Next-to-leading order spin-orbit effects in the equations of motion, energy loss, and phase evolution of binaries of compact bodies in the effective field theory approach

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We compute spin-orbit effects in the equations of motion, binding energy, and energy loss of binary systems of compact objects at the next-to-leading order in the post-Newtonian approximation in the effective field theory (EFT) framework. We then use these quantities to compute the evolution of the orbital frequency and accumulated orbital phase including spin-orbit effects beyond the dominant order. To obtain the results presented in this paper, we make use of known ingredients in the EFT literature, such as the potential and the multipole moments with spin effects at next-to-leading order, and which are given in the linearized harmonic gauge and with the spins in the locally flat frame. We also obtain the correction to the center-of-mass frame caused by spin-orbit effects at next-to-leading order. We demonstrate the equivalence between our EFT results and those which were obtained elsewhere using different formalisms. The results presented in this paper provide us with the final ingredients for the construction of theoretical templates for gravitational waves including next-to-leading order spin-orbit effects, which will be presented in a future publication.

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I. INTRODUCTION

Gravitational wave astronomy is based on high precision experimental and theoretical physics. The main sources of gravitational wave signals which can be detected by the ground-based observatories LIGO and Virgo [1–11] are binary systems of compact objects. During the inspiral stage, those systems can be studied analytically through perturbative approaches such as the post-Newtonian (PN) approximation, which uses the ratio between the relative velocity and the speed of light (v^2/c^2) as the expansion parameter. For accuracy, the calculations have to be carried out to high orders in the expansion parameter in order to be valid up to the late inspiral stage, which is when the theoretical templates are matched onto the numerical ones. In fact, interesting physics can be studied only when we go beyond the leading order, for instance finite size effects such as spin, which plays an important role in understanding the formation and the evolution of binary systems [12–14].

The effective field theory (EFT) framework we use in this paper, called nonrelativistic general relativity (NRGR), originally proposed in [15] and extended to accommodate rotating objects in [16], is an independent approach to the investigation of the dynamics of binaries of compact objects. The current state of the art for the EFT formalism is 4PN order [17–20] in the conservative sector for nonspinning bodies. The spin sector of this formalism—the focus of this paper—has also seen extensive development in the past 15 years. The leading order (LO) spin effects in the conservative dynamics were derived through the NRGR formalism in [16], while the next-to-leading order (NLO) and next-to-next-to-leading order (N^2LO) spin effects were studied in [21–26] and [27,28], respectively. Recently, the next-to-next-to-next-to-leading order (N^3LO) gravitational spin-orbit [29] and quadratic-in-spin [30] interactions were also investigated. Beyond the linear and the quadratic spin effects, the LO cubic and quartic spin interactions [31] and the NLO cubic spin interactions [32] were also explored via the NRGR framework. In the radiative sector of this formalism, spin effects in the multipole moments were obtained in [33,34], and the LO spin effects in the radiation reaction were computed in [35,36].

Although crucial ingredients for the description of the dynamics of binaries of compact bodies including NLO spin effects were previously computed using the NRGR formalism, in particular the spin-orbit potential and the spin evolution in [21], and the multipole moments in [33], other important quantities associated with NLO spin-orbit effects—such as the equations of motion of the compact bodies, the system's binding energy, its energy loss, and the phase evolution—have yet to be derived in the NRGR framework. One of the purposes of this paper is to obtain those quantities, since they play an important role in the investigation of the physics of binary systems. For instance, the acceleration we derive in this paper, which composes a

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2.5PN correction to the system's equations of motion, is used to compute the energy loss associated with the emission of gravitational waves but also to obtain the phase evolution of the binary system. In addition, this acceleration is a necessary ingredient for the construction of theoretical templates of gravitational waves accounting for NLO spin-orbit effects, which shall be presented in a future publication. Another spin-orbit effect that enters at 2.5PN order is the correction to the center-of-mass frame. Although it does not affect the NLO spin-orbit acceleration obtained here, we derive this 2.5PN spin-orbit correction to the center of mass for completeness, with the intent to provide the final pieces related to NLO spin-orbit effects in order to allow the EFT calculations to continue without impediment at higher orders. Furthermore, we provide a discussion between the results obtained in this paper and the ones in the literature [37,38], where different gauge and spin definitions are used while a more traditional PN approach to general relativity is followed, and we show that, through a redefinition of the spin variables, equivalence can be proven even before gauge invariant quantities are computed. This present paper, therefore, also serves as a demonstration of the equivalence between the NRGR methodology and more traditional approaches to general relativity up to NLO regarding spin-orbit effects, in both the conservative and the dissipative sectors.

We organize this paper as follows. In Sec. II, we provide a brief summary of the NRGR formalism (we recommend [39–43] for a comprehensive review). We derive the NLO spin-orbit acceleration in Sec. III by computing the Euler-Lagrange equations of the potential obtained in [21] but also by extracting contributions coming from order reducing terms in lower-order accelerations and from spin precession and constraints. In Sec. IV, we take the Legendre transform of the potential derived in [21] to obtain the NLO spin-orbit effects in the binding energy of the binary system, and we make use of the acceleration computed in Sec. III as well as the multipole moments obtained in [33] to calculate the energy loss due to the emission of gravitational radiation. Then, in Sec. V we use the results obtained in Secs. III and IV to calculate the evolution of the orbital frequency of the binary system and its phase evolution accounting for NLO spin-orbit effects for quasicircular orbits within the adiabatic approximation. In Sec. VI we compute the NLO spin-orbit effects in the 00 component of the binary's pseudotensor, which we use to extract the NLO spin-orbit correction to the center-of-mass frame by Taylor expanding its expression up to the first order in the radiation field momentum. In Sec. VII, we discuss the specific spin transformations which map the main results of this paper to those in the literature obtained from traditional PN approaches. In Sec. VIII, we provide the reader with our final remarks on the contributions of this paper. We compile the known ingredients used to derive the results of this paper in the Appendix for convenience.

A number of conventions and definitions are utilized throughout this paper. The masses m_1 and m_2 of the binary components are used to define the following quantities: $m \equiv m_1 + m_2$, $\nu \equiv m_1 m_2/m^2$, and $\mu \equiv m\nu$. The relative position is defined as $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$ and its unit vector given by $\mathbf{n} \equiv \mathbf{r}/r$; thus $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{a} \equiv \mathbf{a}_1 - \mathbf{a}_2$ are the relative velocity and acceleration, respectively. If those relative quantities appear inside a sum over the compact objects indices $A, B = 1, 2$, they should be considered as depending on those indices instead, e.g., $\mathbf{r} = \mathbf{x}_A - \mathbf{x}_B$. We use the Newtonian orbital angular momentum vector defined by $\mathbf{L} \equiv m\nu \mathbf{r} \times \mathbf{v}$. We use the spins $\mathbf{S}_1$ and $\mathbf{S}_2$ of the bodies to define

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2, \tag{1.1}$$

$$\mathbf{\Sigma} \equiv m\left(\frac{\mathbf{S}_2}{m_2} - \frac{\mathbf{S}_1}{m_1}\right), \tag{1.2}$$

which are two useful quantities to write results in a more elegant way. We adopt the mostly minus signature $(1, -1, -1, -1)$ for the Minkowskian metric $\eta^{\alpha\beta}$. We use $c = 1$ units, and the Planck mass is defined as $m_{\rm Pl} \equiv 1/\sqrt{32\pi G}$.

II. NRGR SETUP

A. Conservative sector

The EFT approach is well suited to investigate the inspiral stage of the binary system, when there is a clear hierarchy between the length scales of the system: the size of the compact objects r_s, the orbital separation r, and the radiation wavelength. The modes of the perturbation $h_{\mu\nu}$ of the gravitational field, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, can be split into two different components: $h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}$, where $H_{\mu\nu}$ are off-shell *potential* modes of the field which mediate gravitational attraction and $\bar{h}_{\mu\nu}$ represent the on-shell propagating *radiation* modes generated by the motion of the compact bodies in the binary [15]. Then, we start with the full theory action

$$S = S_{\rm EH} + S_{\rm gf} + S_{\rm pp} + S_{\rm sg} + \cdots, \tag{2.1}$$

where

$$S_{\rm EH} = -2m_{\rm Pl}\int d^4x\sqrt{-g}g_{\mu\nu}R^{\mu\nu}, \tag{2.2}$$

$$S_{\rm gf} = \int d^4x\sqrt{-\bar{g}}\bar{\Gamma}^\mu\bar{\Gamma}_\mu, \tag{2.3}$$

$$S_{\rm pp} = -\sum_A m_A \int d\tau_A, \tag{2.4}$$

$$S_{\rm sg} = -\frac{1}{2}\sum_A \int dt v_A^\mu \omega_{\mu ab} S_A^{ab}. \tag{2.5}$$

The Einstein-Hilbert action (2.2) represents the purely gravitational interaction terms. We utilize the linearized harmonic gauge fixing action (2.3), which is given in terms of the background field metric $\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \bar{h}_{\mu\nu}$, in order to maintain the diffeomorphism invariance even after the potential modes of the gravitational fields are integrated out. Thus, we have $\bar{\Gamma}_\mu \equiv \bar{\nabla}_\alpha H^\alpha_\mu - \frac{1}{2}\bar{\nabla}_\mu H^\alpha_\alpha$, with $\bar{\nabla}_\mu$ representing the covariant derivative associated with the background metric $\bar{g}_{\mu\nu}$. The point particle approximation is used to describe the two constituents of the binary system, hence (2.4); the index $A = 1, 2$ is a label for the two compact bodies.

The final term (2.5) represents the spin-gravity coupling. Choosing the coordinate time t as the worldline parameter, the spin action is composed of the four-velocity of the compact bodies v^μ_A, the spin connection $\omega_{\mu ab} \equiv e_{\nu b}\nabla_\mu e^\nu_a$,[1] and the antisymmetric spin tensor $S^{ab}_A \equiv S^{\mu\nu}_A e^a_\mu e^b_\nu$ given in the locally flat frame. The vierbien e^a_μ is defined such that $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$ and ∇_μ is the covariant derivative associated with the metric $g_{\mu\nu}$. The locally flat frame retains a residual Lorentz invariance and is equivalent to adding an additional element of the SO(3,1) group to the worldline to implement rotations [16]. Finally, the ellipsis in (2.1) represents other interactions that we are not accounting for in this present paper, including finite size terms which are quadratic or higher in the spins.

The Feynman rules for this EFT theory are obtained after imposing the low velocity limit and the weak field approximation in the full action (2.1). The derivatives of the off-shell potential modes scale as $\partial_0 H_{\mu\nu} \sim (\frac{v}{r})H_{\mu\nu}$ and $\partial_i H_{\mu\nu} \sim (\frac{1}{r})H_{\mu\nu}$ while derivatives of the on-shell radiation modes scale as $\partial_\alpha \bar{h}_{\mu\nu} \sim (\frac{v}{r})\bar{h}_{\mu\nu}$. For maximally rotating objects, we power count the spin as $S \sim Lv$, where L is the angular momentum. Therefore, we can determine how each term in the full action scales with respect to the expansion parameter v^2. This power counting allows us to systematically compute spin or other effects at any desired order in the PN expansion.

After imposing the weak-field approximation using

$$e^a_\mu = \delta^a_\mu + \frac{1}{2}\delta^a_\nu\left(h^\nu{}_\mu - \frac{1}{4}h^\nu{}_\rho h^\rho{}_\mu\right) + \cdots, \tag{2.6}$$

the spin-gravity Lagrangian becomes an infinite series of terms with a single spin tensor contracted with the gravitational field at different orders in its perturbation:

$$L_{\mathrm{sg}} = \sum_{A=1,2}\left[\frac{1}{2m_{\mathrm{Pl}}}\delta^\alpha_a\delta^\beta_b h_{\alpha\gamma,\beta}v^\gamma_A S^{ab}_A + \frac{1}{4m^2_{\mathrm{Pl}}}\delta^\beta_a\delta^\gamma_b h^\lambda_\gamma\left(\frac{1}{2}h_{\beta\lambda,\mu} + h_{\mu\lambda,\beta} - h_{\mu\beta,\lambda}\right)v^\mu_A S^{ab}_A + \cdots\right]. \tag{2.7}$$

[1]This definition differs by a minus sign from the standard convention.

From this Lagrangian, we can extract all the relevant couplings that are needed at the PN order that we consider in this paper. Moreover, when we split the weak field into the two different modes, we can obtain the potential—from which the spin-orbit equations of motion can be derived—by integrating out the potential modes of the gravitational field. Both the potential from [21] and the couplings needed to compute NLO spin-orbit effects, which are 2.5PN corrections in the equations of motion, the binding energy, and the center-of-mass position, are presented in the Appendix.

Using a rank-2 antisymmetric tensor to describe spin in a four-dimensional spacetime comes with a cost: there are a total of 6 independent degrees of freedom to play the role of the three necessary angles to describe the rotation of a body. For the purpose of eliminating the three unphysical components of the spin tensor, we impose constraints known as spin supplementary conditions (SSC). In this paper, we use the covariant SSC, which is given by the contraction of the spin tensor with the linear momentum

$$p_a S^{ab} = 0. \tag{2.8}$$

Even though the bodies label has been suppressed in the equation above, notice that this constraint must be imposed for each of the compact bodies.

B. Radiative sector

The long-wavelength effective theory can be constructed by integrating out the potential modes of the gravitational field. The binary system is then described as a single pointlike object endowed with a series of multipole moments [42,44]:

$$S^{\mathrm{rad}}_{\mathrm{eff}}[\bar{h}, \mathbf{x}_a] = \int dt\sqrt{\bar{g}_{00}}\left[-M(t) + \sum_{l=2}^{\infty}\left(\frac{1}{l!}I^L\nabla_{L-2}E_{i_{l-1}i_l} - \frac{2l}{(2l+1)!}J^L\nabla_{L-2}B_{i_{l-1}i_l}\right)\right]. \tag{2.9}$$

The center of mass of the binary system is placed at the origin and at rest with respect to distant observers, such that $dt\sqrt{\bar{g}_{00}} = d\tau$, while $M(t)$ is the Bondi mass of the binary system. In the action above, the electric and the magnetic components of the Weyl tensor are coupled to the mass and current multipole moments, respectively. Notice that a multi-index representation $L = i_1 \cdots i_l$ is used. The general expressions of the multipole moments I^L and J^L in terms of the components of the pseudotensor of the binary system, which can be found in [45], are determined by matching the effective action (2.9) in the long wavelength limit onto the full action valid below the orbital scale (2.1). On the other hand, the pseudotensor $T^{\mu\nu}$, which satisfies the conservation law $\partial_\mu T^{\mu\nu} = 0$, can be read off from

$$\Gamma[\bar{h}] = -\frac{1}{2m_{\rm Pl}}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}, \tag{2.10}$$

when we integrate out the potential modes in the full action (2.1) for all terms containing a single radiation field.

The knowledge of the components of the pseudotensor and, consequently, of the multipole moments of the binary system, is required in order to determine the energy which is lost in the emission of gravitational waves [45]:

$$\frac{dE}{dt} = -\frac{G}{5}\left(I^{(3)}_{ij}I^{(3)}_{ij} + \frac{16}{9}J^{(3)}_{ij}J^{(3)}_{ij} + \frac{5}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{5}{84}J^{(4)}_{ijk}J^{(4)}_{ijk} + \cdots\right). \tag{2.11}$$

All the necessary multipole moments for the computation of the NLO spin-orbit effects in the energy loss, which we compute in Sec. IV B of this paper, were computed in [33] and are presented in the center-of-mass frame in the Appendix.

III. EQUATIONS OF MOTION

In the PN approximation, the acceleration of the constituents of the binary systems is given as a series of relativistic corrections to the dominant Newtonian gravitational acceleration. If we disregard, for the purposes of this paper, the radiation reaction and effects of quadratic (or higher) order in the spins, the acceleration can be presented as[2]

$$\mathbf{a} = \mathbf{a}^{(0\rm PN)} + \mathbf{a}^{(1\rm PN)} + \mathbf{a}^{(1.5\rm PN)}_{\rm SO} + \mathbf{a}^{(2\rm PN)} + \mathbf{a}^{(2.5\rm PN)}_{\rm SO} + \cdots. \tag{3.1}$$

The expressions for the nonspin accelerations in the right-hand side of the equation above are given in the Appendix. The LO spin-orbit acceleration—a 1.5PN correction to the equation of motion—can be derived from the potential $V^{\rm SO}_{1.5\rm PN}$ given in (A4). Computing the Euler-Lagrange equations using that potential gives

$$\begin{aligned}(\mathbf{a}^i_1)^{V^{(1.5\rm PN)}_{\rm SO}} = \frac{G}{r^3}\Bigg\{&\frac{m_2}{m_1}[(3\mathbf{n}\dot{r} - 2\mathbf{v}_1 + 3\mathbf{v}_2)\times\mathbf{S}_1]^i + [(6\mathbf{n}\dot{r} - 4\mathbf{v}_1 + 3\mathbf{v}_2)\times\mathbf{S}_2]^i \\ &- \left[\mathbf{r}\times\left(\frac{m_2}{m_1}\dot{\mathbf{S}}_1 + 2\dot{\mathbf{S}}_2\right)\right]^i + \left[S^{i0}_2 - \frac{m_2}{m_1}S^{i0}_1 + 3\mathbf{n}^i\mathbf{n}^j\left(\frac{m_2}{m_1}S^{j0}_1 - S^{j0}_2\right)\right]_{\rm cov} \\ &+ 3\mathbf{n}^i\mathbf{n}\cdot\left(\frac{m_2}{m_1}\mathbf{v}_1\times\mathbf{S}_1 - 2\frac{m_2}{m_1}\mathbf{v}_2\times\mathbf{S}_1 + 2\mathbf{v}_1\times\mathbf{S}_2 - \mathbf{v}_2\times\mathbf{S}_2\right)\Bigg\}.\end{aligned} \tag{3.2}$$

Notice that the expression above is given in a general form: it includes time derivatives of the spin vectors, which actually contribute only at orders higher than 1.5PN since $\dot{S}\sim\frac{v^3}{r}S$; it also shows the explicit dependence on the $S^{j0}_{1,2}$ variables, which will be removed by enforcing the covariant SSC (2.8). Although we kept $S^{j0}_{1,2}$ variables to indicate that those terms will also contribute to orders higher than 1.5PN due to PN corrections in the covariant SSC, the result in (3.2) is *only* valid in the covariant SSC and is not general to other choices of constraints.[3] Up to 1PN order, the spin tensors can be written in terms of the spin vectors in the covariant SSC as

$$S^{0i}_A = \mathbf{S}_A\times\mathbf{v}_A + \frac{2Gm_B}{r}\mathbf{S}_A\times\mathbf{v} + \mathcal{O}(\mathbf{S}^2) \tag{3.3}$$

and

$$S^{ij}_A = \epsilon^{ijk}\mathbf{S}^k_A. \tag{3.4}$$

Therefore, after imposing the covariant SSC in (3.2) and keeping only terms which enter at the lowest PN order, we can write the well-defined expression for the LO spin-orbit acceleration [16]:

$$\begin{aligned}(\mathbf{a}^i_1)^{(1.5\rm PN)}_{\rm SO} = \frac{G}{r^3}\Bigg\{&3\frac{m_2}{m_1}[(\mathbf{S}_1\times\mathbf{v})^i - \dot{r}(\mathbf{S}_1\times\mathbf{n})^i - 2\mathbf{S}_1\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n}^i] \\ &+4(\mathbf{S}_2\times\mathbf{v})^i - 6\dot{r}(\mathbf{S}_2\times\mathbf{n})^i - 6\mathbf{S}_2\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n}^i\Bigg\}.\end{aligned} \tag{3.5}$$

The purpose of this section is to advance to the next step, namely, to obtain the equations of motion linear in the spins for the binary system at 1PN beyond Eq. (3.5), which is a 2.5PN correction to the Newtonian acceleration. The result for the NLO spin-orbit acceleration can be presented as the sum of two distinct contributions:

$$(\mathbf{a}^i_1)^{(2.5\rm PN)}_{\rm SO} = (\mathbf{a}^i_1)^{V^{(2.5\rm PN)}_{\rm SO}} + (\mathbf{a}^i_1)^{(\rm Red)}. \tag{3.6}$$

The first term in the right-hand side of the equation above comes from computing the Euler-Lagrange equations of the

[2] The first quadratic spin effects enter at 2PN order, while the radiation reaction enters at 2.5PN order.

[3] If we were working with the Newton-Wigner SSC, for instance, we would have to impose the constraint at the level of the potential before computing the Euler-Lagrange equations. See the discussion presented in Appendix E of [16] for more details.

NLO spin-orbit potential (A5), which was obtained in [21]. The result for this contribution can be conveniently arranged as

$$(\mathbf{a}_1^i)^{V_{\rm SO}^{(2.5\rm PN)}} = \frac{1}{m_1}\sum_{n=0}^{3}\left\{(-1)^{n+1}\left(\frac{d}{dt}\right)^n \frac{\partial}{\partial \mathbf{x}_1^{i(n)}} V_{\rm SO}^{(2.5\rm PN)}\right\} = (\mathbf{A}_1^i)_{S^{i0}}^{\rm cov} + (\mathbf{A}_1^i)_{S^{ij}}, \tag{3.7}$$

where

$$\begin{aligned}
(\mathbf{A}_1^i)_{S^{i0}}^{\rm cov} \equiv \frac{G}{r^3}\bigg\{&\frac{m_2}{m_1}S_1^{j0}\bigg[\delta^{ij}\left(\frac{Gm_1}{r}+2\frac{Gm_2}{r}+2\mathbf{v}\cdot\mathbf{v}_2+\frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}+\frac{3}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right)+\mathbf{v}_2^i(3\mathbf{v}_2\cdot\mathbf{n}\mathbf{n}^j-\mathbf{v}^j)+\frac{1}{2}\mathbf{a}_2^i\mathbf{r}^j\\
&+\mathbf{n}^i\left(-\frac{3}{2}r\mathbf{a}_2^j+3\mathbf{v}_2\cdot\mathbf{n}\mathbf{v}^j-\mathbf{n}^j\left(4\frac{Gm_1}{r}+8\frac{Gm_2}{r}+6\mathbf{v}\cdot\mathbf{v}_2+\frac{3}{2}\mathbf{a}_2\cdot\mathbf{r}+\frac{15}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right)\right)\bigg]\\
&+S_2^{j0}\bigg[\delta^{ij}\left(-2\frac{Gm_1}{r}-\frac{Gm_2}{r}+2\mathbf{v}\cdot\mathbf{v}_1+\frac{1}{2}\mathbf{a}_1\cdot\mathbf{r}-\frac{3}{2}(\mathbf{v}_1\cdot\mathbf{n})^2\right)-\mathbf{v}_1^i(3\mathbf{v}_1\cdot\mathbf{n}\mathbf{n}^j+\mathbf{v}^j)+\frac{1}{2}\mathbf{a}_1^i\mathbf{r}^j\\
&+\mathbf{n}^i\left(-\frac{3}{2}r\mathbf{a}_1^j+3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}^j+\mathbf{n}^j\left(8\frac{Gm_1}{r}+4\frac{Gm_2}{r}-6\mathbf{v}\cdot\mathbf{v}_1-\frac{3}{2}\mathbf{a}_1\cdot\mathbf{r}+\frac{15}{2}(\mathbf{v}_1\cdot\mathbf{n})^2\right)\right)\bigg]\bigg\}\\
&-\frac{d}{dt}\left\{\frac{G}{r^2}\left[\frac{m_2}{m_1}S_1^{j0}(2\mathbf{v}_2^i\mathbf{n}^j-\delta^{ij}\mathbf{v}_2\cdot\mathbf{n})+S_2^{j0}[2\mathbf{n}^j(2\mathbf{v}_1^i-\mathbf{v}_2^i)-\delta^{ij}\mathbf{v}_1\cdot\mathbf{n}-\mathbf{n}^i(\mathbf{v}^j+3\mathbf{v}_1\cdot\mathbf{n}\mathbf{n}^j)]\right]\right\}\\
&+\frac{d^2}{dt^2}\left\{\frac{1}{2}\frac{G}{r}S_2^{j0}(3\delta^{ij}+\mathbf{n}^i\mathbf{n}^j)\right\}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
(\mathbf{A}_1^i)_{S^{ij}} \equiv \frac{G}{r^3}\bigg\{&\frac{m_2}{m_1}S_1^{ij}\bigg[-2\mathbf{v}_2\cdot\mathbf{r}\mathbf{a}_2^j-r^2\dot{\mathbf{a}}_2^j+\mathbf{v}_1^j\left(-\frac{Gm_1}{r}+\frac{1}{2}\frac{Gm_2}{r}+\frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}+\frac{3}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right)\\
&+\mathbf{v}_2^j\left(-\frac{5}{2}\frac{Gm_2}{r}-2\mathbf{v}\cdot\mathbf{v}_2-\mathbf{a}_2\cdot\mathbf{r}-3(\mathbf{v}_2\cdot\mathbf{n})^2\right)\bigg]+S_2^{ij}\bigg[-2\mathbf{v}_1\cdot\mathbf{r}\mathbf{a}_1^j+r^2\dot{\mathbf{a}}_1^j\\
&+\mathbf{v}_1^j\left(\frac{5}{2}\frac{Gm_1}{r}-2\mathbf{v}\cdot\mathbf{v}_1-\mathbf{a}_1\cdot\mathbf{r}+3(\mathbf{v}_1\cdot\mathbf{n})^2\right)+\mathbf{v}_2^j\left(-\frac{1}{2}\frac{Gm_1}{r}+\frac{Gm_2}{r}+\frac{1}{2}\mathbf{a}_1\cdot\mathbf{r}-\frac{3}{2}(\mathbf{v}_1\cdot\mathbf{n})^2\right)\bigg]\\
&+\mathbf{n}^i\bigg[\frac{m_2}{m_1}S_1^{kj}\bigg(\left(-4\frac{Gm_1}{r}+2\frac{Gm_2}{r}+\frac{3}{2}\mathbf{a}_2\cdot\mathbf{r}+\frac{15}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right)\mathbf{v}_1^k\mathbf{n}^j-3\mathbf{v}_2\cdot\mathbf{n}(\mathbf{v}_1^k\mathbf{v}_2^j+2\mathbf{a}_2^k\mathbf{r}^j)\\
&-\left(10\frac{Gm_2}{r}+6\mathbf{v}\cdot\mathbf{v}_2+3\mathbf{a}_2\cdot\mathbf{r}+15(\mathbf{v}_2\cdot\mathbf{n})^2\right)\mathbf{v}_2^k\mathbf{n}^j+\frac{1}{2}r\mathbf{a}_2^k\mathbf{v}_1^j+r\mathbf{a}_2^k\mathbf{v}_2^j+r\mathbf{r}^k\dot{\mathbf{a}}_2^j\bigg)\\
&+S_2^{kj}\bigg(\left(10\frac{Gm_1}{r}-6\mathbf{v}\cdot\mathbf{v}_1-3\mathbf{a}_1\cdot\mathbf{r}+15(\mathbf{v}_1\cdot\mathbf{n})^2\right)\mathbf{v}_1^k\mathbf{n}^j-3\mathbf{v}_1\cdot\mathbf{n}(\mathbf{v}_1^k\mathbf{v}_2^j+2\mathbf{a}_1^k\mathbf{r}^j)\\
&+\left(-2\frac{Gm_1}{r}+4\frac{Gm_2}{r}+\frac{3}{2}\mathbf{a}_1\cdot\mathbf{r}-\frac{15}{2}(\mathbf{v}_1\cdot\mathbf{n})^2\right)\mathbf{v}_2^k\mathbf{n}^j+r\mathbf{a}_1^k\mathbf{v}_1^j+\frac{1}{2}r\mathbf{a}_1^k\mathbf{v}_2^j-r\mathbf{r}^k\dot{\mathbf{a}}_1^j\bigg)\bigg]\\
&+\mathbf{v}_1^iS_2^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j+2\mathbf{a}_1^k\mathbf{r}^j+3\mathbf{v}_1\cdot\mathbf{n}(\mathbf{v}_2^k-2\mathbf{v}_1^k)\mathbf{n}^j]+\mathbf{a}_1^iS_2^{kj}\left[\mathbf{v}_1^k\mathbf{r}^j-\frac{1}{2}\mathbf{v}_2^k\mathbf{r}^j\right]\\
&+\mathbf{v}_2^i\frac{m_2}{m_1}S_1^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j+2\mathbf{a}_2^k\mathbf{r}^j+3\mathbf{v}_2\cdot\mathbf{n}(2\mathbf{v}_2^k-\mathbf{v}_1^k)\mathbf{n}^j]+\mathbf{a}_2^i\frac{m_2}{m_1}S_1^{kj}\left[-\frac{1}{2}\mathbf{v}_1^k\mathbf{r}^j+\mathbf{v}_2^k\mathbf{r}^j\right]\bigg\}\\
&-\frac{d}{dt}\bigg\{\frac{G}{r^2}\bigg\{\frac{m_2}{m_1}S_1^{ij}\left[\frac{1}{2}r\mathbf{a}_2^j+\mathbf{v}_2\cdot\mathbf{n}\mathbf{v}_2^j+\mathbf{n}^j\left(\frac{Gm_1}{r}-\frac{1}{2}\frac{Gm_2}{r}-\frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}-\frac{3}{2}(\mathbf{v}_2\cdot\mathbf{n})^2\right)\right]\\
&+S_2^{ij}\left[r\mathbf{a}_1^j+\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2^j+\mathbf{n}^j\left(-\frac{5}{2}\frac{Gm_1}{r}+2\mathbf{v}\cdot\mathbf{v}_1+\mathbf{a}_1\cdot\mathbf{r}-3(\mathbf{v}_1\cdot\mathbf{n})^2\right)\right]+4\mathbf{v}_1^iS_2^{kj}\mathbf{v}_1^k\mathbf{n}^j\\
&+2\mathbf{v}_2^i\left[\frac{m_2}{m_1}S_1^{kj}\mathbf{v}_2^k\mathbf{n}^j-S_2^{kj}\mathbf{v}_1^k\mathbf{n}^j\right]+\mathbf{n}^iS_2^{kj}[\mathbf{v}_1^k\mathbf{v}_2^j+2\mathbf{a}_1^k\mathbf{r}^j+3\mathbf{v}_1\cdot\mathbf{n}\mathbf{n}^j(\mathbf{v}_2^k-2\mathbf{v}_1^k)]\bigg\}\bigg\}\\
&+\frac{d^2}{dt^2}\left\{\frac{G}{r}\left[S_2^{ij}\left(2\mathbf{v}_1\cdot\mathbf{n}\mathbf{n}^j-\mathbf{v}_1^j-\frac{1}{2}\mathbf{v}_2^j\right)+\mathbf{n}^i\mathbf{n}^jS_2^{kj}\left(\mathbf{v}_1^k-\frac{1}{2}\mathbf{v}_2^k\right)\right]\right\}+\frac{d^3}{dt^3}\{GS_2^{ij}\mathbf{n}^j\}.
\end{aligned} \tag{3.9}$$

The second term in the right-hand side of (3.6) accounts for 2.5PN order terms coming from order reduction of lower PN order accelerations, which can be concisely presented as

$$\begin{aligned}(\mathbf{a}_1^i)^{(\mathrm{Red})} = &\left[\frac{1}{2}\frac{Gm_2}{r}\mathbf{a}_2\cdot\mathbf{n}\mathbf{n}^i - \mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_1^i - \mathbf{a}_1^i\left(3\frac{Gm_2}{r}+\frac{1}{2}\mathbf{v}_1^2\right)+\frac{7}{2}\frac{Gm_2}{r}\mathbf{a}_2^i\right]_{\mathbf{a}_{\mathrm{SO}}^{(1.5\mathrm{PN})}}\\ &+\left[\frac{G}{r^3}\left(-\frac{m_2}{m_1}S_1^{i0}+3\frac{m_2}{m_1}S_1^{j0}\mathbf{n}^j\mathbf{n}^i+S_2^{i0}-3S_2^{j0}\mathbf{n}^j\mathbf{n}^i\right)\right]_{\mathrm{cov}(1\mathrm{PN})}\\ &+\left[-\frac{G}{r^3}\left(\frac{m_2}{m_1}\dot{S}_1^{ij}\mathbf{r}^j+2\dot{S}_2^{ij}\mathbf{r}^j\right)\right]_{\dot{S}_{\mathrm{LO}}}. \end{aligned}\tag{3.10}$$

The expression above includes three contributions from lower-order accelerations: reduced contributions from substituting the LO spin-orbit acceleration (3.5) in the acceleration terms present in the 1PN correction to the equations of motion (A2); frame corrections from imposing the covariant SSC (3.3) in (3.2), and also, in that same equation, terms from reducing spin derivatives. At 2.5PN order, we only need the LO spin derivative term given by [16]

$$\frac{d\mathbf{S}_1}{dt} = \frac{Gm_2}{r^3}[2(\mathbf{r}\times\mathbf{v})\times\mathbf{S}_1 + (\mathbf{S}_1\times\mathbf{r})\times\mathbf{v}_1].\tag{3.11}$$

After imposing the covariant SSC and order reducing the accelerations in order to obtain a fixed order result at 2.5PN, Eq. (3.6) becomes

$$\begin{aligned}(\mathbf{a}_1^i)_{\mathrm{SO}}^{(2.5\mathrm{PN})} = \frac{G}{r^3}\Bigg\{&-\frac{\mathbf{n}^i}{m\nu r}\left[\frac{m_2}{m_1}\mathbf{S}_1\cdot\mathbf{L}\left(\frac{G}{r}(26m_1+22m_2)+12\mathbf{v}\cdot\mathbf{v}_2+3\mathbf{v}_2^2+3\mathbf{v}^2+15(\mathbf{v}_2\cdot\mathbf{n})^2\right)\right.\\ &\left.+\mathbf{S}_2\cdot\mathbf{L}\left(\frac{G}{r}\left(\frac{61}{2}m_1+20m_2\right)+6\mathbf{v}\cdot\mathbf{v}_2+3\mathbf{v}_2^2+15(\mathbf{v}_2\cdot\mathbf{n})^2\right)\right]\\ &+\mathbf{v}_1^i\left[-\frac{3m_2}{m_1}\left(\frac{1}{m\nu r}\mathbf{S}_1\cdot\mathbf{L}(2\mathbf{v}_2\cdot\mathbf{n}+\dot{r})+\dot{r}\mathbf{S}_1\cdot(\mathbf{v}_2\times\mathbf{n})+\mathbf{S}_1\cdot(\mathbf{v}\times\mathbf{v}_2)\right)\right.\\ &\left.-2\left(\frac{3}{m\nu r}\mathbf{S}_2\cdot\mathbf{L}(\mathbf{v}_2\cdot\mathbf{n}+\dot{r})+3\dot{r}\mathbf{S}_2\cdot(\mathbf{v}_2\times\mathbf{n})+2\mathbf{S}_2\cdot(\mathbf{v}\times\mathbf{v}_2)\right)\right]\\ &+\mathbf{v}_2^i\left[\frac{6}{m\nu r}\left(\frac{m_2}{m_1}\mathbf{S}_1+\mathbf{S}_2\right)\cdot\mathbf{L}(\mathbf{v}_2\cdot\mathbf{n}+\dot{r})\right]-\frac{2}{m\nu r}\mathbf{L}^i\left[\frac{G}{r}\left(\frac{m_2^2}{m_1}\mathbf{S}_1\cdot\mathbf{n}+2m_1\mathbf{S}_2\cdot\mathbf{n}\right)\right]\\ &+\frac{m_2}{m_1}(\mathbf{S}_1\times\mathbf{n})^i\left[\dot{r}\frac{G}{r}(14m_1+10m_2)+\frac{3}{2}\dot{r}(\mathbf{v}_1^2+5(\mathbf{v}_2\cdot\mathbf{n})^2)-3\mathbf{v}\cdot\mathbf{v}_2\mathbf{v}_2\cdot\mathbf{n}\right]\\ &+(\mathbf{S}_2\times\mathbf{n})^i\left[\dot{r}\frac{G}{r}\left(\frac{47}{2}m_1+16m_2\right)-2\mathbf{v}_2\cdot\mathbf{n}\left(\frac{Gm_1}{r}+3\mathbf{v}\cdot\mathbf{v}_2\right)+3\dot{r}(2\mathbf{v}\cdot\mathbf{v}_2+\mathbf{v}_2^2+5(\mathbf{v}_2\cdot\mathbf{n})^2)\right]\\ &-\frac{m_2}{m_1}(\mathbf{S}_1\times\mathbf{v})^i\left[\frac{G}{r}(14m_1+10m_2)+6\mathbf{v}\cdot\mathbf{v}_2+\frac{3}{2}\mathbf{v}_2^2+\frac{3}{2}\mathbf{v}^2+\frac{9}{2}(\mathbf{v}_2\cdot\mathbf{n})^2-3\dot{r}\mathbf{v}_2\cdot\mathbf{n}\right]\\ &-(\mathbf{S}_2\times\mathbf{v})^i\left[\frac{G}{r}\left(\frac{31}{2}m_1+12m_2\right)+4\mathbf{v}\cdot\mathbf{v}_2+2\mathbf{v}_2^2+6(\mathbf{v}_2\cdot\mathbf{n})^2\right]\Bigg\}.\end{aligned}\tag{3.12}$$

Note that the spin vector used in this expression is defined in the locally flat frame; see Sec. VII for a discussion of alternative spin definitions.

We also present the NLO spin-orbit acceleration in the center-of-mass frame. In the latter, the expressions for $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of the relative coordinate $\mathbf{r}$ are given by

$$\mathbf{x}_1 = \frac{m_2}{m}\mathbf{r}+\delta\mathbf{r},\tag{3.13}$$

$$\mathbf{x}_2 = -\frac{m_1}{m}\mathbf{r}+\delta\mathbf{r},\tag{3.14}$$

where, considering only corrections up to 1.5PN order,

$$\delta\mathbf{r} = \nu\frac{\delta m}{2m}\left(\mathbf{v}^2 - \frac{Gm}{r}\right)\mathbf{r} + \frac{\nu}{m}\mathbf{v}\times\mathbf{\Sigma}. \tag{3.15}$$

When these PN corrections to the center-of-mass frame are considered in the 1PN acceleration (A2), they yield contributions to the equations of motion at the 2.5PN order. In principle, one must consider 1PN center-of-mass corrections in the LO spin-orbit acceleration (3.5) as well, but these vanish because this acceleration only depends on relative coordinates and velocities; this is also the reason why we do not need to consider the 2.5PN spin-orbit correction to the center of mass in the Newtonian acceleration (A1). Therefore, the final expression for the NLO spin-orbit acceleration in the center-of-mass frame comes solely from considering (3.13) and (3.14) in (3.12) and (A2); the result is

$$\begin{aligned}
(\mathbf{a}^i)^{(2.5\mathrm{PN})}_{\mathrm{SO}} = \frac{G}{m\nu r^4}\Bigg\{&\mathbf{n}^i\Bigg[\mathbf{S}\cdot\mathbf{L}\left(-\frac{Gm}{r}(42+29\nu)+3(-1+10\nu)\mathbf{v}^2-30\nu\dot{r}^2\right)\\
&-\frac{\delta m}{m}\mathbf{\Sigma}\cdot\mathbf{L}\left(\frac{Gm}{r}\left(22+\frac{33}{2}\nu\right)+3(1-5\nu)\mathbf{v}^2+15\nu\dot{r}^2\right)\Bigg]\\
&+3\dot{r}\mathbf{v}^i\left[3\mathbf{S}\cdot\mathbf{L}(-1+\nu)+\frac{\delta m}{m}\mathbf{\Sigma}\cdot\mathbf{L}(-1+2\nu)\right]\\
&-2\frac{Gm}{r}\mathbf{L}^i\left[\mathbf{S}\cdot\mathbf{n}(1+2\nu)+\frac{\delta m}{m}\mathbf{\Sigma}\cdot\mathbf{n}(1+\nu)\right]\Bigg\}\\
&+\frac{G}{r^3}\Bigg\{(\mathbf{S}\times\mathbf{n})^i\dot{r}\left[\frac{Gm}{r}(26+25\nu)+\frac{3}{2}(1-15\nu)\mathbf{v}^2+\frac{45}{2}\nu\dot{r}^2\right]\\
&+\frac{\delta m}{m}(\mathbf{\Sigma}\times\mathbf{n})^i\dot{r}\left[\frac{Gm}{r}\left(10+\frac{27}{2}\nu\right)+\left(\frac{3}{2}-12\nu\right)\mathbf{v}^2+15\nu\dot{r}^2\right]\\
&+(\mathbf{S}\times\mathbf{v})^i\left[-\frac{Gm}{r}(22+15\nu)+\frac{3}{2}(-1+11\nu)\mathbf{v}^2-\frac{33}{2}\nu\dot{r}^2\right]\\
&-\frac{\delta m}{m}(\mathbf{\Sigma}\times\mathbf{v})^i\left[\frac{Gm}{r}\left(10+\frac{15}{2}\nu\right)+\left(\frac{3}{2}-8\nu\right)\mathbf{v}^2+9\nu\dot{r}^2\right]\Bigg\}.
\end{aligned} \tag{3.16}$$

This expression is valid for general orbits and for arbitrary spin orientations within the region of validity of the NRGR formalism. In the next section, we compute the binding energy and the energy loss. For the latter, we need the result (3.16) as well as (A1), (A2), and (3.5) to order reduce the time derivatives of the multipole moments.

IV. BINDING ENERGY AND ENERGY LOSS

A. Binding energy

The LO spin-orbit energy—a 1.5PN correction to the Newtonian binding energy—can be obtained from the potential (A4); it is given by

$$\begin{aligned}
E^{(1.5\mathrm{PN})}_{\mathrm{SO}} &= \frac{G}{r^3}\mathbf{r}^i(m_2 S_1^{i0} - m_1 S_2^{i0})_{\mathrm{cov}}\\
&= \frac{Gm_2}{r^2}\mathbf{S}_1\cdot(\mathbf{n}\times\mathbf{v}_1) + 1\leftrightarrow 2,
\end{aligned} \tag{4.1}$$

where we have imposed the covariant SSC in the second expression. In this section, we obtain the 1PN correction to the LO spin-orbit binding energy:

$$E^{(2.5\mathrm{PN})}_{\mathrm{SO}} = \sum_{A=1}^{2}\sum_{n=0}^{2}\mathbf{p}_{\mathbf{x}_A^{(n)}}\cdot\mathbf{x}_A^{(n+1)} + V^{(2.5\mathrm{PN})}_{\mathrm{SO}} + E^{(\mathrm{Red})}, \tag{4.2}$$

$$\mathbf{p}_{q^{(n)}} = -\sum_{A=1}^{2}\sum_{k=n+1}^{3}\left(-\frac{d}{dt}\right)^{k-n-1}\frac{\partial V^{(2.5\mathrm{PN})}_{\mathrm{SO}}}{\partial\mathbf{x}_A^{(k)}}, \tag{4.3}$$

where the notation $\mathbf{x}_A^{(k)}$ is a compact way to express $\frac{d^k\mathbf{x}_A}{dt^k}$. For the spin-orbit energy at the 2.5PN order, we have two contributions: one from the NLO spin-orbit potential (A5), and another from frame corrections when applying the covariant SSC to the LO spin-orbit energy (4.1), which we represent by $E^{(\mathrm{Red})}$ in (4.2). The sum of the two contributions gives

$$E_{\mathrm{SO}}^{(2.5\mathrm{PN})} = \frac{Gm_2}{r^2}\Big\{(\mathbf{v}_1\cdot\mathbf{n} + 2\mathbf{v}_2\cdot\mathbf{n})\mathbf{S}_1\cdot(\mathbf{v}_1\times\mathbf{v}_2) + \left(\mathbf{v}_2^2 - \mathbf{v}_1\cdot\mathbf{v}_2 - \frac{3}{2}(\mathbf{v}_2\cdot\mathbf{n})^2 - 3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n} + 2\frac{Gm_1}{r}\right)\mathbf{S}_1\cdot(\mathbf{v}_1\times\mathbf{n}) + \left(-2\mathbf{v}_2^2 + 3\mathbf{v}_1\cdot\mathbf{v}_2 - \mathbf{v}_1^2 + 3(\mathbf{v}_1\cdot\mathbf{n})^2 + 3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n} + 3\frac{Gm_2}{r}\right)\mathbf{S}_1\cdot(\mathbf{v}_2\times\mathbf{n})\Big\} + 1\leftrightarrow 2. \tag{4.4}$$

Transforming to the center-of-mass frame as in Sec. III, we have

$$E_{\mathrm{SO}}^{(2.5\mathrm{PN})} = \frac{Gm}{r}\left\{\left(2\nu\frac{Gm}{r} - 2\mathbf{v}^2 - \frac{3}{2}\nu\dot{r}^2\right)\frac{\mathbf{L}\cdot\mathbf{S}}{mr^2} + \frac{\delta m}{m}\left(\frac{3}{2}\nu\frac{Gm}{r} - \frac{3}{2}\nu\mathbf{v}^2\right)\frac{\mathbf{L}\cdot\mathbf{\Sigma}}{mr^2}\right\}. \tag{4.5}$$

Next we calculate the time-averaged energy loss, which completes the pieces necessary to compute the orbital phase evolution in Sec. V.

B. Energy loss

The binary system's energy loss due to the emission of gravitational waves can be computed directly from the one-graviton emission amplitude in the effective theory (for a detailed discussion, see [44,45]), and its general form is given at (2.11). All the necessary multipole moments to compute the NLO spin-orbit effects in the energy loss are presented in (A6)–(A18). Using the equations of motion (3.1) to order reduce the acceleration terms generated by the time derivatives applied to the multipole moments, we obtain a final expression for the NLO spin-orbit energy loss:

$$\left.\frac{dE}{dt}\right|_{\mathrm{SO}}^{(2.5\mathrm{PN})} = -\frac{2G^3m^3\nu}{105r^4}\Big\{\frac{\mathbf{L}\cdot\mathbf{S}}{mr^2}\Big[(3776+1560\nu)\frac{G^2m^2}{r^2} + (-12892+2024\nu)\frac{Gm}{r}\dot{r}^2 + (15164-560\nu)\frac{Gm}{r}\mathbf{v}^2 + (-8976+12576\nu)\dot{r}^4 + (13362-18252\nu)\dot{r}^2\mathbf{v}^2 + (-4226+5952\nu)\mathbf{v}^4\Big] + \frac{\delta m}{m}\frac{\mathbf{L}\cdot\mathbf{\Sigma}}{mr^2}\Big[(-548+952\nu)\frac{G^2m^2}{r^2} + (-14654+4796\nu)\frac{Gm}{r}\dot{r}^2 + (10718-1708\nu)\frac{Gm}{r}\mathbf{v}^2 + (-7941+10704\nu)\dot{r}^4 + (8742-13434\nu)\dot{r}^2\mathbf{v}^2 + (-2001+3474\nu)\mathbf{v}^4\Big]\Big\}. \tag{4.6}$$

Along with the expressions for the acceleration (3.16) and conserved energy (4.5), the above result is the final piece needed to compute the orbital phase evolution of the binary system, in the quasicircular orbit approximation, accounting for NLO spin-orbit effects in the NRGR framework.

V. PHASE EVOLUTION

Until this point, our results are valid for general orbits and arbitrary spin configurations. However, as is well known, the emission of gravitational waves tends to efficiently circularize orbits well before entering the observable frequency band of gravitational wave detectors [46]. Although alternative methods such as the dynamical renormalization group approach [47,48] may be used for more general systems, we will restrict our analysis to circular orbits here. We can then apply an adiabatic approximation in which orbits are approximately circular on an orbital timescale and orbit decay occurs on a radiation-reaction timescale. In this approximation, the expressions above can be expressed as coordinate-independent quantities as functions of a single orbital frequency ω, the orbital angular momentum L, and the spin vectors, and are gauge invariant under coordinate transformations. In our subsequent analysis, we neglect spin-spin [16], tail [17,44], and radiation-reaction [49,50] terms in the orbital frequency, since in this paper we are only investigating spin-orbit effects; those other effects do not mix with our results and thus can be included independently later on.

For nonspinning objects, the procedure of computing the phase evolution of the binary system is unambiguous because the orientation of the orbital plane is constant in time; for spinning objects, the choice of spin vector is crucial because the spin vector may evolve by radiation reaction for an inappropriate choice. For spinning systems, we choose a spin vector with a conserved norm; this allows us to work with orbit averaged spin vectors and to use energy balance arguments to compute the orbital phase [51,52]. We transform from the locally flat spin vectors to conserved norm spin vectors using the relation

$$\mathbf{S}_A \to \left(1+\frac{1}{2}\mathbf{v}_A^2\right)\mathbf{S}_A^c - \frac{1}{2}\mathbf{v}_A(\mathbf{S}_A^c\cdot\mathbf{v}_A)+\cdots. \tag{5.1}$$

See [21] and Sec. VII for details regarding the significance of this redefinition.

For (quasi)circular orbits, we use the relations

$$r\omega^2 = -\langle \mathbf{n}\cdot\mathbf{a}\rangle, \tag{5.2}$$

$$|\mathbf{v}| = r\omega, \tag{5.3}$$

$$\dot{r} = 0, \tag{5.4}$$

and perform the spin transformation to conserved norm spin vectors, which gives us, for instance,

$$E_{\mathrm{SO}}^c = \frac{G}{r^3}\left\{\left[1+2\nu\frac{Gm}{r}-\frac{3}{2}(1+\nu)\mathbf{v}^2\right]\mathbf{L}\cdot\mathbf{S}^c + \frac{\delta m}{m}\left[1+\frac{3}{2}\nu\frac{Gm}{r}+\frac{1}{2}(1-5\nu)\mathbf{v}^2\right]\mathbf{L}\cdot\mathbf{\Sigma}^c\right\} \tag{5.5}$$

and

$$\begin{aligned}\frac{dE^c}{dt}\bigg|_{\mathrm{SO}} = \nu\frac{G^3m^2}{105r^6}\bigg\{&\mathbf{L}\cdot\mathbf{S}^c\left[448\frac{Gm}{r}+4480\mathbf{v}^2-(7552+3120\nu)\frac{G^2m^2}{r^2}-(30440+1792\nu)\frac{Gm}{r}\mathbf{v}^2+(9656-14480\nu)\mathbf{v}^4\right]\\ &+\frac{\delta m}{m}\mathbf{L}\cdot\mathbf{\Sigma}^c\left[-224\frac{Gm}{r}+2408\mathbf{v}^2+(1906-1904\nu)\frac{G^2m^2}{r^2}-(21548-3976\nu)\frac{Gm}{r}\mathbf{v}^2+(5206-8320\nu)\mathbf{v}^4\right]\bigg\}.\end{aligned} \tag{5.6}$$

To write these in terms of the orbital frequency ω, we use Eq. (5.2) and solve order by order in the PN expansion for ω; note that the expression for the acceleration (3.16) must also be rewritten with the conserved norm spin vectors. Then, we find that the orbital frequency is given by

$$\begin{aligned}\omega^2 = \frac{Gm}{r^3}\bigg\{&1+\frac{Gm}{r}(-3+\nu)-\left(\frac{Gm}{r}\right)^{9/2}\left[5\frac{S_\ell^c}{Gm^2}+3\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}\right]\\ &+\left(\frac{Gm}{r}\right)^2\left[\frac{41}{4}\nu+\nu^2\right]+\left(\frac{Gm}{r}\right)^{11/2}\left[\left(\frac{27}{2}-\frac{13}{2}\nu\right)\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}+\left(\frac{45}{2}-\frac{27}{2}\nu\right)\frac{S_\ell^c}{Gm^2}\right]\bigg\}+\cdots,\end{aligned} \tag{5.7}$$

where $S_\ell^c \equiv \hat{\ell}\cdot\mathbf{S}^c$, $\Sigma_\ell^c \equiv \hat{\ell}\cdot\mathbf{\Sigma}^c$, and $\hat{\ell} = \mathbf{L}/|\mathbf{L}|$. We can write Eqs. (5.5) and (5.6) in terms of the orbital separation r to give

$$\begin{aligned}E^c(r) = -\frac{1}{2}\frac{Gm^2\nu}{r}\bigg\{&1+\frac{Gm}{r}\left[-\frac{7}{4}+\frac{1}{4}\nu\right]+\left(\frac{Gm}{r}\right)^{3/2}\left[\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}+3\frac{S_\ell^c}{Gm^2}\right]\\ &+\left(\frac{Gm}{r}\right)^2\left[-\frac{23}{8}+\frac{49}{8}\nu+\frac{1}{8}\nu^2\right]+\left(\frac{Gm}{r}\right)^{5/2}\left[(2-3\nu)\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}+(6-6\nu)\frac{S_\ell^c}{Gm^2}\right]\bigg\}\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\frac{dE^c(r)}{dt} = -\frac{32G^4m^5\nu^2}{5r^5}\bigg\{&1+\frac{Gm}{r}\left[-\frac{2927}{336}-\frac{5}{4}\nu\right]+\left(\frac{Gm}{r}\right)^{3/2}\left[-\frac{25}{4}\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}-\frac{37}{3}\frac{S_\ell^c}{Gm^2}\right]\\ &+\left(\frac{Gm}{r}\right)^2\left[\frac{202663}{9072}+\frac{380}{9}\nu\right]+\left(\frac{Gm}{r}\right)^{5/2}\left[\left(\frac{6953}{112}+\frac{91}{8}\nu\right)\frac{\delta m}{m}\frac{\Sigma_\ell^c}{Gm^2}+\left(\frac{18947}{168}+\frac{68}{3}\nu\right)\frac{S_\ell^c}{Gm^2}\right]\bigg\}.\end{aligned} \tag{5.9}$$

These two expressions depend on the coordinate separation r and are therefore gauge dependent. Inverting our expression for ω^2, we find

$$\frac{Gm}{r} = x + x^2\left[1 - \frac{1}{3}\nu\right] + \frac{x^{5/2}}{Gm^2}\left[\frac{\delta m}{m}\Sigma_\ell^c + \frac{5}{3}S_\ell^c\right] + x^3\left[3 - \frac{65}{12}\nu\right] + \frac{x^{7/2}}{Gm^2}\left[2\frac{\delta m}{m}\Sigma_\ell^c + \left(\frac{10}{3} + \frac{8}{9}\nu\right)S_\ell^c\right], \tag{5.10}$$

where the PN parameter $x \equiv (Gm\omega)^{2/3}$ is formally of order v^2. We can now write the energy and energy loss as gauge independent expressions. They are

$$\begin{aligned} E^c(x) = -\frac{1}{2}m\nu x\bigg\{1 &+ x\left[-\frac{3}{4} - \frac{1}{12}\nu\right] + \frac{x^{3/2}}{Gm^2}\left[2\frac{\delta m}{m}\Sigma_\ell^c + \frac{14}{3}S_\ell^c\right] \\ &+ x^2\left[-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right] + \frac{x^{5/2}}{Gm^2}\left[\left(3 - \frac{10}{3}\nu\right)\frac{\delta m}{m}\Sigma_\ell^c + \left(11 - \frac{61}{9}\nu\right)S_\ell^c\right]\bigg\} \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \frac{dE^c(x)}{dt} = -\frac{32x^5\nu^2}{5G}\bigg\{1 &+ x\left[-\frac{1247}{336} - \frac{35}{12}\nu\right] + \frac{x^{3/2}}{Gm^2}\left[-\frac{5}{4}\frac{\delta m}{m}\Sigma_\ell^c - 4S_\ell^c\right] \\ &+ x^2\left[-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right] + \frac{x^{5/2}}{Gm^2}\left[\left(-\frac{13}{16} + \frac{43}{4}\nu\right)\frac{\delta m}{m}\Sigma_\ell^c + \left(-\frac{9}{2} + \frac{272}{9}\nu\right)S_\ell^c\right]\bigg\}. \end{aligned} \tag{5.12}$$

The coefficients in these expressions are still dependent on the particular definition of the spins; with our choice of conserved norm spin vectors, Eqs. (5.11) and (5.12) yield perfect agreement with the corresponding expressions in [38]. We now proceed to find an expression for the phase evolution of the binary system using energy balance arguments. We first obtain a dimensionless adiabatic parameter (also called the orbital frequency evolution [53]) representing the orbital decay, given by

$$\begin{aligned} \frac{\dot\omega}{\omega^2} = \frac{96}{5}\nu x^{5/2}\bigg\{1 &+ x\left[-\frac{743}{336} - \frac{11}{4}\nu\right] + \frac{x^{3/2}}{Gm^2}\left[-\frac{25}{4}\frac{\delta m}{m}\Sigma_\ell^c - \frac{47}{3}S_\ell^c\right] \\ &+ x^2\left[\frac{34103}{18144} + \frac{13661}{2016}\nu + \frac{59}{18}\nu^2\right] + \frac{x^{5/2}}{Gm^2}\left[\left(-\frac{809}{84} + \frac{281}{8}\nu\right)\frac{\delta m}{m}\Sigma_\ell^c + \left(-\frac{5861}{144} + \frac{1001}{12}\nu\right)S_\ell^c\right]\bigg\}. \end{aligned} \tag{5.13}$$

The orbital phase can then be computed in this adiabatic approximation, where the gravitational wave phase contains two contributions. The first comes from the evolution of the carrier phase, while the second arises due to the precession of the orbital plane due to spin effects. This can schematically be written as $\Phi_{\rm GW} = \phi_{\rm GW} + \delta\phi$ using the notation of [38]. The carrier phase given by $\phi_{\rm GW} = 2\phi$ can be computed using

$$\phi = \int dt\omega = \int d\omega\frac{\omega}{\dot\omega}. \tag{5.14}$$

In general, the carrier phase may be computed numerically for arbitrary spin alignments. However, for spins aligned or antialigned with the binary orbital angular momentum, this can be computed analytically using Eq. (5.13) to yield

$$\begin{aligned} \phi = \phi_0 - \frac{32}{\nu}\bigg\{&x^{-5/2} + x^{-3/2}\left[\frac{3715}{1008} + \frac{55}{12}\nu\right] + \frac{x^{-1}}{Gm^2}\left[\frac{125}{8}\frac{\delta m}{m}\Sigma_\ell^c + \frac{235}{6}S_\ell^c\right] \\ &+ x^{-1/2}\left[\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right] - \frac{\log x}{Gm^2}\left[\left(\frac{41745}{448} - \frac{15}{8}\nu\right)\frac{\delta m}{m}\Sigma_\ell^c + \left(\frac{554345}{2016} + \frac{55}{8}\nu\right)S_\ell^c\right]\bigg\}, \end{aligned} \tag{5.15}$$

for which we find perfect agreement with [38].

VI. CENTER-OF-MASS CORRECTION

We now proceed to compute the center-of-mass correction at 2.5PN order due to NLO spin-orbit effects. But before proceeding to the details of its computation, notice that this correction should have, in principle, entered in the calculation of the quantities derived in the previous sections, namely the NLO spin-orbit acceleration (3.16), the binding energy (4.5), and the energy loss (4.6). The reason why this correction does not affect the result for the NLO spin-orbit acceleration, as previously explained in Sec. III, is that the Newtonian acceleration (A1) is naturally given in terms of relative coordinates. This argument does not hold for the Newtonian energy, but it turns out that the 2.5PN contribution that would arise from it cancels out due to its symmetry:

$$E^{(0\mathrm{PN})} = \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2} - \frac{Gm_1m_2}{r} \xrightarrow{2.5\mathrm{PN}} \frac{m_1}{2}\frac{m_2}{m}\mathbf{v}\cdot\delta\dot{\mathbf{r}}_{\mathrm{SO}}^{(2.5\mathrm{PN})} - \frac{m_2}{2}\frac{m_1}{m}\mathbf{v}\cdot\delta\dot{\mathbf{r}}_{\mathrm{SO}}^{(2.5\mathrm{PN})} = 0. \tag{6.1}$$

The same happens to the LO mass quadrupole moment $I^{ij}_{0\mathrm{PN}} = \sum_a m_a[\mathbf{x}^i_a\mathbf{x}^j_a]_{\mathrm{TF}}$ when we try to extract its 2.5PN contribution going to the center-of-mass frame, and consequently the energy loss due to NLO spin-orbit effects is not affected by the correction to the center of mass at this order. Despite these facts, the NLO spin-orbit correction to the center of mass, which is an effect that enters at 2.5PN order, itself is a nonzero quantity and must be obtained, since it will lead to nonzero contributions in future computations at N^{2}LO order. Below, we present how we proceed to obtain this quantity via the NRGR framework.

The center-of-mass position is defined as

$$\mathbf{r}^i_{\mathrm{cm}} = \frac{1}{m}\int d^3x\mathbf{x}^iT^{00}(\mathbf{x},t). \tag{6.2}$$

As previously mentioned in Sec. II B, we can extract the stress-energy pseudotensor $T^{\mu\nu}(\mathbf{x},t)$ from matching onto the effective action (2.10) by integrating out potential modes from the full theory action in Eq. (2.1). Introducing the partial Fourier transform of the stress-energy pseudotensor and taking the long-wavelength limit, we find

$$T^{\mu\nu}(\mathbf{q},t) = \int d^3xT^{\mu\nu}(\mathbf{x},t)e^{-i\mathbf{q}\cdot\mathbf{x}} \tag{6.3}$$

$$= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\left(\int d^3xT^{\mu\nu}(\mathbf{x},t)\mathbf{x}^{i_1}\cdots\mathbf{x}^{i_n}\right)\mathbf{q}_{i_1}\cdots\mathbf{q}_{i_n}. \tag{6.4}$$

Comparing Eqs. (6.2) and (6.4), we can read off the center-of-mass correction from the $\mathcal{O}(\mathbf{q})$ term in $T^{\mu\nu}(\mathbf{q},t)$ in the effective theory.

The diagrams that contribute to the NLO spin-orbit center-of-mass correction are given in Fig. 1. Figure 1(a) comes from a single insertion of the vertex (A25). Imposing the covariant SSC gives a LO spin-orbit term and a 1PN correction given by

$$T^{00}_{1\mathrm{a}}(t,\mathbf{q}) = \sum_{A\neq B}S^{0i}_A(i\mathbf{q}^i)e^{-i\mathbf{q}\cdot\mathbf{x}_A} \xrightarrow{(\mathrm{cov})} \sum_{A\neq B}S^{ij}_A(i\mathbf{q}^j)\left(\mathbf{v}^i_A + \frac{2Gm_B}{r}\mathbf{v}^i\right)e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.5}$$

At the order we are working, Fig. 1(b) is composed of two different contributions, as we show next. Contracting (A26) with (A20), we find

$$T^{00}_{1\mathrm{b},1}(t,\mathbf{q}) = \sum_{A\neq B}\left[-\frac{2G_Nm_B}{r}S^{ij}_A\mathbf{v}^i_B(i\mathbf{q}^j)\right]e^{-i\mathbf{q}\cdot\mathbf{x}_A}, \tag{6.6}$$

and contracting (A27) with (A19) gives

$$T^{00}_{1\mathrm{b},2}(t,\mathbf{q}) = \sum_{A\neq B}\frac{Gm_BS^{0j}_A\mathbf{r}^j}{r^3}e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.7}$$

Figure 1(c) also accounts for two distinct contributions. Contracting (A23) with (A22), we have

$$T^{00}_{1\mathrm{c},1}(t,\mathbf{q}) = \sum_{A\neq B}\frac{2Gm_AS^{ij}_B\mathbf{v}^i_A\mathbf{r}^j}{r^3}e^{-i\mathbf{q}\cdot\mathbf{x}_A}, \tag{6.8}$$

and contracting (A24) with (A21) gives

$$T^{00}_{1\mathrm{c},2}(t,\mathbf{q}) = \sum_{A\neq B}\left[\frac{Gm_A}{r^3}(-S^{0i}_B\mathbf{r}^i - S^{ij}_B\mathbf{v}^i_B\mathbf{r}^j)\right]e^{i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.9}$$

Finally, Fig. 1(d) comes from three different contractions. The first contribution, constructed from (A19) and (A24) together with the LO 3-point vertex gives

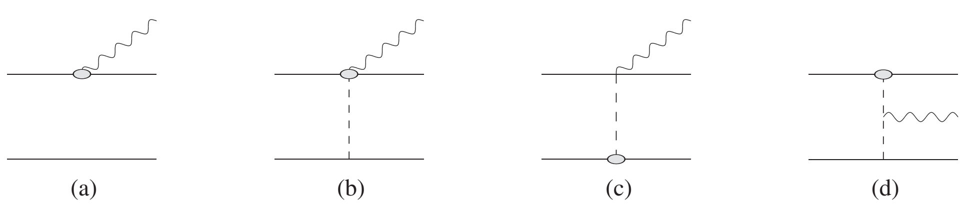

FIG. 1. Diagrams contributing to the 2.5PN spin-orbit center-of-mass correction.

$$T^{00}_{1\mathrm{d},1}(t,\mathbf{q}) = \sum_{A\neq B}\left[\frac{3}{2}\frac{Gm_A}{r^3}S_B^{0j}\mathbf{r}^j - \frac{3}{2}\frac{Gm_B}{r^3}S_A^{0j}\mathbf{r}^j - \frac{3}{2}\frac{Gm_B}{r}S_A^{0j}(i\mathbf{q}^j) + \frac{3}{2}\frac{Gm_A}{r^3}S_B^{ij}\mathbf{v}_B^i\mathbf{r}^j + \frac{1}{2}\frac{Gm_B}{r^3}S_A^{ij}\mathbf{v}_A^i\mathbf{r}^j + \frac{1}{2}\frac{Gm_B}{r}S_A^{ij}\mathbf{v}_A^i(i\mathbf{q}^j)\right]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.10}$$

The second, constructed from (A20) and (A23) with the LO 3-point vertex, is

$$T^{00}_{1\mathrm{d},2}(t,\mathbf{q}) = \sum_{A\neq B}\left[-\frac{Gm_A}{r^3}S_B^{ij}\mathbf{v}_A^i\mathbf{r}^j + \frac{Gm_B}{r^3}S_A^{ij}\mathbf{v}_B^i\mathbf{r}^j + \frac{Gm_B}{r}S_A^{ij}\mathbf{v}_B^i(i\mathbf{q}^j)\right]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.11}$$

The third, constructed from (A19) and (A23) with the 3-point vertex at $\mathcal{O}(v^1)$, reads

$$T^{00}_{1\mathrm{d},3}(t,\mathbf{q}) = \sum_{A\neq B}\left[\frac{Gm_A}{r}S_B^{ij}(\mathbf{v}_A^i+\mathbf{v}_B^i)(i\mathbf{q}^j) - \frac{Gm_A}{r^3}S_B^{ij}\mathbf{r}^i(i\mathbf{q}^j)\mathbf{r}\cdot(\mathbf{v}_A+\mathbf{v}_B)\right]e^{-i\mathbf{q}\cdot\mathbf{x}_A}. \tag{6.12}$$

Now, putting all the contributions above together, we write the final expression for the 00 component of the stress-pseudotensor accounting for NLO spin-orbit terms:

$$\begin{aligned}T^{00}_{\mathrm{SO}}(t,\mathbf{q}) = \sum_{A\neq B}\Bigg\{&S_A^{0i}(i\mathbf{q}^i) + \frac{G}{r^3}\Bigg[\frac{1}{2}m_A S_B^{0j}\mathbf{r}^j - \frac{1}{2}m_B S_A^{0j}\mathbf{r}^j \\ &+ m_B\left(\mathbf{v}_B^i + \frac{1}{2}\mathbf{v}_A^i\right)\mathbf{r}^j S_A^{ij} + m_A\left(\frac{1}{2}\mathbf{v}_B^i + \mathbf{v}_A^i\right)\mathbf{r}^j S_B^{ij} \\ &+ \Bigg(-\frac{3}{2}m_B S_A^{0j} + m_B\left(\frac{1}{2}\mathbf{v}_A^i - \mathbf{v}_B^i\right)S_A^{ij} \\ &+ m_A(\mathbf{v}_A^k + \mathbf{v}_B^k)(\delta^{ik} - \mathbf{n}^i\mathbf{n}^k)S_B^{ij}\Bigg)r^2(i\mathbf{q}^j)\Bigg]\Bigg\}e^{-i\mathbf{q}\cdot\mathbf{x}_A}.\end{aligned} \tag{6.13}$$

We can extract some information regarding the binary system from the expression above when we take the long-wavelength limit by Taylor expanding it around $\mathbf{q}=0$. For instance, the zeroth order terms in the Taylor expansion give us the LO spin-orbit energy

$$E^{(1.5\mathrm{PN})}_{\mathrm{SO}} = \int d^3x T^{00}(\mathbf{x},t) = -\sum_{A\neq B}\frac{Gm_B}{r^3}S_A^{0j}\mathbf{r}^j, \tag{6.14}$$

and this serves as a self-consistency check, since (6.14) agrees with Eq. (4.1), which we calculated from the LO spin-orbit potential. Next, the terms linear in $\mathbf{q}$ yield the center-of-mass position (6.2), which is also conveniently expressed through[4] $\mathbf{G}\equiv m\mathbf{r}_{\mathrm{cm}}$:

$$\mathbf{G}^k_{(1.5\mathrm{PN})} = -\sum_{A=1}^{2}S_A^{0k} = -\sum_{A=1}^{2}S_A^{ik}\mathbf{v}_A^i, \tag{6.15}$$

$$\mathbf{G}^k_{(2.5\mathrm{PN})} = \sum_{A\neq B}\frac{Gm_B}{r^3}[S_A^{ij}\mathbf{r}^j(\mathbf{v}_B^i\mathbf{r}^k - \mathbf{v}_A^i\mathbf{x}_B^k) - S_A^{ik}(2r^2\mathbf{v}^i - \mathbf{r}^i\mathbf{r}\cdot(\mathbf{v}_A+\mathbf{v}_B))]. \tag{6.16}$$

Now, in order to extract its corrections, we put the center-of-mass at the origin, meaning $\mathbf{G}=0$, and iteratively solve for $\mathbf{x}_1$, $\mathbf{x}_2$. Writing

$$\mathbf{x}_1 = \frac{m_2}{m}\mathbf{r} + \delta\mathbf{r}^{(1\mathrm{PN})} + \delta\mathbf{r}^{(1.5\mathrm{PN})}_{\mathrm{SO}} + \delta\mathbf{r}^{(2\mathrm{PN})} + \delta\mathbf{r}^{(2.5\mathrm{PN})}_{\mathrm{SO}} + \cdots, \tag{6.17}$$

$$\mathbf{x}_2 = -\frac{m_1}{m}\mathbf{r} + \delta\mathbf{r}^{(1\mathrm{PN})} + \delta\mathbf{r}^{(1.5\mathrm{PN})}_{\mathrm{SO}} + \delta\mathbf{r}^{(2\mathrm{PN})} + \delta\mathbf{r}^{(2.5\mathrm{PN})}_{\mathrm{SO}} + \cdots, \tag{6.18}$$

we can determine PN corrections to the center-of-mass order by order. The corrections $\delta\mathbf{r}^{(1\mathrm{PN})}$ and $\delta\mathbf{r}^{(1.5\mathrm{PN})}_{\mathrm{SO}}$ can be found in [44,53] and are presented in Eq. (3.13), while the nonspin[5] $\delta\mathbf{r}^{(2\mathrm{PN})}$ can be found in [54]. The NLO correction, with covariant SSC enforced, is

$$\begin{aligned}\delta\mathbf{r}^{(2.5\mathrm{PN})}_{\mathrm{SO}} = \frac{\nu}{2m}\Bigg\{&\left[\nu\mathbf{v}^2 - \frac{Gm}{r}(4+2\nu)\right]\mathbf{\Sigma}\times\mathbf{v} \\ &+ \frac{\delta m}{m}\left[\mathbf{v}^2 - \frac{Gm}{r}\right]\mathbf{S}\times\mathbf{v} \\ &+ \frac{2Gm}{r}\Bigg[\frac{\delta m}{m}\mathbf{S}\cdot(\mathbf{v}\times\mathbf{n})\mathbf{n} + \frac{3}{2}\frac{\delta m}{m}\dot{r}(\mathbf{S}\times\mathbf{n}) \\ &+ (1-4\nu)\dot{r}(\mathbf{\Sigma}\times\mathbf{n})\Bigg]\Bigg\}.\end{aligned} \tag{6.19}$$

VII. CORRESPONDENCE WITH OTHER FORMALISMS

At this point, we note that the expressions for the acceleration (3.16), the binding energy (4.5), the NLO spin-orbit multipole moments (A9), (A16), and the center-of-mass correction (6.19) take a different form than the corresponding results given in the literature [37,38,52,55].

[4]The expression for $\mathbf{G}$ can be expanded order by order as $\mathbf{G} = \mathbf{G}^{(0\mathrm{PN})} + \mathbf{G}^{(1\mathrm{PN})} + \mathbf{G}^{(1.5\mathrm{PN})}_{\mathrm{SO}} + \mathbf{G}^{(2\mathrm{PN})} + \mathbf{G}^{(2.5\mathrm{PN})}_{\mathrm{SO}} + \cdots$; the LO and 1PN corrections can be found in [44], while the 2PN correction was computed in [54].

[5]There is no spin correction to the center-of-mass position at 2PN order.

As emphasized throughout this paper, we work with spins defined in the locally flat frame. We would expect, then, that an appropriate spin transformation coupled with a coordinate transformation should give agreement with existing results; the difficulty reduces to finding the appropriate set of transformations. As was discussed in [21], it is possible to construct an equivalent Hamiltonian to those in [37,56], and thus the equations of motion were expected to agree. In particular, there are two sets of results we would like to show agreement with: those for spin written in the PN frame as in [37,38], and those with spins of constant magnitude as in [52,55].

The relationship between the locally flat spin vectors and the PN spin vectors was shown in [22]. In the locally flat frame, we chose the relation between the spin tensor and spin vector in (3.4). A natural definition of the spin tensor in terms of the spin vector in the PN frame is

$$S^{\mu\nu} = -\frac{1}{m\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}p_\rho S_\sigma, \tag{7.1}$$

which clearly preserves the covariant SSC, and which in the locally flat frame reduces to (3.4). We fix the spin vector by imposing the additional condition used in [57] given by

$$S^\mu p_\mu = 0. \tag{7.2}$$

From these definitions, it was shown in [22] that the transformation from the locally flat spin vectors to the PN spin vectors $\bar{\mathbf{S}}_A$ to 1PN order is given by

$$\mathbf{S}_A \to \left(1 + \frac{\mathbf{v}_A^2}{2} + \frac{Gm_B}{r}\right)\bar{\mathbf{S}}_A - \mathbf{v}_A(\bar{\mathbf{S}}_A \cdot \mathbf{v}_A). \tag{7.3}$$

This transformation induces a 1PN correction to the spins, and was used in that paper to show equivalence between the spin evolution equations in [22] and [37,57]. Note that to leading order in the spins, the locally flat and PN frames are equivalent; corrections only enter at 1PN order. For NLO spin-orbit effects, there is a contribution that leads to different expressions for the accelerations, energy, mass quadrupole, current quadrupole, energy loss, and center-of-mass correction. With this spin transformation, the acceleration (3.16), the binding energy (4.5), the multipole moments (A9), (A16), the energy loss (4.6), and the center-of-mass correction (6.19) agree completely with the corresponding results in [37,38]. Importantly, the general expressions for these quantities agree exactly even *before* writing gauge invariant quantities. Of particular interest, the multipole moments agree completely with those in [38], showing that the EFT formalism used in this paper agrees with the literature, when spin-orbit effects are considered beyond the dominant order, not only in the conservative but also in the dissipative sector.

We also present the transformation to constant magnitude spin vectors as used in computing the orbital phase (5.15). This spin choice was used in [38,52,55] and as discussed in Sec. V is the proper choice when computing quantities in the adiabatic approximation. As shown in [21], the transformation to 1PN order is given by (5.1). This puts the spin evolution equations into a spin precession form [21,37], i.e.,

$$\frac{d\mathbf{S}_A^c}{dt} = \mathbf{\Omega}_A \times \mathbf{S}_A^c, \tag{7.4}$$

where $\mathbf{\Omega}_A$ is the precessional frequency. This spin transformation takes us from the covariant SSC to the Newton-Wigner SSC, with one important caveat. Completing the transformation to the Newton-Wigner SSC requires a change of coordinates that accounts for the shift in the center of mass of each binary constituent (see [21,53,58] for a detailed discussion). In fact, this spin redefinition coupled with the coordinate transformation to the Newton-Wigner SSC is the only possible choice if one wants to work with canonical variables [59]. However, to show the equivalence between our results and those in the literature, we forego the coordinate transformation and find that our results for the acceleration (3.16), the binding energy (4.5), the multipole moments (A9), (A16), and the center-of-mass correction (6.19) agree completely with the corresponding results in [38,52,55] with conserved norm spins.

VIII. FINAL REMARKS

We used the potential obtained in [21] via the NRGR formalism [15,16] to compute the NLO spin-orbit correction to the equations of motion and to the binding energy of a binary system of compact bodies in its inspiral stage. This correction to the equation of motion, which is a 2.5PN acceleration, was used together with the multipole moments computed in [33] to calculate the NLO spin-orbit terms in the energy lost by the system due to the emission of gravitational waves. Then, we utilized these results to compute the evolution of the orbital frequency and, consequently, of the orbital phase of the binary system accounting for spin-orbit effects beyond the dominant order, considering quasicircular orbits within the adiabatic approximation. In performing these computations, we have made extensive use of the *Mathematica* package xAct [60]. In addition, we calculated the 2.5PN spin-orbit terms of the 00 component of the pseudotensor of the system in order to extract the correction to the center of mass associated with NLO spin-orbit effects.

Although the results of this paper—the NLO spin-orbit effects in the equations of motion, center-of-mass frame, binding energy, energy loss, orbital evolution, and phase evolution—only now were obtained in the NRGR framework, they had been previously computed through other formalisms that follow more conventional approaches to general relativity. Therefore, we provided a discussion in which we explained that our EFT results and those found in the literature [37,38,52,55] are in perfect agreement once appropriate spin transformations are considered. While the equivalence between the EFT formalism and other methods was demonstrated in [21] in the conservative sector regarding NLO spin-orbit effects, we have shown now full agreement also in the radiation sector.

Moreover, while inviting for the completion of higher-order spin computations, the results obtained in this paper provide the final missing pieces needed to compute wave-forms that include subleading spin-orbit effects entirely within the NRGR formalism, which will be presented in a future publication.

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APPENDIX: TOOLKIT

1. Nonspin accelerations

The PN corrections to the Newtonian acceleration of one of the bodies—let us choose body 1—in the binary system are given below. In the EFT formalism, the 1PN correction to the LO gravitational acceleration

$$(\mathbf{a}_1^i)^{(0\mathrm{PN})} = -\frac{Gm_2}{r^2}\mathbf{n}^i \tag{A1}$$

can be derived from the Lagrangian obtained in [15], and it reads as

$$\begin{aligned}(\mathbf{a}_1^i)^{(1\mathrm{PN})} = \frac{Gm_2}{2r^2}\bigg\{&\mathbf{n}^i\left[\frac{2Gm}{r} - 3(\mathbf{v}_1^2+\mathbf{v}_2^2) + 7\mathbf{v}_1\cdot\mathbf{v}_2 + 3\mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2\cdot\mathbf{n}\right] - \mathbf{v}_2\cdot\mathbf{n}\mathbf{v}_1^i - \mathbf{v}_1\cdot\mathbf{n}\mathbf{v}_2^i + \dot{r}(6\mathbf{v}_1^i - 7\mathbf{v}_2^i - \mathbf{n}^i\mathbf{v}_2\cdot\mathbf{n})\\ &- 6r\mathbf{a}_1^i + 7r\mathbf{a}_2^i + (\mathbf{v}^i - \mathbf{n}^i\dot{r})\mathbf{v}_2\cdot\mathbf{n} + r\mathbf{a}_2\cdot\mathbf{n}\mathbf{n}^i + \mathbf{n}^i(\mathbf{v}_2\cdot(\mathbf{v}-\mathbf{n}\dot{r}))\bigg\} - \frac{1}{2}\mathbf{a}_1^i\mathbf{v}_1^2 - \mathbf{v}_1^i\mathbf{v}_1\cdot\mathbf{a}_1.\end{aligned} \tag{A2}$$

The second PN correction to the gravitational acceleration was derived in [54] considering the EFT theory in the linearized harmonic gauge, and it is given as follows:

$$\begin{aligned}(\mathbf{a}_1^i)^{(2\mathrm{PN})} = \frac{1}{8}\frac{Gm_2}{r^3}\mathbf{r}^i\bigg\{&\frac{G^2}{r^2}(-2m_1^2 - 20m_1m_2 + 16m_2^2) + \frac{G}{r}\bigg[(18m_1+56m_2)\mathbf{v}_1^2\\ &- (84m_1+128m_2)\mathbf{v}_1\cdot\mathbf{v}_2 + (58m_1+64m_2)\mathbf{v}_2^2 + 30m_1\mathbf{a}_1\cdot\mathbf{r} - 12m\mathbf{a}_2\cdot\mathbf{r}\\ &+ \frac{28}{r^2}(m_1-4m_2)\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_1\cdot\mathbf{r} - 2\mathbf{v}_2\cdot\mathbf{r}) - \frac{1}{r^2}(56m_1+176m_2)(\mathbf{v}_2\cdot\mathbf{r})^2\bigg]\\ &+ 2\mathbf{v}_1^4 - 16(\mathbf{v}_1\cdot\mathbf{v}_2)^2 - 16\mathbf{v}_2^4 + 32\mathbf{v}_1\cdot\mathbf{v}_2\mathbf{v}_2^2 - 2\mathbf{v}_1^2\mathbf{a}_2\cdot\mathbf{r} - 2\mathbf{v}_2^2\mathbf{a}_2\cdot\mathbf{r}\\ &- 4\mathbf{a}_2\cdot\mathbf{v}_2\mathbf{v}_2\cdot\mathbf{r} + \frac{(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2}(12\mathbf{v}_1^2 - 48\mathbf{v}_1\cdot\mathbf{v}_2 + 36\mathbf{v}_2^2) - 15\frac{(\mathbf{v}_2\cdot\mathbf{r})^4}{r^4}\bigg\}\\ &+ \frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_1^i\bigg\{\frac{G}{r}\bigg[(48m_2-15m_1)\mathbf{v}_1\cdot\mathbf{r} + (23m_1-40m_2)\mathbf{v}_2\cdot\mathbf{r}\bigg]\\ &+ \mathbf{v}_2\cdot\mathbf{r}(4\mathbf{v}_1^2 + 16\mathbf{v}_1\cdot\mathbf{v}_2 - 20\mathbf{v}_2^2) - 24\frac{\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2} + 18\frac{(\mathbf{v}_2\cdot\mathbf{r})^3}{r^2}\\ &+ \mathbf{v}_1\cdot\mathbf{r}(8\mathbf{v}_1^2 - 16\mathbf{v}_1\cdot\mathbf{v}_2 + 16\mathbf{v}_2^2 - 2\mathbf{a}_2\cdot\mathbf{r}) + 2r^2(12\mathbf{a}_1 - 7\mathbf{a}_2)\cdot\mathbf{v}_1\bigg\}\\ &+ 2\mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_1^2\mathbf{v}_1^i + \frac{1}{4}\mathbf{a}_1^i\left(49\frac{G^2m_1m_2}{r^2} + 36\frac{G^2m_2^2}{r^2} + 12\frac{Gm_2}{r}\mathbf{v}_1^2 + \mathbf{v}_1^4\right)\\ &+ \frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_2^i\bigg\{\frac{G}{r}\bigg[(31m_1-24m_2)\mathbf{v}_1\cdot\mathbf{r} + (40m_2-9m_1)\mathbf{v}_2\cdot\mathbf{r}\bigg]\\ &+ \mathbf{v}_2\cdot\mathbf{r}(-4\mathbf{v}_1^2 - 16\mathbf{v}_1\cdot\mathbf{v}_2 + 20\mathbf{v}_2^2) + 24\frac{\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2} - 18\frac{(\mathbf{v}_2\cdot\mathbf{r})^3}{r^2}\\ &+ \mathbf{v}_1\cdot\mathbf{r}(16\mathbf{v}_1\cdot\mathbf{v}_2 - 16\mathbf{v}_2^2) - 14r^2\mathbf{a}_2\cdot\mathbf{v}_2\bigg\} - \frac{7}{4}\frac{Gm_2}{r}\mathbf{a}_2^i\left(6\frac{Gm}{r} + \mathbf{v}_1^2 + \mathbf{v}_2^2\right).\end{aligned} \tag{A3}$$

2. Spin-orbit potentials

The LO and NLO spin-orbit potentials [16,21]—from which the LO and NLO spin-orbit accelerations and binding energies are computed—read, respectively, as

$$V_{\rm SO}^{(1.5{\rm PN})} = \frac{G\mathbf{r}^j}{r^3}\{m_2(S_1^{j0} + S_1^{jk}\mathbf{v}_1^k - 2S_1^{jk}\mathbf{v}_2^k) - m_1(S_2^{j0} + S_2^{jk}\mathbf{v}_2^k - 2S_2^{jk}\mathbf{v}_1^k)\}, \tag{A4}$$

$$\begin{aligned}V_{\rm SO}^{2.5{\rm PN}} = \frac{Gm_2}{r^3}\Bigg\{&\Bigg[S_1^{i0}\left(2\mathbf{v}_2^2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{3}{2r^2}(\mathbf{v}_2\cdot\mathbf{r})^2 - \frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}\right) + \left(2\mathbf{v}_1\cdot\mathbf{v}_2 + \frac{3(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2} - 2\mathbf{v}_2^2 + \mathbf{a}_2\cdot\mathbf{r}\right)S_1^{ij}\mathbf{v}_2^j\\ &- \left(\frac{3}{2r^2}(\mathbf{v}_2\cdot\mathbf{r})^2 + \frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}\right)S_1^{ij}\mathbf{v}_1^j + 2S_1^{ij}\mathbf{a}_2^j\mathbf{v}_2\cdot\mathbf{r} + r^2S_1^{ij}\dot{\mathbf{a}}_2^j\Bigg]\mathbf{r}^i\\ &+ S_1^{i0}\left((\mathbf{v}_1-\mathbf{v}_2)^i\mathbf{v}_2\cdot\mathbf{r} - \frac{3}{2}\mathbf{a}_2^i r^2\right) + S_1^{ij}\left(\mathbf{v}_2^i\mathbf{v}_1^j\mathbf{v}_2\cdot\mathbf{r} - r^2\mathbf{a}_2^j\mathbf{v}_2^i - \frac{1}{2}r^2\mathbf{a}_2^j\mathbf{v}_1^i\right)\Bigg\}\\ &+ \frac{G^2m_2}{r^4}\mathbf{r}^i\left[-(m_1+2m_2)S_1^{i0} + \left(m_1 - \frac{m_2}{2}\right)S_1^{ij}\mathbf{v}_1^j + \frac{5m_2}{2}S_1^{ij}\mathbf{v}_2^j\right] + 1\leftrightarrow 2.\end{aligned} \tag{A5}$$

3. Multipole moments

The multipole moments needed to compute the energy loss at 2.5PN were obtained in [33,44]. We present them here, written in the center-of-mass frame and with the covariant SSC imposed. The spin vector is defined in the locally flat frame. The mass quadrupole moments are

$$I^{ij}_{(0{\rm PN})} = m\nu\{\mathbf{r}^i\mathbf{r}^j\}_{\rm TF}, \tag{A6}$$

$$I^{ij}_{(1{\rm PN})} = m\nu\left\{\left[\left(-\frac{5}{7}+\frac{8}{7}\nu\right)\frac{Gm}{r} + \left(\frac{29}{42}-\frac{29}{14}\nu\right)\mathbf{v}^2\right]\mathbf{r}^i\mathbf{r}^j + \left(\frac{11}{21}-\frac{11}{7}\nu\right)r^2\mathbf{v}^i\mathbf{v}^j + \left(-\frac{4}{7}+\frac{12}{7}\nu\right)r\dot{r}\mathbf{v}^j\mathbf{r}^i\right\}_{\rm STF}, \tag{A7}$$

$$I^{ij}_{(1.5{\rm PN})} = \nu\left\{\frac{8}{3}(\mathbf{v}\times\mathbf{S})^i\mathbf{r}^j - \frac{4}{3}(\mathbf{r}\times\mathbf{S})^i\mathbf{v}^j + \frac{8}{3}\frac{\delta m}{m}(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{r}^j - \frac{4}{3}\frac{\delta m}{m}(\mathbf{r}\times\boldsymbol{\Sigma})^i\mathbf{v}^j\right\}_{\rm STF}, \tag{A8}$$

$$\begin{aligned}I^{ij}_{(2.5{\rm PN})} = \nu\Bigg\{&\left[\left(\frac{5}{21}-\frac{5}{7}\nu\right)\mathbf{v}\cdot(\mathbf{r}\times\mathbf{S}) + \left(\frac{5}{21}+\frac{4}{7}\nu\right)\frac{\delta m}{m}\mathbf{v}\cdot(\mathbf{r}\times\boldsymbol{\Sigma})\right]\mathbf{v}^i\mathbf{v}^j\\ &+ \left[\left(-\frac{52}{21}+\frac{10}{7}\nu\right)\mathbf{v}\cdot(\mathbf{n}\times\mathbf{S}) + \left(-\frac{62}{21}+\frac{18}{7}\nu\right)\frac{\delta m}{m}\mathbf{v}\cdot(\mathbf{n}\times\boldsymbol{\Sigma})\right]\frac{Gm}{r}\mathbf{n}^i\mathbf{r}^j\\ &+ \left[\left(\frac{19}{21}+\frac{167}{21}\nu\right)\frac{Gm}{r} + \left(-\frac{2}{21}+\frac{2}{7}\nu\right)\mathbf{v}^2\right](\mathbf{v}\times\mathbf{S})^i\mathbf{r}^j\\ &+ \left[\left(-\frac{1}{3}+\frac{20}{3}\nu\right)\frac{Gm}{r} + \left(-\frac{2}{21}-\frac{20}{7}\nu\right)\mathbf{v}^2\right]\frac{\delta m}{m}(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{r}^j\\ &+ \left[\left(-\frac{22}{3}-\frac{10}{3}\nu\right)\frac{Gm}{r} + \left(-\frac{4}{21}+\frac{4}{7}\nu\right)\mathbf{v}^2\right](\mathbf{r}\times\mathbf{S})^i\mathbf{v}^j\\ &+ \left[\left(-\frac{8}{3}-\frac{34}{21}\nu\right)\frac{Gm}{r} + \left(-\frac{4}{21}+\frac{12}{7}\nu\right)\mathbf{v}^2\right]\frac{\delta m}{m}(\mathbf{r}\times\boldsymbol{\Sigma})^i\mathbf{v}^j\\ &+ \left[\left(\frac{8}{3}-\frac{16}{3}\nu\right)\mathbf{S}\cdot\mathbf{n} + \left(\frac{8}{3}-\frac{8}{3}\nu\right)\frac{\delta m}{m}\boldsymbol{\Sigma}\cdot\mathbf{n}\right]\frac{Gm}{r}(\mathbf{v}\times\mathbf{n})^i\mathbf{r}^j\\ &+ \left(\frac{10}{21}-\frac{10}{7}\nu\right)r\dot{r}(\mathbf{v}\times\mathbf{S})^j\mathbf{v}^i + \left(\frac{10}{21}-\frac{8}{21}\nu\right)\frac{\delta m}{m}r\dot{r}(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{v}^j\\ &+ \left(\frac{31}{21}+\frac{19}{21}\nu\right)\frac{Gm}{r}\dot{r}(\mathbf{n}\times\mathbf{S})^j\mathbf{r}^i + \left(\frac{5}{3}+\frac{2}{7}\nu\right)\frac{Gm}{r}\frac{\delta m}{m}\dot{r}(\mathbf{n}\times\boldsymbol{\Sigma})^i\mathbf{r}^j\Bigg\}_{\rm STF}.\end{aligned} \tag{A9}$$

The mass octupole moments are

$$I^{ijk}_{(0\mathrm{PN})} = -\delta m\nu\{\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\}_{\mathrm{TF}}, \tag{A10}$$

$$I^{ijk}_{(1\mathrm{PN})} = -\delta m\nu\left\{\left[\left(-\frac{5}{6}+\frac{13\nu}{6}\right)\frac{Gm}{r}+\left(\frac{5}{6}-\frac{19}{6}\nu\right)v^2\right]\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k + (-1+2\nu)r\dot{r}\mathbf{r}^i\mathbf{r}^j\mathbf{v}^k + (1-2\nu)r^2\mathbf{r}^i\mathbf{v}^j\mathbf{v}^k\right\}_{\mathrm{STF}}, \tag{A11}$$

$$I^{ijk}_{(1.5\mathrm{PN})} = \nu\left\{-\frac{9}{2}\frac{\delta m}{m}(\mathbf{v}\times\mathbf{S})^i\mathbf{r}^j\mathbf{r}^k + \left(-\frac{9}{2}+\frac{33}{2}\nu\right)(\mathbf{v}\times\boldsymbol{\Sigma})^i\mathbf{r}^j\mathbf{r}^k + 3\frac{\delta m}{m}(\mathbf{r}\times\mathbf{S})^i\mathbf{r}^j\mathbf{v}^k + (3-9\nu)(\mathbf{r}\times\boldsymbol{\Sigma})^i\mathbf{r}^j\mathbf{v}^k\right\}_{\mathrm{STF}}. \tag{A12}$$

The current quadrupole moments are

$$J^{ij}_{(0\mathrm{PN})} = \nu\delta m\{(\mathbf{v}\times\mathbf{r})^i\mathbf{r}^j\}_{\mathrm{STF}}, \tag{A13}$$

$$J^{ij}_{(0.5\mathrm{PN})} = -\frac{3}{2}\nu\{\boldsymbol{\Sigma}^i\mathbf{r}^j\}_{\mathrm{STF}}, \tag{A14}$$

$$J^{ij}_{(1\mathrm{PN})} = \nu\delta m\left\{\left[\left(\frac{27}{14}+\frac{15}{7}\nu\right)\frac{Gm}{r}+\left(\frac{13}{28}-\frac{17}{7}\nu\right)\mathbf{v}^2\right](\mathbf{v}\times\mathbf{r})^i\mathbf{r}^j + \left(\frac{5}{28}-\frac{5}{14}\nu\right)r\dot{r}(\mathbf{v}\times\mathbf{r})^i\mathbf{v}^j\right\}_{\mathrm{STF}}, \tag{A15}$$

$$\begin{aligned} J^{ij}_{(1.5\mathrm{PN})} = \nu\Bigg\{&\left[\left(\frac{61}{28}-\frac{71}{28}\nu\right)\frac{Gm}{r}+\left(-\frac{2}{7}+\frac{20}{7}\nu\right)\mathbf{v}^2\right]\boldsymbol{\Sigma}^i\mathbf{r}^j + \left[\frac{10}{7}\frac{Gm}{r}+\frac{13}{28}\mathbf{v}^2\right]\frac{\delta m}{m}\mathbf{S}^i\mathbf{r}^j \\ &+\left[-\frac{11}{14}\frac{\delta m}{m}\mathbf{S}\cdot\mathbf{r}+\left(-\frac{11}{14}+\frac{47}{14}\nu\right)\boldsymbol{\Sigma}\cdot\mathbf{r}\right]\mathbf{v}^i\mathbf{v}^j + \left[\frac{3}{7}\frac{\delta m}{m}\mathbf{S}\cdot\mathbf{v}+\left(\frac{3}{7}-\frac{23}{7}\nu\right)\boldsymbol{\Sigma}\cdot\mathbf{v}\right]\mathbf{v}^i\mathbf{x}^j \\ &+\left[-\frac{29}{14}\frac{\delta m}{m}\mathbf{S}\cdot\mathbf{n}+\left(-\frac{4}{7}+\frac{31}{14}\nu\right)\boldsymbol{\Sigma}\cdot\mathbf{n}\right]\frac{Gm}{r}\mathbf{n}^i\mathbf{r}^j + \frac{3}{7}\frac{\delta m}{m}r\dot{r}\mathbf{S}^j\mathbf{v}^i + \left(\frac{3}{7}-\frac{16}{7}\nu\right)r\dot{r}\boldsymbol{\Sigma}^i\mathbf{v}^j\Bigg\}_{\mathrm{STF}}. \end{aligned} \tag{A16}$$

The current octupole moments are

$$J^{ijk}_{(0\mathrm{PN})} = -m\nu(1-3\nu)\{(\mathbf{v}\times\mathbf{r})^i\mathbf{r}^j\mathbf{r}^k\}_{\mathrm{STF}}, \tag{A17}$$

$$J^{ijk}_{(0.5\mathrm{PN})} = 2\nu\left\{\mathbf{S}^i\mathbf{r}^j\mathbf{r}^k + \frac{\delta m}{m}\boldsymbol{\Sigma}^i\mathbf{r}^j\mathbf{r}^k\right\}_{\mathrm{STF}}. \tag{A18}$$

4. NRGR vertices

The vertices needed to compute the 2.5PN center-of-mass correction [16,42] are

$$S_H^{v^0} = -\sum_A \frac{m_A}{2m_{\mathrm{Pl}}}\int dt_A H_{00}(x_A), \tag{A19}$$

$$S_H^{v^1} = -\sum_A \frac{m_A}{m_{\mathrm{Pl}}}\int dt_A v_A^i H_{0i}(x_A), \tag{A20}$$

$$S_{H\bar{h}_{00}}^{v^0} = \sum_A \frac{m_A}{4m_{\mathrm{Pl}}^2}\int dt_A H_{00}(x_A)\bar{h}_{00}(x_A), \tag{A21}$$

$$S_{H\bar{h}_{00}}^{v^1} = \sum_A \frac{m_A}{2m_{\mathrm{Pl}}^2}\int dt_A v_A^i H_{0i}(x_A)\bar{h}_{00}(x_A), \tag{A22}$$

$$S_H^{Sv^0} = \sum_A \frac{1}{2m_{\mathrm{Pl}}}\int dt_A H_{i0,k}(x_A)S_A^{ik}, \tag{A23}$$

$$S_H^{Sv^1} = \sum_A \frac{1}{2m_{\rm Pl}} \int dt_A [H_{ij,k}(x_A) S_A^{ik} v_A^j + H_{00,k}(x_A) S_A^{0k}], \tag{A24}$$

$$S_{\bar{h}_{00}}^{Sv^1} = \sum_A \frac{1}{2m_{\rm Pl}} \int dt_A \bar{h}_{00,k}(x_A) S_A^{0k}, \tag{A25}$$

$$S_{H\bar{h}_{00}}^{Sv^0} = \sum_A \frac{1}{4m_{\rm Pl}^2} \int dt_A S_A^{ij} H_j{}^0(x_A) \bar{h}_{00,i}(x_A), \tag{A26}$$

$$S_{H\bar{h}_{00}}^{Sv^1} = \sum_A \frac{1}{4m_{\rm Pl}^2} \int dt_A S_A^{i0} [H_{00}(x_A) \bar{h}_{00,i}(x_A) + \bar{h}_{00}(x_A) H_{00,i}(x_A) + H^l{}_i(x_A) \bar{h}_{00,l}(x_A)]. \tag{A27}$$

Vertices are expressed using the Minkowski metric.

[1] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. **116**, 061102 (2016).

[2] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[3] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017); Phys. Rev. Lett. **121**, 129901(E) (2018).

[4] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW170608: Observation of a 19-solar-mass binary black hole coalescence, Astrophys. J. **851**, L35 (2017).

[5] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, Phys. Rev. Lett. **119**, 141101 (2017).

[6] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119**, 161101 (2017).

[7] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo During the First and Second Observing Runs, Phys. Rev. X **9**, 031040 (2019).

[8] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW190425: Observation of a compact binary coalescence with total mass $\sim 3.4 M_\odot$, Astrophys. J. Lett. **892**, L3 (2020).

[9] R. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW190412: Observation of a binary-black-hole coalescence with asymmetric masses, Phys. Rev. D **102**, 043015 (2020).

[10] R. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW190814: Gravitational waves from the coalescence of a 23 solar mass black hole with a 2.6 solar mass compact object, Astrophys. J. Lett. **896**, L44 (2020).

[11] R. Abbott *et al.* (LIGO Scientific, Virgo Collaborations), GW190521: A Binary Black Hole Merger with a Total Mass of $150 M_\odot$, Phys. Rev. Lett. **125**, 101102 (2020).

[12] S. S. R. Offner, M. M. Dunham, K. I. Lee, H. G. Arce, and D. B. Fielding, The turbulent origin of outflow and spin misalignment in multiple star systems, Astrophys. J. **827**, L11 (2016).

[13] M. A. Sedda and M. Benacquista, Using final black hole spins and masses to infer the formation history of the observed population of gravitational wave sources, Mon. Not. R. Astron. Soc. **482**, 2991 (2019).

[14] M. A. Sedda, M. Mapelli, M. Spera, M. Benacquista, and N. Giacobbo, Fingerprints of binary black hole formation channels encoded in the mass and spin of merger remnants, Astrophys. J. **894**, 133 (2020).

[15] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[16] R. A. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRGR, Phys. Rev. D **73**, 104031 (2006).

[17] C. R. Galley, A. K. Leibovich, R. A. Porto, and A. Ross, Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution, Phys. Rev. D **93**, 124010 (2016).

[18] S. Foffa and R. Sturani, Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian, Phys. Rev. D **100**, 024047 (2019).

[19] R. A. Porto and I. Z. Rothstein, Apparent ambiguities in the post-Newtonian expansion for binary systems, Phys. Rev. D **96**, 024062 (2017).

[20] S. Foffa, R. A. Porto, I. Rothstein, and R. Sturani, Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian, Phys. Rev. D **100**, 024048 (2019).

[21] R. A. Porto, Next to leading order spin-orbit effects in the motion of inspiralling compact binaries, Classical Quantum Gravity **27**, 205001 (2010).

[22] R. A. Porto and I. Z. Rothstein, Spin(1)spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion, Phys. Rev. D **78**, 044012 (2008); Erratum, Phys. Rev. D **81**, 029904 (2010).

[23] R. A. Porto and I. Z. Rothstein, Next to leading order spin(1) spin(1) effects in the motion of inspiralling compact binaries, Phys. Rev. D **78**, 044013 (2008); Erratum, Phys. Rev. D **81**, 029905 (2010).

[24] R. A. Porto and I. Z. Rothstein, Calculation of the First Nonlinear Contribution to the General-Relativistic Spin-Spin Interaction for Binary Systems, Phys. Rev. Lett. **97**, 021101 (2006).

[25] M. Levi, Next to leading order gravitational spin1-spin2 coupling with Kaluza–Klein reduction, Phys. Rev. D **82**, 064029 (2010).

[26] M. Levi, Next to leading order gravitational spin-orbit coupling in an effective field theory approach, Phys. Rev. D **82**, 104004 (2010).

[27] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (2016) 011.

[28] M. Levi and J. Steinhoff, Next-to-next-to-leading order gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme, J. Cosmol. Astropart. Phys. 01 (**2016**) 008.

[29] M. Levi, A. J. Mcleod, and M. Von Hippel, NNNLO gravitational spin-orbit coupling at the quartic order in G, arXiv:2003.02827.

[30] M. Levi, A. J. Mcleod, and M. Von Hippel, NNNLO gravitational quadratic-in-spin interactions at the quartic order in G, arXiv:2003.07890.

[31] M. Levi and J. Steinhoff, Leading order finite size effects with spins for inspiralling compact binaries, J. High Energy Phys. 06 (2015) 059.

[32] M. Levi, S. Mougiakakos, and M. Vieira, Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order, arXiv:1912.06276.

[33] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave flux from binary inspirals to third post-Newtonian order, J. Cosmol. Astropart. Phys. 03 (2011) 009.

[34] R. A. Porto, A. Ross, and I. Z. Rothstein, Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 post-Newtonian order, J. Cosmol. Astropart. Phys. 09 (**2012**) 028.

[35] N. T. Maia, C. R. Galley, A. K. Leibovich, and R. A. Porto, Radiation reaction for spinning bodies in effective field theory I: Spin-orbit effects, Phys. Rev. D **96**, 084064 (2017).

[36] N. T. Maia, C. R. Galley, A. K. Leibovich, and R. A. Porto, Radiation reaction for spinning bodies in effective field theory II: Spin-spin effects, Phys. Rev. D **96**, 084065 (2017).

[37] G. Faye, L. Blanchet, and A. Buonanno, Higher-order spin effects in the dynamics of compact binaries. I. Equations of motion, Phys. Rev. D **74**, 104033 (2006).

[38] L. Blanchet, A. Buonanno, and G. Faye, Higher-order spin effects in the dynamics of compact binaries. II. Radiation field, Phys. Rev. D **74**, 104034 (2006); Erratum, Phys. Rev. D **75**, 049903 (2007); Erratum, Phys. Rev. D **81**, 089901 (2010).

[39] W. D. Goldberger, Course 7 - Effective field theories and gravitational radiation, in *Les Houches*, edited by F. Bernardeau, C. Grojean, and J. Dalibard (Elsevier, New York, 2007), Vol. 86, pp. 351–396, https://doi.org/10.1016/S0924-8099(07)80033-3.

[40] I. Z. Rothstein, Progress in effective field theory approach to the binary inspiral problem, Gen. Relativ. Gravit. **46**, 1726 (2014).

[41] S. Foffa and R. Sturani, Effective field theory methods to model compact binaries, Classical Quantum Gravity **31**, 043001 (2014).

[42] R. A. Porto, The effective field theorist's approach to gravitational dynamics, Phys. Rep. **633**, 1 (2016).

[43] M. Levi, Effective field theories of post-Newtonian gravity: A comprehensive review, Rep. Prog. Phys. **83**, 075901 (2020).

[44] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. D **81**, 124015 (2010).

[45] A. Ross, Multipole expansion at the level of the action, Phys. Rev. D **85**, 125033 (2012).

[46] C. W. Lincoln and C. M. Will, Coalescing binary systems of compact objects to (post)$^{5/2}$-Newtonian order: Late-time evolution and gravitational-radiation emission, Phys. Rev. D **42**, 1123 (1990).

[47] C. R. Galley and I. Z. Rothstein, Deriving analytic solutions for compact binary inspirals without recourse to adiabatic approximations, Phys. Rev. D **95**, 104054 (2017).

[48] Z. Yang and A. K. Leibovich, Analytic solutions to compact binary inspirals with leading order spin-orbit contribution using the dynamical renormalization group, Phys. Rev. D **100**, 084021 (2019).

[49] C. R. Galley and M. Tiglio, Radiation reaction and gravitational waves in the effective field theory approach, Phys. Rev. D **79**, 124027 (2009).

[50] C. R. Galley and A. K. Leibovich, Radiation reaction at 3.5 post-Newtonian order in effective field theory, Phys. Rev. D **86**, 044029 (2012).

[51] C. M. Will, Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. III. Radiation reaction for binary systems with spinning bodies, Phys. Rev. D **71**, 084027 (2005).

[52] A. Bohé, S. Marsat, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the gravitational wave flux and orbital phasing of compact binaries, Classical Quantum Gravity **30**, 135009 (2013).

[53] L. E. Kidder, Coalescing binary systems of compact objects to (post)$^{5/2}$-Newtonian order. V. Spin effects, Phys. Rev. D **52**, 821 (1995).

[54] A. K. Leibovich, N. T. Maia, I. Z. Rothstein, and Z. Yang, Second post-Newtonian order radiative dynamics of inspiralling compact binaries in the effective field theory approach, Phys. Rev. D **101**, 084058 (2020).

[55] S. Marsat, A. Bohé, G. Faye, and L. Blanchet, Next-to-next-to-leading order spin-orbit effects in the equations of motion of compact binary systems, Classical Quantum Gravity **30**, 055007 (2013).

[56] T. Damour, P. Jaranowski, and G. Schäfer, Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling, Phys. Rev. D **77**, 064032 (2008).

[57] H. Tagoshi, A. Ohashi, and B. J. Owen, Gravitational field and equations of motion of spinning compact binaries to 2.5 post-Newtonian order, Phys. Rev. D **63**, 044006 (2001).

[58] B. M. Barker and R. F. O'Connell, Nongeodesic motion in general relativity, Gen. Relativ. Gravit. **5**, 539 (1974).

[59] J. Steinhoff, Canonical formulation of spin in general relativity, Ann. Phys. (Berlin) **523**, 296 (2011).

[60] J. M. Martín-García, xAct: Efficient tensor computer algebra for the Wolfram language, www.xAct.es.