Reconstructing k-essence: Unifying the attractor $n_S(N)$ and the swampland criteria

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The reconstruction of a k-essence inflationary universe, considering the unification between the swampland criteria and the attractor given by the scalar spectral index $n_S(N)$ together with the slow-roll parameter $\epsilon(N)$ in terms of the number of e-folds N is studied. In the context of a coupling of the form $L(\phi)X$ in the k-essence model, we find the effective potential V and the coupling parameter L in terms of the scalar spectral index and the slow-roll parameter under a general formalism. To apply the unification in our model, we consider some examples in order to rebuild the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$ as a function of the inflaton field ϕ. Here, we find that the reconstruction gives rise to an exponential potential and also to natural and hyperbolic inflation, respectively. Thus, in this article we show that it is possible to unify the theoretical foundations from the swampland criteria and the observational parameters corroborated by observations, in the reconstruction of an inflationary universe.

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I. INTRODUCTION

It is well known that the evolution of the early universe can be described by the standard hot big bang model; however, this hot model presents some cosmological problems that the inflationary stage or inflation solves through an accelerated expansion previous to the radiation era [1–4]. Nevertheless, the importance of inflation is that this scenario gives an account of the large-scale structure [5,6], and it also provides a causal description of the anisotropies observed in the cosmic microwave background radiation [7–12].

In the literature, we can find different models that give an account of the inflationary evolution of the early universe. In this context, we can distinguish the inflationary models where inflation is driven for a canonical or noncanonical scalar field; see, e.g., [13–15]. In this sense, we can stand out the k-essence inflationary model, where the description of the k-essence is through an action or Lagrangian density that includes a nonstandard higher order kinetic term associated with the scalar field [16,17]. An important consideration to take into account of the k-essence models is the fact that the speed gravitational waves is equal to the speed of light, coinciding with the speed obtained from the detection of gravitational waves by GW170817 and the γ-ray burst [18–20]. Additionally, the k-essence models give the possibility that the value of the speed of sound of the scalar perturbations is smaller than one or equal to one [16]. In this form, the k-essence model is consistent with these observational data, since the speed of gravitational waves is equal to the speed of light and we can also have the possibility that the speed of sound associated with scalar perturbations could be less than or equal to one, depending on the Lagrangian density associated with the k-essence model. Thus, the k-essence model can be interesting to study the early (inflation) and current (dark energy) [21] universe. In particular, in the context of inflation, different effective potentials associated with a scalar field have been studied under the slow-roll approximation [22–24].

On the other hand, the reconstruction of the background variables, such as the effective potential, the coupling functions, and the scale factor associated with the inflationary models, from the observational parameters, such as the scalar spectrum, the scalar spectral index, and the tensor to scalar ratio, have been analyzed by several authors [25–31]. In this sense, a possible methodology for the reconstruction of inflation under the slow-roll approximation can be developed by means of the parametrization of these cosmological parameters or attractors, in terms of the number of e-foldings N.

As an example of this methodology, we have the scalar spectral index $n_S(N)$ as a function of the number N. In particular, the simple parametrization or attractor $n_S(N) = 1–2/N$ is well corroborated by Planck data [32], when the number of e-foldings $N \simeq 50–60$. Here we consider that the number of e-foldings $N \simeq 50–60$ corresponds to the comoving scale k that crossed the Hubble radius; i.e., $k = aH$ during the inflationary epoch.

In the framework of the general relativity (GR), the reconstruction of inflation under this procedure gives the

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origin for different inflationary models according to the attractor point $n_S(N)$ given by $n_S(N) = 1–2/N$, during the slow-roll scenario for large N. In this way, we can have the following: the hyperbolic tangent model or T-model [33], the E-model [34], the R^2-model [1], the chaotic inflationary model [3], the study of Higgs inflation [35,36], etc. In the reconstruction of two background variables as the case of warm inflation, it was necessary to consider the attractors $n_S(N)$ and $r(N)$ to rebuild the effective potential and the dissipation coefficient as a function of the scalar field, respectively [37]. Similarly, for the reconstruction of G-inflation, the spectral index $n_S(N)$ together with the tensor to scalar ratio $r(N)$ was required, in order to reconstruct the potential and the coupling parameter in terms of the inflaton field (see Ref. [38]).

Additionally, we mention that in the literature it is possible to find other methodologies to rebuild the variables as the scalar potential, the scalar spectral index, and the tensor to scalar ratio under the slow-roll approximation. For example, we have the parametrization of the slow-roll parameter $\epsilon(N)$, in terms of the number of e-folds N [29,39,40]. Similarly, the reconstruction of the scalar potential and spectral index from two slow-roll parameters $\epsilon(N)$ and $\eta(N)$ was studied in Ref. [41]. Also, the reconstruction of the scalar potential, considering as ansatz the velocity of the scalar field as a function of the number N, in a model of k-essence inflation was developed in Ref. [42]. For other reconstruction methodologies in the scenario of inflation, see Refs. [43–45].

On the other hand, in the context of the theoretical foundations of the early and present universe from an effective field theory, there are some criteria or conjectures that have emerged recently in the literature. These criteria are related with the consistency between the effective field theory and superstring theory, in order to describe the universe from one or various scalar fields. In this sense, we have the swampland criteria or conjectures (SC) [46,47] and are related to the conditions on the range of inflaton field during its dynamic evolution and also on the effective potential (derivatives) associated with the inflaton field, in order to permit an embedding in the framework of superstring theory [46]. This first criterion establishes that the range of inflaton field values $\Delta\phi$ is smaller than the Planckian scale during the dynamic of the inflationary epoch. This first conjecture supposes that the effective field theory is consistent with the string theory, if the range of inflaton field values satisfies $\Delta\phi < \Delta M_p$, where Δ denotes a constant of the order $\mathcal{O}(1)$ and M_p denotes the Planck mass [46]. In relation to the condition on the effective potential and its derivatives, we have that the slope of the potential has to be larger to explain that the fields coming from the frame of string theory (see Refs. [46,47]). Thus, the condition on the slope of the effective potential $V(\phi)$ (called the second swampland conjecture) can be written as $V_\phi/V > c/M_p$, where $V_\phi = \partial V/\partial\phi$ and c denotes another constant of the order one as Δ. Additionally, we can have that the above condition cannot be satisfied when the fields are around the maximum (local) of the potential, with which we can also consider that $V_{\phi\phi}/V < -c_1/M_p^2$, in which c_1 denotes another constant of the order one [48]. However, we mention that recently it was shown that these constants may be somewhat less than unity (see, e.g., Ref. [49]). In this sense, under the theoretical description of inflation in the framework of GR, we can find a direct tension from the second SC and the utilization of the slow-roll approximation, since the slow-roll parameter $\epsilon \propto (V_\phi/V)^2$ must be smaller than one during inflation; i.e., $\epsilon \ll 1$. In this way, the imposed conditions by the SC have questioned whether slow-roll inflation is described by an effective field theory.

In this respect, we mention that the SC do not exclude all inflationary models in the context of the slow-roll approximation. In order to describe inflation under the slow-roll approximation, we have some models that can survive to the requirements of the SC. In particular, we can mention that for the case of a single scalar field, the model of warm inflation satisfies the criteria imposed by SC [50] (see also Ref. [51]). Also, the SC for a single field with a chaotic potential in the framework of brane inflation was developed in Ref. [52], and this model showed to be compatible with the SC (see also Ref. [53] for other potentials). The case of quintessential brane inflation and its compatibility with the SC introducing deviations from the Bunch-Davies initial state was studied in [54]. A curvatonlike mechanism is another possibility used in order to conciliate the SC from a single field [55] and for multifield models, and its compatibly with the SC was analyzed in Ref. [56].

Additionally, we comment that another conjecture studied in the literature is known as the trans Planckian censorship conjecture (TCC) [57] (see also Ref. [58]). The TCC is established on the concept that in a suitable quantum theory of gravity the sub-Planckian quantum fluctuations should persist on a quantum scale and never become larger than the Hubble horizon, and then these fluctuations never freeze during the expansion of the universe (see also Refs. [59,60]).

The goal of this investigation is to reconstruct the k-essence inflationary model, considering the unification between the attractor or parametrization of the scalar spectral index and the slow-roll parameter as a function of the e-foldings together with the SC. In this context, we investigate how the k-essence inflationary model, in which the Lagrangian density $\mathcal{L}(\phi, X)$ with a new term given by $L(\phi)X$ modifies the reconstructions of the background variables, such as the scalar potential $V(\phi)$ and the coupling parameter $L(\phi)$, and simultaneously satisfies the SC. In this sense, we will determine the structure of the coupling parameter $L(\phi)$ and of the effective potential

$V(\phi)$, in order to satisfy the swampland criteria and also the attractor point associated with the scalar spectral index n_S from the observations.

In order to satisfy the observational data and the swampland criteria, we consider a general formalism to rebuild the effective potential V and the coupling parameter L, from the parametrization of the cosmological attractor $n_S(N)$ and the slow-roll parameter $\epsilon(N)$, under the slow-roll approximation.

As an application to the developed formalism, we will study different examples in order to analyze the SC considering the slow-roll parameter $\epsilon(N)$ and also assuming the simplest attractor point for the scalar spectral index $n_S - 1 = -2/N$. In this respect, we will reconstruct the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$ as a function of the inflaton field ϕ. Additionally, we will find different constraints on the parameters in our k-essence model from the unification of the observational data and the SC.

The outline of the paper is as follows: In Sec. II we give a brief description of the model of the k-essence. The background equation and cosmological perturbations are shown. In Sec. III, we elaborate a general formalism to rebuild the scalar potential and coupling parameter in terms of the observable or attractor $n_S(N)$ and the slow-roll parameter. Later in Sec. IV, we apply the methodology for different examples in order to obtain the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$, as a function of the scalar field ϕ. In the end, in Sec. V we give our conclusions. We chose units so that $c = \hbar = M_p = 8\pi = 1$.

II. THE k-ESSENCE MODEL

As a brief description of the scenario of the k-essence model, we begin with the four-dimensional action S for this theory given by [16,17]

$$S = \int \sqrt{-g_4}d^4x\left(\frac{1}{2}R + \mathcal{L}(\phi, X)\right), \tag{1}$$

where g_4 corresponds to the determinant of the spacetime metric $g_{\mu\nu}$, R denotes the Ricci scalar, and the quantity $\mathcal{L}(\phi, X)$ represents the Lagrangian density associated with the scalar field ϕ and X. Here the quantity X corresponds to the kinetic energy of the field ϕ defined as $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$.

By assuming that the energy momentum corresponds to a perfect fluid, then it is possible to identify from the action (1) that the energy density ρ and the pressure p associated with the scalar field ϕ and X are given by [16,17]

$$\rho(\phi, X) = 2X\frac{\partial\mathcal{L}(\phi, X)}{\partial X} - \mathcal{L}(\phi, X) \tag{2}$$

and

$$p(\phi, X) = \mathcal{L}(\phi, X), \tag{3}$$

respectively. In particular, for the specific case in which the Lagrangian $\mathcal{L}(\phi, X) = X - V(\phi)$, we recovered the expressions for the energy density and pressure in the framework of the GR. Here the quantity $V(\phi)$ denotes the effective potential associated with the scalar field ϕ.

In this context and in order to develop the reconstruction for the k-essence model of inflation, we will study the specific case in which the Lagrangian density $\mathcal{L}(\phi, X)$ is given by [16,17]

$$\mathcal{L}(\phi, X) = X + 2L(\phi)X - V(\phi), \tag{4}$$

where $L(\phi)$ is a coupling function that depends exclusively on the scalar field ϕ. We note that in the limit in which this coupling parameter $L(\phi) \to 0$, we recovered the standard GR.

To analyze this inflationary model, we consider a spatially flat Friedmann Robertson Walker (FRW) metric, together with a homogeneous scalar field, such that the field $\phi = \phi(t)$. In this sense, we have that the Friedmann equation can be written as

$$3H^2 = \rho, \tag{5}$$

where $H = \frac{\dot{a}}{a}$ denotes the Hubble rate and the quantity a represents the scale factor. In the following, the dots denote differentiation with respect to the time.

Thus, from relations (2), (3), and (4), we can rewrite the continuity equation for the perfect fluid $\dot{\rho} + 3H(\rho + p) = 0$ as

$$\ddot{\phi} + 3H\dot{\phi} + \frac{V_\phi + L_\phi\dot{\phi}^2}{1 + 2L} = 0, \tag{6}$$

and also Eq. (5) can be rewritten as

$$3H^2 = \frac{\dot{\phi}^2}{2} + V(\phi) + L\dot{\phi}^2. \tag{7}$$

Additionally, combining Eqs. (6) and (7) we have

$$2\dot{H} + 3H^2 + \frac{1}{2}\dot{\phi}^2 - V(\phi) + L\dot{\phi}^2 = 0. \tag{8}$$

In the following, we will assume that the notation $V_\phi = \partial V/\partial\phi$, L_ϕ corresponds to $L_\phi = \partial L/\partial\phi$, $V_{\phi\phi}$ to $V_{\phi\phi} = \partial^2 V/\partial\phi^2$, etc.

Introducing the slow-roll parameters ϵ_1, ϵ_2, ϵ_3, and ϵ_4 defined as [16,17]

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = -\frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = -\frac{LX}{H^2}, \quad \epsilon_4 = -\frac{2L_\phi X}{V_\phi}, \tag{9}$$

and assuming that the slow-roll parameters ϵ_1 $|\epsilon_2|$, $|\epsilon_3|$, $|\epsilon_4| \ll 1$ during the inflationary regime, then the Friedmann equation (5) reduces to [16,17]

$$3H^2 \approx V(\phi), \tag{10}$$

and Eq. (6) results in

$$3H\dot{\phi}(1+2L) \approx -V_{\phi}. \tag{11}$$

Note that combining Eqs. (7) and (8) we have that the first slow-roll parameter ϵ_1 can be rewritten as $\epsilon_1 = X(1+2L)/H^2$.

In general, in order to give a measure of the inflationary expansion in an inflationary model, we can define the number of e-folds N as

$$N = \int_t^{t_e} H dt' = \int_{\phi}^{\phi_e} H \frac{d\phi'}{\dot{\phi}} \simeq \int_{\phi_e}^{\phi} V \left[\frac{1+2L}{V_{\phi'}}\right] d\phi', \tag{12}$$

where the quantities t and t_e correspond to two different values of cosmological time in which the time t_e indicates the end of the inflationary stage and additionally we have assumed that the number of e-folds at the end of inflation is defined as $N(t = t_e) = 0$.

On the other hand, in the context of the cosmological perturbations, the action for the curvature perturbation ζ for the k-essence model can be written as [15,16,61]

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x z^2[\mathcal{G}(\zeta')^2 - \mathcal{F}(\vec{\nabla}\zeta)^2], \tag{13}$$

where the quantity $\mathcal{G} = \mathcal{F} = 1 + 2L$ and the variable z is defined as $z = a\dot{\phi}/H$. Here the prime corresponds to the derivative with respect to the conformal time $\eta = \int dt/a$, and from the Lagrangian density given by Eq. (4), we note that the speed of sound associated with perturbations $c_S^2 = p_X/\rho_X = 1$.

From Eq. (13), the scalar power spectrum of the primordial curvature perturbation is given by [15,16,61]

$$\mathcal{P}_{\mathcal{S}} = \frac{H^4}{4\pi^2\dot{\phi}^2(1+2L)} \simeq \frac{V^3}{12\pi^2 V_{\phi}^2}(1+2L). \tag{14}$$

Since the scalar spectral index n_S is defined in terms of the power spectrum $\mathcal{P}_{\mathcal{S}}$ as $n_S = d\ln\mathcal{P}_{\mathcal{S}}/d\ln k$, we have that the index n_S as a function of the standard slow-roll parameters ϵ and η can be written as [15,16,61]

$$n_S - 1 \simeq \frac{1}{1+2L}\left[2\eta - 6\epsilon + \frac{2L_{\phi}}{1+2L}\sqrt{2\epsilon}\right], \tag{15}$$

where the standard parameters ϵ and η are defined by

$$\epsilon = \frac{1}{2}\left(\frac{V_{\phi}}{V}\right)^2 \quad \text{and} \quad \eta = \frac{V_{\phi\phi}}{V}. \tag{16}$$

Note that in the limit in which the coupling parameter $L \to 0$, the spectral index n_S given by Eq. (15) reduces to the standard spectral index of the GR, where $n_S - 1 \simeq 2\eta - 6\epsilon$. Additionally, we have that in the context of the slow-roll approximation, the relation between the parameters ϵ_1 and ϵ is given by

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \simeq \frac{1}{1+2L}\epsilon. \tag{17}$$

For the case of the tensorial perturbation, the amplitude of the tensor mode in the k-essence model of inflation is not modified, and its expression is equivalent to the standard GR, where the tensor spectrum $\mathcal{P}_{\mathcal{T}}$ is defined as $\mathcal{P}_{\mathcal{T}} \simeq (H^2/2\pi^2)$. Thus, we have that the tensor to scalar ratio r in the framework of the k-essence model can be written as [15,16,61]

$$r = \frac{\mathcal{P}_{\mathcal{T}}}{\mathcal{P}_{\mathcal{S}}} = \frac{16X(1+2L)}{H^2} = 16\epsilon_1. \tag{18}$$

Also, we can obtain that the tensor to scalar ratio can be rewritten in terms of the standard slow-roll parameter ϵ as

$$r = \frac{\mathcal{P}_{\mathcal{T}}}{\mathcal{P}_{\mathcal{S}}} = \frac{16X(1+2L)}{H^2} = \frac{16}{1+2L}\epsilon. \tag{19}$$

Here we have considered Eqs. (17) and (18), respectively.

In the following we will study the reconstruction of the background variables from the unification of the observable parameters together with the swampland conjectures or criteria.

III. RECONSTRUCTING k-ESSENCE MODEL

In this section we will consider the methodology used to reconstruct the background variables, such as the scalar potential $V(\phi)$ and the coupling parameter $L(\phi)$, from the attractor $n_S(N)$ and the slow-roll parameter $\epsilon(N)$, together with the swampland conjectures.

In the context of the reconstruction, we rewrite the slow-roll parameters, the spectral index, and the tensor to scalar ratio as a function of the number of e-foldings N and its derivatives [30]. From these expressions and from considering an attractor point $n_S = n_S(N)$ together with a parameter $\epsilon(N)$, we should obtain the effective potential V and the coupling parameter L in terms of the number N; i.e., $V = V(N)$ and $L = L(N)$. Now, from Eq. (12) we should find analytically the number of e-folds N as a function of the inflaton field ϕ, and then we should reconstruct the scalar potential $V(\phi)$ and the coupling function $L(\phi)$.

In this framework, we start rewriting the derivatives of the potential V, the coupling function L, and the slow-roll parameters in terms of the number N, as

$$V_\phi = \frac{dV}{d\phi} = \frac{V(1+2L)}{V_\phi} V_N,$$

and then we get

$$V_\phi^2 = [V(1+2L)V_N]. \tag{20}$$

In the following, we will assume that $V_N > 0$ and then the quantity $1 + 2L > 0$. Also, in the following we will consider the notation $V_N = dV/dN$, $V_{NN} = d^2V/dN^2$, L_N to $L_N = dL/dN$, etc.

For the quantity $V_{\phi\phi}$, we have

$$V_{\phi\phi} = \frac{1}{2V_N}[(1+2L)[V_N^2 + VV_{NN}] + 2L_N VV_N], \tag{21}$$

and for the coupling parameter L_ϕ, we get

$$L_\phi = \sqrt{\frac{V(1+2L)}{V_N}} L_N. \tag{22}$$

From these relations, we find that the standard slow-roll parameters ϵ and η given by Eq. (16) can be rewritten as

$$\epsilon = \frac{1}{2}(1+2L)\frac{V_N}{V} \tag{23}$$

and

$$\eta = \frac{(1+2L)}{2}\left[\frac{V_N}{V} + \frac{V_{NN}}{V_N}\right] + L_N, \tag{24}$$

respectively.

Also, from Eq. (12) we obtain that the relationship between the e-folding N and the inflaton ϕ becomes

$$\int \left[\frac{V_N}{(1+2L)V}\right]^{1/2} dN = \int d\phi. \tag{25}$$

In relation to the observables, we find that the power spectrum of the primordial curvature perturbation $\mathcal{P}_S$ as a function of the number of e-folds becomes

$$\mathcal{P}_S = \frac{1}{12\pi^2}\frac{V^2}{V_N}, \tag{26}$$

and we note this quantity does not depend on the coupling parameter L under this formalism.

Also, from Eqs. (15), (23), and (24), we obtain that the scalar spectral index n_S can be rewritten in terms of the e-folds N as

$$n_S - 1 = -2\frac{V_N}{V} + \frac{V_{NN}}{V_N} = \left[\ln\left(\frac{V_N}{V^2}\right)\right]_N, \tag{27}$$

and by considering Eq. (18) the tensor to scalar ratio can be rewritten as

$$r = \frac{\mathcal{P}_\mathcal{T}}{\mathcal{P}_S} = 8\frac{V_N}{V}. \tag{28}$$

Here, we note that the expressions for the power spectrum $\mathcal{P}_S(N)$, the scalar spectral index $n_S(N)$, and the tensor to scalar ratio $r(N)$ are the same as the one obtained in the standard GR [30]; these observables depend exclusively on the effective potential and its derivative with respect to N. This suggests that the coupling parameter $L(\phi)$ cannot be rebuilt from the one attractor, such as the scalar spectrum or the scalar spectral index $n_S(N)$, or from the tensor to scalar ratio $r(N)$, as can be seen from Eq. (26), (27), or (28). However, we observe from Eq. (25) that the relation between the number of e-folds and the inflaton field depends on the coupling parameter L as well as V, and then in order to reconstruct the background in our model, we should know the function $L(N)$.

Thus, as in the case of GR the effective potential V in terms of the number N can be obtained from the attractor $n_S(N)$ as

$$V(N) = -1 \Big/ \int \left(\exp\left[\int (n_S - 1)dN\right]\right) dN, \tag{29}$$

or also giving the attractor $r(N)$ with which we have that the effective potential becomes

$$V(N) = \exp\left[\frac{1}{8}\int r dN\right]. \tag{30}$$

Here, we emphasize that the reconstruction of the effective potential in terms of the scalar field ϕ, i.e., $V(\phi)$, cannot be determined without the help of the coupling parameter $L(N)$ [see Eq. (25)]. In this sense, we make the difference with respect to the standard GR, although the expressions for the observable parameters $n_S(N)$ and $r(N)$ are the same to GR, as can be seen in Eqs. (27) and (28).

Thus, a possible solution to rebuild the background variables can be through an ansatz on the slow-roll parameter ϵ in terms of the number N, i.e., $\epsilon(N)$. This ansatz on the slow-roll parameter ϵ would have to take into account the swampland conjecture in which $V_\phi/V \sim \mathcal{O}(1)$, since this parameter $\epsilon \propto (V_\phi/V)^2$. In this respect, the coupling parameter $L(N)$ can be determined considering Eq. (23) in which

$$L(N) = \left[\frac{\epsilon V}{V_N} - \frac{1}{2}\right], \tag{31}$$

where the slow-roll parameter $\epsilon = \epsilon(N)$.

Note that the relation between the number N and the inflaton ϕ given by Eq. (25) can be rewritten as

$$\int \frac{V_N}{V}\left[\frac{1}{2\epsilon}\right]^{1/2} dN = \int d\phi. \tag{32}$$

This suggests that the first swampland criterion on the field range traversed by the scalar field during the slow-roll epoch in an effective field theory $\Delta\phi < \Delta \sim \mathcal{O}(1)$ can be written as

$$\phi - \phi_0 = \Delta\phi = \int_{N_0}^{N} \frac{V_{N'}}{V}\left[\frac{1}{2\epsilon}\right]^{1/2} dN' = \int_{N_0}^{N} \sqrt{\frac{r}{8(1+2L)}}dN' < \Delta \sim \mathcal{O}(1), \tag{33}$$

where $N(\phi = \phi_0) = N_0$ corresponds to the number of e-folds during the slow-roll epoch and its value is such that $0 < N_0 < N$. Recall that during the slow-roll scenario we can consider that the number of e-folds is large $[\mathcal{O}(10) \sim \mathcal{O}(10^2)]$; see Refs. [30,37]. Also, we note that in the special case in which the ratio r and L are constants, then we have the relation $\Delta\phi \propto \Delta N < \Delta$, in which $\Delta N = N - N_0 > 0$.

In the context of the second swampland conjecture we consider that this criterion is related to the slope of the effective potential, and then we can associate the parameter ϵ with this conjecture, such as the parameter $\sqrt{2\epsilon} = (V_\phi/V) > c$, where the constant c from the swampland conjecture is of order one; i.e., $c \sim \mathcal{O}(1)$ (in Planck units) as we mentioned before.

In this sense, from Eqs. (19) and (28) [or Eq. (23)], we have

$$(1 + 2L) = 2\epsilon\left(\frac{V}{V_N}\right), \tag{34}$$

and considering the second swampland conjecture in which $\epsilon > c^2/2$, we obtain a lower bound for the coupling parameter L given by

$$L > c^2\left(\frac{V}{2V_N}\right) - \frac{1}{2}. \tag{35}$$

In this form, we can obtain some constraints on the parameter space from the swampland criteria in order to rebuild the k-essence model.

In the following, under the slow-roll approximation we will study some specific examples in order to reconstruct the scalar potential $V(\phi)$ and coupling parameter $L(\phi)$, from the attractor $n_S(N)$ and $\epsilon(N)$ together with the swampland criteria.

IV. UNIFYING SWAMPLAND CRITERIA AND ATTRACTOR POINT

In this section we will apply the formalism of above in order to rebuild the background variables in our k-essence model. In this context, we shall use the simplest attractor for the scalar spectral index $n_S(N)$ together with some examples for the slow-roll parameter $\epsilon(N)$, in order to find analytically the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$ in terms of the inflaton field ϕ. Thus, following Refs. [30,33] we consider that for large N (slow-roll regime), the simplest attractor point for the scalar spectral index in terms of the number of e-foldings can be written as

$$n_S - 1 = -\frac{2}{N}, \tag{36}$$

in which for $N = 60$ the scalar spectral index is well corroborated from the Planck data. As we mentioned before, here large N corresponds to values of the number of e-foldings $\sim\mathcal{O}(10) \sim \mathcal{O}(10^2)$, during the slow-roll stage [30,33], and the point $N = 0$ is not allowed. In this sense, this parametrization on the observable $n_S(N)$ does not pretend to describe the end of the inflation where the number $N = 0$, but it characterizes the slow-roll regime in which the number of e-folds N is large. In the following, we will consider the value of the number of e-folds $N = 60$, in order to evaluate the observational parameters, and we will only analyze the attractor point $n_S(N)$ given by relation (36) for two different slow-roll parameters $\epsilon(N)$.

In this way, replacing Eq. (36) into Eq. (27) and integrating we have [30]

$$\frac{V_N}{V^2} = \frac{\alpha}{N^2},$$

where the parameter α corresponds to an integration constant and as we have assumed that $V_N > 0$, then the parameter $\alpha > 0$. However, from Eq. (26) we can note that this integration constant α can be fixed considering the scalar power spectrum $\mathcal{P}_S$, when the wavelength of the perturbation crosses the Hubble radius (at $N = 60$) results,

$$\alpha = \frac{N^2}{12\pi^2\mathcal{P}_S}. \tag{37}$$

In this sense, for the specific case in which the number $N = 60$ and the scalar power spectrum $\mathcal{P}_S \simeq 2.2 \times 10^{-9}$, we have that the value of $\alpha \simeq 10^{10}$.

Now, from Eq. (29) the effective potential in terms of the number of e-folds N can be written as

$$V(N) = \frac{N}{\alpha + \beta N}, \tag{38}$$

where the quantity β corresponds to a second integration constant and rigorously this constant can be chosen to be positive, negative, or zero. However, we can obtain an estimate of this constant considering the tensor to scalar ratio (28) in which

$$\beta = \frac{\alpha}{N}\left[\frac{8}{Nr} - 1\right], \quad \text{for } \beta > 0, \quad \text{and}$$
$$\beta = \frac{\alpha}{N}\left[1 - \frac{8}{Nr}\right], \quad \text{for } \beta < 0. \tag{39}$$

Thus, for the case in which $\beta > 0$, using the values $N = 60$, $\alpha = 10^{10}$, and considering that the tensor to scalar ratio $r < 0.07$ from observational data, we obtain a lower limit for the integration constant β given by $\beta > 1.5 \times 10^8$. For the case in which the integration constant β is negative, we find an upper bound given by $\beta < -1.5 \times 10^8$.

In order to rebuild the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$, we should know the parameter $L(N)$ to perform the integral given by Eq. (25); i.e., we should find the relation between the number of e-folds and the inflaton field $[N = N(\phi)]$ for the reconstruction of the functions $V(\phi)$ and $L(\phi)$.

In fact, to find the coupling parameter $L(N)$ and then $N = N(\phi)$, we can consider an ansatz on the slow-roll parameter $\epsilon = \epsilon(N)$ together with the attractor point $n_S(N)$ from the relation given by Eq. (31). The motivation to consider this ansatz on the slow-roll parameter $\epsilon(N)$ is associated with satisfying the second swampland conjecture, since the parameter ϵ is proportional to $(V_\phi/V)^2$.

As a first ansatz on the variable $\epsilon(N)$, we consider the simplest situation in which the slow-roll parameter ϵ is equal to a constant

$$\epsilon(N) = \epsilon_0 = cte. \tag{40}$$

In order to satisfy the second swampland conjecture, we can assume that this constant ϵ_0 satisfies the lower bound $\epsilon_0 > c^2/2 \sim 1/2$.

Immediately, we can see that from Eq. (16) and this ansatz for the slow-roll parameter, the effective potential as a function of the inflaton field corresponds to an exponential potential with which

$$V(\phi) = \exp[A_1\phi + C] = \exp[\tilde{\phi}], \tag{41}$$

where the field $\tilde{\phi}$ is defined as $\tilde{\phi} = A_1\phi + C$, in which the quantity $A_1 = \sqrt{2\epsilon_0}$ and C corresponds to an integration constant. Thus, we note that the attractor given by Eq. (36) is not necessary for the reconstruction of the $V(\phi)$. However, in order to rebuild the coupling parameter $L(\phi)$, we need an ansatz on the attractor point $n_S(N)$.

In this context, from Eq. (31) we obtain the coupling parameter $L(N)$ in terms of the number of e-folds N becomes,

$$L(N) = \frac{\epsilon_0 N(\alpha + \beta N)}{\alpha} - \frac{1}{2}. \tag{42}$$

Here we have used the relation for the potential $V(N)$ given by Eq. (38) obtained from attractor $n_S(N)$ given by Eq. (36).

Now, considering the relation between the number N and the scalar field given by Eq. (32) and assuming the special case in which $\beta > 0$, we have that the number $N(\phi)$ is given by

$$N(\phi) = \frac{\alpha \exp[\tilde{\phi}]}{\alpha - \beta \exp[\tilde{\phi}]}. \tag{43}$$

Here we can note that replacing Eq. (43) into Eq. (38) we recover the effective potential $V(\phi) = e^{\tilde{\phi}}$ given by Eq. (41). Also, we observe that the number of e-folds has a pole at $\tilde{\phi} = \ln(\alpha/\beta)$ when $\beta > 0$, and then we can only consider that the range for the scalar field $\tilde{\phi}$ is given by $\tilde{\phi} < \ln(\alpha/\beta)$, since the number $N > 0$ [see Eq. (43)].

Additionally, we can mention that for the case in which the integration constant $\beta < 0$ the number $N = N(\phi)$ results $N(\phi) = \frac{\alpha \exp[\tilde{\phi}]}{\alpha + (-\beta)\exp[\tilde{\phi}]}$ and this number N does not contain a pole.

Also, for the specific case in which the constant $\beta = 0$, we have that the number of e-foldings evolves exponentially as $N(\phi) = e^{\tilde{\phi}}$, and we note that the number of e-foldings does not depend on the integration constant α. Here, we mention that in this specific case in which the integration constant $\beta = 0$, we obtain that the effective potential is reduced to $V(N) = N/\alpha$ and then combining Eqs. (28) and (36), the consistency relation $r = r(n_S)$ is given by $r = 4(1 - n_S)$. In this sense, for the value $n_S = 0.967$ the tensor to scalar ratio $r \simeq 0.13$ and this value is disapproved from the Planck data. Furthermore for $\beta = 0$, we note that this consistency relation $r = 4(1 - n_S)$ becomes independent of the chosen parameters $\epsilon(N)$, and then this rebuilt model does not work.

Thus for $\beta > 0$, we find that the reconstruction for the coupling function $L(\phi)$ by combining Eqs. (42) and (43) becomes

$$L(\phi) = \frac{\epsilon_0 \alpha^2 e^{\tilde{\phi}}}{(\alpha - \beta e^{\tilde{\phi}})^2} - \frac{1}{2} = \frac{\epsilon_0 \alpha^2 V(\phi)}{[\alpha - \beta V(\phi)]^2} - \frac{1}{2}, \tag{44}$$

with $\alpha > \beta V(\phi)$, since the number of e-foldings N is defined as positive.

For the case in which the integration constant $\beta < 0$, we have

$$L(\phi) = \frac{\epsilon_0 \alpha^2 e^{\tilde{\phi}}}{(\alpha + (-\beta)e^{\tilde{\phi}})^2} - \frac{1}{2} = \frac{\epsilon_0 \alpha^2 V(\phi)}{[\alpha + (-\beta)V(\phi)]^2} - \frac{1}{2}. \quad (45)$$

For the situation in which the quantity $\beta = 0$, we find that the reconstruction for the coupling parameter $L(\phi)$ is given by

$$L(\phi) = \epsilon_0 e^{\tilde{\phi}} - \frac{1}{2}. \quad (46)$$

In relation to the unification between the observational parameters and the SC, we consider the first swampland conjecture in which $\Delta\phi < \Delta \sim \mathcal{O}(1)$, for our first example. In this sense, we can rewrite Eq. (43) for the case $\beta > 0$ in which the integration constant C is defined as $C = \ln[N_0/(\alpha + \beta N_0)] - \sqrt{2\epsilon_0}\phi_0$, wherewith the range of the inflaton field during the slow-roll regime results as

$$\Delta\phi = \phi - \phi_0 = \frac{1}{\sqrt{2\epsilon_0}} \ln\left[\frac{N(\alpha + \beta N_0)}{N_0(\alpha + \beta N)}\right] < \Delta \sim 1, \quad (47)$$

as before $N(\phi = \phi_0) = N_0$ and its value is $0 < N_0 < N$ [see Eq. (33)]. Thus, from this conjecture and for the case $\beta > 0$, we can find a lower bound for the value ϵ_0 given by

$$\epsilon_0 > \frac{1}{2} \ln^2\left[\frac{N(\alpha + \beta[N - \Delta N])}{[N - \Delta N](\alpha + \beta N)}\right], \quad (48)$$

where we recall that the quantity ΔN is defined as $\Delta N = N - N_0 > 0$.

In particular for the case in which $\Delta N = 50$ (or $N_0 = 10$) and $\alpha = 10^{10}$, we find that the integration constant β becomes $\beta < 3.2 \times 10^8$ according to Eq. (48) together with the constraint $\epsilon_0 > c^2/2 \sim 0.5$ (second swampland conjecture). However, from the tensor to scalar ratio in which $r < 0.07$ at $N = 60$ we have found that $\beta > 1.5 \times 10^8$ [see relation given by Eq. (39)]. In this form, we obtain that the range for the parameter β unifying the observational data together with the swampland conjectures becomes

$$1.5 \times 10^8 < \beta < 3.2 \times 10^8. \quad (49)$$

In this sense, we observe that the range for the parameter β is very narrow if we want to satisfy the observational and theoretical conditions. Similarly, for the situation in which the integration constant β is negative, we find that the range for this parameter is given by

$$-1.5 \times 10^8 > \beta > -3.2 \times 10^8. \quad (50)$$

As a second ansatz on the variable $\epsilon(N)$, we can assume that for large N, the slow-roll parameter $\epsilon(N)$ can be considered as [39,40,62]

$$\epsilon(N) = \frac{\gamma}{N}, \quad (51)$$

where γ corresponds to a constant. We note that from Ref. [40], we can recognize that for large N, the constant γ is equal to $\gamma = 1/4$ for this parametrization on $\epsilon(N)$. However, applying the second swampland conjecture, we have that the slow-roll parameter $\epsilon > c^2/2$, and then the constant γ can satisfy the lower bound given by $\gamma > Nc^2/2$. Thus, for the case in which the number of e-folds $N = 60$ and $c \sim 1$, we get that the lower limit for γ results in $\gamma > 30$.

From this ansatz on $\epsilon(N)$, we find that the coupling parameter L as a function of the number of e-folds results in

$$L(N) = \frac{\gamma(\alpha + \beta N)}{\alpha} - \frac{1}{2}. \quad (52)$$

Now, from Eq. (32) and assuming the case in which the constant β is positive, we get that the number of e-foldings in terms of the scalar field becomes

$$N(\phi) = \frac{\alpha}{\beta} \tan^2\left[\frac{1}{2}\sqrt{\alpha\beta}(A_2\phi + C)\right], \quad (53)$$

where A_2 is defined as $A_2 = \sqrt{2\gamma}/\alpha$ and C corresponds to an integration constant. Here we note that in order to evade the singularity on the number $N = N(\phi)$ given by Eq. (53), we have that $0 < \sqrt{\alpha\beta}(A_2\phi + C) \lesssim \pi$. For the situation in which the integration constant $\beta < 0$, we obtain that the relation between N and ϕ can be written as

$$N(\phi) = \frac{\alpha}{(-\beta)} \tanh^2\left[\frac{1}{2}\sqrt{\alpha(-\beta)}(A_2\phi + C)\right]. \quad (54)$$

In particular for the special case in which $\beta = 0$, we find that the relation $N = N(\phi)$ is given by

$$N(\phi) = \frac{1}{4}[\sqrt{2\gamma}\phi + C]^2. \quad (55)$$

Again, we note that this relation $N = N(\phi)$ for the case $\beta = 0$ does not depend on the integration constant α.

Thus, for the case $\beta > 0$, we obtain that the reconstruction for the effective potential $V(\phi)$ combining Eqs. (38) and (53) results in

$$V(\phi) = \frac{1}{\beta} \sin^2\left[\frac{1}{2}\sqrt{\alpha\beta}(A_2\phi + C)\right] = \frac{1}{2\beta}[1 - \cos(\sqrt{\alpha\beta}[A_2\phi + C])], \quad (56)$$

and this effective potential corresponds to natural inflation [63,64] (see also [65]). In this framework, the scalar field is associated with a pseudo–Nambu-Goldstone boson (pNGb) with a pNGb potential given by (56). Here the constants

$\beta^{-1/4}$ and $1/\sqrt{\alpha\beta}A_2$ can be associated with the mass scales from the particle physics models; for more details see [63,64].

For the case in which the integration constant β is negative, we have that the reconstruction of the effective potential as a function of the scalar field becomes

$$V(\phi) = \frac{1}{(-\beta)}\sinh^2\left[\frac{1}{2}\sqrt{\alpha(-\beta)}(A_2\phi + C)\right], \quad (57)$$

and this potential corresponds to hyperbolic inflation [66]. Also, we mention that this potential has an interesting application to describe the dark energy and then the later time acceleration of the present universe [67]. Additionally, we have that this hyperbolic potential has a behavior of an exponential or power law potential, depending on the specific limits taken for the scalar field. In this way, in the case in which $|\sqrt{\alpha(-\beta)}(A_2\phi + C)/2| \gg 1$, we have an exponential potential $V(\phi) \propto e^{-\sqrt{\alpha(-\beta)}A_2\phi}$, and in the opposite limit $|\sqrt{\alpha(-\beta)}(A_2\phi + C)/2| \ll 1$, we get a quadratic potential $V(\phi) \propto \phi^2$.

In the special case in which the integration constant $\beta = 0$, we obtain that the reconstruction for the effective potential is given by

$$V(\phi) = \frac{1}{4\alpha}[\sqrt{2\gamma}\phi + C]^2, \quad (58)$$

and it corresponds to a quadratic potential (chaotic potential), i.e., $V(\phi) \propto \phi^2$, and it coincides with the potential given by Eq. (57) in the limit $|\sqrt{\alpha(-\beta)}(A_2\phi + C)/2| \ll 1$.

For the reconstruction of the coupling parameter $L(\phi)$, we can combine Eqs. (52) and (53) for the case in which $\beta > 0$ obtaining

$$\begin{aligned} L(\phi) &= \gamma\left(1 + \tan^2\left[\frac{1}{2}\sqrt{\alpha\beta}(A_2\phi + C)\right]\right) - \frac{1}{2} \\ &= \frac{\gamma}{(1 - \beta V(\phi))} - \frac{1}{2}, \end{aligned} \quad (59)$$

and in order to have $1 + 2L > 0$, then it is necessary that during the inflationary epoch $\beta V < 1$.

In the situation in which the integration constant β is negative we have

$$\begin{aligned} L(\phi) &= \gamma\left(1 + \tanh^2\left[\frac{1}{2}\sqrt{\alpha(-\beta)}(A_2\phi + C)\right]\right) - \frac{1}{2} \\ &= \frac{\gamma}{(1 + (-\beta)V(\phi))} - \frac{1}{2}. \end{aligned} \quad (60)$$

Note that in this case we have the possibility to consider the regime in which $1 < (-\beta)V(\phi)$, for the function $L(\phi)$ that increases as $L(\phi) \propto 1/V(\phi)$, when the potential decreases during the slow-roll scenario.

Now, for the situation in which the integration constant β is equal to zero, we obtain that the reconstruction of the coupling parameter $L(\phi) = \gamma - 1/2 = \text{const}$ and positive, since $\gamma > 30$ in order to satisfy the SC. Again, we mention that in this specific case in which the integration constant $\beta = 0$, the tensor to scalar ratio r is disapproved from the data and then this reconstructed model does not work, since the consistency relation $r = r(n_S)$ becomes $r = 4(1 - n_S)$.

As before, in order to consider the first conjecture in the reconstruction of k-essence inflation, we rewrite Eq. (53), and then the range $\Delta\phi$ in terms of the variation $\Delta N = N - N_0$ can be written as

$$\Delta\phi = \frac{2}{\sqrt{\alpha\beta}A_2}\arctan\left[\sqrt{\frac{\beta\Delta N}{\alpha}}\right] < \Delta \sim 1. \quad (61)$$

In this form, from Eq. (61) we find that a lower bound for the parameter γ is given by

$$\gamma > \left(\frac{2\alpha}{\beta}\right)\arctan^2\left[\sqrt{\frac{\beta\Delta N}{\alpha}}\right]. \quad (62)$$

From the lower bound for γ given by Eq. (62) and under the condition $\gamma > Nc^2/2 \simeq 30$ imposed by the second SC, we find numerically that the upper bound for the integration constant β is given by $\beta < 8.3 \times 10^8$. Here we have used the values $\alpha = 10^{10}$ and $\Delta N = 50$.

In this way, we obtain that the range for the integration constant β under the unification of the upper bound for the ratio $r < 0.07$ and the swampland criteria is

$$1.5 \times 10^8 < \beta < 8.3 \times 10^8. \quad (63)$$

Thus, to satisfy the observational data together with the swampland criteria, we note that this range for the parameter β is not that narrow as the previous case in which $\epsilon(N) = \epsilon_0 = \text{const}$. Also, we note that this result suggests that the mass scale $\beta^{-1/4} \sim \mathcal{O}(10^{-2})$ (in units of Planck mass) is similar to that obtained in the framework of GR, where $\beta^{-1/4}\sim$ grand unified theory scale [$\sim\mathcal{O}(10^{-4})$] [64].

Analogously, for the situation in which the parameter β is negative, we find that the range for β is given by $-10^8 > \beta > -10^9$, assuming the values of $\alpha = 10^{10}$ and the variation $\Delta N = N - N_0 = 50$.

V. CONCLUSIONS

In this article we have investigated the reconstruction of the background variables in the k-essence inflationary model from the Lagrangian density given by Eq. (4). To rebuild this scenario, we have unified the swampland criteria together with the attractor point associated with the scalar spectral index $n_S(N)$ and the slow-roll parameter $\epsilon(N)$, in which N denotes the number of e-folds. From a coupling of the form $L(\phi)X$ in the k-essence model, we have found a general treatment of reconstruction in the framework of slow-roll approximation. In this sense, we have obtained integrable results for the scalar potential $V(N)$ and the coupling parameter $L(N)$, in terms of the scalar spectral index $n_S(N)$ and the slow-roll parameter $\epsilon(N)$. Interestingly, we have found that the effective potential as a function of the number N, i.e., $V(N)$, coincides with the expression obtained in the framework of GR [see Eq. (29)] and it does not depend on the coupling parameter L. However, in order to rebuild the effective potential in terms of the scalar field $V(\phi)$, we need to consider an expression for the coupling parameter L as a function of the number of e-foldings N, i.e., $L(N)$. In this respect, we make the difference with respect to the reconstruction of the effective potential $V(\phi)$ in the framework of the GR. Additionally, in this general analysis we have obtained from the slow-roll parameter $\epsilon(N)$ and the attractor point $n_S(N)$ an expression for the coupling parameter $L(N)$ [see Eq. (31)].

In order to apply this reconstruction methodology during the slow-roll regime, we have considered the simplest example for the scalar spectral index $n_S(N)$ given by $n_S = 1\text{–}2/N$, together with two specific ansatze for the slow-roll parameter $\epsilon(N)$ under the assumption of large N. In this sense, we have utilized the special cases in which the slow-roll parameter $\epsilon(N)$ is a constant and when $\epsilon(N) \propto N^{-1}$, in order to rebuild the effective potential $V(\phi)$ and the coupling parameter $L(\phi)$.

In the case in which the slow-roll parameter is a constant, we have found that the effective potential as a function of the scalar field corresponds to an exponential potential, as can be seen of Eq. (41). In this situation, we have noted that the attractor $n_S(N)$ was not necessary to rebuild the exponential potential $V(\phi)$, since from the definition of the slow-roll parameter it is possible to obtain the effective potential. In relation to the coupling parameter $L(\phi)$ when $\epsilon(N) = \text{const}$, we have found that the reconstruction of this parameter is given by Eq. (44) for the case in which the integration constant β is positive, and for the case in which $\beta < 0$, the coupling parameter $L(\phi)$ is given by Eq. (45). Also, we have obtained that in order to satisfy that the number of e-folds $N > 0$, we have imposed the condition $\alpha > \beta V$.

To unify these results with the swampland criteria, we have found a lower bound for the ansatz $\epsilon(N) = \epsilon_0 = \text{const}$ given by Eq. (48), together with the fact that the second swampland conjecture $\epsilon_0 > c^2/2 \sim 1/2$. In particular for the case in which $\Delta N = 50$ and from the observational data in which $\alpha = 10^{10}$, we have obtained a range for the integration constant β when $\beta > 0$ is given by Eq. (49) and for $\beta < 0$, we have obtained the range described by Eq. (50). Here from this unification, we have found that the range for the parameter β is very narrow, in order to satisfy the observational data and swampland criteria. In particular for the specific case in which the integration constant $\beta = 0$, we have obtained that the effective potential $V(N) \propto N$ and by combining Eqs. (28) and (36), the consistency relation $r = r(n_S)$ becomes $r = 4(1 - n_S)$. Here we have found that the reconstruction of the model does not work when the constant $\beta = 0$.

Another ansatz on the slow-roll parameter $\epsilon(N)$ that we have studied for large N, is given by Eq. (51), in which the parameter $\epsilon(N) \propto 1/N$. Interestingly in this situation, we have found that the reconstruction for the effective potential $V(\phi)$ for $\beta > 0$ coincides with the natural inflation. In this sense, we have obtained a pNGb potential given by Eq. (56) in the reconstruction of this k-essence model. Here we have recognized the constants $\beta^{-1/4}$ and $\sqrt{\alpha/(\beta\gamma)}$ as parameters associated with mass scales from the particle physics models. For the case in which the integration constant $\beta < 0$, we have found the effective potential given by Eq. (57), and this potential corresponds to the hyperbolic inflation. Here the hyperbolic potential has a behavior of an exponential or power law potential (quadratic potential) depending on the limits applied to the scalar field. Additionally, we have obtained that for the specific case in which the integration constant $\beta = 0$, the effective potential corresponds to a chaotic potential in which $V(\phi) \propto \phi^2$. Also, we have found that the reconstruction for the coupling parameter $L(\phi)$ in the case in which the integration constant $\beta > 0$ is given by Eq. (59) and for $\beta < 0$ by Eq. (60).

As well, from the first swampland conjecture we have obtained a lower limit for the parameter γ given by Eq. (62) and by considering the second conjecture we have found the limit $\gamma > Nc^2/2$. In particular for the situation in which $\Delta N = 50$, we have obtained numerically an upper bound for the constant β given by $\beta < 8.3 \times 10^8$ from both criteria. On the other hand, considering the observational data for the tensor to scalar ratio in which $r < 0.07$, we have obtained a lower limit, and then we have found that the range for the integration constant β given by $1.5 \times 10^8 < \beta < 8.3 \times 10^8$. As before, we have found that the range for the parameter β is very narrow from the unification of the observational data and swampland criteria. In this respect, these narrow ranges on the parameter β found from this unification suggest that the amplitude of primordial gravitational waves must have a lower limit different from zero and similar to its upper bound. In this sense, the results obtained from the unification of the observational data and swampland criteria predict that the amplitude of primordial gravitational waves can be detected in the future.

Thus, we have shown that it is possible to unify the theoretical foundations from the SC and the observational parameters corroborated by observations in the reconstruction of the early universe (inflation epoch). In relation to this point, we mention that the reconstruction of inflation only from the observational attractors does not guarantee that the SC are satisfied (see, e.g., [37,38]). In particular, it is possible for an adaptation to the first swampland criterion to be associated with the range of the inflaton $\Delta\phi$; however, this methodology of reconstruction only from the observational counterpart does not ensure that the second conjecture is satisfied.

Finally in this article, we have not addressed the reconstruction to another attractor point $n_S(N)$ or slow-roll parameter $\epsilon(N)$. Also, we have not included the TCC in our analysis. We hope to return to these points in the near future.

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