Generation of rotating solutions in Einstein-scalar gravity

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We revisit the problem of the rotating generalization of the Fisher-Janis-Newman-Winicour solution of the minimal Einstein-scalar theory proving that previously known metrics do not satisfy the equations of motion. The same is shown for a putative rotating solution of the Brans-Dicke theory. We then use various generating techniques to derive correct spinning solutions with the scalar charge, in particular, those endowed with oblate deformations.

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I. INTRODUCTION

Einstein's general theory of relativity with a minimally coupled massless scalar field (MES) has recently sparked new interest in view of various dualities that connect this theory with nonminimal scalar-tensor theories such as Horndeski and DHOST [1,2]. This interest gives new life to famous Fisher's solution of MES [3], repeatedly rediscovered by many authors [4–11], in particular, by Janis *et al.* [6], and now often abbreviated as FJNW. For more recent discussion of this solution, see [12]. It has a singular horizon, but the solution may correspond to a regular black hole or a fully regular metric in the dual frame of some nonminimal theory [1,13,14]. Therefore, it is interesting to find new physically relevant solutions to the minimal Einstein-scalar theory and, above all, to endow Fisher's solution with rotation. This turned out to be a nontrivial task.

An earlier attempt to introduce the angular momentum into FJNW was made in [15] using the Janis-Newman (JN) algorithm [16] (for a detailed discussion of this method, see [17]). Due to simplicity of the solution obtained in the Ref. [15], it was repeatedly applied in the astrophysical context; see, e.g., [18–21]. But, it is worth noting that the JN method was originally proposed simply as a formal trick, which generates the Kerr metric from the Schwarzschild solution. Although this algorithm was later tested in various other theories [17], no rigorous mathematical proof was given of its validity in the general case, especially in the scalar-tensor theories. An explicit check [22] of the fulfillment of some of Einstein's equations for the metric of Ref. [15] led to a negative conclusion (see also [23]). However, since this solution is still used in applications [20,21], we again return to the problem of its validity here, confirming the result of the Ref. [22].

Other stationary solutions of the minimal Einstein-scalar theory were suggested recently in the Ref. [24]; one of which is asymptotically flat. It has a Kerr-like metric, but tends toward the nonspherical Penney solution [5] in the static limit, so it cannot be considered as a true rotating FJNW solution. Astorino [25] has found the rotating generalization of the Bocharova-Bronnikov-Melnikov-Bekenstein solution (BBMB) using the generating technique developed by Hoenselaers *et al.* [26]. He also gave a scalarized subfamily of Cosgrove's generalization of the Tomimatsu-Sato vacuum metric [27]. Other known rotating solutions contain scalarly charged strings at the polar axes [28,29].

Let us briefly mention some other earlier results. In the framework of the Brans-Dicke theory (BD), several attempts were undertaken to construct a rotating solution [and more general ones, endowed with the Newman-Unti-Tamburino (NUT) parameter] [30–32] using the Kinnersley form of the BD field equations. These solutions do not reproduce the FJNW solution in the Einstein frame. A rotating version of the BD analogue of the FJNW solution was also suggested in [33] again using the JN trick. We test this solution here.

Several mathematically rigorous methods for solving Einstein's equations are based on the hidden symmetries of their dimensional reduction. Hidden symmetries of static MES equations were discovered long ago [12,34,35]. Here, we give them a modern interpretation in terms of a three-dimensional σ model derived previously in the context of a more general Einstein-Maxwell-dilaton theory (EMD) [36]. Various related generation methods for the minimal Einstein-scalar and Einstein-Maxwell-scalar systems have been proposed in the past based on further dimensional reduction to two dimensions. We recall some of them here and then expand their set to include a technique suggested by Clément [37] (CT), which generates the vacuum Kerr solution from Schwarzschild exploiting the symmetries of the Einstein-Maxwell theory. In this approach, the Maxwell field is introduced as an auxiliary one at an intermediate

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stage of the calculations. Generalizing this approach to incorporate a minimal scalar field, we obtain some new rotating generalizations of the FJNW solution. It turns out that for the successful application of CT in the presence of the scalar field, it is necessary to combine the FJNW with the Zipoy-Voorhees (ZV) [38,39] solution, which has a similar structure in spheroidal coordinates. We also use technique proposed by Eriş and Gürses [40] (EG transformation) who noticed that stationary axisymmetric EMS solutions can be divided into purely electrovacuum part plus some additional terms. Applying this technique to a rotating vacuum axisymmetric solution, we obtain its generalization with some nontrivial scalar field. A similar method was used in [25] to find the scalarized version of Cosgrove metrics.

The static FJNW solution was the starting point for other generalizations: to arbitrary dimensions [41], to the Einstein-Maxwell theory [7], to the EMD theory [42], always leading to singular metrics. Note that the objects with singular horizons cannot appear in vacuum [43] but typically occur in the theories with a scalar field.

In this paper, we define the MES action as

$$\mathcal{S} = \int d^4x\sqrt{-g}(R - 2(\partial_\mu\phi)(\partial^\mu\phi)), \tag{1.1}$$

which corresponds to the equations of motion,

$$R_{\mu\nu} = 2(\partial_\mu\phi)(\partial_\nu\phi), \tag{1.2a}$$

$$\nabla_\mu\nabla^\mu\phi = 0. \tag{1.2b}$$

The outline for the rest of the article is as follows. In the Sec. II, we combine the ZV and FJNW solutions using the σ-model technique. In Sec. III, we review and apply Clément's transformations to the solution obtained. As a result, we get the rotating generalization of the combined ZV-FJNW metric with an oblateness parameter as a function of mass and scalar charge. In Sec. IV, we discuss the relation between the FJNW rotating solution and the Tomimatsu-Sato (TS) metric [44], and give an example of a rotating FJNW without oblateness supported by a phantom scalar field.

II. STATIC REINCARNATIONS

Stationary sector of the model (1.1) admits a three-dimensional σ-model representation, related to an ansatz for stationary metrics,

$$ds^2 = -f(dt - \omega_i dx^i)^2 + f^{-1}h_{ij}dx^i dx^j, \tag{2.1}$$

where the function f, the one-form ω_i and the three-metric h_{ij} are functions of space coordinates x^i, $i = 1, 2, 3$. Indices of the three-metric are supposed to be lowered and raised with h_{ij} and an inverse metric h^{ij}. The one-form $\omega_i dx^i$ can be expressed in terms of the twist potential χ,

$$\partial_i\chi = -\frac{f^2}{\sqrt{h}}h_{ij}\epsilon^{jkl}\partial_k\omega_l, \tag{2.2}$$

where $\epsilon^{jkl} = \pm 1$, which then enter into the set of three-dimensional scalar potentials $\Phi^A = \{\psi, \chi, \phi\}$, with $\psi = \frac{1}{2}\ln f$, in the action,

$$\mathcal{S} = \int d^3x\sqrt{h}h^{ij}(R^{(3)}_{ij} - \mathcal{G}_{AB}\partial_i\Phi^A\partial_j\Phi^B). \tag{2.3}$$

The target space metric $\mathcal{G}_{AB}$ is given by

$$\mathcal{G}_{AB}d\Phi^A d\Phi^B = 2(d\phi^2 + d\psi^2) + \frac{1}{2}e^{-4\psi}d\chi^2, \tag{2.4}$$

where $R^{(3)}_{ij}$ is the Ricci tensor constructed with the metric h_{ij}. Note that the MES theory can be considered as a truncation of EMD with trivial electromagnetic field [36].

The target-space metric $\mathcal{G}_{AB}$ admits three gauge isometries,

$$I\colon\ \phi \to \phi + \lambda_\phi \tag{2.5a}$$

$$II\colon\ \chi \to \chi + \lambda_\chi \tag{2.5b}$$

$$III\colon\ \psi \to \psi + \lambda_\psi, \qquad \chi \to e^{2\lambda_\psi}\chi, \tag{2.5c}$$

with the constant parameters λ_ϕ, λ_χ, λ_ψ, and a nontrivial Ehlers transformation [45],

$$\frac{1}{z} \to \frac{1}{z'} = \frac{1}{z} + i\lambda_E, \qquad z = f + i\chi, \tag{2.6}$$

with the parameter λ_E.

For static truncation $\chi = 0$, there is also $SO(2)$-rotational symmetry in the plane (ψ, ϕ),

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix} \to \begin{pmatrix}\psi'\\ \phi'\end{pmatrix} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix}, \tag{2.7}$$

parametrized by the angle β. The transformation (2.7) is the σ model equivalent of the transformation found by Buchdahl [35], which was also rediscovered in [12].

A. Generation of FJNW

Any vacuum solution of general relativity satisfies the equations of motion (1.2) with a constant scalar field. Considering the Schwarzschild solution as a seed and applying the transformation (2.7) with $\cos\beta = S$, we can restore the FJNW solution in the form (2.1) with

$$h_{ij}dx^i dx^j = dr^2 + r^2F(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.8a}$$

$$\psi = \frac{S}{2}\ln F, \qquad \phi = \phi_\infty - \frac{\Sigma S}{2M}\ln F,$$
$$\omega_i = 0, \qquad S = \frac{M}{\sqrt{M^2+\Sigma^2}}, \tag{2.8b}$$

where M, Σ are the ADM mass and the scalar charge, respectively, and the function F has the form,

$$F(r) = 1 - \frac{2M}{rS}. \tag{2.9}$$

Setting $\Sigma \to 0$ brings us back to the Schwarzschild solution.

For further purposes of this article, it is convenient to use the prolate spheroidal coordinates x and y, defined as

$$x = \frac{r}{k} - \tilde{k}, \qquad y = \cos\theta, \tag{2.10}$$

where k and $\tilde{k}$ are constants, chosen so that $g_{\varphi\varphi} = k^2(x^2-1)(1-y^2)$. The function F and the three-metric of the solution (2.8) in the prolate spheroidal coordinates with $k = M/S$, $\tilde{k} = 1$ read

$$h_{ij}dx^i dx^j = k^2\left(dx^2 + \frac{x^2-1}{1-y^2}dy^2 + (x^2-1)(1-y^2)d\varphi^2\right), \tag{2.11}$$

$$F(x) = \frac{x-1}{x+1}. \tag{2.12}$$

This definition of F will be used in further calculations.

B. Generation of ZV with scalar charge

ZV solution in the form (2.1) reads

$$\psi = \frac{\delta}{2}\ln F, \qquad \phi = 0, \qquad \omega_i = 0, \tag{2.13a}$$

$$h_{ij}dx^i dx^j = k^2\left(H_{\rm ZV}(x,y)\left(dx^2 + \frac{x^2-1}{1-y^2}dy^2\right) + (x^2-1)(1-y^2)d\varphi^2\right), \tag{2.13b}$$

$$H_{\rm ZV}(x,y) = \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2-1}, \tag{2.13c}$$

where $k = M/\delta$.

It is possible to represent the gravitational potential ψ of the solution FJNW (2.8b) in the same form as ZV (2.13a), up to the permutation of the constants S and δ. This suggests that FJNW and ZV can naturally be combined into one solution using transformations (2.7). Application of the $SO(2)$ transformation to the solution (2.13a) leads to the ZV metric with scalar charge, which we denote by FZV. The corresponding potentials read

$$\psi = \frac{S\delta}{2}\ln F, \tag{2.14a}$$

$$\phi = \phi_\infty - \frac{\Sigma S\delta}{2M}\ln F, \tag{2.14b}$$

the three-metric is the same (2.13b), and the constant $k = M/S\delta$, where M is the Arnowitt-Deser-Misner (ADM) mass. The only difference between the ZV and FZV solutions is the replacement of the parameters $\delta \to S\delta$ in the gravitational potential ψ and the constant k. This modification of the solution will be used to generate angular momentum in the next section.

C. Generation of NUT

For completeness, add the NUT parameter to the solution (2.14). To do that, we successively apply the Ehlers transformation (2.6) and the gauge transformation (2.5c) to ensure $g_{tt} \to -1$ for $r \to \infty$,

$$\psi = \frac{1}{2}\ln\frac{(1+\lambda^2)F^{S\delta}}{1+\lambda^2F^{2S\delta}}, \tag{2.15a}$$

$$\omega_i dx^i = 2Nyd\varphi, \tag{2.15b}$$

$$\phi = \phi_\infty - \frac{\Sigma}{2k}\ln F, \tag{2.15c}$$

where λ is the parameter of the Ehlers transformation (2.6). The three-metric of the solution with the NUT parameter has the form (2.13b). The ADM mass M, the scalar charge Σ, and the NUT parameter N are

$$M = kS\delta\frac{1-\lambda^2}{1+\lambda^2}, \qquad \Sigma^2 = k^2\delta^2(1-S^2), \qquad N = \frac{2\delta k\lambda S}{\lambda^2+1}, \tag{2.16}$$

which can be resolved in the following form:

$$k = \frac{\sqrt{M^2(1+\lambda^2)^2 + \Sigma^2(1-\lambda^2)^2}}{\delta(1-\lambda^2)},$$
$$S = \frac{M(1+\lambda^2)}{\sqrt{M^2(1+\lambda^2)^2 + \Sigma^2(1-\lambda^2)^2}}. \tag{2.17}$$

The solution (2.15) with $\Sigma = 0$ represents the vacuum ZV solution with the NUT parameter, which was given in [46]. For $\delta = 1$, we obtain the FJNW solution with the NUT found in [47].

D. Singularities

The solution (2.15) is the most general of all obtained earlier. The scalar curvature R of this solution can be found from the equation of motion (1.2a) as simple as

$$R = 2g^{xx}(\partial_x\phi)^2 = \frac{2\Sigma^2}{k^4}\frac{(1+\lambda^2)}{1+\lambda^2 F^{2S\delta}}(x-1)^{S\delta-1-\delta^2} \times (x+1)^{-S\delta-1-\delta^2}(x^2-y^2)^{\delta^2-1}. \quad (2.18)$$

One can see that the parameter λ does not influence the divergence in the exponent of $(x-1)$ and (x^2-y^2), which depends only on S and δ. For $y \neq \pm 1$, the metric is singular for $S\delta - 1 - \delta^2 < 0$, while at the polar axis $y = \pm 1$, the condition is $S\delta < 2$. The "horizon" is regular only when both $S\delta \geq 2$ and $S\delta \geq 1 + \delta^2$ are satisfied. For $|S| < 1$, the regularity condition cannot be achieved for any δ.

E. Chazy-Curzon limit

The ZV solution admits the limit $\delta \to \infty$ leading to the Chazy-Curzon solution [48,49]. The solutions (2.14) and (2.15) have the same limiting form. The three-metric, the scalar field, and the function $F^{S\delta}$ will be

$$h_{ij}dx^i dx^j = \exp\left\{-\frac{M^2\sin^2\theta}{S^2 r^2}\right\}(dr^2 + r^2 d\theta^2) + r^2\sin^2\theta d\varphi^2, \quad (2.19a)$$

$$F^{S\delta} \to e^{-2M/r}, \qquad \phi \to \phi_\infty + \Sigma/r. \quad (2.19b)$$

III. CLÉMENT TRANSFORMATION

Here, we apply another generating technique suggested by Clément and designed for the framework of the Einstein-Maxwell theory [37] (Clément's transformation, CT). In the original paper, it is a two-step procedure with the initial and final solutions being vacuum, but the Maxwell field is auxiliary. Since the Maxwell field plays an important role in CT, we have to extend our model to

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R - 2(\partial\phi)^2 + F^2], \quad (3.1)$$

where $F = dA$ is the Maxwell two-form. It was shown in [37] that the application of CT to the ZV solution with an oblateness parameter δ does not lead to any vacuum metric at the end because of insufficient number of free parameters. But, one can hope to obtain the desired result applying CT to the combined FZV metric with the replacement $\delta \to S\delta$, which gives a new parameter. We expect to get a rotating generalization of the FZV metric imposing the constraint $S\delta = 1$.

First, we generalize the sigma-model to include the Maxwell field, introducing the electric and magnetic potentials v, u via

$$F_{i0} = \frac{1}{\sqrt{2}}\partial_i v, \qquad F^{ij} = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u. \quad (3.2)$$

The other components of the electromagnetic tensor in terms of (3.2) read

$$F^{i0} = F^{ij}\omega_j - h^{ij}F_{j0}, \quad F_{ij} = f^{-2}h_{ik}h_{jl}F^{kl} + F_{0i}\omega_j - F_{0j}\omega_i, \quad (3.3)$$

where h^{ij} is a three-inverse metric tensor of h_{ij}. We also modify the equations for the twist potential χ as

$$\partial_i\chi = -f^2 h^{-1/2}h_{ij}\epsilon^{jkl}\partial_k\omega_l + u\partial_i v - v\partial_i u. \quad (3.4)$$

This representation in terms of the scalar potentials f, χ, u, v, ϕ was derived for the EMD system in [36], generalizing the result of [50]. In our case, we have to put $\alpha = 0$,

$$dl^2_{\mathrm{EMS}} = \frac{1}{2f^2}(df^2 + (d\chi + vdu - udv)^2) - \frac{1}{f}(du^2 + dv^2) + 2d\phi^2. \quad (3.5)$$

Here, the scalar field is decoupled from the other potentials; therefore, all symmetries of the Einstein-Maxwell model are preserved, allowing us to apply generating techniques described in [51,52]. Following [37], we pass to the complex Ernst $(\mathcal{E}, Q)$ and Kinnersley (U, V, W) potentials,

$$\mathcal{E} = f + i\chi - \bar{Q}Q = \frac{U-W}{U+W}, \qquad Q = \frac{v+iu}{\sqrt{2}} = \frac{V}{U+W}, \quad (3.6)$$

with one of the Kinnersley potentials being redundant. The scalar-free sector of the σ model (3.5) possesses the $SU(2,1)$ isometry group, which acts on the complex vector space (U, V, W) leaving the norm $\bar{U}U + \bar{V}V - \bar{W}W$ invariant.

For the reader's convenience, we briefly recall the CT transformation, which is a triple $\mathcal{R} = \Pi^{-1}R\Pi$ with the target space discrete map,

$$\Pi\colon\ U \leftrightarrow V, \quad (3.7)$$

followed by the coordinate transformation,

$$R\colon\ \varphi \to \varphi + \Omega t, \quad (3.8)$$

and another target space map. Both the target-space and the coordinate transformations do not change the scalar field ϕ.

Starting with the static vacuum seed solution $\mathcal{E}_1 \in \mathbb{R}$, $Q_1 = 0$, we can take

$$V_1 = 0, \qquad U_1 = -1 - \mathcal{E}_1, \qquad W_1 = -1 + \mathcal{E}_1, \tag{3.9}$$

where the indices number the steps of the procedure. After the first Π transformation $U \leftrightarrow V$, the Ernst potentials become

$$\mathcal{E}_2 = -1, \qquad Q_2 = \frac{1+\mathcal{E}_1}{1-\mathcal{E}_1}, \tag{3.10}$$

and the new functions f, χ, u, v read from (3.6) as

$$f_2 = \frac{4\mathcal{E}_1}{(\mathcal{E}_1 - 1)^2}, \qquad \chi_2 = 0, \qquad \omega_2 = 0,$$
$$v_2 = \sqrt{2}\frac{1+\mathcal{E}_1}{1-\mathcal{E}_1}, \qquad u_2 = 0. \tag{3.11}$$

The corresponding spacetime is not asymptotically flat. Next, perform the global coordinate transformation (3.8) to a uniformly rotating frame $\varphi = \tilde{\varphi} + \Omega t$. Acting with (3.8) on the metric in the Weyl-Papapetrou parametrization,

$$ds^2 = -f(dt - \omega d\varphi)^2 - f^{-1}(\gamma_{mn}dx^m dx^n + \rho^2 d\varphi^2), \tag{3.12}$$

one obtains

$$f' = f[1 - 2\Omega\omega + \Omega^2(\omega^2 - f^{-2}\rho^2)],$$
$$\omega' = \frac{\omega - \Omega(\omega^2 - f^{-2}\rho^2)}{1 - 2\Omega\omega + \Omega^2(\omega^2 - f^{-2}\rho^2)}, \tag{3.13a}$$

$$\gamma'_{mn} = \frac{f'}{f}\gamma_{mn}, \qquad \rho' = \rho, \tag{3.13b}$$

$$\partial_m v' = (1 - \Omega\omega)\partial_m v - \Omega f^{-1}\rho\tilde{\partial}_m u,$$
$$\partial_m u' = (1 - \Omega\omega)\partial_m u + \Omega f^{-1}\rho\tilde{\partial}_m v, \tag{3.13c}$$

where $\tilde{\partial}_m = \gamma^{-1/2}\gamma_{mn}\epsilon^{np}\partial_p$. Applying these transformations to (3.11), the transformed functions are simplified to

$$f_3 = f_2(1-w^2), \quad \omega_3 = \frac{\Omega^{-1}w^2}{1-w^2}, \quad \gamma_{3mn} = (1-w^2)\gamma_{2mn},$$
$$\rho_3 = \rho_2, \quad v_3 = v_2, \quad \partial_m u_3 = w\tilde{\partial}_m v_2, \quad w = \Omega\rho/f_2. \tag{3.14}$$

The Ernst potentials are then rescaled by a constant $\mathcal{E} \to p^2\mathcal{E}$, $Q \to pQ$, which corresponds to the solution invariance with respect to the following transformations: $t \to pt$, $f \to p^{-2}f$, $\omega \to p\omega$, $h_{ij} \to p^{-2}h_{ij}$, $u \to p^{-1}u$, $v \to p^{-1}v$. The need for this transformation will be disclosed below. Applying all the above transformations to the solution (2.14) and putting $\mathcal{E}_1 = F^{S\delta}$, one obtains

$$\mathcal{E}_3 = p^2\Bigg[-1 - (ky\Omega S\delta)^2 - \frac{k^2\Omega^2(x^2-1)(1-y^2)(F^{S\delta}-1)^2}{4F^{S\delta}} + 2iky\Omega\left(S\delta\frac{F^{S\delta}+1}{F^{S\delta}-1} + x\right)\Bigg],$$
$$Q_3 = p\left(\frac{1+F^{S\delta}}{1-F^{S\delta}} + ikyS\delta\Omega\right). \tag{3.15}$$

The last transformation Π^{-1} leads to the final solution with the Ernst potentials $\mathcal{E}_4$, Q_4 in the form,

$$\mathcal{E}_4 = \frac{2Q_3 + \mathcal{E}_3 - 1}{2Q_3 - \mathcal{E}_3 + 1}, \qquad Q_4 = \frac{1+\mathcal{E}_3}{2Q_3 - \mathcal{E}_3 + 1}. \tag{3.16}$$

To get the final solution with a zero electromagnetic field, it is necessary to find such parameters, that set Q_4 equal to zero, which can be achieved with $\mathcal{E}_3 = -1$. This is possible only for $S\delta = 1$ and $p = (1 + k^2\Omega^2)^{-1/2}$, which leads us to the final expression for Ernst potentials,

$$\mathcal{E}_4 = \frac{px + iqy - 1}{px + iqy + 1}, \qquad Q_4 = 0, \tag{3.17}$$

where $q = k\Omega p$ is a constant ($p^2 + q^2 = 1$). Using the transformation from prolate to spherical coordinates in the form $x \to (r - M)/Mp$ and $y \to \cos\theta$, with the redefined constants,

$$\Omega = aM/(M^2 - a^2)^{3/2}, \qquad k = (M^2 - a^2)/M, \tag{3.18}$$

and properly rescaling the scalar charge Σ, the solution will be written in the Kerr-like form,

$$f(r,\theta) = \frac{\Delta - a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}, \qquad \omega(r,\theta) = -\frac{2aMr\sin^2\theta}{\Delta - a^2\sin^2\theta},$$
$$h_{ij}dx^i dx^j = H(r,\theta)(dr^2 + \Delta d\theta^2) + \Delta\sin^2\theta d\varphi^2,$$
$$\Delta(r) = (r - M)^2 - b^2, \tag{3.19}$$

with the following scalar field ϕ and the function H:

$$\phi(r) = \phi_\infty + \frac{\Sigma}{2b}\log\frac{r - M + b}{r - M - b}, \tag{3.20a}$$

$$H(r,\theta) = \frac{\Delta - a^2\sin^2\theta}{\Delta}\left(1 + \frac{b^2}{\Delta}\sin^2\theta\right)^{-\Sigma^2/b^2}, \tag{3.20b}$$

where $b = \sqrt{M^2 - a^2}$. Naturally, for $\Sigma = 0$, the solution coincides with the vacuum Kerr solution. One can also guess the generalization with the NUT parameter by taking Kerr-NUT solution and putting ϕ and H from (3.20) with the constant $b = \sqrt{M^2 + N^2 - a^2}$. The correctness of this guess was checked with the equations of motion. Later in this section, we will consider the solution without the NUT charge.

The Ricci scalar, following from the equations of motion (1.2a), reads

$$R = 2(\partial_r \phi)^2 g^{rr} = \frac{2\Sigma^2}{\Delta(r^2 + a^2\cos^2\theta)} \left(1 + \frac{b^2}{\Delta}\sin^2\theta\right)^{\Sigma^2/b^2}. \tag{3.21}$$

The curvature scalar R diverges at $\Delta = 0$, $b^2 \geq 0$. The solution also possesses the ring singularities in the equatorial plane at $r = 0$ for any b and $r = M$ for $b^2 < 0$.

Recently, some new rotating solutions were given by Chauvineau [24]. One of them is asymptotically flat, generalizing Penney's solution [5]. It also has the form (3.19) with the following ϕ and H:

$$\phi_N = \frac{\Lambda}{\sqrt{\Delta + (M^2 - a^2)\cos^2\theta}}, \tag{3.22a}$$

$$H_N = \frac{\Delta - a^2\sin^2\theta}{\Delta} \exp\left\{-\frac{\Lambda^2 \Delta \sin^2\theta}{(\Delta + (M^2 - a^2)\cos^2\theta)^2}\right\}, \tag{3.22b}$$

where $\Delta = r^2 - 2Mr + a^2$ is the same as in (3.19). Here, the subscript N stands for "Newtonian" as in [24], and we changed the definition of the constant Λ, which was $\Lambda\sqrt{M^2 - a^2}$ in [24]. The scalar curvature for this solution diverges on equator of the Killing horizon.

To make contact with the results of Chauvineau, consider the limit $M^2 - a^2 \to 0$ in our solution (3.19). One finds the following expressions for the metric function H and the scalar field:

$$H = \frac{\Delta - a^2\sin^2\theta}{\Delta} \exp\left\{-\frac{\Sigma^2\sin^2\theta}{\Delta}\right\}, \tag{3.23a}$$

$$\phi = \phi_\infty + \frac{\Sigma}{\sqrt{\Delta}}, \tag{3.23b}$$

which coincide with the same limit $M^2 - a^2 \to 0$ of the functions (3.22). Outside of this limit, our solutions are different.

IV. ERIŞ-GÜRSES TRANSFORMATION

Further simplification of the equations of motion can be achieved assuming the axial symmetry. Then one uses the Weyl-Papapetrou ansatz for the metric,

$$ds^2 = -\exp(2\psi)(dt - \omega d\varphi)^2 + \exp(-2\psi)[\exp(2\gamma)(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \tag{4.1}$$

where ψ, ω and γ are functions of the coordinates $\rho = k\sqrt{(x^2 - 1)(1 - y^2)}$ and $z = kxy$. The equations of motion will read

$$\Delta\psi = -\frac{1}{2} e^{4\psi} \rho^{-2} (\nabla\omega)^2, \tag{4.2a}$$

$$\nabla(e^{4\psi}\rho^{-2}\nabla\omega) = 0, \tag{4.2b}$$

$$\gamma_{,\rho} = \rho\left(\psi_{,\rho}^2 - \psi_{,z}^2 + \phi_{,\rho}^2 - \phi_{,z}^2 - \frac{1}{4} e^{4\psi}\rho^{-2}(\omega_{,\rho}^2 - \omega_{,z}^2)\right), \tag{4.2c}$$

$$\gamma_{,z} = 2\rho\left(\psi_{,\rho}\psi_{,z} + \phi_{,\rho}\phi_{,z} - \frac{1}{4} e^{4\psi}\rho^{-2}(\omega_{,\rho}\omega_{,z})\right), \tag{4.2d}$$

$$\Delta\phi = 0, \tag{4.2e}$$

where the vector operators act in the same way as in the cylindrical coordinates of the flat space (ρ, z, φ). Eriş and Gürses [40] suggested to split the equations into vacuum and scalar parts. For that, one has to present the function γ as a sum of two terms $\gamma = \gamma^\psi + \gamma^\phi$. Then the following theorem holds.

Theorem. If the functions ψ, ω, and $\gamma = \gamma^\psi$ fulfill the vacuum field equations, then the functions ψ, ω, and $\gamma = \gamma^\psi + \gamma^\phi$ satisfy the Eqs. (4.2), provided the scalar field satisfies the Eq. (4.2e) and

$$\gamma_{,\rho}^\phi = \rho(\phi_{,\rho}^2 - \phi_{,z}^2), \qquad \gamma_{,z}^\phi = 2\rho\phi_{,\rho}\phi_{,z}. \tag{4.3}$$

The integrability condition of the Eqs. (4.3) is given by (4.2e).

This theorem allows us to generate a solution with a nontrivial scalar field from a vacuum solution. We will abbreviate this procedure as EG transformation. Also, EG duality can be applied in the opposite direction to get rid of the scalar field. For the static case, the composition of the $SO(2)$ transformation (2.7) and the EG transformations, with an appropriate choice of the parameters, transforms one vacuum solution to another. This combined transformation coincides with the Zipoy-Voorhees transformation $\psi \to s\psi$, $\gamma^\psi \to s^2\gamma^\psi$.

Application of the EG duality to the Kerr metric gives the solutions (3.19) and (3.22) if we choose ϕ in the form (3.20a), (3.22a), correspondingly. Note that we are not limited to using ϕ depending on the radial coordinate r only, but this seems to be the most relevant case.

As the result of the $SO(2)$ symmetry (2.7), in the static case $\omega = 0$, the equations of motion for γ^ψ and ψ have the form (4.3), (4.14) similar to γ^ϕ and ϕ,

$$\gamma^{\psi}_{,\rho} = \rho(\psi^2_{,\rho} - \psi^2_{,z}), \qquad \gamma^{\psi}_{,z} = 2\rho\psi_{,\rho}\psi_{,z}, \tag{4.4a}$$

$$\partial_\rho(\rho\partial_\rho\psi) + \rho\partial_z^2\psi = 0. \tag{4.4b}$$

A. FJNW rotating generalization

Consider a stationary generalization of the ZV solution with arbitrary δ, angular momentum J and some potentials ψ, γ^ψ, ω. We will split γ^ψ and ψ into two parts: the static part γ^s, ψ^s and the rotational part γ^ω, ψ^ω,

$$\gamma^s = \lim_{J\to 0}\gamma^\psi = \delta^2\gamma_{\mathrm{Sch}}, \qquad \psi^s = \lim_{J\to 0}\psi = \delta\psi_{\mathrm{Sch}},$$
$$\gamma^\omega = \gamma^\psi - \gamma^s, \qquad \psi^\omega = \psi - \psi^s, \tag{4.5}$$

where $\gamma_{\mathrm{Sch}} = \frac{1}{2}\ln\frac{L^2-k^2}{l_+l_-}$, $\psi_{\mathrm{Sch}} = \frac{1}{2}\ln\frac{L-k}{L+k}$ are the potentials of the Schwarzschild solution (note that γ_{Sch} corresponds to FJNW solution as well), $L = \frac{1}{2}(l_+ + l_-)$, $l_\pm = \sqrt{\rho^2 + (z \pm k)^2}$, with k being a constant entering the spheroidal coordinates. Since ψ^s, γ^s and ϕ, γ^ϕ satisfy the same equations, we can introduce a scalar field into our solution in the form,

$$\gamma^\phi = c^2\gamma_{\mathrm{Sch}}, \qquad \phi = c\psi_{\mathrm{Sch}}. \tag{4.6}$$

The final metric is described by the functions $\psi = \delta\psi_{\mathrm{Sch}} + \psi^\omega$, $\phi = c\psi_{\mathrm{Sch}}$, ω, $\gamma = (\delta^2 + c^2)\psi_{\mathrm{Sch}} + \gamma^\omega$. Setting $\delta^2 + c^2 = 1$, we get rid of the three-metric deformation of ZV kind. Thus, on one hand, in the static limit, $\omega, \psi^\omega, \gamma^\omega \to 0$, such a solution exactly corresponds to FJNW (this was noticed by Eriş and Gürses [40]). For the zero scalar $c = 0$, the constraint gives $\delta = 1$, and the solution represents the rotating Schwarzschild solution, i.e., the Kerr metric. Such a solution can be considered as a full-fledged rotating generalization of FJNW. On the other hand, one can take the scalar field in the form $\phi = c\psi_{\mathrm{Sch}} + \mathcal{Q}(J)\tilde{\phi}$, where $\mathcal{Q}(0) = 0$ is some function of the angular momentum J, and $\tilde{\phi}$ is an arbitrary solution of (4.14). This solution will have the same properties for static and scalarless limits. Here, we will not consider this case.

The general rotating axisymmetric vacuum solution was considered in [53]. The asymptotically flat rotating generalization of the ZV metric was found by Tomimatsu and Sato (TS) for the integer deformation parameter $\delta = 1, 2, 3, 4$ in [44]. The further generalization to an arbitrary integer $\delta \in \mathbb{Z}_+$ was given by Hori [54]. Then, Cosgrove extended TS solutions to an arbitrary real δ in the six-parametric family [55]. Another attempt to interpolate TS for real δ was given by Hori without evidence of its correctness with respect to the equations of motion [56]. Eriş and Gürses mentioned without details the possibility to apply their transformations to the TS metric. A similar idea of EG-transformations was applied by Astorino for some metrics in [25]. Among them, there is a subfamily of Cosgrove's metrics from [27].

From the constraint on δ and c, it follows that $c^2 < 0$ for $\delta > 1$, so we can construct the rotating FJNW from the Tomimatsu-Sato solutions with the integer δ for a phantom scalar field only. We will consider the case $\delta = 2$ with

$$e^{2\psi} = \frac{A}{B}, \quad e^{2\gamma^\psi} = \frac{A}{p^4(x^2-y^2)^4}, \quad e^{2\gamma^\phi} = \left(\frac{x^2-y^2}{x^2-1}\right)^3,$$
$$\omega = \frac{2qM(1-y^2)C}{A}, \tag{4.7}$$

where the constants satisfy the constraint $p^2 + q^2 = 1$ and the functions A, B, C are

$$A = p^4(x^2-1)^4 + q^4(1-y^2)^4 - 2p^2q^2(x^2-1)(1-y^2)[2(x^2-1)^2 + 2(1-y^2)^2 + 3(x^2-1)(1-y^2)], \tag{4.8}$$

$$B = [p^2(x^2+1)(x^2-1) - q^2(y^2+1)(1-y^2) + 2px(x^2-1)]^2 + 4q^2y^2[px(x^2-1) + (px+1)(1-y^2)]^2, \tag{4.9}$$

$$C = -p^3x(x^2-1)[2(x^2+1)(x^2-1) + (x^2+3)(1-y^2)]$$
$$- p^2(x^2-1)[4x^2(x^2-1) + (3x^2+1)(1-y^2)] + q^2(px+1)(1-y^2)^3. \tag{4.10}$$

The final solution reads

$$ds^2 = -\frac{A}{B}(dt - \omega d\varphi)^2 + \frac{B}{A}k^2(H(dx^2 + h^{-1}dy^2) + (x^2-1)(1-y^2)d\varphi^2),$$
$$H = \frac{A}{p^4(x^2-1)^4} = \left(1 + \frac{q^2}{p^2}h^2\right)^2 - 4\frac{q^2}{p^2}h(h+1)^2,$$
$$\phi = \pm\frac{i}{2}\sqrt{3}\ln\frac{x-1}{x+1}, \qquad h = \frac{1-y^2}{x^2-1}, \qquad k = Mp/2, \qquad q = J/M^2, \tag{4.11}$$

where J is the angular momentum and M is the mass. Note that $H = e^{2\gamma}(x^2 - y^2)/(x^2 - 1)$ under the coordinate transformation from ρ, z to x, y. The EG transformation allows adding the nontrivial scalar field to a vacuum solution by the correction to the function γ, leaving the corresponding Ernst potential unchanged. From the definition of the scalar charge $\phi \approx \phi_\infty + \Sigma/r$ at infinity, one finds the imaginary value $\Sigma = \mp i\sqrt{3}Mp$, corresponding to a phantom scalar. The metric is asymptotically flat since the Ernst potential of the solution (4.11) is the same as for the TS solution with $\delta = 2$. Note, that our scalar field coincides with that given by Astorino [25],[1] but the metric of the solution presented in [25] is not asymptotically flat, as we explain in detail in Appendix C.

The solution (4.11) has a horizon at $x = \pm 1$. Its ergoregion is defined by the equation $H = 0$, which can be resolved as $h = \tilde{h}(q/p)$, where $\tilde{h}$ is some function of p/q. The location of singularities corresponds to divergence of the scalar curvature,

$$R = 2g^{xx}(\partial_x\phi)^2 = \frac{-6A}{BHk^2(x^2-1)^2} = \frac{-24p^2}{M^2}\frac{(x^2-1)^2}{B}. \tag{4.12}$$

From this expression, it follows that singularities correspond to $B = 0$, with B being a sum of two squares. Therefore, both squared expressions must be zero for the singular point,

$$y(px(x^2-1) + (px+1)(1-y^2)) = 0, \tag{4.13a}$$

$$p^2(x^2+1)(x^2-1) - q^2(y^2+1)(1-y^2) + 2px(x^2-1) = 0. \tag{4.13b}$$

The Eq. (4.13a) holds if one of the following conditions are met:

(i) $y^2 = (1+px^3)/(1+px)$

(ii) $y = 0$.

Substituting the solution (i) into the Eq. (4.13b), we get a condition,

$$\frac{p(x^2-1)(4p^2x^3 + p(x^4+6x^2+1) + 4x)}{(px+1)^2} = 0.$$

The first root $x^2 = 1$ does not lead to divergence due to the presence of $(x^2-1)^2$ in the numerator of Ricci scalar (4.12). The second bracket is positive for $p > 0$, $x > 0$. Therefore, in this case, the singularity can be located only under the horizon $x = 1$.

The solution (ii) substituted into the Eq. (4.13b) gives a condition,

$$p^2x^4 + 2px(x^2-1) - 1 = 0.$$

For $x = 1$, the lhs is negative $p^2 - 1 < 0$, while in the limit of large $x \to \infty$, the lhs tends to $p^2x^4 > 0$, so it has at least one root in the outer region $x > 1$. Numerically, one can show that there is always exactly one root in the region $x > 1$. Thus, the solution represents a black hole with a regular event horizon and a singular ring in the equatorial plane. The scalar field diverges on the horizon, so the horizon has a scalar charge, which is typical for the static FJNW solution with $S > 1$. Moreover, the scalar field is regular in the ring singularity, so the ring does not carry the scalar charge.

B. Oblate rotating solutions

Metric functions depend on p^2, q^2, x^2, y^2, px and allow an analytic continuation $x \to ix$, $p \to ip$, (i.e., $a > M$), in which the metric remains physical with the same signature, but the scalar field becomes real $\phi = \pm\sqrt{3}\arctan x$ (up to an additive constant). In this case, $x \geq 0$ and $-1 \leq y \leq +1$ will represent the oblate spheroidal coordinate system, and the scalar field will have a cusp at the disk $x = 0$. Therefore, this solution cannot be considered as a desired rotating generalization of FJNW. But it is interesting since analytically continuing the coordinate x to the whole real line $x \in \mathbb{R}$, one gets a wormhole without the scalar cusp. Such objects with wormhole interpretation were described by Gibbons and Volkov in [57,58].

C. Generation of higher scalar multipoles

The equation of motion of the scalar field (4.2e) does not contain explicitly the coordinate z,

$$\partial_\rho(\rho\partial_\rho\phi) + \rho\partial_z^2\phi = 0. \tag{4.14}$$

Therefore, if $\phi_0(\rho, z)$ is a solution of the Eq. (4.14), then $\phi_n = \partial_z^n\phi_0$ is a solution too, and the function γ^ϕ has a new form defined from the Eq. (4.3). As for the problem of Kerr solutions with scalar fields, such transformations were discussed in [24]. For example, acting with $\partial_z = r_{,z}\partial_r + \theta_{,z}\partial_\theta$ on (3.20a) and using the coordinate transformations $\rho = \sqrt{\Delta}\sin\theta$, $z = (r-M)\cos\theta$, after lengthy calculations, one can obtain

$$\phi_{(1)} = \frac{\Lambda_{(1)}\cos\theta}{\Delta + b^2\sin^2\theta}, \tag{4.15a}$$

$$\gamma^\phi_{(1)} = \frac{-\Lambda^2_{(1)}}{8b^4(\Delta + b^2\sin^2\theta)^2}\left[\Delta^2 + 2b^2\Delta\sin^2\theta - b^4\sin^4\theta + \frac{4b^4\sin^2(2\theta)(r-M)^2\Delta}{(\Delta + b^2\sin^2\theta)^2}\right], \tag{4.15b}$$

[1] We thank the anonymous referee for indicating this.

where $\Lambda_{(1)}$ is a constant. The asymptotic behavior of the scalar field $\phi \approx \Lambda_{(1)} \cos\theta / r^2$ suggests that the solution describes the rotating source with a scalar dipole moment. The general multipolar expansion of the scalar field in MES and conformally coupled theory is considered in Appendix D of Ref. [25].

V. CONCLUSIONS

Let us briefly summarize the results. First, using the sigma-model formulation of the field equations, we obtained a new static generalization of FJNW, endowed with oblateness and NUT parameters. The combined ZV-FJNW solution opened up a way to apply Clement's technique to generate rotation, obtaining a nontrivial rotating solution of MES. The solution is simple and can be used as a legal rotating metric to study physics outside of the Kerr paradigm. In the extreme limit, it coincides with one of the solutions recently found by Chauvineau. But in this solution, the oblateness can not be eliminated due to the internal constraint that exists in this technique.

In search of rotating scalar-tensor configurations with no additional parameters, we have resorted to the Eriş and Gürses method. By applying the EG duality, we could reproduce the result obtained with CT. Using the EG transformation, we put forward the argument that rotating vacuum solutions of ZV family are dual to scalar rotating solutions. As an example, we have obtained a new rotating solution, which is dual to the Tomimatsu-Sato vacuum solution with $\delta = 2$, which has a regular horizon surrounded by a naked ring singularity. It is supported by a phantom scalar field. Using some complex transformations, the scalar field can be made real, but it will be no longer a generalization of the FJNW solution and can be interpreted as a disk with scalar charge or a wormhole with ring singularity.

We also obtained a new solution using the generating technique suggested by Chauvineau for Kerr-like metrics. Using EG transformations, this technique was extended to an arbitrary axisymmetric solution in the Weyl-Papapetrou form. Following Chauvineau, the action of the differential operator ∂_z on a scalar field leaves the equations of motion satisfied. Due to the theorem of EG, the presence of a scalar field leads to an additional term γ^ϕ in the metric, which can be found from the first-order partial differential equations. Therefore, a new generated scalar field redefines the term γ^ϕ, which is more or less easy to find. We applied this technique to the solution obtained through the Clément transformations.

Both generalizations found with Clément and EG-transformations are described by the Ernst potentials corresponding to the Kerr and Tomimatsu-Sato solutions. All solutions we obtained in this paper can be considered as the valid alternatives to false solutions which arise from application of the JN algorithm within MES and BD theories.

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APPENDIX A: FALSE ROTATING FJWN

Consider the metric, obtained in [15] by application of the JN trick to FJNW. In terms of the sigma-model variables (2.1), it reads

$$f = \frac{R^2}{\sigma^2}(\Delta - \tilde{\omega}^2 \sin^2\theta), \qquad \omega_i dx^i = -f^{-1}\tilde{\omega}\sin^2\theta d\varphi, \tag{A1a}$$

$$h_{ij}dx^i dx^j = \frac{fR^2}{\Delta}(dr^2 + \Delta d\theta^2) + \Delta \sin^2\theta d\varphi^2, \tag{A1b}$$

with the scalar field,

$$\phi(r) = \frac{\Sigma}{2\sqrt{\eta^2 - a^2}} \ln\left(1 - \frac{\eta + \sqrt{\eta^2 - a^2}}{r}\right), \qquad \eta = \sqrt{M^2 + \Sigma^2}, \tag{A2}$$

where

$$\tilde{\omega} = \frac{a(R^2 + a^2\sin^2\theta - \Delta)}{R^2},$$
$$R^2 = (r^2 + a^2\cos^2\theta)\left(1 - \frac{2\eta r}{r^2 + a^2\cos^2\theta}\right)^{1-M/\eta},$$
$$\Delta = r^2 - 2\eta r + a^2,$$
$$\sigma^2 = (R^2 + a^2\sin^2\theta)^2 - \Delta a^2 \sin^2\theta.$$

For $\Sigma = 0$, $\eta = M$, and we recover the Kerr solution. This metric looks simple, and it became a popular model for describing possible deviations from general relativity in astrophysical observations [18–21]. Pirogov [22] checked part of Einstein's equations and found that they do not hold. This statement was supported in [23]. Here, we check the validity of the sigma-model equations. Considering the equation $\Box\phi = 0$ for ϕ depending on r only, taking into account $\sqrt{-g} = R^2 \sin\theta$, one can derive the equation,

$$\partial_r(\Delta \partial_r \phi) = 0, \tag{A3}$$

which can be solved with $\phi = \mathrm{const}\ln(r - \eta + \sqrt{\eta^2 - a^2})/(r - \eta - \sqrt{\eta^2 - a^2})$, but not (A2). Still this does not mean that the metric (A1) is incorrect.

The σ model (2.4) implies the following equation for ψ:

$$\Delta\psi + \frac{1}{2}e^{-4\psi}(\partial\chi)^2 = 0, \quad (A4)$$

where Δ and the contraction over indices relate to the three-metric,

$$(\partial\chi)^2 = e^{8\psi}((\partial_i\omega_j)(\partial^i\omega^j) - (\partial_i\omega_j)(\partial^j\omega^i)), \quad (A5)$$

for $\omega_i dx^i = \omega(r,\theta)d\varphi$, and diagonal three-metric the second term $(\partial_i\omega_j)(\partial^j\omega^i)$ is zero. The first term we will write as $(\partial\omega)^2$. Then, the equation is

$$\Delta\psi + \frac{1}{2}e^{4\psi}(\partial\omega)^2 = 0. \quad (A6)$$

It can be expanded as

$$\partial_r(\Delta\partial_r\psi) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{e^{4\psi}}{2\sin^2\theta}((\partial_r\omega)^2 + (\partial_r\omega)^2/\Delta) = 0. \quad (A7)$$

Substituting the functions f and ω and expanding as $r\to\infty$, we find the nonzero term,

$$\frac{a^2M(3\cos(2\theta)+5)(M-\eta)}{r^4} + \mathcal{O}(r^{-5}) = 0. \quad (A8)$$

This can be fulfilled for $a = 0$ (static FJNW solution) or $\eta = M$ (Kerr). So we confirm the results of [22,23].

APPENDIX B: BRANS-DICKE FALSE ROTATING SOLUTION

Another rotating solution with the scalar field generated with JN algorithm was derived within the Brans-Dicke theory [33]. The Brans-Dicke equations of motion read

$$\Box\Phi = 0, \quad (B1a)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\omega}{\Phi^2}\left(\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{;\lambda}\Phi^{;\lambda}\right) + \frac{1}{\Phi}(\Phi_{;\mu\nu} - g_{\mu\nu}\Box\Phi). \quad (B1b)$$

Taking into account (B1a), one can find Ricci tensor,

$$R_{\mu\nu} = \frac{\omega}{\Phi^2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{\Phi}\Phi_{;\mu\nu}. \quad (B2)$$

The theory can be formulated in the Jordan and Einstein frames, the latter corresponding to MES. The explicit conversion to Einstein's frame reads

$$g^E_{\mu\nu} = \Phi g^J_{\mu\nu}, \qquad \phi = \frac{1}{2}\sqrt{2\omega+3}\ln\Phi. \quad (B3)$$

The FJNW solution in the Jordan frame is

$$ds^2 = -F^{-\sigma+S}dt^2 + F^{-\sigma-S}(dr^2 + r^2F(d\theta^2 + \sin^2\theta d\varphi^2)), \quad \Phi = \Phi_0 F^\sigma \quad (B4)$$

$$F = 1 - \frac{2M}{Sr}, \quad S = \frac{M}{\sqrt{M^2+\Sigma^2}}, \quad \sigma = -\frac{\Sigma S}{M\sqrt{2\omega+3}}. \quad (B5)$$

Application of the JN trick [33] leads to

$$ds^2 = f_K^\eta(dt - \omega_K d\varphi)^2 - f_K^\xi\rho(dr^2/\Delta + d\theta^2 + \sin^2\theta d\varphi^2) + 2f_K^\sigma\omega_K(dt - \omega_K d\varphi)d\varphi, \quad (B6a)$$

$$\Phi = \Phi_0 f_K^\sigma, \quad (B6b)$$

where

$$f_K = 1 - 2r_0r/\rho, \qquad \rho = r^2 + a^2\cos^2\theta, \qquad \omega_K = a\sin^2\theta, \qquad \Delta = r(r-2r_0) + a^2 \quad (B7)$$

and

$$\sigma = (\eta+\xi-1)/2 = -c/2\lambda, \quad \eta = 1/\lambda, \quad \xi = (\lambda - c - 1)/\lambda \quad (B8)$$

with the free parameters λ and c. The static limit of the solution (B6b) should coincide with (B4) up to the definition of constants. The solution (B4) possesses a property that $\ln|g_{tt}g_{rr}| = -2\ln\Phi + \text{const}$. For the solution (B6), we find

$$\ln|g_{tt}g_{rr}| = (\eta+\xi-1)\ln f_K = 2\ln\Phi + \text{const};$$

thus, the scalar field is incorrect, and the correct one is $\Phi = \Phi_0 f_K^{-\sigma}$.

For the tt component, the Einstein equation is

$$R_{tt} = \frac{1}{\Phi}\Phi_{;tt} = -\Gamma^r_{tt}\partial_r\ln\Phi. \quad (B9)$$

Let us calculate an asymptotic behavior of the quantity $X = R_{tt} - \Phi_{;tt}/\Phi$ up to the seventh order for the solution (B6). The first nonzero term of the Taylor series starts from the fourth order. The fourth and the fifth terms are zero if we use a corrected definition of the scalar field. Then, the term of the sixth order reads

$$X \approx \frac{a^2r_0^2((c+2)^2 - 4\lambda)(5+3\cos2\theta)}{4\lambda^2r^6} + \mathcal{O}(r^{-7}), \quad (B10)$$

which is zero for $a = 0$ or $\lambda = (c+2)^2/4$. The first case brings us back to the static solution and satisfies the Eq. (B9) exactly. Substituting the second case into the seventh order gives

$$X \approx \frac{64a^2 c r_0^3 \cos^2\theta}{(c+2)^3 r^7} + \mathcal{O}(r^{-8}) \tag{B11}$$

and requires either c or r_0 to be zero, which guarantees the trivial form of the scalar field. Therefore, the solution found with JN algorithm in [33] is incorrect as well.

APPENDIX C: TOMIMATSU-SATO SOLUTIONS AND COSGROVE'S SUBFAMILY

In this Appendix, we will show that Cosgrove's subfamily considered in [25] are not asymptotically flat, and it does not include TS.

The first physically interesting rotating generalizations of ZV were found by Tomimatsu and Sato in Ref. [44] for the integer deformation parameter $\delta = 2, 3, 4$. To construct these solutions, the authors represented the potential $\xi = (1+\mathcal{E})/(1-\mathcal{E})$ as a rational fraction $\xi = \alpha/\beta$ and made use of seven formal rules found empirically. One of these rules is that the functions α and β are polynomials in x, y with powers δ^2 and $\delta^2 - 1$, respectively. This rule can be easily verified for $\delta = 2$,

$$\begin{aligned} \alpha &= p^2x^4 + q^2y^4 - 1 - 2ipqxy(x^2 - y^2), \\ \beta &= 2px(x^2-1) - 2iqy(1-y^2). \end{aligned} \tag{C1}$$

Tomimatsu-Sato solutions are asymptotically flat, which follows from $\mathcal{E} \to 1$ for $x \to \infty$.

The potential for the subfamily of the Cosgrove solutions considered in [25] can be represented in a similar form $\xi = \alpha/\beta$ with

$$\begin{aligned} \alpha = {} & \frac{p}{2}(x^2-1)^m[(x+1)^{m+1}(1-y)^m + (x-1)^{m+1}(1+y)^m] \\ & + \frac{iq}{2}(1-y^2)^m[(1-y)^{m+1}(x+1)^m - (1+y)^{m+1}(x-1)^m], \end{aligned} \tag{C2a}$$

$$\begin{aligned} \beta = {} & \frac{p}{2}(x^2-1)^m[(x+1)^{m+1}(1-y)^m - (x-1)^{m+1}(1+y)^m] \\ & + \frac{iq}{2}(1-y^2)^m[(1-y)^{m+1}(x+1)^m + (1+y)^{m+1}(x-1)^m], \end{aligned} \tag{C2b}$$

where $m = \delta - 1$. Solutions (C2) do not include TS metrics found in [44,54] (except the Kerr case $\delta = 1$). This can be seen from the fact that the solution (C2) does not satisfy the aforementioned formal rule: if m is a positive integer, the power index of polynomials α, β is $4m+1$ (or $4\delta - 3$). Furthermore, solutions (C2) are not asymptotically flat,

$$\mathcal{E} \approx \frac{Y_m^+ e^{i\tau} - Y_m^-}{Y_m^+ e^{i\tau} + Y_m^-} + \mathcal{O}(x^{-1}), \tag{C3}$$

where $Y_m^\pm$ are functions of y

$$Y_m^\pm = \begin{cases} (1-y)^m \pm (1+y)^m, & m > -1/2 \\ (p \mp iq)(1-y)^m \pm (p \pm iq)(1+y)^m, & m = -1/2\ , \\ (1-y)^{m+1} \mp (1+y)^{m+1}, & m < -1/2 \end{cases} \tag{C4}$$

and τ is an additional Ehlers transformation parameter introduced for completeness (see [25], Appendix B for details). The expression (C3) is not a constant for any set of parameters m and τ (except the Kerr case $m = 0$), and thus (C2) is not asymptotically flat.

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