become negative for particular parameter values, and thus ultracold black holes may be formed. Finally, we examined the thermodynamic stability of the solutions by examining the sign of the heat capacity, extracting the corresponding conditions.

In summary, we showed that even small deviations from general relativity and/or from linear electrodynamics may lead to novel spherically symmetric solution branches, with novel properties that do not appear in standard general relativity. Since these properties may be embedded in the gravitational waves profiles, they could serve as a smoking gun of this subclass of gravitational modification.

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