Radiative contribution to classical gravitational scattering at the third order in *G*

Thibault Damour[*]

Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

Working within the post-Minkowskian approach to general relativity, we prove that the radiation-reaction to the emission of gravitational waves during the large-impact-parameter scattering of two (classical) point masses modifies the conservative scattering angle by an additional contribution of order G^3 which involves a high-energy (or massless) logarithmic divergence of opposite sign to the one contained in the third-post-Minkowskian result of Bern *et al.* [Phys. Rev. Lett. **122**, 201603 (2019)]. The high-energy limit of the resulting radiation-reaction-corrected (classical) scattering angle is finite, and is found to agree with the one following from the (quantum) eikonal-phase result of Amati *et al.* [Nucl. Phys. **B347**, 550 (1990)].

I. INTRODUCTION

Post-Minkowskian (PM) perturbative gravity, i.e., the Poincaré-covariant perturbative approach to the classical gravitational dynamics of binary systems, was initiated long ago [1–6], and has been recently revived through the use of new frameworks, notably the effective one-body method [7–10], and the use of dictionaries relating quantum scattering amplitudes to classical PM dynamics [8,11–25]. Here, we shall focus on the interaction between structureless bodies; see, e.g., Refs. [26–34] for PM-type works on spinning bodies.

A recent milestone has been the derivation by Bern *et al.* [16,17] of the 3PM-accurate, $O(G^3)$, conservative gravitational dynamics of binary systems from a two-loop scattering amplitude. The 3PM dynamics of Refs. [16,17] have been checked at the sixth post-Newtonian (6PN) level [35–37] and rederived in different ways [22,25]. However, the $O(G^3)$ dynamics of Refs. [16,17] has several puzzling features (see, e.g., the discussion in Ref. [10]).

One of the puzzling features of the 3PM-accurate scattering angle derived in Refs. [16,17] is that it involves a logarithmically divergent contribution $\propto \ln(s/(m_1 m_2))$ that becomes large both in the massless limit ($m_1 \to 0$, $m_2 \to 0$) and in the high-energy limit ($s \to \infty$). By contrast, previous work by Amati *et al.* (ACV) has extracted a finite scattering angle from the high-energy (trans-Planckian), two-loop scattering of two massless particles in the (quasiclassical) eikonal-phase approximation [38]. It was argued in Refs. [16,17] that this discontinuity in the scattering angle was linked to the need of imposing the inequality $q \sim \frac{\hbar}{b} \ll m_1, m_2$ in order to extract the classical

limit from the quantum scattering amplitude of two massive particles. The validity of the high-energy eikonal ACV scattering angle (as well as its universality in two-derivative gravity theories) has been recently established in Refs. [39–41]. In addition, Ref. [41] brought a new light on the puzzling issue of the relation between the ultrarelativistically singular classical 3PM dynamics of Refs. [16,17] and the ultrarelativistic limit of the quantum scattering of massive particles (in any two-derivative gravity theory). They presented two different approaches and showed that in both cases it is crucial to include radiative effects for recovering ultrarelativistic finiteness and continuity with the ACV eikonal scattering phase [38]. In particular, in their second approach (involving the evaluation of a four-point two-loop amplitude in $\mathcal{N} = 8$ supergravity) they emphasized the need to integrate the graviton contribution over the full soft region, rather than only over the potential region.

The results of Ref. [41] do not, by themselves, clarify the precise way in which the inclusion of *classical* gravitational-radiation effects can reconcile the (ultrarelativistically singular) *conservative* 3PM dynamics of two *classical particles* in general relativity (GR) with the finite eikonal ACV scattering angle. The present work will clarify this issue by proving (by a PM-gravity computation in GR) that the classical radiation-reaction to the emission of gravitational radiation during the large-impact-parameter scattering of two classical point masses modifies the conservative scattering angle, χ^{cons}, by an additional contribution, say χ^{rad}, which: (i) is of order G^3; (ii) cancels the logarithmically divergent high-energy contribution $\propto \ln(s/(m_1 m_2))$ present in χ^{cons}; and (iii) yields a finite high-energy 3PM-accurate scattering angle that precisely agree with the one following [40,42] from the ACV eikonal phase [38].

---
[*]damour@ihes.fr

Amusingly, the resolution offered here of the 3PM high-energy puzzle uses in a crucial way two ideas that were first discussed many years ago in a completely different context. Indeed, soon after the discovery of the Hulse-Taylor binary pulsar [43] (which brought, after a few years, direct evidence for the reality of gravitational radiation reaction [44]) several authors emphasized the lack of a consistent derivation of radiation-reaction effects in GR [45]. One of the reasons for doubt was that the then extant (heuristic) derivations of radiation-reaction effects were based on formal *post-Newtonian* (PN) expansions involving infrared ambiguities because of a lack of clear matching between the nearzone gravitational field ("potential modes") and the wavezone gravitational field ("soft modes"). The answer to this doubt was brought by the use of a PM framework in which the gravitational binary dynamics was described by (nonlinearly iterated) *retarded-propagator* interactions, thereby evacuating the issue of the nearzone/wavezone matching [4,5,46]. This led to the derivation of equations of motion including (on the same footing) both conservative and radiation-reaction effects and leading to observable predictions for the binary dynamics that agreed with binary-pulsar observations [47]. Within this PM framework, it was emphasized in Ref. [48], that the retarded PM equations of motion (which included radiation-reaction effects) implied a loss of the mechanical angular momentum of the system of order G^2/c^5, i.e., at the 2PM level, while they implied a loss of the mechanical energy of the system at the 3PM level [namely $O(G^3/c^5)$]. Our derivation below of the radiation-reaction correction $\chi^{\rm rad}$ to the scattering angle will be crucially based on a PM computation of the radiative loss of angular momentum at the $O(G^2)$ level. Another crucial ingredient of our computation will be to use a general result [49] linking $\chi^{\rm rad}$ to the radiative losses of angular momentum and energy.

We leave to our concluding section below a discussion of the possible consequences of our results for the general program of developing PM gravity to higher orders in G.

II. GRAVITATIONAL WAVEFORM AT $O(G^2)$

The radiative part of the GR gravitational field is asymptotically measured by the waveform, $f_{ij}(u,\theta,\phi)$ (where $u \equiv t - r$), defined as the transverse-traceless (TT) projection of the $O(1/r)$ part of the metric[1] $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in a suitable Bondi-Sachs-type coordinate system,

$$h_{ij}^{\rm TT} = \frac{f_{ij}(t-r,\theta,\phi)}{r} + O\left(\frac{1}{r^2}\right). \quad (2.1)$$

Here, and below, latin indices, $a, b, \ldots, i, j, k$, refer to Cartesian-like spatial directions. The waveform f_{ij} satisfies

the algebraic constraints $f_{ij}n^j = 0$ and $\delta^{ij}f_{ij} = 0$ [where $n^i(\theta,\phi) \equiv x^i/r$ is the radial unit vector].

The outgoing wave field $h_{ij}^{\rm TT}$ carries away both energy-momentum, $P_\mu^{\rm rad}$, and angular momentum $J_{\mu\nu}^{\rm rad}$. There are subtleties in the definition of $J_{\mu\nu}^{\rm rad}$ linked to nonlinear memory effects, but they do not affect our $O(G^2)$ computation below. Indeed, one of the crucial elements underlying our computation is that, while $P_\mu^{\rm rad} = O(G^3)$, the radiated angular momentum is of lower PM order, namely: $J_{\mu\nu}^{\rm rad} = O(G^2)$. Expressions for the radiated angular-momentum[2] in terms of the asymptotic waveform $f_{ij}(u,\theta,\phi)$ have been given in Refs. [50–53] (see also [54] for $J_{0i}^{\rm rad}$),

$$J_k^{\rm rad} = \frac{\epsilon_{kij}}{16\pi G} \int du d\Omega \left[f_{ia}\partial_u f_{ja} - \frac{1}{2}x^i\partial_j f_{ab}\partial_u f_{ab} \right]. \quad (2.2)$$

Note that the integrand of Eq. (2.2) is bilinear in f_{ij} (or its angular derivatives) and in $\dot{f}_{ij} \equiv \partial_u f_{ij}$. By contrast, the radiated energy-momentum, $P_\mu^{\rm rad}$, is quadratic in $\dot{f}_{ij}$, namely,

$$P_{\rm rad}^\mu = \frac{1}{32\pi G} \int du d\Omega [\partial_u f_{ab}\partial_u f_{ab}]n^\mu, \quad (2.3)$$

where $n^\mu = (1, x^i/r)$. The PM perturbation theory of gravitationally interacting point masses [5,6,55,56] has shown that the waveform has an expansion in powers of G of the form,

$$f_{ij}(u,\theta,\phi) = Gf_{ij}^{(1)}(\theta,\phi) + G^2 f_{ij}^{(2)}(u,\theta,\phi) + O(G^3), \quad (2.4)$$

where the 1PM ($O(G)$) contribution is independent[3] of the retarded time u, so that

$$\dot{f}_{ij} = G^2 \dot{f}_{ij}^{(2)} + O(G^3), \quad (2.5)$$

is of order $O(G^2)$. This simple fact immediately shows, in view of Eqs. (2.2), (2.3), that $J_{\mu\nu}^{\rm rad} = O(G^2)$, while $P_\mu^{\rm rad} = O(G^3)$.

The static nature of $f_{ij}^{(1)}$ allows one to perform the time integration in Eq. (2.2) and to express the radiated angular momentum as

---

$$J_k^{\rm rad} = \frac{\epsilon_{kij}}{16\pi} \int d\Omega \left[f_{ia}^{(1)} \Delta f_{ja} - \frac{1}{2} x^i \partial_j f_{ab}^{(1)} \Delta f_{ab} \right], \quad (2.6)$$

where

$$\Delta f_{ij}(\theta, \phi) \equiv \int_{-\infty}^{+\infty} du \partial_u f_{ij} = [f_{ij}]_{u=-\infty}^{u=+\infty}, \quad (2.7)$$

is the *gravitational wave memory*, i.e., the global change in the waveform. To compute the waveform memory, $\Delta f_{ij}(\theta, \phi)$, it is better to use the form of the PM expansion employed, e.g., in Ref. [5], where the gothic-metric perturbation, $\bar{h}^{\mu\nu} \equiv -(\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu})$, i.e., $\bar{h}^{\mu\nu} = \eta^{\mu\mu'} \eta^{\nu\nu'} h_{\mu'\nu'} - \frac{1}{2} h \eta^{\mu\nu} + O(h^2)$, is decomposed as $\bar{h}^{\mu\nu} = \bar{h}_{\rm lin}^{\mu\nu} + \bar{h}_{\rm nonlin}^{\mu\nu}$. Here the linear piece, $\bar{h}_{\rm lin}^{\mu\nu}$, is the one generated by $T^{\mu\nu}$, i.e., $\bar{h}_{\rm lin}^{\mu\nu} = -16\pi G \Box_{\rm ret}^{-1}(T^{\mu\nu})$, while $\bar{h}_{\rm nonlin}^{\mu\nu}$ is sourced by the nonlinear contributions $T_{\rm nonlin}^{\mu\nu} = \partial^2 hh + \partial^2 hhh + \cdots$ appearing on the right-hand side of the PM-expanded Einstein's equations (in harmonic coordinates). The linear contribution generated by a system of pointlike bodies reads

$$\bar{h}_{\rm lin}^{\mu\nu}(x) = \sum_A 4Gm_A \left(\frac{u_A^\mu u_A^\nu}{r_A} \right)_{\rm ret}. \quad (2.8)$$

Here: the index A labels the gravitationally interacting bodies ($A = 1, 2$ for our present case); $u_A^\mu = dz_A^\mu/d\tau_A$ denotes the (Minkowski-normalized) four velocities; $r_A \equiv -\eta_{\mu\nu}(x^\mu - z_A^\mu) u_A^\nu$; and the subscript "ret" indicates that z_A is the retarded foot of the field point x on the Ath worldline [such that $\eta_{\mu\nu}(x^\mu - z_A^\mu)(x^\nu - z_A^\nu) = 0$ and $x^0 - z_A^0 > 0$]. In addition, the worldlines used in the definition Eq. (2.8) are taken to be the exact worldlines, or, at least, 1PM-accurate worldlines, curved by the linearized gravitational interaction.

When using such a decomposition, it is physically clear that the nonlinear memory $\Delta f_{ij}^{\rm nonlin}$ induced by the nonlinear effective source, i.e., by the splash of gravitational-wave energy emitted by the collision, will be of order $G \times P_\mu^{\rm rad}$, and therefore $O(G^4)$ (as was explicitly shown in Ref. [57]). We therefore conclude that the waveform memory Δf_{ij} to be used in Eq. (2.6) is given with sufficient accuracy by the simple linear formula,

$$\Delta f_{ij}(\mathbf{n}) = \sum_A 4G \left[\frac{(p_A^i p_A^j)^{\rm TT}}{E_A - \mathbf{n} \cdot \mathbf{p}_A} \right]_{-\infty}^{+\infty} + O(G^4). \quad (2.9)$$

Here: $\mathbf{n} = n^i(\theta, \phi)$ is the unit vector parametrizing the direction of gravitational-wave emission; $p_A^\mu = (E_A, \mathbf{p}_A)$, with $E_A = \sqrt{m_A^2 + \mathbf{p}_A^2}$, is the four-momentum of the Ath particle in the incoming ($-\infty$) or outgoing ($+\infty$) state [with $[f]_{-\infty}^{+\infty} \equiv f(+\infty) - f(-\infty)$]; and $(S_{ij})^{\rm TT}$ denotes (as usual) the TT projection of a (symmetric) three-dimensional tensor S_{ij} in the two-plane orthogonal to $\mathbf{n} = n^i(\theta, \phi)$, i.e., the two-plane tangent to the sphere at infinity.

Note that it is crucial to insert in the expression Eq. (2.9) values for $p_A^\mu(\pm\infty)$ that take into account the effect of the gravitational scattering. For computing the $O(G^2)$ angular-momentum loss, it is enough to use the $O(G)$ (1PM-accurate) gravitational deflection, i.e., (for $A = 1, 2$),

$$\Delta p_1^\mu = -\Delta p_2^\mu = -\beta \frac{b^\mu}{b} + O(G^2), \quad (2.10)$$

where we defined

$$\beta \equiv \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{Gm_1 m_2}{b}. \quad (2.11)$$

Here,

$$\gamma \equiv -(u_1 \cdot u_2)_{-\infty} = -\left[\frac{p_1 \cdot p_2}{m_1 m_2} \right]_{-\infty} \quad (2.12)$$

denotes the Lorentz factor between the two incoming worldlines, while b^μ (of magnitude b) denotes the vectorial impact parameter, i.e., the value of the four-vector $z_{1(0)}^\mu(\tau_1) - z_{2(0)}^\mu(\tau_2)$ *orthogonally* connecting the incoming (unperturbed) worldlines. [b^μ is orthogonal both to $p_1^\mu(-\infty)$ and to $p_2^\mu(-\infty)$; see [28] for details.]

III. WAVEFORM AND WAVEFORM-MEMORY IN THE CENTER-OF-MASS FRAME

We are interested in computing the change of angular momentum of the binary system in the (incoming) center-of-mass (c.m.) frame, i.e., with $\mathbf{p}_1(-\infty) = -\mathbf{p}_2(-\infty) \equiv \mathbf{P}$, and $E_A(-\infty) = \sqrt{m_A^2 + \mathbf{P}^2}$. We recall that $P_{\rm rad}^\mu = O(G^3)$, so that we have also $E_A(+\infty) = E_A(-\infty) + O(G^3)$, while $\mathbf{p}_1(+\infty) = -\mathbf{p}_2(+\infty) + O(G^3)$ is the deflected value of $\mathbf{P}$, obtained by rotating it by the c.m. deflection angle χ. The magnitude of the incoming c.m. angular momentum is $J(-\infty) = bP$, where $P \equiv |\mathbf{P}|$.

Using Eq. (2.8) the waveform is given, to $O(G)$ accuracy, by the time-independent TT tensor,

$$f_{ij}(\mathbf{n}) = 4G \left[\frac{1}{E_2 + \mathbf{n} \cdot \mathbf{P}} + \frac{1}{E_1 - \mathbf{n} \cdot \mathbf{P}} \right] (P^i P^j)^{\rm TT} + O(G^2). \quad (3.1)$$

On the other hand, the waveform memory is obtained by inserting the changes (2.10) in the expression (2.9), with the result,

$$\Delta f_{ij}(\mathbf{n}) = -\beta \frac{4G}{b} \left[\frac{1}{E_2 + \mathbf{n} \cdot \mathbf{P}} + \frac{1}{E_1 - \mathbf{n} \cdot \mathbf{P}} \right] (P^i b^j + P^j b^i)^{\rm TT}$$
$$+ \beta \frac{4G}{b} \left[\frac{\mathbf{n} \cdot \mathbf{b}}{(E_2 + \mathbf{n} \cdot \mathbf{P})^2} - \frac{\mathbf{n} \cdot \mathbf{b}}{(E_1 - \mathbf{n} \cdot \mathbf{P})^2} \right] (P^i P^j)^{\rm TT}. \quad (3.2)$$

IV. RADIATED ANGULAR MOMENTUM

In order to explicitly compute the angular integral (2.2) giving the radiated angular momentum, it is convenient to choose a system of polar coordinates adapted to the angular dependence of $f_{ij}(\mathbf{n})$ and $\Delta f_{ij}(\mathbf{n})$. We choose $\mathbf{e}_z$ in the direction of $\mathbf{P}$ (i.e., $\mathbf{P} = +P\mathbf{e}_z$) and $\mathbf{e}_y$ in the direction of $\mathbf{b}$ (i.e., $\mathbf{b} = +b\mathbf{e}_y$). We then define polar coordinates θ, ϕ in the usual way i.e., $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, as well as a standard orthonormal two-frame tangent to the sphere, namely $\mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ and $\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0)$. The two independent components of the TT projection of a tensor S_{ij} with respect to the frame $\mathbf{e}_\theta, \mathbf{e}_\phi$ are then $S_+ \equiv \frac{1}{2}(S_{\theta\theta} - S_{\phi\phi})$ and $S_\times \equiv \frac{1}{2}(S_{\theta\phi} + S_{\phi\theta})$.

Inserting $\mathbf{n} \cdot \mathbf{P} = P\cos\theta$, and introducing the short-hand notations,

$$A \equiv \frac{P}{E_2 + P\cos\theta} + \frac{P}{E_1 - P\cos\theta},$$
$$B \equiv \frac{P^2}{(E_2 + P\cos\theta)^2} - \frac{P^2}{(E_1 - P\cos\theta)^2}, \quad (4.1)$$

the $+$ and $\times$ components of f_{ij}, Eq. (3.1), and of Δf_{ij}, Eq. (3.2), read

$$f_+ = 2GPA\sin^2\theta + O(G^2); \qquad f_\times = O(G^2), \quad (4.2)$$

and

$$\Delta f_+ = 4G\beta\left[A\sin\theta\cos\theta\sin\phi + \frac{B}{2}\sin^3\theta\sin\phi\right],$$
$$\Delta f_\times = 4G\beta A\sin\theta\cos\phi. \quad (4.3)$$

Reexpressing Eq. (2.6) in terms of $f_{+,\times}$ yields, for the relevant component $J_x^{\rm rad} = J_{yz}^{\rm rad}$ in the direction of the initial angular momentum $J_{yz}^{(0)} = bP$, the expression [53],

$$J_{yz}^{\rm rad} = \int \frac{d\Omega}{16\pi G\sin\theta}[(\mathcal{D}f_+ - 2\cos\phi f_\times)\Delta f_+$$
$$+ (\mathcal{D}f_\times + 2\cos\phi f_+)\Delta f_\times], \quad (4.4)$$

where $\mathcal{D} \equiv \sin\theta\sin\phi\partial_\theta + \cos\theta\cos\phi\partial_\phi$.

Inserting Eqs. (4.2), (4.3) in Eq. (4.4) yields an explicit integral which can be performed without difficulty. (The integral over ϕ is elementary, while the integral over θ becomes elementary when rewritten as an integral over $c \equiv \cos\theta$.) Though intermediate results depend on the c.m. velocities of the two bodies, i.e., $v_1 = P/E_1$ and $v_2 = P/E_2$, the final result only depends on the (relativistic) relative velocity v between the two bodies, namely,

$$v \equiv \frac{v_1 + v_2}{1 + v_1 v_2} = \sqrt{1 - \frac{1}{\gamma^2}}, \quad (4.5)$$

and takes the relatively simple form,

$$J_{yz}^{\rm rad} = \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}}\frac{G^2 m_1 m_2 P}{b}\mathcal{I}(v) + O(G^3). \quad (4.6)$$

Here $\mathcal{I}(v)$ denotes the following (even) function:

$$\mathcal{I}(v) = -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2 - 1)}{v^3}\mathcal{A}(v), \quad (4.7)$$

where $\mathcal{A}$ is a short-hand notation for the arctanh function, i.e.,

$$\mathcal{A}(v) \equiv \operatorname{arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v} = 2\operatorname{arcsinh}\sqrt{\frac{\gamma - 1}{2}}. \quad (4.8)$$

The last form relates $\mathcal{A}(v)$ to the (crucial) arcsinh function entering the 3PM results of Refs. [16,17]. [The latter works use the notation σ for the Lorentz factor here denoted $\gamma = -(u_1 \cdot u_2)_{-\infty}$.] Due to cancellations between the various terms in Eq. (4.7), the small-v expansion of $\mathcal{I}(v)$ starts as $\mathcal{I}(v) = \frac{8}{5}v^2 + O(v^4)$.

The ratio between the radiated angular momentum, and the incoming mechanical angular momentum of the binary system, takes the following simple form:

$$\frac{J^{\rm rad}}{J} = \frac{J_{yz}^{\rm rad}}{bP} = \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}}\frac{G^2 m_1 m_2}{b^2}\mathcal{I}(v) + O(G^3). \quad (4.9)$$

The PN (slow velocity) expansion of Eq. (4.9) begins as

$$\frac{J^{\rm rad}}{J} = \frac{G^2 m_1 m_2}{c^5 b^2}\left(\frac{16}{5}v + \frac{232}{35}\frac{v^3}{c^2} + \frac{2146}{315}\frac{v^5}{c^4} + \cdots\right), \quad (4.10)$$

where we added the (dimensionally determined) powers of $\frac{1}{c}$ to emphasize that this is a 2.5PN $[O(G^2/c^5)]$ effect. The leading-order term in $\frac{J^{\rm rad}}{J}$ agrees with the result of Ref. [48] (where it was directly derived from the G^2-accurate retarded equations of motion of the binary system). We have also checked that the next-to-leading-order term is compatible with the fractionally 1PN-accurate computation [58] of the angular momentum radiated to gravitational waves during an hyperbolic encounter. The ratio $J^{\rm rad}/J$ is a positive, and monotonically growing, function of v in the interval $0 < v < 1$.

V. RADIATION-REACTION CONTRIBUTION TO THE SCATTERING ANGLE

Having in hand the leading-order radiative loss of angular momentum, we can deduce from it the corresponding radiative contribution to the scattering angle. Indeed, Ref. [49] has derived a general formula, namely Eq. (5.74) there,[4] yielding the radiation-reaction contribution to the scattering angle (considered as a first-order correction to the conservative-dynamics value of the scattering angle). Namely,

$$\chi^{\text{rad}}(E,J) = -\frac{1}{2}\frac{\partial\chi^{\text{cons}}}{\partial E}E^{\text{rad}} - \frac{1}{2}\frac{\partial\chi^{\text{cons}}}{\partial J}J^{\text{rad}}. \quad (5.1)$$

Note here the factor $\frac{1}{2}$ and the negative sign, because E^{rad} and J^{rad} denote the (positive) energy and angular momentum radiated away. (E and J are both measured in the c.m. frame.)

As $E^{\text{rad}} = O(G^3)$, while $J^{\text{rad}} = O(G^2)$, the leading-PM-order contribution to $\chi^{\text{rad}}(E,J)$ will be the one induced by the angular-momentum loss. In addition, the leading-PM-order contribution to $\chi^{\text{cons}}(E,J)$ is $\propto GJ^{-1}$, so that the leading-PM-order contribution to $\chi^{\text{rad}}(E,J)$ is *positive*, of order $O(G^3)$, and given by

$$\chi^{\text{rad}}(E,J) = +\frac{1}{2}\chi^{\text{cons}}_{\text{LO}}\frac{J^{\text{rad}}}{J}. \quad (5.2)$$

Inserting our result (4.9), and working (as is often convenient) with the half-scattering angle,

$$\frac{1}{2}\chi^{\text{cons}}(E,J) = \frac{\chi_1(\gamma)}{j} + \frac{\chi_2(\gamma)}{j^2} + \frac{\chi_3(\gamma)}{j^3} + \cdots, \quad (5.3)$$

where $j \equiv J/(Gm_1m_2)$ and

$$\chi_1(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \quad (5.4)$$

we get the explicit result,

$$\frac{1}{2}\chi^{\text{rad}} = +\frac{(2\gamma^2 - 1)^2}{\gamma^2 - 1}\mathcal{I}(v)\frac{G^2m_1m_2}{b^2j} + O(G^4). \quad (5.5)$$

Using the relation [8],

$$\frac{GE}{b} \equiv \frac{GMh(\gamma,\nu)}{b} = \frac{\sqrt{\gamma^2 - 1}}{j}, \quad (5.6)$$

where $h(\gamma,\nu) \equiv \frac{E}{M} = \sqrt{1 + 2\nu(\gamma - 1)}$, we can rewrite (5.5) either in terms of b or of j. Its expression in terms of j reads

---

[4]As indicated around Eq. (5.98) of Ref. [49] this formula has a general validity.

$$\frac{1}{2}\chi^{\text{rad}}(\gamma,j,\nu) = +\frac{\nu}{h^2(\gamma,\nu)j^3}(2\gamma^2 - 1)^2\mathcal{I}(v) + O(G^4), \quad (5.7)$$

where $\nu \equiv \frac{m_1m_2}{(m_1+m_2)^2}$ denotes the symmetric mass ratio.

VI. RADIATION-REACTION CORRECTED SCATTERING ANGLE

The $O(G^3)$ contribution, χ_3/j^3, to the (half) *conservative* scattering angle [see Eq. (5.3)] has been computed by Bern *et al.* [16,17] (see also [25]) with the result,

$$\chi_3^{\text{cons}}(\gamma,\nu) = \chi_3^{\text{Schw}}(\gamma) - \frac{2\nu p_\infty}{h^2(\gamma,\nu)}\bar{C}^{\text{cons}}(\gamma), \quad (6.1)$$

where (denoting $p_\infty \equiv \sqrt{\gamma^2 - 1} = \frac{v}{\sqrt{1-v^2}}$)

$$\chi_3^{\text{Schw}}(\gamma) = \frac{64p_\infty^6 + 72p_\infty^4 + 12p_\infty^2 - 1}{3p_\infty^3}, \quad (6.2)$$

and

$$\bar{C}^{\text{cons}}(\gamma) = \frac{2}{3}\gamma(14\gamma^2 + 25)$$
$$+ 2(4\gamma^4 - 12\gamma^2 - 3)\frac{\mathcal{A}(v)}{\sqrt{\gamma^2 - 1}}. \quad (6.3)$$

In the latter expression we have replaced the function $\text{arcsinh}\sqrt{\frac{\gamma-1}{2}}$ used in Refs. [16,17] by the (half) arctanh function $\frac{1}{2}\mathcal{A}(v)$ [see Eq. (4.8)].

The physical, $O(G^3)$-accurate, total scattering angle, i.e., the sum of the conservative contribution and of the leading-PM-order radiation-reaction correction (5.7), then reads

$$\frac{1}{2}\chi^{\text{tot}}(E,J) = \frac{\chi_1(\gamma)}{j} + \frac{\chi_2(\gamma)}{j^2} + \frac{\chi_3^{\text{tot}}(\gamma)}{j^3} + O(G^4), \quad (6.4)$$

where

$$\chi_3^{\text{tot}}(\gamma,\nu) = \chi_3^{\text{cons}}(\gamma,\nu) + \chi_3^{\text{rad}}(\gamma,\nu), \quad (6.5)$$

with

$$\chi_3^{\text{rad}}(\gamma,\nu) = +\frac{\nu}{h^2(\gamma,\nu)}(2\gamma^2 - 1)^2\mathcal{I}(v), \quad (6.6)$$

or more explicitly,

$$\chi_3^{\text{rad}}(\gamma,\nu) = +\frac{2\nu}{h^2(\gamma,\nu)}\frac{(2\gamma^2 - 1)^2}{3(\gamma^2 - 1)}$$
$$\times\left(8 - 5\gamma^2 + (6\gamma^2 - 9)\frac{\mathcal{A}(v)}{v}\right). \quad (6.7)$$

Let us note that the mass-ratio dependence of $\chi_3^{\mathrm{rad}}(\gamma, \nu)$ satisfies the general rule pointed out in Ref. [10], namely $h^2(\gamma, \nu)\chi_3(\gamma, \nu)$ is a linear function of ν. In other words, we can write $\chi_3^{\mathrm{rad}}(\gamma, \nu)$ in the same form as $\chi_3^{\mathrm{cons}}(\gamma, \nu) - \chi_3^{\mathrm{Schw}}(\gamma)$, namely,

$$\chi_3^{\mathrm{rad}}(\gamma, \nu) = -\frac{2\nu p_\infty}{h^2(\gamma, \nu)}\bar{C}^{\mathrm{rad}}(\gamma), \qquad (6.8)$$

where

$$\bar{C}^{\mathrm{rad}}(\gamma) = -\frac{(2\gamma^2 - 1)^2}{2\sqrt{\gamma^2 - 1}}\mathcal{I}(v)$$

$$= -\frac{(2\gamma^2 - 1)^2}{\sqrt{\gamma^2 - 1}}\left(-\frac{8}{3} + \frac{1}{v^2} + \frac{3v^2 - 1}{v^3}\mathcal{A}(v)\right). \qquad (6.9)$$

VII. LOW-ENERGY AND HIGH-ENERGY LIMITS OF THE RADIATION-REACTION CORRECTED SCATTERING ANGLE

A. Low-velocity limit

In the low-kinetic-energy (or low-velocity) limit $v \to 0$ (or $\gamma \to 1$), the v-expansion of $\bar{C}^{\mathrm{cons}}(\gamma)$ reads

$$\bar{C}^{\mathrm{cons}}(\gamma) = 4 + 18v^2 + \frac{271}{10}v^4 + \frac{4999}{140}v^6 + \frac{440273}{10080}v^8 + \cdots \tag{7.1}$$

The first term of this expansion corresponds to the 2PN level. The last term we explicitly wrote belongs to the 6PN level. All those terms have been explicitly checked by PN-based computations [35–37].

The corresponding PN expansion of the complementary radiation-reaction contribution $\bar{C}^{\mathrm{rad}}(\gamma)$ reads

$$\bar{C}^{\mathrm{rad}}(\gamma) = -\frac{4}{5}v - \frac{114}{35}v^3 - \frac{4169}{630}v^5 - \frac{138451}{13860}v^7 - \cdots \tag{7.2}$$

Here, the first term of this expansion corresponds to the 2.5PN level (which is indeed the leading level for radiation-reaction in GR). The corresponding leading-order (both in the PN and the PM senses), $O(G^3/c^5)$, contribution to the scattering angle, namely,

$$\frac{1}{2}\chi^{\mathrm{rad}} = +\frac{8G^3}{5c^5}\frac{m_1^3 m_2^3}{J^3}\nu v^2 + \cdots \tag{7.3}$$

agrees with the large-eccentricity limit of Eq. (5.116) in Ref. [49].

The radiation-reaction character of $\bar{C}^{\mathrm{rad}}(\gamma)$ shows up in the fact that it involves only odd powers of the relative velocity v. In other words, $\bar{C}^{\mathrm{rad}}(\gamma)$ is an odd function of v, while $\bar{C}^{\mathrm{cons}}(\gamma)$ is an even function of v (as is easily checked on their exact expressions).

B. High-energy (or massless) limit

Let us now consider the high-energy (HE) limit ($\gamma \to +\infty$ or $v \to 1$) of the scattering angle. This limit is taken together with the massless limit, $m_1 \to 0$, $m_2 \to 0$, keeping fixed the c.m. linear momentum P, so that $\sqrt{s} = E = \sqrt{P^2 + m_1^2} + \sqrt{P^2 + m_2^2} \to 2P$. A natural PM expansion parameter in this limit is then

$$\alpha \equiv \frac{\gamma}{j} = \frac{G}{2}\frac{s - m_1^2 - m_2^2}{J}. \tag{7.4}$$

In the HE limit we can write

$$\alpha \overset{\mathrm{HE}}{=} \frac{GE}{b}. \tag{7.5}$$

The HE limit of the 3PM-accurate conservative scattering angle Eq. (5.3) [with (6.1)] reads

$$\frac{1}{2}\chi^{\mathrm{cons}} \overset{\mathrm{HE}}{=} 2\alpha + \left(\frac{64}{3} - \frac{28}{3} - 8\ln(2\gamma)\right)\alpha^3 + O(G^4),$$

$$= 2\alpha + (12 - 8\ln(2\gamma))\alpha^3 + O(G^4), \tag{7.6}$$

where the $\ln(2\gamma)$ term introduces a logarithmic divergence of $\frac{1}{2}\chi^{\mathrm{cons}}(\alpha)$ in the HE limit. On the first line of Eq. (7.6), we have indicated the (positive) contribution from the 3PM-level Schwarzschild term $\chi_3^{\mathrm{Schw}}(\gamma)$, and the (negative) contribution from the ν-dependent, last term in Eq. (6.1).

On the other hand, the HE limit of the radiative correction to $\frac{1}{2}\chi$ reads

$$\frac{1}{2}\chi^{\mathrm{rad}} \overset{\mathrm{HE}}{=} + \left(-\frac{20}{3} + 8\ln(2\gamma)\right)\alpha^3 + O(G^4). \tag{7.7}$$

As we see, the HE limit of $\frac{1}{2}\chi^{\mathrm{rad}}$ contains exactly the opposite of the logarithmic divergence contained in $\frac{1}{2}\chi^{\mathrm{cons}}$. The radiation-reaction corrected scattering angle is therefore finite in the HE limit and equal to

$$\frac{1}{2}\chi^{\mathrm{tot}} \overset{\mathrm{HE}}{=} 2\alpha + \frac{16}{3}\alpha^3 + O(G^4). \tag{7.8}$$

Remarkably, this HE limit of the radiation-reaction corrected scattering angle agrees with the ACV eikonal-approximation two-loop result [38], namely (see [40,42]),

$$\frac{1}{2}\chi^{\mathrm{eikonal}} \overset{\mathrm{HE}}{=} 2\alpha + \frac{16}{3}\alpha^3 + O(\alpha^5). \tag{7.9}$$

The eikonal result (7.9) was initially derived in GR. Its validity in GR was recently confirmed [40,41]. In addition, it was proven to hold in all supergravity theories [39–41].

The (finiteness and) equality, at the G^3 level, between the HE limit of the *classical* radiation-reaction corrected scattering angle (7.8), and the *quantum-derived*, two-loop

eikonal massless scattering angle (7.9) is the main result of the present work. Various aspects of this result are discussed next.

VIII. DISCUSSION

Let us summarize the context, and meaning, of our derivation. We worked within a purely *classical* GR framework, and our derivation of the radiative correction to the scattering angle used a post-Minkowskian (PM) perturbative approach. By contrast, we are not aware of a fully classical, and fully PM, derivation of the so-called *conservative* 3PM-accurate scattering angle (6.1). The 1PM, $O(G)$, and 2PM, $O(G^2)$, scattering angles have been computed within classical PM frameworks in Refs. [3,7] (1PM) and Refs. [6,28] (2PM). The first derivation [16,17] of the conservative 3PM scattering angle was performed by using a mix of various tools: a selection of quantum scattering amplitude integrands, and an evaluation of the corresponding integrals by expanding the integrand in the so-called *potential region*. The rederivation of Ref. [24] cannot either be considered as a purely classical PM derivation because the propagator of the graviton used in the latter computation is the Feynman one, namely $\propto (k^2 - i0)^{-1}$. Let us recall in this respect that the correct, classical conservative action for PM gravity [59] must be classically defined, *à la* Fokker-Wheeler-Feynman [60,61], by using the *time-symmetric* graviton propagator, whose Fourier-space kernel is

$$G_{\alpha\beta;\alpha'\beta'}(k) = \left(\eta_{\alpha\alpha'}\eta_{\beta\beta'} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\alpha'\beta'} \right) \mathrm{P}\frac{1}{k^2}, \quad (8.1)$$

where P denotes the principal value. By contrast, the Feynman propagator involves

$$\frac{1}{k^2 - i0} = \mathrm{P}\frac{1}{k^2} + i\pi\delta(k^2), \quad (8.2)$$

while the retarded propagator would involve

$$\frac{1}{k^2 - \mathrm{sign}(k^0)i0} = \mathrm{P}\frac{1}{k^2} + i\pi\mathrm{sign}(k^0)\delta(k^2). \quad (8.3)$$

The differences between the coefficients of the $\delta(k^2)$ term are crucial here and start making a difference already at the 2PM, $O(G^2)$, level. Indeed, the 2PM-accurate *retarded* equations of motion derived in Refs. [4,5] lead (as was explicitly shown in Ref. [48]) to an $O(G^2)$ loss of the *mechanical* angular momentum of the binary system, which was checked to balance the *radiated* angular momentum *at lowest PN order*. [We have used here the fact that this balance must formally hold at the full (PN-exact) 2PM level, leaving a direct technical check to future work.]

At the 3PM level, the effects linked to using either a time-symmetric propagator, a Feynman one, or a retarded one should be even more drastic. The first derivation [16,17] of the conservative 3PM dynamics (as well as the rederivation of Ref. [24]) selected a graviton propagator by expanding the graviton propagator in the so-called potential region, defined by the inequality $k^0 \ll |\mathbf{k}|$. At face value, it seems that the use of such an expansion, namely (with $\omega \equiv k^0$),

$$\frac{1}{k^2 + i\epsilon} = \frac{1}{\mathbf{k}^2 - \omega^2 + i\epsilon} = \frac{1}{\mathbf{k}^2} + \frac{\omega^2}{\mathbf{k}^4} + \frac{\omega^4}{\mathbf{k}^6} + \cdots \quad (8.4)$$

is not only independent of the choice of contour around the poles in ω (i.e. of the choice of $i\epsilon$), but is equivalent to the PN-expansion of the x-space time-symmetric massless propagator,

$$\Box_{\mathrm{sym}}^{-1} = (\Delta - \partial_0^2)_{\mathrm{sym}}^{-1} = \Delta^{-1} + \partial_0^2\Delta^{-2} + \partial_0^4\Delta^{-3} + \cdots \quad (8.5)$$

This suggests that computing the dynamics by expanding in the potential region is equivalent to using the *PN-expanded time-symmetric* graviton propagator. As genuine nonlocal-in-time effects enter only at the 4PN and 4PM $[O(G^4/c^8)]$ level [62], this indicates that the potential-region expansion should yield the PN-expanded version of the conservative dynamics up to the accuracy $O(G^3)$ but will encounter subtle nonlocal-in-time effects starting at the $O(G^4)$ level. For our present purposes, it confirms that the current potential-region derivations of the 3PM scattering angle (6.1) do capture the classical (Fokker-Wheeler-Feynman-type) time-symmetric conservative dynamics. It would, however, be instructive to give both an *ab initio* classical derivation of the 3PM conservative dynamics, as well as an *ab initio* classical derivation of the retarded 3PM dynamics, so as to directly check that the radiation-reaction corrected scattering angle (6.4) does directly follow from the retarded 3PM dynamics, without appealing (as we did) to the balance between mechanical and radiated angular momentum.

As already mentioned in the Introduction, the motivation for the present work was Ref. [41] which pointed out two different, but interrelated, facts: (i) analyticity, crossing properties, and impact-parameter-space exponentiation of the HE quantum scattering amplitude allows one to relate the real part of the two-loop eikonal phase to its imaginary part; the latter being then derived from the phase-space integral of a three-particle cut; and (ii) the explicit evaluation of the four-point two-loop amplitude in $\mathcal{N} = 8$ supergravity using integration over the full soft-graviton region $k^0 \sim |\mathbf{k}| \ll \sqrt{s}$. Their first approach led to a rederivation of the ACV HE eikonal phase that emphasized its link with the *inelastic* tree-level amplitude describing the emission of a graviton in a certain HE double-Regge limit. Their second approach confirmed (in a particular setting)

that radiative corrections (soft region, rather than potential region) were crucial for getting a finite HE eikonal phase. Our present work has completed, on the classical side, the quantum-based results of Ref. [41] by showing in detail how classical radiative corrections to the classical conservative dynamics[5] resolve the puzzle of the logarithmic divergence in the HE (or massless limit) of the conservative 3PM scattering angle, by establishing, by a purely classical computation, that the ACV eikonal scattering does agree with the massless limit of the 3PM scattering of two classical masses.

Let us note that, independently of the issue of the high-energy behavior, our results show that even the 2PM $[O(G^2)]$ dynamics cannot be considered as being naturally conservative [as had been assumed so far because of the $O(G^3)$ order of the radiative energy-momentum loss $P_\mu^{\rm rad}$]. Indeed, the 2PM-accurate effective one-body Hamiltonian description [8] (or the equivalent [10] 2PM effective-field-theory formulation [13]) involves the total c.m. angular momentum J of the binary system and crucially assumes that J is conserved. Our explicit PM result (4.9) above shows that this is not true for the physical (retarded) PM interaction but only applies to the time-symmetric 2PM dynamics. Another consequence of our result is that, contrary to what had been hitherto assumed, the classical ACV scattering angle should not be thought as being a conservative quantity, but rather as a strongly radiation-reacted quantity. The ACV scattering angle should not therefore be transcribed into some high-energy Hamiltonian (as was done, e.g., in [8]). At the formal level, one could use the dictionary given in Refs. [8,10,37] to transcribe, for any finite values of the masses and of γ, the 3PM-level radiation-corrected total scattering angle $\chi_3^{\rm tot}$ into a corresponding value for the 3PM-level effective one-body potential (either in the Q Finsler-like form or the W potential one). The resulting potential would be of the form $\hat{Q}^E = q_2(\gamma,\nu)(GM/R)^2 + q_3^{\rm tot}(\gamma,\nu)(GM/R)^3 + O(G^4)$, with $q_2(\gamma,\nu)$ given in Eq. (5.12) of [8], and $q_3^{\rm tot}(\gamma,\nu) = q_3^{\rm cons}(\gamma,\nu) + q_3^{\rm rad}(\gamma,\nu)$, where $q_3^{\rm cons}(\gamma,\nu)$ is the 3PM-level conservative effective one-body potential (see [9,10]), and where the radiative correction would explicitly read

$$q_3^{\rm rad}(\gamma,\nu) = \frac{2\nu}{h^2(\gamma,\nu)}\bar{C}^{\rm rad}(\gamma)$$
$$= -\frac{2\nu}{1+2\nu(\gamma-1)}\frac{(2\gamma^2-1)^2}{\sqrt{\gamma^2-1}}$$
$$\times\left(-\frac{8}{3}+\frac{1}{v^2}+\frac{3v^2-1}{v^3}\mathcal{A}(v)\right). \quad (8.6)$$

The latter radiative contribution to the Q potential would start at the 2.5PN order [namely $q_3^{\rm rad} = -\frac{8}{5}\nu(GM/R)^3 \times (v+O(v^3))$] and would be odd in the relative velocity v. It would be (somewhat analogously to the use of an optical potential) a way of encoding the effect of radiation-reaction on χ in an effective potential. But it would miss the fact that the angular momentum of the system is not conserved at the G^2 level. For concreteness, let us recall that the leading-order *radiation-reaction force* (to be added to the conservative acceleration of the first body) is of order G^2/c^5, and reads [see Eq. (5a) in [48]]

$$\mathbf{a}_1^{\rm rad} = -\frac{4}{5}\frac{G^2}{c^5}m_1 m_2 \frac{v_{12}^2}{r_{12}^3}[\mathbf{v}_{12}-3(\mathbf{v}_{12}\cdot\mathbf{n}_{12})\mathbf{n}_{12}], \quad (8.7)$$

where $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{n}_{12} = (\mathbf{x}_1 - \mathbf{x}_2)/r_{12}$.

The time-symmetric dynamics of gravitationally interacting systems does admit a perturbatively defined Poincaré-invariant action [59], with Noether-associated conservation laws for energy-momentum and (relativistic) angular momentum (see, e.g., [61] for the electromagnetic case). We recalled in the Introduction that in the 1980's it had been important to use a (physical, retarded) PM framework in order to establish, without using any ill-defined PN-type nearzone expansion, that GR did predict nearzone radiation-reaction effects in accord with the heuristic expectation of a global balance between the mechanical energy and angular momentum of the binary system, and the energy and angular momentum radiated in gravitational waves. However, the classical GR community stopped pursuing to higher orders the physical (retarded) PM approach because of the following combination of facts: (i) technical difficulties in explicitly evaluating retarded PM equations of motion at the 3PM, $O(G^3)$, level (which could only be done by neglecting some terms explicitly involving the relative velocities between the two bodies [63]); (ii) confidence built by the PM result in the validity of the nearzone-based PN expansion (whose breakdown was, moreover, shown to occur only at the G^4/c^8 level [62]); and (iii) superior technical efficiency of the PN approximation for computing higher-order effects [64–66]. Instead of using a physical, retarded PM approach, it was found efficient to rely on a dual approximation where the equations of motion of binary systems are decomposed in a conservative part and a radiation-reaction one, both parts being computed by using PN methods (amplified by a PN-matched multipolar-post-Minkowskian approach, see, e.g., [67] for a review). (Such a dual approximation is notably used in the effective one-body approach[6][68,69], and has been recently pushed to very high orders [70].)

---

[5]In this sense, we do not agree with the statement of Ref. [41] that "the real part of [the two-loop eikonal phase δ_2] captures the conservative dynamics." Our calculation clearly shows that the J-derivative of the HE eikonal phase is the full, radiation-reaction-corrected scattering angle.

---

[6]Let us, however, note that Ref. [68] initially advocated to describe radiation-reaction effects through an additional contribution to the effective one-body metric.

It is only recently that the PM approach (especially in its quantum-amplitude version) was rekindled with the assumption [13,16,17] that the combined use of quasi-classical-type approximations, and potential-region truncation, would give an efficient (and scalable) way to compute higher-orders in PM gravity. This enthusing program is, however, now facing several types of difficulties highlighted by various recent results (notably Refs. [35,41] and the results presented here). We have in mind here both the subtleties linked to the differences between the various graviton propagators, as displayed in Eqs. (8.1), (8.2), (8.3), and the breakdown of the potential-region propagator Eq. (8.4) [or its x-space, PN-expanded, analog Eq. (8.4)] starting at the G^4/c^8 level, where time-nonlocality becomes essential [62]. Now that these subtleties have been more clearly identified, we hope that improved ways of tapping the deep knowledge of quantum perturbative gravity brought by many years of work (notably related to string theory) for deriving classical observable quantities of direct importance for gravitational-wave physics will be explored.

The dual, conservative-plus-radiation-reaction, *PM-based* approach (instead of the usual *PN-based* one), exemplified by our computation above, is *a priori* scalable, i.e., can be extended to higher PM orders. We leave to future work an application of our method to the $O(G^4)$ level.

Let us finally note that it is tempting (as suggested in Ref. [41]) to assume that Weinberg's quantum result [71] about the absence of $\ln m_A$ divergences in the massless limit ($m_A \to 0$) of perturbative quantum-gravity amplitudes implies, at the classical level, that the physical, retarded (radiation-reaction corrected) scattering angle admits, at each order of the (classical) PM perturbative gravity expansion, a finite massless limit, and therefore a well-defined HE limit. We wish, however, to recall that Sec. VI D of Ref. [10] has presented an argument (based on the results of Refs. [72,73]) showing how radiative (tail) effects give a *conservative* contribution to the scattering angle of order G^4 at finite γ, which becomes of order $G^3 \ln G$ in the HE limit $\gamma \to \infty$.

[1] B. Bertotti, On gravitational motion, Nuovo Cimento **4**, 898 (1956).

[2] P. Havas and J. N. Goldberg, Lorentz-invariant equations of motion of point masses in the general theory of relativity, Phys. Rev. **128**, 398 (1962).

[3] M. Portilla, Scattering of two gravitating particles: Classical approach, J. Phys. A **13**, 3677 (1980).

[4] K. Westpfahl and M. Goller, Gravitational scattering of two relativistic particles in postlinear approximation, Lett. Nuovo Cimento **26**, 573 (1979).

[5] L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity, Gen. Relativ. Gravit. **13**, 963 (1981).

[6] K. Westpfahl, High-speed scattering of charged and uncharged particles in general relativity, Fortschr. Phys. **33**, 417 (1985).

[7] T. Damour, Gravitational scattering, post-Minkowskian approximation and effective one-body theory, Phys. Rev. D **94**, 104015 (2016).

[8] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).

[9] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. D **99**, 104004 (2019).

[10] T. Damour, Classical and quantum scattering in post-Minkowskian gravity, Phys. Rev. D **102**, 024060 (2020).

[11] A. Guevara, Holomorphic classical limit for spin effects in gravitational and electromagnetic scattering, J. High Energy Phys. 04 (2019) 033.

[12] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. **121**, 171601 (2018).

[13] C. Cheung, I. Z. Rothstein, and M. P. Solon, From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion, Phys. Rev. Lett. **121**, 251101 (2018).

[14] A. Koemans Collado, P. Di Vecchia, and R. Russo, Revisiting the second post-Minkowskian eikonal and the dynamics of binary black holes, Phys. Rev. D **100**, 066028 (2019).

[15] D. A. Kosower, B. Maybee, and D. O'Connell, Amplitudes, observables, and classical scattering, J. High Energy Phys. 02 (2019) 137.

[16] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. **122**, 201603 (2019).

[17] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.

[18] G. Kälin and R. A. Porto, From boundary data to bound states, J. High Energy Phys. 01 (2020) 072.

[19] G. Kälin and R. A. Porto, From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist), J. High Energy Phys. 02 (2020) 120.

[20] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Post-Minkowskian Hamiltonians in general relativity, Phys. Rev. D **100**, 084040 (2019).

[21] N. E. J. Bjerrum-Bohr, A. Cristofoli, and P. H. Damgaard, Post-Minkowskian scattering angle in Einstein gravity, J. High Energy Phys. 08 (2020) 038.

[22] C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams, J. High Energy Phys. 06 (2020) 144.

[23] J. Parra-Martinez, M. S. Ruf, and M. Zeng, Extremal black hole scattering at $\mathcal{O}(G^3)$: Graviton dominance, eikonal exponentiation, and differential equations, arXiv:2005.04236.

[24] G. Kälin and R. A. Porto, Post-Minkowskian effective field theory for conservative binary dynamics, arXiv:2006.01184.

[25] G. Kälin, Z. Liu, and R. A. Porto, Conservative dynamics of binary systems to third post-Minkowskian order from the effective field theory approach, arXiv:2007.04977.

[26] D. Bini and T. Damour, Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory, Phys. Rev. D **96,** 104038 (2017).

[27] J. Vines, Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings, Classical Quantum Gravity **35,** 084002 (2018).

[28] D. Bini and T. Damour, Gravitational spin-orbit coupling in binary systems at the second post-Minkowskian approximation, Phys. Rev. D **98,** 044036 (2018).

[29] J. Vines, J. Steinhoff, and A. Buonanno, Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order, Phys. Rev. D **99,** 064054 (2019).

[30] A. Guevara, A. Ochirov, and J. Vines, Scattering of spinning black holes from exponentiated soft factors, J. High Energy Phys. 09 (2019) 056.

[31] M. Z. Chung, Y. T. Huang, J. W. Kim, and S. Lee, The simplest massive S-matrix: From minimal coupling to Black Holes, J. High Energy Phys. 04 (2019) 156.

[32] A. Guevara, A. Ochirov, and J. Vines, Black-hole scattering with general spin directions from minimal-coupling amplitudes, Phys. Rev. D **100,** 104024 (2019).

[33] N. Arkani-Hamed, Y. T. Huang, and D. O'Connell, Kerr black holes as elementary particles, J. High Energy Phys. 01 (2020) 046.

[34] N. Siemonsen and J. Vines, Test black holes, scattering amplitudes and perturbations of Kerr spacetime, Phys. Rev. D **101,** 064066 (2020).

[35] D. Bini, T. Damour, and A. Geralico, Novel Approach to Binary Dynamics: Application to the Fifth Post-Newtonian Level, Phys. Rev. Lett. **123,** 231104 (2019).

[36] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, Testing binary dynamics in gravity at the sixth post-Newtonian level, Phys. Lett. B **807,** 135496 (2020).

[37] D. Bini, T. Damour, and A. Geralico, Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders, Phys. Rev. D **102,** 024062 (2020).

[38] D. Amati, M. Ciafaloni, and G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in planckian energy superstring collisions, Nucl. Phys. **B347,** 550 (1990).

[39] P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano, and C. D. White, A tale of two exponentiations in $\mathcal{N} = 8$ supergravity at subleading level, J. High Energy Phys. 03 (2020) 173.

[40] Z. Bern, H. Ita, J. Parra-Martinez, and M. S. Ruf, Universality in the Classical Limit of Massless Gravitational Scattering, Phys. Rev. Lett. **125,** 031601 (2020).

[41] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, arXiv:2008.12743.

[42] M. Ciafaloni and D. Colferai, Rescattering corrections and self-consistent metric in Planckian scattering, J. High Energy Phys. 10 (2014) 085.

[43] R. A. Hulse and J. H. Taylor, Discovery of a pulsar in a binary system, Astrophys. J. Lett. **195,** L51 (1975).

[44] J. H. Taylor, L. A. Fowler, and P. M. McCulloch, Measurements of general relativistic effects in the binary pulsar PSR 1913 + 16, Nature (London) **277,** 437 (1979).

[45] J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas, Comments on gravitational radiation damping and energy loss in binary systems, Astrophys. J. Lett. **208,** L77 (1976).

[46] T. Damour, Gravitational radiation and the motion of compact bodies, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland Company, Amsterdam, 1983), pp. 59–144.

[47] T. Damour, Gravitational Radiation Reaction in the Binary Pulsar and the Quadrupole Formula Controversy, Phys. Rev. Lett. **51,** 1019 (1983).

[48] T. Damour and N. Deruelle, Radiation reaction and angular momentum loss in small angle gravitational scattering, Phys. Lett. A **87,** 81 (1981).

[49] D. Bini and T. Damour, Gravitational radiation reaction along general orbits in the effective one-body formalism, Phys. Rev. D **86,** 124012 (2012).

[50] P. C. Peters, Gravitational radiation and the motion of two point masses, Phys. Rev. **136,** B1224 (1964).

[51] B. DeWitt and S. M. Christensen, Bryce DeWitt's lectures on gravitation, Lect. Notes Phys. **826,** 1 (2011).

[52] K. S. Thorne, Multipole expansions of gravitational radiation, Rev. Mod. Phys. **52,** 299 (1980).

[53] B. Bonga and E. Poisson, Coulombic contribution to angular momentum flux in general relativity, Phys. Rev. D **99,** 064024 (2019).

[54] L. Blanchet and G. Faye, Flux-balance equations for linear momentum and center-of-mass position of self-gravitating post-Newtonian systems, Classical Quantum Gravity **36,** 085003 (2019).

[55] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 3. Derivation of bremsstrahlung formulas, Astrophys. J. **217,** 252 (1977).

[56] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 4. Bremsstrahlung, Astrophys. J. **224,** 62 (1978).

[57] A. G. Wiseman and C. M. Will, Christodoulou's nonlinear gravitational wave memory: Evaluation in the quadrupole approximation, Phys. Rev. D **44,** R2945 (1991).

[58] W. Junker and G. Schäfer, Binary systems—Higher order gravitational radiation damping and wave emission, Mon. Not. R. Astron. Soc. **254,** 146 (1992).

[59] T. Damour and G. Esposito-Farèse, Testing gravity to second postNewtonian order: A field theory approach, Phys. Rev. D **53,** 5541 (1996).

[60] A. D. Fokker, Ein invarianter Variationssatz für die Bewegung mehrerer elektrischer Massenteilchen, Z. Phys. **58,** 386 (1929).

[61] J. A. Wheeler and R. P. Feynman, Classical electrodynamics in terms of direct interparticle action, Rev. Mod. Phys. **21,** 425 (1949).

[62] L. Blanchet and T. Damour, Tail transported temporal correlations in the dynamics of a gravitating system, Phys. Rev. D **37,** 1410 (1988).

[63] T. Damour, Radiation damping in general relativity, in *Proceedings of the Third Marcel Grossmann Meeting on General Relativity*, edited by H. Ning (Science Press, Beijing, and North-Holland, Amsterdam, 1983), part A, pp. 583–597.

[64] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman, Gravitational Radiation Damping of Compact Binary Systems to Second postNewtonian Order, Phys. Rev. Lett. **74,** 3515 (1995).

[65] P. Jaranowski and G. Schäfer, Third postNewtonian higher order ADM Hamilton dynamics for two-body point mass systems, Phys. Rev. D **57,** 7274 (1998); Erratum, Phys. Rev. D**63,** 029902 (2000).

[66] L. Blanchet and G. Faye, General relativistic dynamics of compact binaries at the third postNewtonian order, Phys. Rev. D **63,** 062005 (2001).

[67] L. Blanchet, Gravitational radiation from post-newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17,** 2 (2014).

[68] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59,** 084006 (1999).

[69] A. Buonanno and T. Damour, Transition from inspiral to plunge in binary black hole coalescences, Phys. Rev. D **62,** 064015 (2000).

[70] D. Bini, T. Damour, and A. Geralico, Sixth post-Newtonian local-in-time dynamics of binary systems, Phys. Rev. D **102,** 024061 (2020).

[71] S. Weinberg, Infrared photons and gravitons, Phys. Rev. **140,** B516 (1965).

[72] A. Gruzinov and G. Veneziano, Gravitational radiation from massless particle collisions, Classical Quantum Gravity **33,** 125012 (2016).

[73] M. Ciafaloni, D. Colferai, F. Coradeschi, and G. Veneziano, Unified limiting form of graviton radiation at extreme energies, Phys. Rev. D **93,** 044052 (2016).