Kerr–Vaidya black holes

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Kerr–Vaidya metrics are the simplest nonstationary extensions of the Kerr metric. We explore their properties and compare them with the near-horizon limits of the spherically symmetric self-consistent solutions (the ingoing Vaidya metric with decreasing mass and the outgoing Vaidya metric with increasing mass) for the evaporating and accreting physical black holes. The Newman–Janis transformation relates the corresponding Vaidya and Kerr-Vaidya metrics. For nonzero angular momentum, the energy-momentum tensor violates the null energy condition (NEC). However, we show that its structure differs from the standard form of the NEC-violating tensors. The apparent horizon in the outgoing Kerr–Vaidya metric coincides with that of the Kerr black hole. For the ingoing metric, its location is different. We derive the ordinary differential equation for this surface and locate it numerically. A spherically symmetric accreting black hole leads to a firewall—a divergent energy density, pressure, and flux as perceived by an infalling observer. We show that this is also true for the outgoing Kerr–Vaidya metric.

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I. INTRODUCTION

Black holes are described both as "the most perfect macroscopic objects in the universe" [1] and as one of the "most mysterious concepts conceived by the human mind" [2]. Thanks to the successes of the gravitational wave astronomy and direct observations of ultracompact objects (UCOs), the old debate about the physical relevance of black hole solutions [3] has been reframed as a question about the nature of UCOs [4–6].

Diversity of opinions about black holes and their significance are matched by absence of a universally accepted definition [7]. However, the core idea of a black hole as a spacetime region from which nothing can escape is formalized in the notion of a trapped region. Gravity there is so strong that both ingoing and outgoing future-directed null geodesics originating at a spacelike two-dimensional surface with spherical topology have negative expansion [8–10]. The apparent horizon is its evolving outer boundary.

A physical black hole contains such a trapped region [11]. To be relevant to distant observers with a finite lifespan it has to be formed in finite time according to their clocks [12]. Otherwise, black hole solutions can have only approximate or asymptotic meaning. A physical black hole may possess other classical features, such as an event horizon and a singularity, or be a singularity-free regular black hole. One of the issues at stake is whether the observed astrophysical black hole candidates contain light-trapping regions, i.e., they are black holes or do not, and thus they are horizonless UCOs.

Quantum effects make the black hole physics particularly interesting [9,10,13–17]. On the one hand, an apparent horizon is accessible to an observer at infinity (Bob) only if the classical energy conditions [8,18–20] are violated [8]. The Hawking radiation [9,13,14] has precisely this property. On the other hand, the Hawking radiation precipitates the infamous information loss paradox [16,17]. One way to resolve the paradox is to have a horizonless UCO or a regular black hole as the final product of the gravitational collapse [4,5]. These objects also require a violation of the energy conditions for their existence. Another resolution of the information loss paradox posits that the infalling observer (Alice) does not see a vacuum at the black hole horizon, but instead encounters a large number of high-energy modes [16,21], known as the firewall.

A self-consistent approach [12,22–24] starts with the assumption that physical black holes do form. Once the assumption of formation of a singularity-free apparent horizon is translated into mathematical statements, it allows to obtain a number of concrete results. In spherical symmetry there are only two possible classes of black hole solutions, and it is possible to identify the amount of violation of the energy conditions that they require. Accreting physical black hole solutions lead to divergent energy density, pressure, and flux as experienced by Alice, while the curvature scalars remain finite.

Real astrophysical objects are rotating. Hence it is important to verify that the firewall is not an artifact of the spherical symmetry. The Kerr metric is the asymptotic result of the classical collapse [1,8,9]. The simplest models that allow for an axially symmetric variable mass distribution are given by the so-called Kerr–Vaidya metrics [25–27]. In Sec. II we review the relevant properties of the spherically symmetric solutions. In Sec. III we discuss their axially symmetric counterparts, focusing on the violation of the energy conditions, location of the apparent horizons and presence of a firewall.

II. NEAR-HORIZON REGIONS OF SPHERICALLY SYMMETRIC BLACK HOLES

Working in the framework of semiclassical gravity [15,28,29] we use classical notions (horizons, trajectories, etc.), and describe dynamics via the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where the Einstein tensor $G_{\mu\nu}$ is equated to the expectation value $T_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle_\omega$ of the renormalized energy-momentum tensor (EMT). For simplicity we consider an asymptotically flat space. We do not make any specific assumptions apart from (i) the apparent horizon was formed at some finite time of Bob, (ii) it is regular, i.e., the curvature scalars, such as $\mathrm{T} \coloneqq T^\mu{}_\mu \equiv -\mathcal{R}/8\pi$ and $\mathfrak{T} \coloneqq T^{\mu\nu}T_{\mu\nu} \equiv R^{\mu\nu}R_{\mu\nu}/64\pi^2$ are finite at the horizon. (Here $R_{\mu\nu}$ and $\mathcal{R} \coloneqq R^\mu{}_\mu$ are the Ricci tensor and the Ricci scalar, respectively).

A general spherically symmetric metric in the Schwarzschild coordinates is given by

$$ds^2 = -e^{2h(t,r)} f(t,r) dt^2 + f(t,r)^{-1} dr^2 + r^2 d\Omega, \quad (1)$$

where r is the areal radius. The Misner-Sharp mass [10,30] $M(t,r)$ is invariantly defined via $1 - 2M/r \coloneqq \partial_\mu r \partial^\mu r$, and $f(t,r) = 1–2M(t,r)/r$. The apparent horizon is located at the Schwarzschild radius r_g that is the largest root of $f(t,r) = 0$ [10,31].

Only two near-horizon forms of the EMT and the metric are consistent with the above two assumptions. Here we consider the generic form that agrees with the *ab initio* calculations of the EMT on the background of the Schwarzschild solution [32]. In this case the leading terms in the expansion of the metric functions in terms of $x \coloneqq r - r_g(t)$ are

$$2M(t,r) = r_g - w\sqrt{x} + \mathcal{O}(x), \quad (2)$$

$$h(t,r) = -\frac{1}{2}\ln\frac{x}{\xi} + \mathcal{O}(\sqrt{x}), \quad (3)$$

where the function $\xi(t)$ is determined by the choice of the time variable (and requires for its determination knowledge of the full solution of the Einstein equations), and $w^2 \coloneqq 16\pi\Upsilon^2 r_g^3$ characterizes the leading behavior of the EMT [12].

In particular, in the orthonormal basis the $(\hat{t}\,\hat{r})$ block of the EMT near the apparent horizon is given by

$$T_{\hat{a}\,\hat{b}} = -\frac{\Upsilon^2}{f}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}. \quad (4)$$

The upper (lower) signs of $T_{\hat{t}\,\hat{r}}$ correspond to evaporation (growth) of the physical black hole. Consistency of the Einstein equations results in the relation

$$r_g'/\sqrt{\xi} = \pm 4\sqrt{\pi}\Upsilon\sqrt{r_g} = \pm w/r_g. \quad (5)$$

The null energy condition (NEC) requires $T_{\mu\nu}l^\mu l^\nu \geq 0$ for all null vectors l^μ [8,19,20]. It is violated by radial vectors $l^{\hat{a}} = (1, \mp 1, 0, 0)$ for the evaporating and the accreting solutions, respectively [12].

Null coordinates allow to represent the near-horizon geometry in a simpler form. The advanced null coordinate v,

$$dt = e^{-h}(e^{h_+} dv - f^{-1} dr), \quad (6)$$

is useful in the case $r_g' < 0$. A general spherically symmetric metric in (v,r) coordinates is given by

$$ds^2 = -e^{2h_+}\left(1 - \frac{C_+}{r}\right) dv^2 + 2e^{h_+} dv dr + r^2 d\Omega. \quad (7)$$

Using the Einstein equations and the relationships between components of the EMT in two coordinates systems [24] one can show that

$$C_+(v,r) = r_+(v) + w_1(v)x + \dots, \quad (8)$$

$$h_+(v,r) = \chi_1(v)x + \dots, \quad (9)$$

where $r_+(v)$ is the radial coordinate of the apparent horizon, $C_+(v, r_+) \equiv r_+$, $x \eqqcolon r - r_+(v)$. As a result, at the apparent horizon both the metric corresponds to the Vaidya geometry with $C_+'(v) = 2M'(v) < 0$.

If $r_g' > 0$ it is useful to switch to the retarded null coordinate u. The near-horizon geometry is then described by the Vaidya metric with $C_-'(u) > 0$.

A static observer finds that the energy density $\varrho = T_{\mu\nu}u^\mu u^\nu = -T^t{}_t$, the pressure $p = T_{\mu\nu}n^\mu n^\nu = T^r{}_r$, and the flux $\phi \coloneqq T_{\mu\nu}u^\mu n^\nu$ (where u^μ is the four-velocity and n^μ is the outward-pointing radial spacelike vector), diverge at the apparent horizon. A radially infalling Alice moves on a trajectory $x_{\mathrm{A}}^\mu(\tau) = (T(\tau), R(\tau), 0, 0)$. Horizon crossing happens not only at some finite proper time τ_0, $r_g(T(\tau_0)) = R(\tau_0)$, but thanks to the form of the metric also at a finite time $T(\tau_0)$ of Bob.

However, experiences of Alice are different at the apparent horizon of an evaporating and accreting physical black holes. For an evaporating black hole, $r_g' < 0$, energy density, pressure and flux are finite. For example, if we approximate the near-horizon geometry by a pure outgoing Vaidya metric with $M'(v) < 0$, Alice's energy density at the horizon crossing is

$$\varrho_{\mathrm{A}}^{<} = p_{\mathrm{A}}^{<} = \phi_{\mathrm{A}}^{<} = -\frac{\Upsilon^2}{4\dot{R}^2}, \qquad (10)$$

at $r_g = R$. For an accreting black hole, $r_g' > 0$, Alice experiences the divergent values of energy density, pressure and flux,

$$\varrho_{\mathrm{A}}^{>} = p_{\mathrm{A}}^{>} = -\phi_{\mathrm{A}}^{>} = -\frac{2\dot{R}^2\Upsilon^2}{F^2} + \mathcal{O}(F^{-1}), \qquad (11)$$

in the vicinity of the apparent horizon, as $F \coloneqq f(T,R) \to 0$.

Thus an expanding trapped region is accompanied by a firewall—a region of unbounded energy density, pressure, and flux—that is perceived by an infalling observer. Unlike the firewall from the eponymous paradox, it appears as a consequence of regularity of the expanding apparent horizon and its finite formation time. The divergent energy density leads to a violation [23,24] of the inequality that bounds the amount of negative energy along a timelike trajectory in a moderately curved spacetime [33]. As a result a physical black hole, once formed, can only evaporate. Another possibility is that the semiclassical physics breaks down at the horizon scale.

III. KERR–VAIDYA METRIC

A general time-dependent axisymmetric metric contains seven functions of three variables (say, t, r and θ) that enter the Einstein equations via six independent combinatons [1]. However, to verify that certain predictions of the self-consistent approach (such as a firewall at an expanding apparent horizon) are not an artefact of the spherical symmetry, it is enough to consider a simpler geometry.

The Kerr metric can be represented using either the ingoing [34]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dv^2 + 2dvdr - \frac{4aMr\sin^2\theta}{\rho^2}dvd\psi - 2a\sin^2\theta drd\psi + \rho^2 d\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\psi^2, \qquad (12)$$

or the outgoing null congruences [1],

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)du^2 - 2dudr - \frac{4aMr\sin^2\theta}{\rho^2}dud\psi + 2a\sin^2\theta d\psi dr + \rho^2 d\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\psi^2, \qquad (13)$$

where $\rho^2 \coloneqq r^2 + a^2\cos^2\theta$, $\Delta \coloneqq r^2 - 2Mr + a^2$, and $a = J/M$ is the angular momentum per unit mass.

The easiest way to obtain this result is to follow the complex-valued Newman–Janis transformation [35] starting with the Schwartzschild metric written in the ingoing or the outgoing Eddington–Finkelstein coordinates. The simplest nonstationary generalizations of the Kerr metric are obtained by introducing evolving masses $M(v)$ and $M(u)$. The metric of Eq. (13) with a variable $M(u)$ is obtained from the retarded Vaidya metric [36,37]. By using the advanced Vaidya metric of Eq. (7) as the seed metric, the metric of Eq. (12) can be obtained following the same procedure (Appendix A).

A. Energy conditions

A schematic form of the EMT in both cases is

$$T_{\mu\nu} = \begin{pmatrix} T_{oo} & 0 & T_{o\theta} & T_{o\psi} \\ 0 & 0 & 0 & 0 \\ T_{o\theta} & 0 & 0 & T_{\theta\psi} \\ T_{o\psi} & 0 & T_{\theta\psi} & T_{\psi\psi} \end{pmatrix}, \qquad (14)$$

where $o = u, v$. Using the null vector $k^\mu = (0,1,0,0)$ [25] the EMT can be represented as

$$T_{\mu\nu} = T_{oo}k_\mu k_\nu + q_\mu k_\nu + q_\nu k_\mu, \qquad (15)$$

where the components of $T_{\mu\nu}$ and of the auxiliary vector q_μ, $q_\mu k^\mu = 0$, for both cases are given in Appendix B. The EMT [for the metric Eq. (13)] was identified in Ref. [38] as belonging to the type $[(1,3)]$ in the Segre classification [18], i.e., to the type III of the Hawking–Ellis classification [8,19], indicating that the NEC is violated for any $a \neq 0$.

A detailed investigation reveals some interesting properties of this EMT. We use a tetrad in which the null eigenvector $k^\mu = k^{\hat{a}} e^\mu_{\hat{a}}$ has the components $k^{\hat{a}} = (1,1,0,0)$, the third vector $e_{\hat{2}} \propto \partial_\theta$ and the remaining vector $e_{\hat{3}}$ is found by completing the basis, the EMT takes the form

$$T^{\hat{a}\hat{b}} = \left(\begin{array}{cc|cc} \nu & \nu & q^{\hat{2}} & q^{\hat{3}} \\ \nu & \nu & q^{\hat{2}} & q^{\hat{3}} \\ \hline q^{\hat{2}} & q^{\hat{2}} & 0 & 0 \\ q^{\hat{3}} & q^{\hat{3}} & 0 & 0 \end{array}\right). \qquad (16)$$

Explicit expressions for the tetrad vectors and the matrix elements are given in Appendix B.

For an arbitrary null vector $l_{\hat{a}} = (-1, n_{\hat{a}})$, $n_{\hat{a}} = (\cos\alpha, \sin\alpha\cos\beta, \sin\alpha\sin\beta)$ the NEC becomes

$$\nu(1-\cos\alpha) + 2\sin\alpha(q^{\hat{2}}\cos\beta + q^{\hat{3}}\sin\beta) \geq 0. \quad (17)$$

This inequality is satisfied if and only if $\nu \geq 0$ and $q^{\hat{2}} = q^{\hat{3}} = 0$. The condition $q^{\hat{2}} = q^{\hat{3}} = 0$ holds only when $a = 0$, so the metric reduces to its Vaidya counterpart and the EMT becomes a type II tensor. Only in this case the NEC may be satisfied.

Each type of the EMT is characterized by its Lorentz-invariant eignevlaues [18,19]. These are the eigenvalues of the matrix $T^{\hat{a}}_{\hat{b}}$, i.e., the roots of the equation

$$\det(T^{\hat{a}\hat{b}} - \lambda\eta^{\hat{a}\hat{b}}) = 0, \qquad \eta^{\hat{a}\hat{b}} = \mathrm{diag}(-1,1,1,1). \quad (18)$$

The EMT of Eq. (16) has a single quadruple-degenerated Lorentz-invariant eigenvalue $\lambda = 0$. On the other hand, two of the eigenvalues $\tilde{\lambda}$ of the matrix $T^{\hat{a}\hat{b}}$ are nonzero,

$$\tilde{\lambda}_{1,2} = \nu \pm \sqrt{2(q_{\hat{2}}^2 + q_{\hat{3}}^2) + \nu^2}. \quad (19)$$

As a result the EMT tensor (16) cannot be brought to a generic type III form by an arbitrary similarity transformations unless $\tilde{\lambda}_1 = -\tilde{\lambda}_2$, which is impossible for $M' \neq 0$ (see Appendix B for the details).

B. Apparent horizon

The apparent horizon of the Kerr black hole coincides with its event horizon. It is located at the largest root of $\Delta = 0$,

$$r_0 \coloneqq M + \sqrt{M^2 - a^2}. \quad (20)$$

For both the ingoing and the outgoing Vaydia metrics the apparent horizon is located at $r_g = r_0 = 2M$. For the metric (13) the relation $r_g = r_0$ also holds [39], but it fails for the metric (12) [27]. In this case the difference $r_g(v,\theta) - r_0(v)$ is of the order $|M_v|$.

We now identity its location in the foliations with the hypersurfaces $v = \text{const}$. The standard approach [40,41] for constructing the ordinary differential equation for the apparent horizon is based on exploiting properties of a spacelike foliation. It cannot be used in this case as the foliating hypersurfaces are timelike. However, since the approximate location of the apparent horizon is known, we obtain the leading correction in M_v by using the methods of analysis of null congruences and hypersurfaces [34].

Assume that at some advanced time v the apparent horizon $S_0(v)$ is located at $r_g = r_0(v) + z(\theta)$, where the function $z(\theta)$ is to be determined. Once the future-directed outgoing null geodesic congruence orthogonal the surface is identified, calculating the expansion ϑ and equating it to zero results in the differential equation for $z(\theta)$. There are at least two equivalent ways to obtain this equation.

The outward and inward-pointing null vectors l^μ and N^μ, respectively, are defined on S_0. They are orthogonal to its tangents and can be normalized by the condition $N_\mu l^\mu = -1$. These vectors can be extended to the fields of tangent vectors to the families of affinely parameterized null geodesics in the bulk.

One approach to calculation of the expansion uses its geometric meaning as a relative rate of change of the two-dimensional cross-section area [8,34]. Consider an infinitesimal geodesic triangle $(x_{\text{in}}x_\theta x_\psi)$ on the surface S_0. It is defined by three vertices

$$x^\mu_{\text{in}}, \qquad x^\mu_\psi = x^\mu_{\text{in}} + b^\mu_\psi\delta\psi, \qquad x^\mu_\theta = x^\mu_{\text{in}} + b^\mu_\theta\delta\theta. \quad (21)$$

The two tangent vectors

$$b^\mu_\psi \coloneqq \left.\frac{\partial x^\mu}{\partial\psi}\right|_{S_0}, \qquad b^\mu_\theta \coloneqq \left.\frac{\partial x^\mu}{\partial\theta}\right|_{S_0}, \quad (22)$$

introduce a metric two-tensor

$$\sigma_{AB} \coloneqq g_{\mu\nu}b^\mu_A b^\nu_B, \qquad A, B = \theta, \psi. \quad (23)$$

The area of the triangle $(x_{\text{in}}x_\theta x_\psi)$ is $\delta A = \frac{1}{2}\sqrt{\sigma}\delta\theta\delta\psi$, where $\sigma = \sigma_{11}\sigma_{22} - \sigma_{12}^2$ is the determinant of the two-dimensional metric. Under the geodesic flow $x^\mu \to x^\mu + l^\mu(x)d\lambda$ the coordinates θ and ψ are comoving and thus constant for the three vertices, but both the metric g and the vectors b_A evolve with λ.

Calculating the ratio of the first-order area change $\delta A(\lambda + d\lambda) - \delta A(\lambda)$ to the initial area of the triangle allows to obtain the expansion by using the relation

$$\vartheta = \frac{1}{\sqrt{\sigma}}\frac{d\sqrt{\sigma}}{d\lambda}, \quad (24)$$

where λ is the affine parameter. Solution of the second-order differential equation $\vartheta(z) = 0$ gives the desired function $z(\theta)$. An alternative derivation is based on the direct evaluation of $\vartheta = l^\mu_{;\mu}$ on S_0 (see Appendix C for the details).

If both $z(\theta)$ and its derivatives are much smaller than r_0, we obtain a linear ordinary differential equation for z,

$$\begin{aligned} &8r_0^2(a^2 + r_0^2)^2(z'' + \cot\theta z') \\ &\quad - (r_0^2 - a^2)(a^4 + 7a^2r_0^2 + 8r_0^4 + a^2(r_0^2 - a^2)\cos 2\theta)z \\ &\quad - 8r_0^3a^2(a^2 + r_0^2)\sin^2\theta M_v = 0, \end{aligned} \quad (25)$$

where the regular singular term $(\cot\theta z' + z'')$ is a standard feature of the apparent horizon equations [40,41].

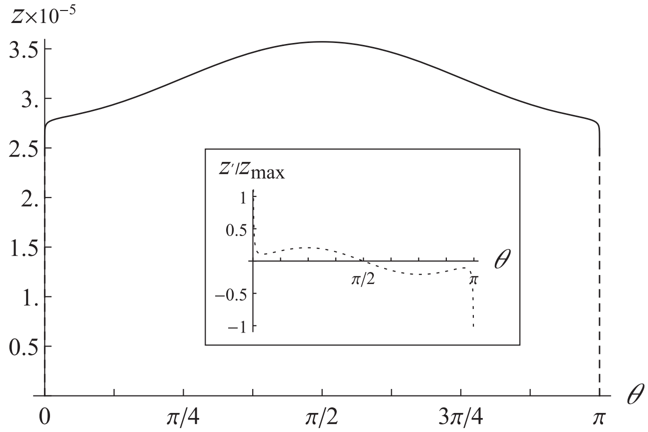

FIG. 1. Location of the apparent horizon relative to r_0 for $M = 1$, $a = 0.1$, $M_v = -\kappa/M^2 = 0.01$. The equation was first solved as a boundary value problem $z(0) = 0$, $z'(\pi/2) = 0$, resulting in $z_m := z(\pi/2) \approx 3.57\times10^{-5}$. Solution of the initial value problem $z(\pi/2) = z_m$, $z'(\pi/2) = 0$ coincides with the previous one within the relative precision of 2×10^{-15} outside $\delta = 10^{-6}$ interval from the poles.

Symmetry considerations lead to the condition $z(0)=z(\pi)=0$. The second initial condition $z'(\pi) = 0$ ensures that the surface of the apparent horizon is smooth [40].

A typical result of the numerical solution is depicted in Fig. 1. It was obtained by imposing the boundary conditions $z(0) = 0$ and $z'(\pi/2) = 0$, that enforces the equatorial symmetry. The assumption of $|z'| \ll r_0$ fails near the poles, where $z = 0$. This is not an artefact of the approximation. Using the series solutions of Eq. (25) with the conventional initial conditions $z(0) = 0$, $z'(0) = 0$ [40,41], i.e., in the regime where the assumption $|z'| \ll r_0$ is clearly valid, leads to

$$z_{\rm ser} = \frac{a^2 M_v r_0}{16(a^2 + r_0^2)}\theta^4 + \mathcal{O}(\theta^5). \tag{26}$$

For $M_v > 0$ it implies that at least near the poles $r_g > r_0$, i.e., at $r = r_0$ the expansion is still negative. However, this is impossible: at the poles the null congruence that is orthogonal to the two-dimensional surface $r = r_0$ [27] has $\vartheta > 0$. Moreover, using this solution to provide the initial values $z(\theta)$, $z'(\theta)$ at some $\theta = \epsilon \ll 1$ leads to inconsistencies.

We investigated stability of this result in numerical experiments. For a fixed $M_v = -\kappa/M^2$ the initial value problem $z(\pi/2) = z_0$, $z'(\pi/2) = 0$, where z_0 is some number, leads to a well-behaved numerical solution. However, the conditions $z(0) = z(\pi) = 0$ are satisfied within a prescribed tolerance only for a very narrow range of the values z_0 around $z_m = z(\pi/2)$ of the numerical solution of the above boundary value problem. We will provide a full analysis of the apparent horizon in a future work.

C. Firewall

All components of the EMT (14) are finite at the apparent horizon. Divergences of the comoving parameters can appear only as a result of divergences in the components of the four-velocity of Alice. We now show that similarly to the spherically symmetric geometries density and pressure in Alice's frame are finite if Alice crosses the apparent horizon in the metric of Eq (12), but diverge for the metric of Eq. (13).

In the spherically symmetric case Alice was a zero angular momentum observer (ZAMO) [9,34]. In axially symmetric spacetimes the ZAMO condition results in a nontrivial angular velocity Ψ_Z. We begin with the retarded Kerr–Vaidya metric, where the apparent horizon is located at $r_g = r_0$. Alice's four-velocity is

$$u^\mu_{\rm A} = (\dot U, \dot R, \dot\Theta, \dot\Psi_Z), \tag{27}$$

where the ZAMO condition $\xi_\psi \cdot u_{\rm A} = 0$, where the Killing vector $\xi_\psi = \partial_\psi$, implies $\dot\Psi_Z = -(g_{u\psi}\dot U + g_{r\psi}\dot R)/g_{\psi\psi}$. During the fall $\dot R < 0$. The velocity component $\dot U > 0$ is obtained from the normalization condition $u^2_{\rm A} = -1$,

$$\dot U = -\frac{\dot R^2}{\Delta}(r^2+a^2) + \frac{1}{\Delta\rho}\sqrt{(\Delta(1+\rho^2\dot\Theta^2)+\rho^2\dot R^2)\Sigma}, \tag{28}$$

where $\Sigma = (a^2+r^2)\rho^2 + 2a^2 rM\sin^2\theta$. As $X := R(\tau) - r_0(U(\tau)) \to 0$ the derivative $\dot U$ diverges as Δ^{-1},

$$\dot U = -\frac{2r_0\dot R M}{X(r_0 - M)} + \mathcal{O}(X). \tag{29}$$

The energy density in Alice's frame is given then by

$$\rho_{\rm A} = \left(T_{uu} + T_{\psi\psi}\left(\frac{g_{u\psi}}{g_{\psi\psi}}\right)^2 - 2T_{u\psi}\frac{g_{u\psi}}{g_{\psi\psi}}\right)\dot U^2 + \mathcal{O}(\Delta^{-1}), \tag{30}$$

resulting in

$$\rho_{\rm A} \approx \frac{(-2M' - (2M - r_0)\sin^2\theta M'')r_0^2\dot R^2}{8\pi X^2(r_0 - M)^2} = \frac{(-2r_0M' - a^2\sin^2\theta M'')r_0\dot R^2}{8\pi X^2(r_0 - M)^2}. \tag{31}$$

We choose the spacelike direction analogously to the spherically symmetric case,

$$n^{\rm A}_\mu = (-\dot R, \dot U, 0, 0). \tag{32}$$

Then (after setting $\dot\Theta = 0$),

$$p_{\rm A} = T_{\mu\nu}n^\mu_{\rm A}n^\nu_{\rm A} \approx \frac{(-2r_0M' - a^2\sin^2\theta M'')r_0\dot R^2}{8\pi X^2(r_0 - M)^2}. \tag{33}$$

It is easy to see that for $a = 0$ we recover the firewall of the outgoing Vaidya metric.

Violations of the NEC are bounded by quantum energy inequalities (QEIs) [20,42]. For spacetimes of small curvature explicit expressions that bound time-averaged energy density for a geodesic observer were derived in Ref. [33]. For any Hadamard state ω and a sampling function $\mathfrak{f}(\tau)$ of compact support, negativity of the expectation value of the energy density $\rho = \langle \hat{T}_{\mu\nu} \rangle_\omega u^\mu u^\nu$ as seen by a geodesic observer that moves on a trajectory $\gamma(\tau)$ is bounded by

$$\int_\gamma \mathfrak{f}^2(\tau)\rho d\tau \geq -B(R, \mathfrak{f}, \gamma), \tag{34}$$

where $B > 0$ is a bounded function that depends on the trajectory, the Ricci scalar and the sampling function [33].

Consider a growing apparent horizon, $r_0'(u) > 0$. For simplicity we consider a polar trajectory $\theta = 0$. For a macroscopic black hole the curvature at the apparent horizon is low and thus Eq. (34) is applicable. Horizon radius (and mass, as in this model $a = \text{const}$), do not appreciably change while Alice moves in its vicinity. Hence $dM/d\tau = M'(U)\dot{U} \approx \text{const}$ and $\dot{X} \approx \dot{R}$. Given Alice's trajectory we can choose $\mathfrak{f} \approx 1$ at the horizon crossing and $\mathfrak{f} \to 0$ within the NEC-violating domain (as Eq. (12) can be valid only in the vicinity of the horizon). As the trajectory passes through $X_0 + r_g \to r_g$ the lhs of Eq. (34) behaves as

$$\int_\gamma \mathfrak{f}^2 \rho_A d\tau \approx -\int_\gamma \frac{M' r_0^2 \dot{R}^2 d\tau}{4\pi X^2 (r_0 - M)^2} \approx \int_\gamma \frac{M_\tau r_0 dX}{8\pi M (r_0 - M) X} \propto \log X_0 \to -\infty, \tag{35}$$

where $M_\tau = M'\dot{U}$ and we used $\dot{R} \sim \text{const}$. The rhs of Eq. (34) remains finite, and thus the QEI is violated. This violation indicates that the apparent horizon cannot expand, similarly to the spherically symmetric case.

On the other hand, nothing dramatic happens to the comoving density and pressure in the ingoing Kerr–Vaidya metric. Following the same steps we find that, e.g., the comoving energy density for the motion in the equatorial plane ($\Theta = \pi/2$, $\dot{\Theta} = 0$), is

$$\rho_A = \frac{T_{vv}}{4\dot{R}^2} + \mathcal{O}(a^2). \tag{36}$$

This quantity is finite and for $a = 0$ reduces to Eq. (10).

IV. DISCUSSION

Extending the self-consistent approach of horizon analysis to the axially symmetric spacetimes is difficult. Kerr–Vaidya metrics are the simplest nonstationary extension of the Kerr solution. All Kerr–Vaidya metrics violate classical energy conditions. While it could have been previously considered as a drawback, this violation is a necessary condition to describe an object with a trapped region that is accessible, even in principle, to a distant observer. Moreover, Kerr–Vaidya metrics are related by the Newman–Janis transformation to the pure Vaidya metrics that describe the geometry of physical black holes near their apparent horizons.

These simple geometries have several remarkable properties. The EMT of the Kerr-Vaidya metric, while violating the NEC for all $a \neq 0$ is a special case of type III form of the EMT in the Segre–Hawking–Ellis classification. An expanding spherically symmetric apparent horizon leads to a firewall and violates the quantum energy inequality that bounds the amount of negative energy in spacetimes of low curvature. The outgoing Kerr-Vaidya metric has the same property, showing that the firewall is not an artifact of spherical symmetry.

The apparent horizon of the outgoing Kerr-Vaidya metric coincides with the event horizon $r_0 = M + \sqrt{M^2 - a^2}$ of the Kerr metric, $M(u) = \text{const}$. For the ingoing the two surfaces are different. However, the difference $z(\theta) = r_g - r_0$ is small if $|M'(v)| \ll 1$, as in this case $z \propto M'$. However, while at the poles $z(0) = z(\pi) = 0$, a commonly used assumption $z'(0) = 0$ does not hold. As a result, the apparent horizon is not a smooth surface.

The assumption $a = \text{const}$ is incompatible with the continuous eventual evaporation of a physical black hole, as for $M < a$ the equation $\Delta = 0$ has no real roots and the Hawking temperature

$$T = \frac{1}{2\pi}\left(\frac{r_0 - M}{r_0^2 + a^2}\right), \tag{37}$$

that is proportional to the surface gravity, goes to zero as $M \to a$. Moreover, the semiclassical analysis [9] shows that during evaporation a/M decreases faster than M [43,44].

The variability of $a = J/M$ ratio should not affect existence of the firewall for accreting PBHs, as it is exhibited as a result $\Delta \to 0$ effect in (vr) coordinates and holds for $a = 0$. We will drop the assumption $a = \text{const}$ in the future work, and will to use the-self consistent approach to match the semiclassical results [43–45], as it was done in the spherically symmetric case.

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APPENDIX A: THE NEWMAN-JANIS TRANSFORMATION OF THE ADVANCED VAIDYA METRIC

The procedure follows the Newman-Janis prescription [35,37] that is applied to the Vaidya metric in advanced coordinates as the seed metric. We use the null tetrad that consists of a pair of real [1]

$$l^\mu = \delta^\mu_v + \frac{1}{2} f(v,r)\delta^\mu_r, \tag{A1}$$

$$n^\mu = -\delta^\mu_r \tag{A2}$$

and a pair of complex-conjugate vectors

$$m^\mu = \frac{1}{\sqrt{2}r}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\psi\right), \qquad \bar{m}^\mu = (m^\mu)^*, \tag{A3}$$

that satisfy the standard completeness and orthogonality relations,

$$\begin{aligned} l^\mu l_\mu &= l^\mu m_\mu = l^\mu \bar{m}_\mu = 0, \\ n^\mu n_\mu &= n^\mu m_\mu = n^\mu \bar{m}_\mu = m^\mu m_\mu = 0, \\ l^\mu n_\mu &= -m^\mu \bar{m}_\mu = -1. \end{aligned} \tag{A4}$$

The metric

$$ds^2 = -f(v,r)dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2, \tag{A5}$$

where $f(v,r) = 1 - 2M(v)/r$, is rewritten as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu. \tag{A6}$$

We treat r and v is complex-valued coordinates and introduce a real-valued function

$$f = 1 - M\left(\frac{1}{2}(v+v^*)\right)\left(\frac{1}{r} + \frac{1}{r^*}\right), \tag{A7}$$

that coincides with $f(v,r)$ for real values of the coordinates, $v = v^*$, $r = r^*$. The complex coordinate transformation

$$x'^\mu = x^\mu - ia(\delta^\mu_r + \delta^\mu_v)\cos\theta, \tag{A8}$$

i.e.,

$$v' = v - ia\cos\theta, \qquad \theta' = \theta, \tag{A9}$$

$$r' = r - ia\cos\theta, \qquad \psi' = \psi, \tag{A10}$$

leaves M invariant and transforms the tetrad as

$$l'^\mu = \delta^\mu_v + \frac{1}{2}\mathcal{F}(v,r,\theta)\delta^\mu_r, \qquad n'^\mu = -\delta^\mu_r, \tag{A11}$$

$$m'^\mu = \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(ia(\delta^\mu_v + \delta^\mu_r)\sin\theta + \delta^\mu_\theta + \frac{i}{\sin\theta}\delta^\mu_\psi\right), \tag{A12}$$

where after restricting to the real-valued coordinates

$$\mathcal{F} = 1 - 2M(v)r/\rho^2. \tag{A13}$$

Substituting these explicit expressions into the transformed metric

$$g'^{\mu\nu} = -l'^\mu n'^\nu - l'^\nu n'^\mu + m'^\mu \bar{m}'^\nu + m'^\nu \bar{m}'^\mu, \tag{A14}$$

produces the Kerr-Vaidya metric in the advanced coordinates that is given in Eq. (12).

APPENDIX B: ENERGY-MOMENTUM TENSOR AND THE NEC VIOLATION FOR KERR-VAIDYA METRIC

The nonzero components of the energy-momentum tensor for the Kerr–Vaidya metric in advanced coordinates are

$$T_{vv} = \frac{r^2(a^2+r^2) - a^4\cos^2\theta\sin^2\theta}{4\pi\rho^6}M_v - \frac{a^2 r\sin^2\theta}{8\pi\rho^4}M_{vv}, \tag{B1}$$

$$T_{v\theta} = -\frac{a^2 r\sin\theta\cos\theta}{4\pi\rho^4}M_v, \tag{B2}$$

$$T_{v\psi} = -a\sin^2\theta T_{vv} - a\sin^2\theta\frac{r^2 - a^2\cos^2\theta}{8\pi\rho^4}M_v, \tag{B3}$$

$$T_{\theta\psi} = \frac{a^3 r\sin^3\theta\cos\theta}{4\pi\rho^4}M_v, \tag{B4}$$

$$T_{\psi\psi} = a^2\sin^4\theta T_{vv} + a^2\sin^4\theta\frac{r^2 - a^2\cos^2\theta}{4\pi\rho^4}M_v. \tag{B5}$$

The non-zero components of the energy-momentum tensor for the Kerr–Vaidya metric in retarded coordinates are

$$T_{uu} = -\frac{r^2(a^2+r^2) - a^4\cos^2\sin^2\theta}{4\pi\rho^6}M_u - \frac{a^2 r\sin^2\theta}{8\pi\rho^4}M_{uu}, \tag{B6}$$

$$T_{u\theta} = -\frac{2a^2 r\sin\theta\cos\theta}{8\pi\rho^4}M_u, \tag{B7}$$

$$T_{u\psi} = -a\sin^2\theta T_{uu} + a\sin^2\theta\frac{r^2 - a^2\cos^2\theta}{8\pi\rho^4}M_u, \quad \text{(B8)}$$

$$T_{\theta\psi} = \frac{2a^3 r\sin^3\theta\cos\theta}{8\pi\rho^4}M_u, \quad \text{(B9)}$$

$$T_{\psi\psi} = a^2\sin^4\theta T_{uu} - a^2\sin^4\theta\frac{(r^2 - a^2\cos^2\theta)}{4\pi\rho^4}M_u. \quad \text{(B10)}$$

In the advanced coordinate the decomposition (15) of the EMT is obtained with the vectors

$$k_\mu = (1, 0, 0, -a\sin^2\theta), \quad \text{(B11)}$$

and

$$q_\mu = \left(0, 0, T_{v\theta}, -a\sin^2\theta\frac{r^2 - a^2\cos^2\theta}{8\pi\rho^4}M_v\right). \quad \text{(B12)}$$

The orthonormal tetrad with where $k^\mu = e^\mu_{\hat{1}} + e^\mu_{\hat{0}}$ is given by

$$e^\mu_{\hat{0}} = (-1, rM/\rho^2, 0, 0), \quad \text{(B13)}$$

$$e^\mu_{\hat{1}} = (1, 1 - rM/\rho^2, 0, 0) \quad \text{(B14)}$$

$$e^\mu_{\hat{2}} = (0, 0, 1/\rho, 0) \quad \text{(B15)}$$

$$e^\mu_{\hat{3}} = \frac{1}{\rho}(a\sin\theta, a\sin\theta, 0, \csc\theta). \quad \text{(B16)}$$

Hence the EMT is given by Eq. (16) with $\nu = T_{vv}$ and $q^\mu = q^{\hat{2}}e^\mu_{\hat{2}} + q^{\hat{3}}e^\mu_{\hat{3}}$ with

$$q^{\hat{2}} = -\frac{a^2 rM_v}{8\pi\rho^5}\sin 2\theta, \quad \text{(B17)}$$

$$q^{\hat{3}} = -\frac{a(r^2 - a^2\cos^2\theta)M_v}{8\pi\rho^5}\sin\theta. \quad \text{(B18)}$$

A generic form [19] of a type III EMT is

$$T^{\hat{a}\hat{b}} = \left(\begin{array}{ccc|c} \varrho & 0 & \varphi & 0 \\ 0 & -\varrho & \varphi & 0 \\ \varphi & \varphi & -\varrho & 0 \\ \hline 0 & 0 & 0 & p \end{array}\right). \quad \text{(B19)}$$

All four Lorentz-invariant eigenvalues are zero if and only if $\varrho = p = 0$. In this case the nonzero eigenvalues of the matrix $T^{\hat{a}\hat{b}}$ are

$$\tilde{\lambda}_{1,2} = \pm\sqrt{2}\varphi. \quad \text{(B20)}$$

APPENDIX C: APPARENT HORIZON IN THE OUTGOING VAIDYA METRIC

On a hypersurface $v = \text{const}$ we introduce the surface coordinates $(\check{r}, \theta, \phi)$ where the bulk coordinate r is expressed in terms of the coordinates $\check{r}$ and θ as

$$r = \check{r} + z(\theta), \quad \text{(C1)}$$

for some function z. Locating the apparent horizon $r_g(v, \theta)$ is then expressed as a problem of finding the function $z(\theta)$ such that $r_g = r_0(v) + z(\theta)$. While the function z also depends on v, it does not affect the derivation below and this dependence is omitted.

Two spacelike vectors that are tangent to the surface $\check{r} = \text{const}$ are

$$b^\mu_\theta = z'(\theta)\delta^\mu_r + \delta^\mu_\theta, \qquad b^\mu_\psi = \delta^\mu_\psi. \quad \text{(C2)}$$

We obtain the outward- and inward-pointing future-directed null vectors $l^+ \equiv l$ and $l^- \equiv N$ by using the orthogonality condition $l^\pm_\mu b^\mu_A = 0$. Before the rescaling $l^v = 1$ and the normalization $N \cdot l = -1$ the two null vectors are given by

$$l^\pm_\mu \propto (-1, \ell_\pm, -\ell_\pm z'(\theta), 0). \quad \text{(C3)}$$

The two values of $\ell_\pm$ are obtained from the null condition $l^\pm \cdot l^\pm = 0$,

$$\ell_\pm = \frac{1}{\Delta + z'^2}\Big(r^2 + a^2 \pm \sqrt{2a^2 rM\sin^2\theta + \rho^2(a^2 + r^2) - a^2 z'^2\sin^2\theta}\Big). \quad \text{(C4)}$$

After setting $l^v = 1$ the leading order components of the future-directed outward-pointing null vector orthogonal to the two-surface $r = r_0 + z(\theta)$ are

$$l^v = 1, \quad \text{(C5)}$$

$$l^r = \frac{(r_0^2 - a^2)z'}{2r_0(r_0^2 + a^2)}, \quad \text{(C6)}$$

$$l^\theta = -\frac{z'}{r_0^2 + a^2}, \quad \text{(C7)}$$

$$l^\psi = \frac{a}{r_0^2 + a^2} + \frac{a(a^4 - 7a^2 r_0^2 - 10r_0^4 - a^2(r_0^2 - a^2)\cos 2\theta)z}{4r_0(r_0^2 + a^2)}, \quad \text{(C8)}$$

where we assume that $|z| \ll r_0$ and $|z'| \ll r_0$.

We now consider the change in the two-dimensional area after one infinitesimal step $\delta\lambda$ of the evolution $x^\mu_{\text{in}} \to x^\mu_{\text{fn}}$, where

$$x^\mu_{\text{in}} = (v, r_0 + z(\theta), \theta, 0), \tag{C9}$$

$$x^\mu_{\text{fn}} = x^\mu_{\text{in}} + l^\mu(x^\mu_{\text{in}}) d\lambda, \tag{C10}$$

and λ is the affine parameter.

The determinant of the two-dimensional metric σ_{AB} is given by Eq. (23). To obtain the initial area the Kerr-Vaidya metric is evaluated at x_{in} and the vectors b_A are given by Eq. (C2). To calculate the final area we evaluate the four-dimensional metric at the point x_{fn}. In addition, since the points x_ψ and x_θ that are defined by Eq. (21) evolve with the vectors

$$l^\mu(x_\psi) = l^\mu(x_{\text{in}}), \tag{C11}$$

and

$$l^\mu(x_\theta) \approx l^\mu(x_{\text{in}}) + \partial_\theta l^\mu(x_{\text{in}}) \delta\theta, \tag{C12}$$

respectively, the cross section tangents evolve as

$$b^\mu_\psi \to b^\mu_\psi, \qquad b^\mu_\theta \to b^\mu_\theta + \partial_\theta l^\mu(x_{\text{in}}) d\lambda. \tag{C13}$$

The area differential $d\delta A \propto (d\sqrt{\sigma}/d\lambda)d\lambda$ is evaluated by subtracting $\sqrt{\sigma(x_{\text{in}})}$ from the first-order expansion in $d\lambda$ of $\sqrt{\sigma(x_{\text{fn}})}$. The desired Eq. (25) is obtained by setting $d\delta A = 0$.

An alternative derivation is based on extending the vector field l^μ from the hypersurface $v = \text{const}$ to the bulk in such a way that the new field $\mathring{l}^\mu$ satisfies the geodesic equation $\mathring{l}^\mu_{;\nu}\mathring{l}^\nu = 0$. In fact, this needs to be done only on the hypersurface itself, where it is realized by setting

$$\mathring{l}^\mu \coloneqq l^\mu, \tag{C14}$$

and thus $\mathring{l}^0 = l^0 = 1$,

$$\mathring{l}^\mu_{;m} \coloneqq l^\mu_{;m}, \tag{C15}$$

for $m = 1, 2, 3$, and setting the covariant derivative over v as

$$\mathring{l}^\mu_{;0} \coloneqq -l^\mu_{;m} l^m, \tag{C16}$$

For the affinely parametrized geodesic congruence $\vartheta = \mathring{l}^\mu_{;\mu}$, and Eq. (25) follows from

$$\vartheta = -l^0_{;m} l^m + l^m_{;m} = 0. \tag{C17}$$

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