Canonical loop quantization of the lowest-order projectable Horava gravity

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The Hamiltonian formulation of the lowest-order projectable Horava gravity, namely the so-called λ-R gravity, is studied. Since a preferred foliation has been chosen in projectable Horava gravity, there is no local Hamiltonian constraint in the theory. In contrast to general relativity, the constraint algebra of λ-R gravity forms a Lie algebra. By canonical transformations, we further obtain the connection-dynamical formalism of the λ-R gravity theories with real $su(2)$ connections as configuration variables. This formalism enables us to extend the scheme of nonperturbative loop quantum gravity to the λ-R gravity. While the quantum kinematical framework is the same as that for general relativity, the Hamiltonian constraint operator of loop quantum λ-R gravity can be well defined in the diffeomorphism-invariant Hilbert space. Moreover, by introducing a global dust degree of freedom to represent a dynamical time, a physical Hamiltonian operator with respect to the dust can be defined and the physical states satisfying all the constraints are obtained.

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I. INTRODUCTION

It is well known that all the fundamental interactions of Nature, except for gravity, can be described in the framework of quantum field theory (QFT). Since gravity is universally coupled to all the matter fields, the quantum nature of matter field implies that gravity should be also quantized. In addition, around the singularities of the big bang and black holes interior, the space-time curvature becomes divergent. Hence it is generally expected that general relativity (GR), as a classical theory, is no longer valid there, and quantum physics should be taken into account. If a quantum theory of gravity could be available, the singularities would be smoothed out by a certain physically meaningful quantum description. Motivated by the above considerations, to realize the quantization of gravity serves as one of the main driving forces in theoretical physics in the past decades [1], and various approaches have been pursued, including string/M-theory [2] and loop quantum gravity (LQG) [3–6].

As a background-independent approach to quantize GR, LQG has been widely investigated in the past 30 years [3–6]. It is remarkable that, as a nonrenormalizable theory, GR can be nonperturbatively quantized by the loop quantization procedure. This background-independent quantization method relies on the key observation that classical GR can be cast into the connection-dynamical formalism with the structure group of $SU(2)$. The LQG quantization method has been successfully generalized to $f(R)$ gravity [7,8], scalar-tensor gravity [9], and Weyl gravity [10].

The notion of time plays an important role in any quantum gravity theories and on how to implement particular proposals in technical terms [11]. In the Hamiltonian framework of GR, one assumes that a Lorentzian spacetime M is diffeomorphic to a product $M = \mathbb{R} \otimes \Sigma$ with Σ being a smooth spacelike hypersurface, and $\mathbb{R}$ being a preferred time direction following from the usual requirement of global hyperbolicity, which ensures that the causal structure of spacetime is sufficiently well behaved. The spacetime diffeomorphism invariance of GR in restored by the diffeomorphism and Hamiltonian constraints in the Hamiltonian framework. Thus, different choices of foliation can be considered as a part of the gauge freedom of GR.

As a different kind of gravity theories, the so-called Hořava-Lifshitz gravity was proposed [12], associated with a preferred foliation of spacetime. As a consequence, these theories are only invariant under a subset of spacetime diffeomorphisms, namely those that do not change the preferred foliation. The remaining invariant group consists of three-dimensional diffeomorphisms acting independently on each leaf Σ_t (labeled by time t) and space-independent time reparametrizations. The most general action of the metric fields which is at most quadratic in derivatives and invariant under this reduced symmetry group is not the concise Einstein-Hilbert action, but in a rather complicated form [12].

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By giving up the space-time covariance, Hořava-Lifshitz gravity becomes renormalizable in QFT perturbative quantization [13–15]. However, from the nonperturbative viewpoint, the LQG quantization method has not been extended to these theories. It is well known that the loop quantization highly relies on the connection-dynamical formalism of the corresponding gravity theories, while the connection-dynamical formalism of the Hořava-Lifshitz gravity is still absent. Note that due to the extremely complicated form of Hořava-Lifshitz gravity theories, one usually performs the quantization procedures in some simpler cases, for example, in lower dimensions [14,15] or in the symmetry-reduced case such as the cosmological situations [16].

The low energy limit of Hořava-Lifshitz gravity, which is suitable for most astrophysical objects as well as cosmological applications [17,18], can be described by the following action:

$$S = \frac{1-\beta}{16\pi G}\int dt \int_\Sigma d^3x N\sqrt{q}\left(K_{ab}K^{ab} - \frac{1+\nu}{1-\beta}K^2 + \frac{1}{1-\beta}R + \frac{\sigma}{1-\beta}a_i a^i\right), \tag{1.1}$$

where G is the gravitational constant, K_{ab} is the extrinsic curvature of a spatial hypersurface Σ, $K \equiv K_{ab}q^{ab}$, R denotes the scalar curvature of the 3-metric q_{ab} induced on Σ, $a_i = \partial_i(\ln N)$, β, σ and ν are coupling constants. The coupling constants must satisfy a series of theoretical requirements, such as the absence of gradient instabilities and ghosts [19–21], as well as experimental constraints, including the absence of vacuum Cherenkov radiation [22], solar system experiments [23,24], gravitational wave propagation bounds from GW170817 [25,26], and cosmological constraints [27–29]. Those constraints suggest that β and σ are vanishingly small as $\beta \le 10^{-15}$ and $\sigma \le 10^{-7}$. However the other coupling constant ν is relatively unconstrained aside from the stability requirements and cosmological bounds [26,29,30] such that $0 \leqq \nu \lesssim 0.01$–$0.1$. Therefore, in this paper, we are going to quantize the four-dimensional simpler model of gravity by setting $\beta = \sigma = 0$ [26,31], which is the lowest-order Horava gravity. This theory is sometimes called the λ-R gravity model [32–35]. Thus the action of λ-R gravity reads [32–35]

$$S = \frac{1}{16\pi G}\int dt \int_\Sigma d^3x\sqrt{q}N(K_{ab}K^{ab} - \lambda K^2 + R) \equiv \int d^4x\mathcal{L} \tag{1.2}$$

with the coupling parameter $\lambda \equiv 1+\nu$. This theory serves as the minimal generalization of GR, since action (1.2) reduces to Einstein-Hilbert action by setting $\lambda = 1$. It was first proposed and investigated in a purely classical context in Ref. [32]. Though it is simpler, the λ-R gravity theory shares the same kinetic term and the symmetry of the Hořava-Lifshitz gravity. It has been shown in Refs. [33–35], that the nonprojectable λ-R gravity models are equivalent to GR in the asymptotically flat case, while the projectable sector of λ-R gravity is inequivalent to GR. More precisely, by choosing a preferred foliation the usual local Hamiltonian constraint of GR was removed. As shown in Refs. [36,37], the absence of the local constraint leads to an additional strongly coupled scalar degree of freedom, which becomes dynamical here. Then the coupling of λ-R gravity to matter would suggest a universal scalar (fifth) force in nature, which has not been seen. Nevertheless, the projectable theory provides a practicable model to test the scheme of LQG. Thus, we will focus on the projectable model of λ-R gravity, where the lapse function N is only a function of time t [34,35].

This paper is organized as follows: We will present a detailed Hamiltonian analysis of λ-R gravity to obtain its connection-dynamical formalism in Sec. II. Then in Sec. III, the λ-R gravity will be nonperturbatively quantized by the LQG method based on the connection dynamics, and the quantum Hamiltonian constraint operator for λ-R gravity will be constructed. In Sec. IV, the nonrotational dust field will be introduced to represent a dynamical time and the physical Hamiltonian operator will be defined so that the physical states can be obtained. Our result will be summarized in the last section. Throughout the paper, we use Latin alphabet $a, b, c, \cdots$ for spatial indices, and $i, j, k, \cdots$ for internal indices, and set $8\pi G = 1$ for simplicity.

II. HAMILTONIAN ANALYSIS

Starting from action (1.2), by Legendre transformation, the momentum conjugate to the dynamical variable q_{ab} reads

$$p^{ab} = \frac{\partial\mathcal{L}}{\partial\dot{q}_{ab}} = \frac{N\sqrt{q}}{2}(K^{ab} - \lambda K q^{ab}). \tag{2.1}$$

The Hamiltonian of λ-R gravity can be derived as a liner combination of constraints [33,34,36],

$$H_{\text{total}} = \int_\Sigma d^3x(N^a C_a + NC), \tag{2.2}$$

where the shift vector N^a is a vector-valued function on Σ, N is a constant in every spatial slice. The smeared diffeomorphism and Hamiltonian constraints read respectively

$$C(\vec{N}) = \int_\Sigma d^3x N^a C_a \equiv \int_\Sigma d^3x N^a(-2D^b(p_{ab})), \tag{2.3}$$

$$\tilde{C}_0 = \int_\Sigma d^3x C \equiv \int_\Sigma d^3x\left(\frac{2}{\sqrt{q}}\left(p_{ab}p^{ab} - \frac{\lambda}{3\lambda-1}p^2\right) - \frac{1}{2}\sqrt{q}R\right) \tag{2.4}$$

where we fix $N = 1$ from now on. Note that the Hamiltonian constraint $\tilde{C}_0$ is a global constraint rather than a local one, which does not generate local gauge transformations. The symplectic structure is given by the following nontrivial Poisson bracket between the canonical variables,

$$\{q_{ab}(x), p^{cd}(y)\} = \delta_a^{(c}\delta_b^{d)}\delta^3(x, y). \tag{2.5}$$

Straightforward calculations show that the constraints (2.3) and (2.4) comprise a first-class system as [36]

$$\{C(\vec{N}), C(\vec{N}')\} = C([\vec{N}, \vec{N}']), \tag{2.6}$$

$$\{\tilde{C}_0, C(\vec{N})\} = 0, \tag{2.7}$$

$$\{\tilde{C}_0, \tilde{C}_0\} = 0. \tag{2.8}$$

This constraint algebra has the nice property of a Lie algebra, and the diffeomorphism constraints also nicely form an ideal. This implies that in the canonical quantization it is possible to define the Hamiltonian constraint operator directly on the diffeomorphism invariant Hilbert space.

To set up the classical foundation of loop quantization, we can employ the canonical transformation technique for metric theories of gravity to obtain the connection dynamical formalism of λ-R gravity. Let

$$\tilde{K}^{ab} = K^{ab} - \frac{1-\lambda}{2} K q^{ab}. \tag{2.9}$$

Then the conjugate momentum p^{ab} of q_{ab} could be rewritten as

$$p^{ab} = \frac{\sqrt{q}}{2}(\tilde{K}^{ab} - \tilde{K} q^{ab}). \tag{2.10}$$

We define the new geometric variables through

$$E_i^a = \sqrt{q} e_i^a, \qquad \tilde{K}_i^a \equiv \tilde{K}^{ab} e_b^j \delta_{ij}, \tag{2.11}$$

where e_i^a is the triad on Σ such that $q_{ab} e_i^a e_j^b = \delta_{ij}$. Now we extend the phase space of the theory to the space consisting of pairs $(E_i^a, \tilde{K}_a^i)$. It is then easy to see that the symplectic structure (2.5) can be derived from the following Poisson brackets:

$$\{\tilde{K}_a^j(x), E_k^b(y)\} = -\delta_a^b \delta_k^j \delta^3(x, y), \tag{2.12}$$

$$\{E_j^a(x), E_k^b(y)\} = 0, \tag{2.13}$$

$$\{\tilde{K}_a^j(x), \tilde{K}_b^k(y)\} = 0. \tag{2.14}$$

Thus there is a direct symplectic reduction from the extended phase space to the original one. In this sense the transformation from conjugate pairs (q_{ab}, p^{cd}) to $(E_i^a, \tilde{K}_b^j)$ is canonical. Note that the symmetry of $\tilde{K}^{ab}$, i.e., $\tilde{K}^{ab} = \tilde{K}^{ba}$, gives rise to an additional constraint in the extend phase space as

$$G_{jk} \equiv \tilde{K}_{a[j} E_{k]}^a = 0. \tag{2.15}$$

So we can make a second canonical transformation by defining [4,6]:

$$A_a^i = \Gamma_a^i + \gamma \tilde{K}_a^i, \tag{2.16}$$

where Γ_a^i is the spin connection determined by the densitized triad E_i^a, and γ is a nonzero real number which is usually called the Barbero-Immirzi parameter in the community of LQG [38]. It is clear that our new variable A_a^i coincides with the Ashtekar-Barbero connection of GR [38,39] when $\lambda = 1$. Therefore our new variable A_a^i serves as an extension of the Ashtekar-Barbero connection for λ-R gravity. The Poisson brackets among the new variables read

$$\{A_a^j(x), E_k^b(y)\} = \gamma \delta_a^b \delta_k^j \delta(x, y), \tag{2.17}$$

$$\{A_a^i(x), A_b^j(y)\} = 0, \tag{2.18}$$

$$\{E_j^a(x), E_k^b(y)\} = 0. \tag{2.19}$$

Now, the phase space of λ-R gravity consists of conjugate pairs (A_a^i, E_j^b). Combining Eq. (2.15) with the compatibility condition,

$$\partial_a E_i^a + \epsilon_{ijk} \Gamma_a^j E^{ak} = 0, \tag{2.20}$$

we obtain the standard Gaussian constraint

$$\mathcal{G}_i = \mathcal{D}_a E_i^a \equiv \partial_a E_i^a + \epsilon_{ijk} A_a^j E^{ak}, \tag{2.21}$$

which justifies A_a^i as an $su(2)$ connection. Note that, had we let $\gamma = \pm i$, the (anti-)self-dual complex connection formalism would be obtained. The original diffeomorphism constraint as well as the Hamiltonian constraint can be expressed in terms of new variables up to Gaussian constraint as

$$C_a^{\lambda R} = \frac{1}{\gamma} F^i_{\ ab} E_i^b = 0, \tag{2.22}$$

$$\begin{aligned} C_0 &= \int_\Sigma d^3x C^{\lambda R} \\ &= \frac{1}{2}\int_\Sigma d^3x \Bigg((F_{ab}^j - (1+\gamma^2)\epsilon_{jmn} \tilde{K}_a^m \tilde{K}_b^n) \frac{\epsilon_{jkl} E_k^a E_l^b}{\sqrt{q}} \\ &\quad + \frac{2-2\lambda}{1-3\lambda} \frac{(\tilde{K}_a^i E_i^a)^2}{\sqrt{q}} \Bigg) = 0, \end{aligned} \tag{2.23}$$

where $F^i_{ab} \equiv 2\partial_{[a}A^i_{b]} + \epsilon^i_{kl}A^k_a A^l_b$ is the curvature of the $su(2)$-connection A^i_a. The total Hamiltonian can be expressed as a linear combination

$$H_{\rm total} = \int_\Sigma d^3x(\Lambda^i \mathcal{G}_i + N^a C^{\lambda R}_a + C^{\lambda R}). \qquad (2.24)$$

It is easy to check that the smeared Gaussian constraint, $\mathcal{G}(\Lambda) \coloneqq \int_\Sigma d^3x\Lambda^i(x)\mathcal{G}_i(x)$, generates $SU(2)$ gauge transformations on the phase space, while the smeared constraint $\mathcal{V}(\vec{N}) \coloneqq \int_\Sigma d^3xN^a(C^{\lambda R}_a - A^i_a\mathcal{G}_i)$ generates spatial diffeomorphism transformations on the phase space. Together with the Hamiltonian constraint, it is straightforward to show that the constraints algebra has the following form:

$$\{\mathcal{G}(\Lambda), \mathcal{G}(\Lambda')\} = \mathcal{G}([\Lambda, \Lambda']), \qquad (2.25)$$

$$\{\mathcal{G}(\Lambda), \mathcal{V}(\vec{N})\} = -\mathcal{G}(\mathcal{L}_{\vec{N}}\Lambda), \qquad (2.26)$$

$$\{\mathcal{G}(\Lambda), C_0\} = 0, \qquad (2.27)$$

$$\{\mathcal{V}(\vec{N}), \mathcal{V}(\vec{N}')\} = \mathcal{V}([\vec{N}, \vec{N}']), \qquad (2.28)$$

$$\{\mathcal{V}(\vec{N}), C_0\} = 0, \qquad (2.29)$$

$$\{C_0, C_0\} = 0. \qquad (2.30)$$

Hence the constraints are all of first class. To summarize, the λ-R gravity has been cast into the $su(2)$-connection dynamical formalism. It is worth noting that in the LQG of GR, although the Hamiltonian constraint is well defined in gauge-invariant Hilbert space $\mathcal{H}_G$, it is difficult to define it directly in the diffeomorphism-invariant Hilbert space $\mathcal{H}_{Diff}$. Moreover, since the constraint algebra of GR does not form a Lie algebra, the quantum anomaly might appear after quantization. In contrast, the diffeomorphism constraints nicely form an ideal in λ-R gravity. Therefore the Hamiltonian constraint operator could be defined directly in $\mathcal{H}_{Diff}$.

III. QUANTIZATION OF λ-R THEORY

Based on the connection dynamical formalism, the nonperturbative loop quantization procedure can be straightforwardly extended to the λ-R gravity. The kinematical structure of λ-R gravity is just the same as that of LQG for GR [5,6]. The kinematical Hilbert space, $\mathcal{H}_{\rm kin} \coloneqq \mathcal{H}^{\rm gr}_{\rm kin}$, of the λ-R gravity is spanned by the spin-network basis $\psi_\alpha(A) = |\alpha, j, i\rangle$ over graphs $\alpha \subset \Sigma$, where j labels the irreducible representations of $SU(2)$ associated to the edges of α and i denotes the intertwiners assigned to the vertices linking the edges. The basic operators are the quantum analog of holonomies, $h_e(A) = \mathcal{P}\exp\int_e A_a$, of connections and densitized triads smeared over 2-surfaces, $E(S, f) \coloneqq \int_S \epsilon_{abc}E^a_i f^i$. Note that the whole construction is background independent, and the spatial geometric operators of LQG, such as the area [40], the volume [41,42], and the length operators [43,44], are still valid here. As in LQG, it is straightforward to promote the Gaussian constraint $\mathcal{G}(\Lambda)$ to a well-defined operator [4,6]. Its kernel is the internal gauge-invariant Hilbert space $\mathcal{H}_G$ with gauge-invariant spin-network basis. Moreover the diffeomorphisms of Σ act covariantly on the cylindrical functions in $\mathcal{H}_G$, and hence the so-called group averaging technique can be employed to solve the diffeomorphism constraint [5,6], which gives rise to the desired gauge and diffeomorphism invariant Hilbert space $\mathcal{H}_{Diff}$ for the λ-R gravity.

The remaining nontrivial task for λ-R gravity is to implement the Hamiltonian constraint (2.23) at quantum level. In order to compare the Hamiltonian constraint of λ-R gravity with that of GR in connection formalism, we write Eq. (2.23) as $C_0 = \sum_{i=1}^3 C_i$, where the terms C_1, C_2 take the same form as the Euclidean and Lorentzian terms in GR [5,6], i.e.,

$$C_1 = H^E(1) = \frac{1}{2}\int_\Sigma d^3xF^j_{ab}\frac{\epsilon_{jkl}E^a_kE^b_l}{\sqrt{q}}, \qquad (3.1)$$

$$C_2 = -\frac{(1+\gamma^2)}{2}\int_\Sigma d^3x\epsilon_{jmn}\tilde{K}^m_a\tilde{K}^n_b\frac{\epsilon_{jkl}E^a_kE^b_l}{\sqrt{q}}. \qquad (3.2)$$

Hence the difference comes from the completely new term,

$$C_3 = \int_\Sigma d^3x\frac{(2-2\lambda)}{1-3\lambda}\frac{(\tilde{K}^i_aE^a_i)^2}{\sqrt{q}}. \qquad (3.3)$$

This term can be treated by the similar regularization techniques developed for the Hamiltonian in the LQG [4]. We may triangulate Σ in adaptation to some graph α underling a cylindrical function in $\mathcal{H}_{\rm kin}$ and reexpress connections by holonomies. To this aim, we first note the following classical identity:

$$\tilde{K} = \int_\Sigma d^3x\tilde{K}^i_aE^a_i = \frac{1}{\gamma^2}\{H^E(1), V\}, \qquad (3.4)$$

where $H^E(1)$ is the Euclidean term and V is the volume [4]. Therefore, one can further regularize Eq. (3.3) by the point-splitting method and obtain

$$\begin{aligned} C_3 &= \lim_{\epsilon\to0}C_3^\epsilon \\ &= \lim_{\epsilon\to0}\int_\Sigma d^3y\int_\Sigma d^3x\frac{(2-2\lambda)}{1-3\lambda}\chi_\epsilon(x-y) \\ &\quad\times\frac{\tilde{K}^i_a(x)E^a_i(x)}{\sqrt{V_{U^\epsilon_x}}}\frac{\tilde{K}^j_b(y)E^b_j(y)}{\sqrt{V_{U^\epsilon_y}}}, \end{aligned} \qquad (3.5)$$

where $\chi_\epsilon(x-y)$ is the characteristic function of a box U_x^ϵ containing x with scale ϵ and satisfies the relation

$$\lim_{\epsilon\to 0}\frac{\chi_\epsilon(x-y)}{\epsilon^3}=\delta^3(x-y), \tag{3.6}$$

and $V_{U_x^\epsilon}$ denotes the volume of U_x^ϵ. Now, we triangulate Σ into elementary tetrahedra Δ with scale ϵ, and denote the triangulation by $\mathcal{T}$. For each Δ, we single out one of its vertices, and call it $v(\Delta)$. Then, as $\Delta\to v(\Delta)$, we have

$$\int_\Delta d^3x\frac{\tilde{K}_a^i(x)E_i^a(x)}{\sqrt{V_{U_x^\epsilon}}}\approx\frac{2}{\gamma^2}\left\{H_\Delta^E,\sqrt{V_{U_{v(\Delta)}^\epsilon}}\right\}, \tag{3.7}$$

where

$$H_\Delta^E=\frac{2}{3\gamma}\epsilon^{IJK}\mathrm{Tr}(h_{\alpha_{IJ}(\Delta)}h_{s_K(\Delta)}\{h_{s_K(\Delta)}^{-1},V_{U_v^\epsilon}\}). \tag{3.8}$$

Here $s_I(\Delta), I=1,\ 2,\ 3,$ denote the three edges of Δ incident at $v(\Delta)$, $(I,J,K)\in\{(1,2,3),(2,3,1),(3,1,2)\}$ such that the triple $(s_I(\Delta),s_J(\Delta),s_K(\Delta))$ has positive orientation induced by Σ, and $\alpha_{IJ}(\Delta)\coloneqq s_I(\Delta)\circ a_{IJ}(\Delta)\circ s_J(\Delta)$ is the loop based at $v(\Delta)$ with $a_{IJ}(\Delta)$ being the edge of Δ connecting those endpoints of $s_I(\Delta)$ and $s_J(\Delta)$ which are distinct from $v(\Delta)$. Thus C_3^ϵ in Eq. (3.5) can be expressed as

$$\begin{aligned}C_3^\epsilon=\frac{4}{\gamma^4}\frac{(2-2\lambda)}{1-3\lambda}\sum_{\Delta,\Delta'\in\mathcal{T}}\chi_\epsilon(v(\Delta)-v(\Delta'))\\ \times\left\{H_\Delta^E,\sqrt{V_{U_{v(\Delta)}^\epsilon}}\right\}\left\{H_{\Delta'}^E,\sqrt{V_{U_{v(\Delta')}^\epsilon}}\right\}.\end{aligned} \tag{3.9}$$

Note that all the terms in (3.9) including the Euclidean term H_Δ^E and volume $V_{U_{v(\Delta)}^\epsilon}$ could be promoted as well-defined operators in the gauge-invariant Hilbert space $\mathcal{H}_G$. Furthermore, for a given graph α, one constructs a triangulation $\mathcal{T}(\alpha)$ of Σ adapted to α [4]. Notice that the volume operator acts only at vertices of α, and for sufficiently small ϵ the function $\chi_\epsilon(v(\Delta),v(\Delta'))=0$ unless $v(\Delta)=v(\Delta')$. Thus (3.9) can also be promoted as a well-defined regularized operator acting on any $\psi_\alpha(A)\in\mathcal{H}_G$ as

$$\begin{aligned}\hat{C}_3^\epsilon\psi_\alpha(A)=\frac{4}{\gamma^4(i\hbar)^2}\frac{(2-2\lambda)}{1-3\lambda}\sum_{v\in V(\alpha)}\frac{8^2}{E(v)^2}\\ \times\sum_{v(\Delta)=v(\Delta')=v}\left[\hat{H}_\Delta^E,\sqrt{\hat{V}_v}\right]\\ \times\left[\hat{H}_{\Delta'}^E,\sqrt{\hat{V}_v}\right]\psi_\alpha(A),\end{aligned} \tag{3.10}$$

where the first summation is over the vertices v of α, the second summation is over Δ with $v(\Delta)=v$, $E(v)=\binom{n(v)}{3}$ is the possible choice of triples for a vertex v with $n(v)$ edges, and

$$\hat{H}_\Delta^E\coloneqq\frac{2}{3i\hbar\gamma}\epsilon^{IJK}\mathrm{Tr}(\hat{h}_{\alpha_{IJ}(\Delta)}\hat{h}_{s_K(\Delta)}[\hat{h}_{s_K(\Delta)}^{-1},\hat{V}_v]). \tag{3.11}$$

In LQG of GR, because the diffeomorphism-invariant Hilbert space $\mathcal{H}_{Diff}$ is not preserved by the Hamiltonian constraint operator, the Hamiltonian operator can only be well defined in $\mathcal{H}_G$ rather than $\mathcal{H}_{Diff}$. However, in λ-R gravity, since the lapse N is a constant, $\mathcal{H}_{Diff}$ would be preserved by the Hamiltonian constraint operator, and hence we can further define the Hamiltonian operator in $\mathcal{H}_{Diff}$. Note that a diffeomorphism-invariant state can be produced from a state $\psi_\alpha(A)\in\mathcal{H}_G$ by the group averaging method as [4–6]

$$\hat{P}_{Diff_\alpha}\psi_\alpha(A)\coloneqq\frac{1}{n_\alpha}\sum_{\varphi\in GS_\alpha}\hat{U}_\varphi\psi_\alpha(A), \tag{3.12}$$

where the operator $\hat{U}_\varphi$ denotes the finite diffeomorphism $\varphi:\Sigma\to\Sigma$, $GS_\alpha=Diff_\alpha/TDiff_\alpha$ is the group of graph symmetries with $Diff_\alpha$ being the group of all diffeomorphisms preserving the graph α, $TDiff_\alpha$ is its subgroup which has trivial action on α, and n_α is the number of the elements in GS_α.

Since the regularized operator $\hat{C}_3^\epsilon$ with different value of ϵ are diffeomorphic to each other, we can naturally define the action of the limit operator $\hat{C}_3=\lim_{\epsilon\to 0}\hat{C}_3^\epsilon$ on the diffeomorphism-invariant state as

$$\hat{C}_3\hat{P}_{Diff_\alpha}\psi_\alpha(A)\coloneqq\lim_{\epsilon\to 0}\frac{1}{n_{\alpha(\epsilon)}}\sum_{\varphi\in GS_{\alpha(\epsilon)}}\hat{U}_\varphi\hat{C}_3^\epsilon\psi_\alpha(A), \tag{3.13}$$

where $\alpha(\epsilon)$ represents the new graphs produced by the action of $\hat{C}_3^\epsilon$ on α. Note that Eq. (3.13) does not depend on ϵ, since all the graphs $\alpha(\epsilon)$ are diffeomorphism equivalent to each other. Similar to the definition of $\hat{C}_3$, it is straightforward to define the whole Hamiltonian constraint operator $\hat{C}_0$ in $\mathcal{H}_{Diff}$ as

$$\begin{aligned}\hat{C}_0\hat{P}_{Diff_\alpha}\psi_\alpha(A)\coloneqq\lim_{\epsilon\to 0}\frac{1}{n_{\alpha(\epsilon)}}\sum_{\varphi\in GS_{\alpha(\epsilon)}}\sum_{i=1,2,3}\hat{U}_\varphi\\ \times\hat{C}_i^\epsilon\psi_\alpha(A),\end{aligned} \tag{3.14}$$

with

$$\hat{C}_1^\epsilon=\sum_{v\in V(\alpha)}\frac{8}{E(v)}\sum_{v(\Delta)=v}\hat{H}_\Delta^E, \tag{3.15}$$

$$\begin{aligned}\hat{C}_2^\epsilon=-\frac{4(1+\gamma^2)}{3(i\hbar\gamma)^3}\sum_{v\in V(\alpha)}\frac{8}{E(v)}\sum_{v(\Delta)=v}\epsilon^{IJK}\\ \times\mathrm{Tr}(\hat{h}_{s_I(\Delta)}[\hat{h}_{s_I(\Delta)}^{-1},\hat{\tilde{K}}_v]\hat{h}_{s_J(\Delta)}[\hat{h}_{s_J(\Delta)}^{-1},\hat{\tilde{K}}_v]\\ \times\hat{h}_{s_K(\Delta)}[\hat{h}_{s_K(\Delta)}^{-1},\hat{V}_v]),\end{aligned} \tag{3.16}$$

where $\hat{\tilde{K}}_v := \frac{1}{i\hbar\gamma^2}[\hat{H}^E_v, \hat{V}_v]$ with $\hat{H}^E_v := \sum_{v(\Delta)=v} \hat{H}^E_\Delta$. Note that, to have a well-defined adjoint operator of $\hat{C}_0$ [45], we used the freedom of choosing the spin representations attached to each new added loop in (3.14) to ensure that the valence of any vertex would not be changed by the action of $\hat{C}_0$.

IV. A PHYSICAL HAMITONIAN AND PHYSICAL STATES

It should be noted that even in projectable λ-R gravity, due to the existence of the global Hamiltonian constraint, there still exists a global gauge freedom corresponding to the global time reparametrization. Thus, in the corresponding quantum theory, the Hamiltonian constraint operator has to vanish on physical states. Therefore, the time problem of quantum gravity is still there. The purpose of this section is to overcome this problem by introducing a single global dust degree of freedom to represent a dynamical time. In a theory of gravity with time reparametrization invariance, in order to pick up a unique time to represent the evolution of physical states [11], one naturally takes the viewpoint of relational evolution [46–49]. This allows one to map the totally constrained theory into a theory with a true nonvanishing Hamiltonian with respect to some chosen dynamical (emergent) time variable. The dynamical "time" can be achieved at the classical level as well as the quantum level. The combination of LQG with the relational evolution framework makes it possible to solve the quantum Hamiltonian constraint.

The action of a nonrotational dust model in a covariant spacetime reads

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}M(g^{ab}\partial_a T\partial_b T + 1), \qquad (4.1)$$

where T is the configuration variable of the nonrotational dust, and M is the rest mass density of the dust field. Its Hamiltonian can be written as [50]

$$H_D = \int d^3x\left[N\sqrt{\pi^2 + q^{ab}C^D_a C^D_b} + N^a C^D_a\right], \qquad (4.2)$$

where π is the conjugate momentum of T and $C^D_a = -\pi\partial_a T$. In order to introduce the nonrotational dust model, which was widely used in LQG literature [50–53], to represent a dynamical time for the λ-R gravity, we consider the case that the dust is adapted to the spacetimes of Horava gravity so that the time foliation of the spacetimes coincides with the hypersurfaces of constant T. In the other words, certain function $t(T)$ of the dust configuration variable T is employed to define the given time foliation of Horava spacetimes. Note that $t(T)$ needs not to be a fixed function. Thus the global time reparametrization freedom still exists.

As the gauge group of Horava theory consists of the foliation-preserving diffeomorphisms, the projectable version of the theory concerns only the case that the lapse function depends only on time t [34]. Note that for the adapted nonrotational dust, we have $q^{ab}\partial_a T = 0$, and hence the dust has no local degrees of freedom. Thus, in the case of projectable λ-R gravity with the adapted nonrotational dust, the global Hamiltonian constraint reads

$$\begin{aligned} C_{\text{total}} &= \int d^3x(\pi(x) + h(x)) \\ &:= \int d^3x(\pi(x) + C^{\lambda R}(x)) = 0. \end{aligned} \qquad (4.3)$$

Hence one can define a physical Hamiltonian $h_{phy} = \int d^3x h(x)$ which generates the evolution of the system with respect to the dynamical time T.

In the quantum theory, one would expect to implement the constraint corresponding to (4.3) through a Schrodinger-like equation

$$i\hbar\frac{\partial}{\partial T}\Phi(A,T) = \hat{h}_{phy}\Phi(A,T) \qquad (4.4)$$

for certain quantum states $\Phi(A,T)$. Note that in certain simplified models of quantum gravity, there are different ideas to treat the Hamiltonian constraint as a true Hamiltonian [54,55]. Since loop quantum λ-R gravity has been constructed in previous sections and the gravitational Hamiltonian constraint $\hat{C}_0$ is well defined by (3.14) on any diffeomorphism-invariant state $\Phi_{[\alpha]}(A) = \hat{P}_{Diff_\alpha}\psi_\alpha(A) \in \mathcal{H}_{Diff}$, it is convenient to define the physical Hamiltonian operator $\hat{h}_{phy}$ as a self-adjoint extension of the symmetric operator $\frac{1}{2}(\hat{C}_0 + \hat{C}_0{}^\dagger)$. Then the general solutions to Eq. (4.4) read

$$\Phi_{[\alpha']}(A,T) = e^{-\frac{i}{\hbar}\hat{h}_{phy}T}\Phi_{[\alpha]}(A), \qquad (4.5)$$

with an arbitrary given $\Phi_{[\alpha]}(A) \in \mathcal{H}_{Diff}$. Thus, the physical Hilbert space of the coupled system is unitarily isomorphic to $\mathcal{H}_{Diff}$.

V. CONCLUSION

In the previous sections, a detailed construction of connection-dynamical formalism of the lowest-order projectable Horava gravity is given. This theory is the so-called λ-R gravity. Since a preferred foliation has been chosen in projectable Horava gravity, there is no local Hamiltonian constraint. We obtain a connection dynamics with real $su(2)$ connections as configuration variables. In contrast to GR, the constraint algebra of λ-R gravity forms a Lie algebra, and the Hamiltonian (2.23) possesses an extra term which would vanish for $\lambda = 1$. This classical connection-dynamical formalism enables us to extend the scheme of nonperturbative loop quantum gravity to the λ-R theories of gravity. While the quantum kinematical framework is the same as that for GR, the global Hamiltonian constraint of loop quantum v gravity has been rigorously constructed as a well-defined operator in the diffeopmorphism-invariant Hilbert space.

To overcome the time problem related to the global time reparametrization freedom of the projectable λ-R gravity, the nonrotating dust adapted to the Horava spacetimes is introduced as a dynamical time. The physical time evolution with respect to the dust is then naturally defined. As a result, the quantum dynamics of the coupled system is dictated by a Schrodinger-like equation. For an arbitrarily given initial diffeomorphism-invariant state, the physical quantum Hamiltonian operator would generate and thus completely determine the forthcoming quantum state with respect to the dynamical time. Moreover, the physical states we obtained satisfy all the constraints, and the physical Hilbert space of the coupled system is unitarily isomorphic to the diffeomorphism-invariant Hilbert space of λ-R gravity. Therefore, we obtained a quantum theory of gravity in which the Dirac algorithm of canonical quantization for a totally constrained system could be completely realized.

There are of course a few issues that deserve further investigating in our loop quantum v theory of gravity. First, it is interesting to study some symmetry-reduced models of our loop quantum λ-R gravity, which might tell us more physical properties of the quantum λ-R gravity. Second, how to extend LQG to the nonprojectable version of λ-R gravity is an interesting issue. Third, if our result could be generalized to the general Hořava-Lifshitz gravity, it would be helpful to get a better understanding on the quantum gravity without Lorentzian invariance.

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