

## Invariant approach to Weyl's unified field theory

T. A. T. Sanomiya,<sup>1,\*</sup> I. P. Lobo,<sup>2,†</sup> J. B. Formiga,<sup>1,‡</sup> F. Dahia,<sup>1,§</sup> and C. Romero<sup>1,||</sup>

<sup>1</sup>*Departamento de Física, Universidade Federal da Paraíba  
Caixa Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil*

<sup>2</sup>*Departamento de Física, Universidade Federal de Lavras,  
Caixa Postal 3037, 37200-000 Lavras-MG, Brazil*

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We revisit Weyl's unified field theory, which arose in 1918, shortly after general relativity was discovered. As is well known, in order to extend the program of geometrization of physics started by Einstein to include the electromagnetic field, H. Weyl developed a new geometry which constitutes a kind of generalization of Riemannian geometry. However, despite its mathematical elegance and beauty, a serious objection was made by Einstein, who considered Weyl's theory not suitable as a physical theory since it seemed to lead to the prediction of a not yet observed effect, the so-called “second clock effect”. In this paper, our aim is to discuss Weyl's proposal anew and examine its consistency and completeness as a physical theory. Finally, we propose new directions and possible conceptual changes in the original work. As an application, we solve the field equations assuming a Friedmann-Robertson-Walker universe and a perfect fluid as its source. Although we have entirely abandoned Weyl's attempt to identify the vector field with the 4-dimensional electromagnetic potentials, which here must be simply viewed as part of the space-time geometry, we believe that in this way we could perhaps be led to a rich and interesting new modified gravity theory.

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### I. INTRODUCTION

In his attempt to unify gravity with electromagnetism H. Weyl discovered a new geometry, which in a certain way, constitutes a kind of generalization of Riemannian geometry [1]. As he wrote in the introduction of his original paper, the insight which led him to a new geometry came from the perception that the Riemann theory of parallel transport was entirely dependent on concepts directly taken from our intuition of the “rigid” Euclidean spaces. Indeed, in this geometric setting the parallel transport of a vector  $V$  along a certain path is required to preserve the length of  $V$ . In more technical terms, this requirement imposed on a manifold endowed with a metric tensor  $g$  and a connection  $\nabla$  arises as a direct consequence of what is known in the literature as the compatibility condition between  $g$  and  $\nabla$ . Then, from Koszul formula, it follows the celebrated Levi-Civita theorem, which states that for torsion-free manifolds there exists a unique connection completely determined by the metric [2]. Weyl, however, found the Riemannian compatibility condition too restrictive, and replaced it by a much weaker form, which then allows for the variation of the length of vectors along parallel transport, this process

being regulated by a new geometric object, namely a 1-form field  $\sigma$ , later to be identified with the electromagnetic four-potential. The introduction of the 1-form field  $\sigma$  leads, in turn, to a new notion of curvature, a sort of “length curvature” (*Streckenkrümmung*) in addition to the “direction curvature” (*Richtungskrümmung*), the latter represented by the Riemann tensor. The length curvature is quantified by the 2-form  $F = d\sigma$ , whose mathematical properties present striking similarities with those possessed by the electromagnetic tensor. After this first development, another important discovery made by Weyl was that his geometric construction exhibited a new kind of symmetry. Indeed, he found that his modified compatibility condition, as well as the length curvature, were both invariant under a certain group of transformations involving  $g$  and  $\nabla$ . It is worth mentioning that the discovery of this new symmetry, later to be called *gauge symmetry* (in addition to the already known general relativistic invariance under space-time diffeomorphisms) is now viewed as a most significant fact in the history of physics: it represents the birth of modern gauge theories [3]. It turned out that *Weyl's Principle of Gauge Invariance* played an essential role in the development of the unified field theory. Indeed, in building the action for the gravitational and the (geometric) electromagnetic fields, Weyl was primarily guided by this principle and chose the simplest of all possible invariants. He also took advantage of the principle to work out the field equations in a particular gauge (the “natural” gauge), where

\*sanomiya@fisica.ufpb.br

†iarley.lobo@ufla.br

‡jformiga@fisica.ufpb.br

§fdahia@fisica.ufpb.br

||cromero@fisica.ufpb.br

the field equations look much more simple. Enriched by the property of gauge symmetry, the geometric structure of space-time in Weyl's theory became more complex, rather similar to what is known as a conformal structure, that is, a manifold equipped with an equivalence class of triples  $\mathcal{M} = \{(g, \nabla, \sigma)\}$ , in which the members of the class are related by Weyl transformations and satisfy a particular compatibility condition. It is to be expected that in such space-time only invariants (in the sense of Weyl's principle) may have physical meaning. (For instance, as Weyl put it clearly, the usual metric concept of length is no longer meaningful [1].) This entirely new framework has far-reaching consequences as far as it selects which physical scenarios are allowed to come in, and it is our aim in the present work to investigate some of these possibilities under the guide of gauge invariance hoping in this way to carry on with Weyl's original program. With regard to the latter point we think two questions must be addressed. First, to what extent did Weyl succeed in constructing a unifying theory of gravity and electromagnetism? Second, is the theory free from inconsistency and/or incompleteness? (For the reader interested in historical and philosophical issues concerning Einstein's critical review of Weyl's unified theory see, for instance, [4], and references therein.)

The paper is organized as follows. In Sec. II, we give a brief summary of Weyl geometry. We then proceed to Sec. III to present the Weyl field equations, both written in an arbitrary and in the *natural gauge*. In Sec. IV, we discuss the field equations in the limit when space-time becomes Riemannian, and give an interpretation for the constant that appears in the natural gauge. Section V contains a discussion of the nature of the geometric electromagnetism introduced by Weyl and the conceptual problems arising from this identification. Section VI is devoted to the notion of time in Weyl theory and the related problem of the second clock effect. In Sec. VII, we touch on the question of how to extend Weyl theory to include matter. In Sec. VIII, we outline the axiomatic structure of the new approach with the aim at defining a gauge-invariant procedure to extend the Weyl field equations to include matter fields. As an application, in Sec. IX we solve the field equations assuming a Friedmann-Robertson-Walker universe and a perfect fluid as its source. We conclude with some remarks in Sec. X.

## II. A BRIEF SUMMARY OF WEYL GEOMETRY

Weyl geometry is perhaps one of the simplest generalizations of Riemannian geometry, the only modification being the fact that the covariant derivative of the metric tensor  $g$  is not zero, but instead given by<sup>1</sup>

<sup>1</sup>This article was written in parallel with the authors' contribution to the Proceedings of the 10th Alexander Friedmann Seminar on Gravitation and Cosmology, and should be considered as a completed and streamlined version of the latter. Therefore identical prose may be found in some parts between the two texts.

$$\nabla_\alpha g_{\beta\lambda} = \sigma_\alpha g_{\beta\lambda}, \quad (1)$$

where  $\sigma_\alpha$  denotes the components of a one-form field  $\sigma$  in a local coordinate basis. This weakening of the Riemannian compatibility condition is entirely equivalent to requiring that the length of a vector field may change when parallel-transported along a curve in the manifold [5]. We shall refer to the triple  $(M, g, \sigma)$  consisting of a differentiable manifold  $M$  endowed with both a metric  $g$  and a 1-form field  $\sigma$  as a *Weyl gauge* (or, *Weyl frame*). Now one important discovery made by Weyl was the following. Suppose we perform the conformal transformation

$$\bar{g} = e^f g, \quad (2)$$

where  $f$  is an arbitrary scalar function defined on  $M$ . Then, the Weyl compatibility condition (1) still holds provided that we let the Weyl field  $\sigma$  transform as

$$\bar{\sigma} = \sigma + df. \quad (3)$$

In other words, the Weyl compatibility condition does not change when we go from one gauge  $(M, g, \sigma)$  to another gauge  $(M, \bar{g}, \bar{\sigma})$  by simultaneous transformations in  $g$  and  $\sigma$ .

If we assume that the Weyl connection  $\nabla$  is symmetric, a straightforward algebra shows that one can express the components of the affine connection in an arbitrary vector basis completely in terms of the components of  $g$  and  $\sigma$ :

$$\Gamma_{\beta\lambda}^\alpha = \{\alpha_{\beta\lambda}\} - \frac{1}{2} g^{\alpha\mu} [g_{\mu\beta} \sigma_\lambda + g_{\mu\lambda} \sigma_\beta - g_{\beta\lambda} \sigma_\mu], \quad (4)$$

where  $\{\alpha_{\beta\lambda}\}$  represents the Christoffel symbols. It is not difficult to see that the connection and, consequently, the geodesic equations are invariant with respect to the transformations (2) and (3).

We now present Weyl's second great discovery. Suppose we are given two vector fields  $V$  and  $U$  parallel-transported along a curve  $\alpha = \alpha(\lambda)$ . Then, (1) clearly leads to the following equation:

$$\frac{d}{d\lambda} g(V, U) = \sigma \left( \frac{d}{d\lambda} \right) g(V, U), \quad (5)$$

where  $\frac{d}{d\lambda}$  denotes the vector tangent to  $\alpha$ . If we integrate this equation along the curve  $\alpha$ , starting from a point  $P_0 = \alpha(\lambda_0)$ , we obtain [5]

$$g(V(\lambda), U(\lambda)) = g(V(\lambda_0), U(\lambda_0)) e^{\int_{\lambda_0}^{\lambda} \sigma \left( \frac{d}{d\rho} \right) d\rho}. \quad (6)$$

Setting  $U = V$  and denoting by  $L(\lambda)$  the length of the vector  $V(\lambda)$  at a point  $P = \alpha(\lambda)$  of the curve, it is easy to verify that in a local coordinate system  $\{x^\alpha\}$  the equation (5) becomes

$$\frac{dL}{d\lambda} = \frac{\sigma_\alpha}{2} \frac{dx^\alpha}{d\lambda} L. \quad (7)$$

Let us now consider the set of all closed curves  $\alpha: [a, b] \in R \rightarrow M$ , i.e., with  $\alpha(a) = \alpha(b)$ . Then, either from (6) or (7) it follows that

$$L = L_0 e^{\frac{1}{2} \oint \sigma_\alpha dx^\alpha},$$

where  $L_0$  and  $L$  denotes the values of  $L(\lambda)$  at  $a$  and  $b$ , respectively. From Stokes's theorem we then can write<sup>2</sup>

$$L = L_0 e^{-\frac{1}{4} \int \int F_{\mu\nu} dx^\mu \wedge dx^\nu},$$

where  $F_{\mu\nu} = \partial_\nu \sigma_\mu - \partial_\mu \sigma_\nu$ . We thus see that, according to the rules of Weyl geometry, the necessary and sufficient condition for a vector to have its original length preserved after being parallel transported along any closed trajectory is that the 2-form  $F = d\sigma = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$  vanishes.

Therefore Weyl realized that in his new geometry there are two kinds of curvature, a *direction curvature* (*Richtungskrummung*) and a *length curvature* (*Streckenkrummung*). The first is responsible for changes in the direction of parallel-transported vectors and is given by the usual curvature tensor  $R^\alpha_{\beta\mu\nu}$ , while the other regulates the changes in their length, and is given by  $F_{\mu\nu}$ . Weyl's second great discovery was that the 2-form  $F$  is invariant under the gauge transformation (3). The analogy with the electromagnetic field is now clear and becomes even more so when we take into account that  $F$  satisfies the identity  $dF = 0$ .<sup>3</sup>

### III. THE FIELD EQUATIONS OF WEYL'S UNIFIED FIELD THEORY

As we know, the Weyl transformations (2) and (3) define a whole equivalence class in the set  $\{(M, g, \sigma)\}$  of all Weyl gauges. It is then natural to expect that, as in conformal geometry the geometrical objects of interest are conformal-invariant, here we should look for those that are gauge-invariant.<sup>4</sup> Surely, these invariants will be fundamental to build the action that is expected to give the field equations of the geometrical unified theory. Some basic invariants are easily found: the affine connection  $\Gamma^\alpha_{\beta\lambda}$ , the curvature tensor  $R^\alpha_{\beta\mu\nu}$ , the Ricci tensor  $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ , and the length curvature

$F_{\mu\nu} = \partial_\nu \sigma_\mu - \partial_\mu \sigma_\nu$ . The simplest invariant scalars, in four-dimensional space-time, that can be constructed out of these are:  $\sqrt{-g}R^2$ ,  $\sqrt{-g}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ ,  $\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}$ , and  $\sqrt{-g}F_{\alpha\beta}F^{\alpha\beta}$ , where  $R = g^{\alpha\beta}R_{\alpha\beta}$  denotes the Ricci scalar calculated with the Weyl affine connection. (Curiously, the first of these invariants appears in the action of some F(R) theories, for instance, in the well known Starobinsky's model of inflation [7].)

For reasons of consistency of his physics with the new geometry, Weyl required his theory to be completely invariant with respect to change between gauges (or frames). On the other hand, he chose the simplest of all possible invariant actions, namely,

$$S = \int d^4x \sqrt{|g|} [\omega R^2 + \omega F_{\mu\nu} F^{\mu\nu}], \quad (8)$$

where  $\omega$  is a constant.<sup>5</sup> This action describes the gravitational-electromagnetic sector only.<sup>6</sup> (Incidentally, it is odd that Weyl did not consider the coupling with matter, which clearly constitutes an element of incompleteness of the theory. We shall return to this point later.) Carrying out variations with respect to  $\sigma_\mu$  and  $g_{\mu\nu}$  will lead, respectively, to the following field equations:

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}) = \frac{3}{2\omega} g^{\mu\nu} (R\sigma_\nu + \partial_\nu R), \quad (9)$$

$$R \left( R_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} R \right) = \omega T_{\mu\nu} - D_{\mu\nu}, \quad (10)$$

where  $R_{(\mu\nu)}$  stands for the symmetric part of  $R_{\mu\nu}$ ,  $T_{\mu\nu} = F_{\mu\alpha} F^\alpha_\nu + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$  and  $D_{\mu\nu} = \nabla_{(\mu} \nabla_{\nu)} R + \frac{1}{2} R (\sigma_{\mu;\nu} + \sigma_{\nu;\mu}) + R\sigma_\mu\sigma_\nu + R_{,\mu}\sigma_\nu + R_{,\nu}\sigma_\mu$ . Note that the presence of the term  $D_{\mu\nu}$  introduces derivatives of third and fourth order in the theory. This fact was readily pointed out by Pauli, who considered it to be a flaw of Weyl theory [9]. (However, as is well known, present-day researchers welcome higher-derivative theories since, as was later shown, they allow renormalizability of divergences in the quantum corrections to the interactions of matter fields [10].)

The above equations are drastically simplified if we choose the so-called *natural gauge*, defined by Weyl as  $R = \Lambda = \text{const} \neq 0$ . In this case (9) and (10) reduce to

<sup>2</sup>Here we are assuming that the region of integration is simply connected.

<sup>3</sup>In a local coordinate system, this identity takes the form  $\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu} = 0$ , which looks identical to one pair of Maxwell's equations.

<sup>4</sup>In conformal geometry, one basic invariant is the Weyl tensor  $W^\alpha_{\beta\mu\nu}$ . In conformal gravity, this tensor is used to form the scalar  $W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu}$ , which, then, defines the gravitation sector of the action [6].

<sup>5</sup>Here we are not considering the matter action.

<sup>6</sup>The question why a term proportional to  $R_{\mu\nu} R^{\mu\nu}$  was not included in the action of the original Weyl theory is interesting and deserves a comment. We believe that Weyl's choice was motivated not only by simplicity, but also because, by using a particular gauge transformation, he could formally rewrite (8) as similar to the Einstein-Hilbert's action coupled to a (massive) vector field, in which the cosmological constant arises in a rather natural way. Let us mention that recently some generalizations of the Weyl action have been considered (See, for instance, [8].)

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \frac{3\Lambda}{2\omega}\sigma^\mu, \quad (11)$$

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}g_{\mu\nu} + \frac{\Lambda}{4}g_{\mu\nu} + \frac{3}{2}\left(\sigma_\mu\sigma_\nu - \frac{1}{2}g_{\mu\nu}\sigma^\alpha\sigma_\alpha\right) = \frac{\omega}{\Lambda}T_{\mu\nu}, \quad (12)$$

where  $\tilde{R}_{\mu\nu}$  and  $\tilde{R}$  are now Riemannian and defined with respect to the metric  $g_{\mu\nu}$ .<sup>7</sup> At this point, let us note that we can reobtain the field equations in a general gauge (9) and (10) in an elegant and straightforward way by using the gauge transformations (2) and (3) only, avoiding the long and tedious calculations involved in the process of carrying out variations in the action (8) (see Appendix).

It is worth mentioning that Weyl's theory correctly predicts the perihelion precession of Mercury as well as the gravitation deflection of light by a massive body [9]. This is a consequence of the fact that all vacuum solutions of Einstein's equations (including the Schwarzschild solution) satisfy (9) and (10) when we set  $\sigma_\mu = 0$ .

Now before we start our discussion of the Einstein's objection to Weyl's theory, we would like to stress that to build his theory Weyl adopted a very strong and, at the same time, rather restrictive principle, the so-called *Principle of Gauge Invariance*, which requires all physical quantities to be invariant under the gauge transformations (2) and (3). This principle was strictly followed by Weyl and guided him to choose the action (8). It should also be noted here that any invariant scalar of this geometry must necessarily be formed from both the metric  $g_{\mu\nu}$  and the Weyl gauge field  $\sigma_\mu$ . These two fields constitute an essential and intrinsic part of the geometry and neither of them can be neglected when we want to construct an invariant scalar, so they are, in this sense, inseparable, and must always appear together.

#### IV. THE GENERAL RELATIVISTIC LIMIT

In this section, let us briefly examine how we can recover general relativity from the Weyl field equations. First, let us assume that in a certain gauge we have  $\sigma = 0$ , which then means that the geometry becomes Riemannian and  $R_{(\mu\nu)} = R_{\mu\nu} = \tilde{R}_{\mu\nu}$ . In this case,  $F_{\mu\nu}$  and  $T_{\mu\nu}$  vanishes, and then from (9) we have  $R = \tilde{R} = \Lambda = \text{constant}$ , which in turn implies  $D_{\mu\nu} = 0$ . Now from (10) we are left with two possibilities:  $\tilde{R} = 0$  or  $\tilde{R}_{\mu\nu} = \frac{1}{4}\Lambda g_{\mu\nu}$ . In the first case, this means that all solutions of Einstein vacuum equations (with vanishing cosmological constant) are included. In the second case, the Weyl vacuum solutions correspond to

<sup>7</sup>Let us recall that in Weyl theory the so-called natural gauge is defined by  $R = \Lambda = \text{const}$ . It should be clear, however, that in this choice here  $R$  refers to the Ricci scalar calculated with the Weyl connection, and not with the Christoffel symbols in general relativity.

spaces of constant Ricci curvature (Einstein spaces), and this seems to make the cosmological constant appear in a natural way, deduced directly from the field equations.<sup>8</sup> (Incidentally, if  $\Lambda > 0$ , one may be tempted to consider this fact as an indication that Weyl theory might naturally lead to the idea that the empty space-time of special relativity should be identified to the de Sitter space, a speculation which has gained more attention recently after the discovery of the acceleration expansion of the Universe [11].) However, if  $\Lambda$  is sufficiently small its effects in the field equations can be neglected, and then Weyl's field equations becomes identical to the Einstein vacuum equations and the results of the so-called solar system tests satisfied by general relativity will be in accordance with Weyl theory. Moreover, if  $\Lambda = 0$  then Weyl's theory includes special relativity as a particular case.

#### V. THE GEOMETRIZED ELECTROMAGNETIC FIELD

It is certainly undeniable that the geometric structure found by Weyl in his attempt to unify gravity and electromagnetism leading in a very natural way to the appearance of the geometric tensor  $F_{\mu\nu}$  whose algebraic and invariant properties exhibit striking similarities with Faraday tensor. However, looking into the field equations derived from the action (8) chosen by Weyl deviations from Maxwell equations become apparent. For instance, let us consider the field equations written in the natural gauge. The equation (11) tells us that the electromagnetic field is coupled to itself, i.e., it acts as its own source. On the other hand, in (12) there are nonlinear terms in  $\sigma$ , which are more characteristic of nonlinear theories of electrodynamics. It is also instructive to have a look at the action (8), which, when it is written in the natural gauge [12] is given by

$$S = \int d^4x\sqrt{-g}\left[\tilde{R} + \frac{\omega}{2\Lambda}F_{\mu\nu}F^{\mu\nu} + \frac{3}{2}\sigma_\mu\sigma^\mu - \frac{\Lambda}{2}\right],$$

which is equivalent to the action of the Proca's neutral spin-1 field in curved space-time with the cosmological constant [13].

Another problem of Weyl's geometrized electromagnetism concerns the motion of neutral and electric charged particles. Because the affine geodesics are the only curves which are gauge-invariant one would expect that they would describe the motion of particles interacting only with the gravitational and electromagnetic field. However, it is clear that from the geodesic equations one cannot obtain the equation of motion for a charged particle moving in a curved space-time, i.e., the Lorentz force equation. Let us recall that in special or general relativity the Lorentz force appears when we perform variations in the action

<sup>8</sup>Note that the same results follow easily from the Weyl equations written in the natural gauge.



$S = \int d^4x \sqrt{-g} A_\mu dx^\mu$  containing the interaction of charged particle with the electromagnetic 4-potential  $A_\mu$ . In Weyl theory, there is no prescription of how matter interacts with the gravitation and the (geometric) electromagnetic field, an element of "incompleteness" that we shall consider later, in Sec. VII.

## VI. THE PROBLEM OF TIME IN WEYL'S THEORY

It has been recognized in recent years that the notion of time in Weyl's theory is rather problematic. To begin with, let us recall Einstein's objection contained in an addendum to Weyl's original paper concerning the dependence of the clock rate of ideal clocks on their paths [1]. This is now referred to as the *second clock effect* [14]. In order to examine Einstein's objection, let us first make more explicit the hypotheses upon which the argument is based, which may be stated as follows:

- (i) The proper time  $\Delta\tau$  measured by a clock travelling along a curve  $\alpha = \alpha(\lambda)$  is given as in general relativity, namely, by the (Riemannian) prescription

$$\Delta\tau = \frac{1}{c} \int [g(V, V)]^{\frac{1}{2}} d\lambda = \frac{1}{c} \int [g_{\mu\nu} V^\mu V^\nu]^{\frac{1}{2}} d\lambda, \quad (13)$$

where  $V$  denotes the vector tangent to the clock's world line and  $c$  is the speed of light. This supposition is known as the *clock hypothesis* and clearly assumes that the proper time only depends on the instantaneous speed of the clock and on the metric field [15]. (Note that the gauge field  $\sigma$ , which is also an essential and unseparable part of the Weyl space-time geometry does not appear in the above expression [15].)

- (ii) The clock rate of a clock (in particular, atomic clocks) is modeled by the (Riemannian) length  $L = \sqrt{g(\Upsilon, \Upsilon)}$  of a certain vector  $\Upsilon$ . As the clock moves in space-time  $\Upsilon$  is parallel-transported along its worldline from a point  $P_0$  to a point  $P$ ; hence,  $L = L_0 e^{\frac{1}{2} \int \sigma_\alpha dx^\alpha}$ ,  $L_0$  and  $L$  denoting the duration of the clock rate of the clock at  $P_0$  and  $P$ , respectively. (Later, this assumption was made explicit by Ehlers, Pirani, and Schild [16].)

Let us now examine more closely these two assumptions. We start with the first hypothesis (A). First, for consistency with the *Principle of Gauge Invariance* proper time should be a gauge-invariant concept, and clearly this requirement is not fulfilled by (13). It turns out, however, that up to this date no such invariant notion of proper time consistent with Weyl's theory (and which does not lead to the second clock effect) is known.<sup>9</sup> Secondly, in the second hypothesis, gauge

invariance is again violated as the concept of clock rate is not modeled as a gauge-invariant physical quantity, let alone the fact that the Weyl geometrical field plays no role in its determination.

Incidentally, it should be mentioned that a new notion of proper time, entirely consistent with the *Principle of Gauge Invariance*, was given by V. Perlick [17]. His line of reasoning is the following. In Riemannian geometry, the compatibility condition between the metric and the connection may be given, as we know, by the equation

$$\nabla_V [g(W, U)] = g(\nabla_V W, U) + g(W, \nabla_V U), \quad (14)$$

where  $V$ ,  $W$  and  $U$  are vector fields. Now consider a curve  $\alpha = \alpha(\lambda)$  and set  $V = W = U = \frac{d}{d\lambda}$ , the vector tangent to  $\alpha$ . Then,  $\frac{d}{d\lambda} g(\frac{d}{d\lambda}, \frac{d}{d\lambda}) = 2g(\nabla_{\frac{d}{d\lambda}} \frac{d}{d\lambda}, \nabla_{\frac{d}{d\lambda}} \frac{d}{d\lambda})$ , and we can say that  $\lambda$  is the arc-length parameter  $s$  of the curve  $\alpha$  (up to an affine reparametrization) if and only if  $g(\nabla_{\frac{d}{d\lambda}} \frac{d}{d\lambda}, \frac{d}{d\lambda}) = 0$ . If this condition, which may be taken to characterize the arc-length parameter in Riemannian geometry, is carried over to Weyl geometry, then we have a definition of proper time which is completely invariant with respect to Weyl transformations. This was, in fact, the starting point of Perlick's definition of proper time, which, amazingly enough, also leads to the second clock effect [18]. By replacing the (noninvariant) general relativistic parametrization condition  $g(\frac{d}{d\lambda}, \frac{d}{d\lambda}) = 1$  by the gauge-invariant equation  $g(\nabla_{\frac{d}{d\lambda}} \frac{d}{d\lambda}) = 0$  it can be shown that the proper time elapsed between two events corresponding to the parameter values  $\lambda_0$  and  $\lambda$  in the curve  $\alpha$  is given by

$$\Delta\tau(\lambda) = \left( \frac{d\tau/d\lambda}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} \right)_{\lambda=\lambda_0} \int_{\lambda_0}^{\lambda} e^{-\frac{1}{2} \int_{u_0}^u \sigma_\rho \dot{x}^\rho ds} [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{1/2} du, \quad (15)$$

where dot means derivative with respect to the curve's parameter [18]. It has also been shown that Perlick's notion of time has all the properties a definition of proper time in a Weyl space-time should have, such as, Weyl invariance, positive definiteness, additivity. Moreover, in the limit in which the length curvature  $F_{\mu\nu}$  goes to zero Perlick's time reduces both to the Einsteinian proper time and to the proper time defined in Weyl geometric scalar-tensor theories [19]. Furthermore, it has been proved the equivalence between Perlick's definition of proper time and the one given in the paper by Ehlers, Pirani, and Schild (EPS) [16,18], which was entirely based on an axiomatic approach [16,18,20].

It is interesting to note that Perlick's proposal leads to a new kind of geometry. Indeed, one can view the equation (1) as a prescription of how to define length of curves in an entire class of Weyl manifolds. In other words, Perlick's proper time endows space-time with new metric properties

<sup>9</sup>In 1986, V. Perlick proposed a new notion of proper time defined in a Weyl manifold that is invariant by Weyl transformations [17] and reduces to the WIST and general relativistic definitions in the appropriate limits. However, it has been shown that Perlick's time also leads to the second clock effect [18].

which are distinct from the Riemannian metric of each member of the class. At this point, one may ask the question: What are the “geodesics” of this geometry? This question is motivated both by geometry and physics. Indeed, one may be interested in knowing whether it is possible or not to define “distance” between points. Or, one may regard the geodesics as describing the paths of freely falling particles. In any case, the equation of the curve which extremizes Perlick’s functional is needed. The answers to these questions have not been found yet as the extremization of the functional (15) does not seem easy to carry out. However, preliminary results yield the following: i) Perlick’s geodesics do not coincide with the affine geodesics of Weyl geometry; ii) they exhibit a nonlocal character in the sense that they depend on the whole past of world line of the particle. In fact, Perlick’s geometry is completely nonlocal and this nonlocality should not be unexpected since the second clock effect may also be viewed as a nonlocal phenomenon. In addition to the fact that Perlick’s time introduces great mathematical difficulties (in spite of being a gauge-invariant notion), it is not operational when it comes to consider matter in Weyl’s theory, a question to be dealt with in the next section.

## VII. MATTER COUPLING IN WEYL’S THEORY

Let us begin by quoting some words by the British mathematician M. Atiyah regarding Einstein’s historical objection to Weyl’s theory: “Given this devastating critique it is remarkable but fortunate that Weyl’s paper was still published... Clearly the beauty of the idea attracted the editor...” [21] Certainly this “devastating” critique was what prevented Weyl from going ahead and completing his elegant and aesthetically appealing theory by adding matter to his universe and get the full field equations in the presence of matter. In what follows we shall briefly discuss this point and suggest a possible way of carrying out this completion while maintaining consistency with the principle of gauge invariance. In this article we shall just outline the general theoretical construction, a preliminary step in this direction, leaving applications for future work.

We start by calling attention of the reader to the fact that in the case of the so-called geometric scalar-tensor gravity theories the definition of proper time is given by the gauge-invariant equation [22]

$$\Delta\tau = \int_a^b e^{-\frac{\phi}{2}} \left( g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda. \quad (16)$$

These theories are framed in a geometric structure known in the literature as *WIST* (Weyl integrable space-time) [19]. It can be viewed as a “weak” version of Weyl’s geometry when the 1-form  $\sigma$  representing the gauge field is exact, i.e.,  $\sigma = d\phi$ . Thus instead of the geometrizing the electromagnetic field we shall geometrize a scalar field. In this case, the Weyl transformations (2) and (3) become  $\bar{g} = e^f g$  and

$\bar{\phi} = \phi + f$ , and the compatibility condition now is given by

$$\nabla_V [g(W, U)] = g(\nabla_V W, U) + g(W, \nabla_V U) + d\phi(V)g(W, U),$$

in which  $U$ ,  $V$ , and  $W$  are vector fields. The relevant point we wish to highlight here is that (16) is nothing more than the clock hypothesis redefined in terms of the *gauge-invariant* metric  $\gamma_{\mu\nu} = e^{-\frac{\phi}{2}} g_{\mu\nu}$ . This gives us a clue to tackle the problem of matter coupling in the nonintegrable case. For this purpose, let us recall the procedure used to define an invariant energy-momentum tensor in geometric scalar-tensor theories. Let  $S^{(m)} = \int d^4x \sqrt{|\eta|} \mathcal{L}_m(\psi_A, \partial\psi_A)$  denote the action of the matter fields  $\psi_A$  (which we want to couple with the geometry) in Minkowski space-time. We next apply the *principle of minimal coupling*  $\eta_{\mu\nu} \rightarrow \gamma_{\mu\nu}$ ,  $\partial\psi_A \rightarrow \nabla\psi_A$ , where  $\nabla$  stands for the covariant derivative with respect to the metric connection determined by  $\gamma_{\mu\nu}$ . We proceed to define the gauge-invariant energy-momentum tensor  $T_{\mu\nu}$  (the source of the gravitational field) by the well-known Hilbert prescription

$$\delta S^{(m)} = k \int d^4x \sqrt{|\gamma|} T_{\mu\nu}^{(m)} \delta\gamma^{\mu\nu}, \quad (17)$$

with  $k$  denoting the coupling constant. In this way we have an invariant procedure to obtain the coupling between matter and space-time.

Our aim at this point is to obtain a gauge-invariant procedure which enables us to construct the coupling between matter and geometry in Weyl theory by following a somehow similar procedure as above. Clearly, the “non-locality” of the functional (15) makes the use of Perlick’s metric virtually impossible. Therefore we need to find out another gauge-invariant metric tensor. It turns out that Weyl’s idea of working out the field equations in a particular gauge, namely, the natural gauge defined by the condition  $R = \Lambda$  may be of great help. Indeed, by just picking up this idea we are now able to define a metric tensor which may be regarded as the representative of the whole conformal structure  $\mathcal{M} = \{(g, \nabla, \sigma)\}$  of the Weyl manifold. Indeed, let  $(g, \nabla, \sigma)$  be an arbitrary member of  $\mathcal{M}$  and define the tensor  $\gamma = \frac{R}{\Lambda} g$  for some  $\Lambda > 0$ .<sup>10</sup> Now suppose that  $(\bar{g}, \bar{\nabla}, \bar{\sigma})$  is another member of  $\mathcal{M}$ . Clearly both members are related by the transformations  $\bar{g}_{\mu\nu} = e^f g_{\mu\nu}$ ,  $\bar{\sigma}_\alpha = \sigma_\alpha + \partial_\alpha f$ , for some function  $f$ . Thus, the fact that  $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \bar{g}^{\mu\nu} R_{\mu\nu} = e^{-f} g^{\mu\nu} R_{\mu\nu} = e^{-f} R$  immediately implies  $\bar{\gamma} = \frac{\bar{R}}{\Lambda} \bar{g} = \frac{R}{\Lambda} g = \gamma$ , and this means that  $\gamma$  is a gauge-invariant object, which can be computed from any member of the conformal structure  $\mathcal{M}$ . Note that in Weyl natural gauge (determined by the chosen  $\Lambda$ ) the metric

<sup>10</sup>If  $R = 0$  it is easy to verify that Weyl field equations become trivial and we have no longer an electromagnetic field.

tensor  $\gamma$  assumes its simplest form, namely,  $\gamma = g$ . The same reasoning leads us to define a second gauge-invariant object, namely, the 1-form given by  $\xi = \sigma + d(\ln R)$ . Therefore, we can regard  $\xi$  as the 1-form representative of  $\mathcal{M}$ . Once we have the gauge-invariant metric tensor  $\gamma$  we can then adopt the same procedure used in WIST theories to obtain a gauge invariant energy-momentum tensor in Weyl theory by defining  $\delta S^{(m)} = \kappa \int d^4x \sqrt{|\gamma|} T_{\mu\nu}^{(m)} \delta\gamma^{\mu\nu}$ . Therefore, our strategy will be to reframe Weyl's theory in terms of  $\gamma$  and  $\xi$ .

### VIII. A NEW APPROACH TO WEYL'S THEORY

In this section we summarize the ideas developed so far in the following set of postulates:

- P1. We still consider space-time modeled by the conformal structure  $\mathcal{M}$ , whose members are related by the group of transformations (2) and (3). However, all relevant geometric objects will be constructed from  $\gamma$  and  $\xi$ .
- P2. The field equations of the theory will be given by varying the action  $S = \int d^4x \sqrt{|\gamma|} [R^2 + \omega F_{\mu\nu} F^{\mu\nu} + \kappa \mathcal{L}_m] d^4x$  with respect to  $\gamma$ ,  $\xi$ , and  $\psi_A$ , where  $\kappa$  is a coupling constant and  $\mathcal{L}_m(\psi_A, \nabla\psi_A)$  denote the Lagrangian of the matter fields. In the Weyl gauge, these variations will have the form

$$\delta S = \delta \int d^4x \sqrt{-g} \left[ R + \frac{\omega}{2\Lambda} F_{\mu\nu} F^{\mu\nu} - \frac{\Lambda}{2} + \kappa \mathcal{L}_m \right] = 0, \quad (18)$$

yielding the equations

$$\begin{aligned} \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} g_{\mu\nu} + \frac{\Lambda}{4} g_{\mu\nu} + \frac{3}{2} \left( \sigma_\mu \sigma_\nu - \frac{1}{2} g_{\mu\nu} \sigma^\alpha \sigma_\alpha \right) \\ = \frac{\omega}{\Lambda} T_{\mu\nu} - \kappa T_{\mu\nu}^{(m)}, \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}) &= \frac{3\Lambda}{2\omega} \sigma^\mu, \\ \Phi^A &= 0, \end{aligned} \quad (20)$$

where  $\delta \int d^4x \sqrt{-g} [\kappa \mathcal{L}_m] = \int d^4x \sqrt{-g} \Phi^A \delta\psi_A$ , and  $\kappa = \frac{\omega}{2\Lambda}$ .

Finally, to complete the theoretical framework we assume the following postulates:

- P3. The motion of free-falling test particles will be given by Riemannian geodesics with respect to the metric tensor  $\gamma$ .
- P4. The gauge-invariant proper time of a standard clock will be given by assuming the usual clock hypothesis, namely, that

$$\Delta\tau = \frac{1}{c} \int [\gamma(V, V)]^{\frac{1}{2}} d\lambda,$$

which in Weyl gauge reduces to (13).

- P5. The clock rate of standard clocks is strictly determined by the metric properties of  $\gamma$  and its corresponding Levi-Civita connection.

As an application of the ideas developed so far, we shall now solve the field equations (18) and (19) assuming a simple cosmological scenario.

### IX. A SIMPLE COSMOLOGICAL SOLUTION

Although general relativity is still considered the best available theory of gravity, and as such has been applied to the study of the universe with enormous success, we are currently seeing a great interest (which can be justified for several reasons) in alternative theoretical proposals, generally referred to as “modified theories of gravity.” This kind of research is particularly connected and stimulated by the recent advances in the field of observational cosmology [23]. We thus thought it could be interesting to apply the present new approach to Weyl's theory in searching for a solution of the field equations in a simple cosmological setting.

The model assumes homogeneity and isotropy both in the metric and the vector field, the first being assumed to be given by a Friedmann-Robertson-Walker line element, in which, for simplicity, we have chosen a flat spatial section ( $k=0$ ). As to the matter distribution of the universe, we admit that it is described by the energy-momentum tensor of a perfect fluid  $T_{\mu\nu}^{(m)} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$ , where  $\rho$ ,  $p$ , and  $u_\mu$  denote the energy-density, the pressure, and the 4-velocity of the fluid (Here we are employing a comoving coordinate system, so that  $u^\mu = \delta_0^\mu$ ). (We do not assume a particular equation of state, leaving it to be determined by the solution of the field equations.)

We thus write  $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$  for the line element, and  $\sigma_\mu = (\phi(t), \theta(t), \theta(t), \theta(t))$  for the Weyl vector field. A direct calculation give us

$$F^{\mu\nu} = \frac{\dot{\theta}}{a^2} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } T_{\mu\nu} = \frac{\dot{\theta}^2}{2} \begin{pmatrix} \frac{3}{a^2} & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -2 & 1 \end{pmatrix},$$

where dot denotes derivative with respect to  $t$ .

Let us now work with the equations in the natural gauge. Setting  $\mu = 0$  in (20) will give us

$$\partial_0(a^3 F^{0\nu}) = \frac{3\Lambda}{2\omega} a^3 \sigma^0 = 0, \quad (21)$$

which then implies  $\phi(t) = 0$ . On the other hand, setting  $\mu = 1$  in the same equation leads us to

$$\ddot{\theta} + \left(\frac{\dot{a}}{a}\right)\dot{\theta} = \frac{3\Lambda}{2\omega}\theta. \quad (22)$$

Now let us consider the equation (19). For  $\mu = 1$  and  $\nu = 2$  we obtain

$$\frac{\dot{\theta}}{\theta} = \pm \sqrt{-\frac{3\Lambda}{2\omega}},$$

which immediately yields the solutions

$$\theta = e^{\pm \sqrt{-\frac{3\Lambda}{2\omega}}t},$$

in which the integration constant was set equal to unity by rescaling the line element.<sup>11</sup> Choosing the negative solution above and inserting it in (22) gives for the scale factor  $a(t)$ <sup>12</sup>

$$a(t) = e^{2\sqrt{-\frac{3\Lambda}{2\omega}}t},$$

in which the integration constants were set equal to unity just by rescaling the line element and choosing appropriate initial conditions. Setting  $\mu = \nu = 0$  in we obtain

$$\rho = -\frac{\Lambda}{\kappa} \left( \frac{18}{\omega} + \frac{1}{4} \right) - \frac{9}{2\kappa} e^{-6\sqrt{-\frac{3\Lambda}{2\omega}}t},$$

whereas putting  $\mu = \nu = 1$  leads to

$$p = \frac{\Lambda}{\kappa} \left( \frac{18}{\omega} + \frac{1}{4} \right).$$

The solution obtained above represents a typical non-singular and expansive model, that is, a de Sitter universe. Solutions of this kind have been found in different contexts suggesting the possibility of describing dark energy in our present universe or inflation in the early universe. It is interesting to note that  $\sigma(t) \rightarrow 0$  when  $t \rightarrow \infty$ . In other words, the Weyl field tends to fade away with the expansion of the universe, while the equation of state of the cosmological fluid becomes  $p = -\rho$ , which is typical of dark energy models. It should be mentioned that inhomogeneous time-dependent equations have also been considered in dark some energy scenarios [24]. Finally, in connection with the simple model outlined above, we would like to call attention of the reader to the fact that, although most versions of inflationary cosmology require a scalar field, previous results found in the literature show that inflation

<sup>11</sup>Because we would like to interpret  $\Lambda$  as the cosmological constant we are restricting ourselves to negative values of  $\omega$ .

<sup>12</sup>Choosing the positive sign here leads to a rather nonphysical scenario, which merely describes a contracting universe.

can also be driven by vector fields, including the particular case of massive fields<sup>13</sup> [25].

## X. FINAL REMARKS

We would like to conclude this work with a few remarks. First of all, it is important to stress the fact that the adoption of a very special set of gauge-invariant tensors playing the role of representatives of the space-time modeled as a conformal structure leads to two unexpected consequences: i) nonlocal effects, such as the second clock effect are no longer predicted; ii) the coupling between space-time and matter is carried out in an invariant way following the traditional prescription contained in the principle of minimal coupling of general relativity.<sup>14</sup>

In deriving the equation for the Weyl field  $\sigma$  when matter fields are present we have implicitly made the assumption that  $\mathcal{L}_m$  does not depend on  $\sigma$ , or, in other words, that the vector field does not couple directly with matter. Surely, if one wishes a more general framework, it is possible to weaken this restriction by just adding a current term  $j_\mu$ , given by  $\delta \int d^4x \sqrt{-g} [\kappa \mathcal{L}_m] = \int d^4x \sqrt{-g} j_\mu \delta \sigma^\mu$ .

Finally, the original identification of the 1-form field  $\sigma$  with the electromagnetic potential is no longer assumed here. Instead, the pair  $(\gamma, \xi)$  constitutes what would we would call the complete gravitational field. In this way, we are simply left with a modified gravity theory instead of a unified theory as in the Weyl's original program.

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## APPENDIX: DERIVATION OF THE WEYL FIELD EQUATIONS

In what follows we show how to obtain the Weyl field equations (9) and (10), written in an arbitrary gauge,

<sup>13</sup>If we rewrite the field equations (12) in the form  $\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}g_{\mu\nu} + \frac{\Lambda}{4}g_{\mu\nu} = \frac{\omega}{\Lambda}T_{\mu\nu} - \frac{3}{2}(\sigma_\mu\sigma_\nu - \frac{1}{2}g_{\mu\nu}\sigma^\alpha\sigma_\alpha) - \kappa T_{\mu\nu}^{(m)}$ , then it is easy to verify that both sides are spatially isotropic tensors. In fact, the right-hand side of the above equation may be viewed as being formed by two tensors: the energy-momentum tensor of matter and the energy-momentum tensor of the (massive) Proca field  $\sigma^\mu$ , and both are isotropic as can readily be seen. Incidentally, it is an interesting fact that, at least in this particular case, it is the mass term of the latter that guarantees the required isotropy.

<sup>14</sup>It should be mentioned that, in principle, extra terms could be added to the Lagrangian of the theory, or more general non-minimal matter couplings could be included (provided that the gauge symmetry is preserved) generating new gravitational theories which obey this additional symmetry principle.



directly from the equations (11) and (12), the latter valid in the natural gauge only.

We start by looking for a Weyl transformation (2) and (3) which allows us to go from an arbitrary gauge  $(M, \bar{g}, \bar{\sigma})$  to a particular gauge  $(M, g, \sigma)$ , in which  $R = \Lambda \neq 0$ , where  $\Lambda$  is an arbitrary constant. From the fact that the Ricci tensor is gauge-invariant it is not difficult to verify that the desired transformation is given by taking  $f = \ln(\frac{\bar{R}}{\Lambda})$ , where  $\bar{R}$  denotes the Ricci scalar in the arbitrary gauge  $(M, \bar{g}, \bar{\sigma})$ . We thus have  $g_{\mu\nu} = \frac{\bar{R}}{\Lambda} \bar{g}_{\mu\nu}$  and  $\sigma_\mu = \bar{\sigma}_\mu + \frac{1}{\bar{R}} \partial_\mu \bar{R}$ . Let us now rewrite the equation (12) in terms of  $R_{\mu\nu}$  and  $R$  (respectively, the Ricci and scalar curvature calculated with the Weyl connection), also recalling the identity which relates the two Ricci tensors  $R_{\mu\nu}$  and  $\bar{R}_{\mu\nu}$ , the first calculated with the Weyl connection and the second with the Christoffel symbols:

$$\begin{aligned} \bar{R}_{\mu\nu} &= R_{(\mu\nu)} + \frac{1}{2} (\tilde{\nabla}_\mu \sigma_\nu + \tilde{\nabla}_\nu \sigma_\mu + g_{\mu\nu} \tilde{\nabla}_\alpha \sigma^\alpha) \\ &\quad + \frac{1}{2} (\sigma_\mu \sigma_\nu - g_{\mu\nu} \sigma_\alpha \sigma^\alpha), \end{aligned} \quad (\text{A1})$$

the symbol  $\tilde{\nabla}$  standing for the Riemannian covariant derivative. Contracting the above equation with  $g^{\mu\nu}$  yields

$$\bar{R} = R - \frac{3}{2} \sigma_\alpha \sigma^\alpha, \quad (\text{A2})$$

where we have used the fact that (11) implies  $\tilde{\nabla}_\alpha \sigma^\alpha = 0$ .

Substituting (A1) and (A2) into (12) leads to

$$R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R + \frac{\Lambda}{4} g_{\mu\nu} + \frac{1}{2} (\nabla_\mu \sigma_\nu + \nabla_\nu \sigma_\mu) + \sigma_\mu \sigma_\nu = \frac{\omega}{\Lambda} T_{\mu\nu}, \quad (\text{A3})$$

in which we have taken into account the following relation between the Weylian and Riemannian covariant derivatives:

$$\nabla_\mu \sigma_\nu = \tilde{\nabla}_\mu \sigma_\nu + \sigma_\mu \sigma_\nu - \frac{1}{2} g_{\mu\nu} \sigma_\alpha \sigma^\alpha.$$

Recalling the expression of  $T_{\mu\nu}$  in Sec. III we now see that all terms in (A3) possess a well-defined transformation under (2) and (3). Therefore, choosing the latter as being given by  $g_{\mu\nu} = \frac{\bar{R}}{\Lambda} \bar{g}_{\mu\nu}$  and  $\sigma_\mu = \bar{\sigma}_\mu + \frac{1}{\bar{R}} \partial_\mu \bar{R}$  we get, after a straightforward calculation,

$$\begin{aligned} \bar{R}_{(\mu\nu)} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} + \nabla_{(\mu} \bar{\sigma}_{\nu)} + \frac{1}{\bar{R}} \nabla_{(\mu} \nabla_{\nu)} \bar{R} \\ + \bar{\sigma}_\mu \bar{\sigma}_\nu + \frac{1}{\bar{R}} \bar{\sigma}_{(\mu} \nabla_{\nu)} \bar{R} = \frac{\omega}{\bar{R}} \bar{T}_{\mu\nu}. \end{aligned} \quad (\text{A4})$$

Multiplying the above equation by  $\bar{R}$  clearly leads to (10). Finally, the equation (9) is directly obtained by the same procedure.

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