

Revisiting the Ramond sector of the $\mathcal{N} = 1$ superconformal minimal models

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Key to the exact solubility of the unitary minimal models in two-dimensional conformal field theory is the organization of their Hilbert space into Verma modules, whereby all eigenstates of the Hamiltonian are obtained by the repeated action of Virasoro lowering operators onto a finite set of highest-weight states. The usual representation-theoretic approach to removing from all modules zero-norm descendant states generated in such a way is based on the assumption that those states form a nested sequence of Verma submodules built upon singular vectors, i.e., descendant highest-weight states. We show that this fundamental assumption breaks down for the Ramond-sector Verma module with highest weight $c/24$ in the even series of $\mathcal{N} = 1$ superconformal minimal models with central charge c . To resolve this impasse, we conjecture, and prove at low orders, the existence of a nested sequence of linear-dependence relations that enables us to compute the character of the irreducible $c/24$ module. Based on this character formula, we argue that imposing modular invariance of the torus partition function requires the introduction of a non-null odd-parity Ramond-sector ground state. This symmetrization of the ground-state manifold allows us to uncover a set of conformally invariant boundary conditions not previously discussed and absent in the odd series of superconformal minimal models, and to derive for the first time a complete set of fusion rules for the even series of those models.

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I. INTRODUCTION

Two-dimensional (2D) conformal field theories (CFTs) [1] play a central role in physics. The presence of additional symmetries besides conformal symmetry can lead to CFTs with a rich mathematical structure. For instance, a minimal generalization of the conformal Virasoro algebra consistent with supersymmetry is the $\mathcal{N} = 1$ super-Virasoro algebra [2,3]. The $\mathcal{N} = 1$ superconformal minimal models (SMMs) [4–7] are an infinite, discrete series of superconformal field theories (SCFTs) corresponding to unitary, irreducible representations of this algebra with central charge $c = \frac{3}{2} [1 - \frac{8}{m(m+2)}]$, $m = 2, 3, 4, \dots$. As opposed to early proposals for the experimental realization of SMMs at classical multicritical points [5], recent advances in condensed matter physics suggest several SMMs may promisingly be realized as quantum critical points or even stable quantum critical phases in a diverse array of platforms,

ranging from anyonic spin chains [8,9] to boundaries of topological superconductors [10,11] and lattice models of interacting Majorana fermions [12–14].

Entanglement properties of critical (1+1)D quantum many-body systems, to which entanglement-based numerical methods such as the density-matrix renormalization group (DMRG) [15] give direct access, can probe various universal quantities of the underlying CFT. In Ref. [10], the central charge $c = 7/10$ of the tricritical Ising universality class was extracted from a DMRG calculation of the ground-state entanglement entropy [16]. Besides the central charge, a fuller characterization of the CFT may be achieved by a study of the low-lying entanglement spectrum, which was argued to match the set of scaling dimensions in boundary CFT [17–19] for a particular choice of boundary conditions (BCs) on the entangling surface. This choice of entanglement BCs in turn singles out one among a set of allowed conformally invariant BCs [20], i.e., a pair of Cardy states [21]. By determining numerically how different entanglement BCs affect degeneracies in the low-lying entanglement spectrum, one may directly probe the entire set of fusion rules of the bulk CFT [21,22]. Motivated by ongoing studies of entanglement properties of quantum-critical systems with emergent supersymmetry, we revisit a basic structure in the representation theory of $\mathcal{N} = 1$ SMMs which is heuristically summarized as follows.

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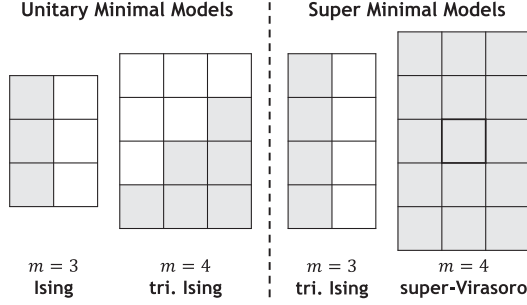


FIG. 1. Examples illustrating four types of Kac tables that characterize unitary minimal models and their $\mathcal{N} = 1$ supersymmetric generalization. The first three types have an even number of entries, half of which are redundant (white boxes), while the Kac table for even- m SMMs has an odd number of entries and no redundancy.

II. PICTORIAL SUMMARY OF THE PAPER

The highest weights (HWs) in a 2D CFT, a set of universal numbers that determines the operator scaling dimensions, form a rectangular Kac table (Fig. 1). For nonsupersymmetric minimal models and the odd- m series of SMMs, the Kac table contains an even number of entries, half of which are redundant. By contrast, the Kac table for even- m SMMs contains an odd number of entries (due to a central box), all of which are intertwined in the corresponding representation theory. This observation indicates that while odd- m SMMs can be understood similarly as unitary minimal models, the even- m series of SMMs calls for a distinct treatment.

The left-hand side of Fig. 2 depicts schematically the commonly held perspective on the Hilbert-space structure for even- m SMMs: the central entry highlighted in the Kac table gives rise to a unique (Ramond) ground state with even fermion parity. As explained below, a representation theory based on this assumption conflicts with a central requirement in CFT, modular invariance [23]. The main result of our paper is a proposed resolution to this problem: the addition of a second Ramond ground state with odd fermion parity (Fig. 2), which restores consistency between representation theory and modular invariance.

More precisely, we show that the standard assumption of representation theory—the nested Verma-submodule

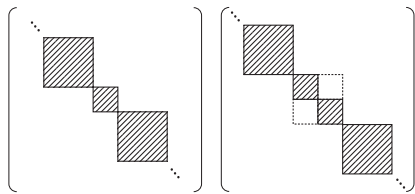


FIG. 2. Schematic Hilbert-space structure for even- m SMMs. The commonly held viewpoint (left) leads to inconsistency with modular invariance, which can be resolved by introducing an additional, odd-parity Ramond ground state (right).

structure of null states—fails for the $c/24$ HW, i.e., the central box in the Kac table of even- m SMMs, demanding an alternate approach to the construction of an irreducible module and the computation of its character. We propose as resolution an infinite hierarchy of linear-dependence relations among null states in an auxiliary module whose character is identical to, but easier to compute than, that of the original module. Based on this newly derived character, we argue that modular invariance of the torus partition function [23] requires the Ramond-sector ground-state manifold to contain states of both fermion parities. We subsequently construct Ishibashi states [24] and solve the Cardy equations [21], finding an extra Cardy state, Eq. (22), beyond those found in the literature [24–26]. Inverting the Cardy equations, we correspondingly find two extra fusion rules, Eqs. (23)–(24) and (29), absent from previous discussions.

III. REVISITING REPRESENTATION THEORY FOR THE $c/24$ MODULE

The $\mathcal{N} = 1$ super-Virasoro algebra is given by $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$, $\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}$, $[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$, where L_m (G_r) are the bosonic (fermionic) Laurent modes of the energy-momentum tensor (supercurrent); $m \in \mathbb{Z}$ while $r \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz (NS) sector and $r \in \mathbb{Z}$ in the Ramond (R) sector. We focus on the holomorphic part of the algebra; identical results are obtained for the antiholomorphic part. The central result in representation theory is the Kac determinant [5], an expression for the determinant of the Gram matrix in the degenerate subspace at level ℓ in a Verma module V^+ (V^-) with even (odd) fermion parity and HW h , $\det V_0^+ = 1$, $\det V_0^- = h - \frac{c}{24}$, $\det V_{\ell>0}^\pm = (h - \frac{c}{24})^{\frac{P_R(\ell)}{2}} \prod_{r,s \geq 1} (h - h_{r,s}(c))^{P_R(\ell - \frac{rs}{2})}$, where $h_{r,s}(c)$ is a prescribed function of c and integers r, s which determines the finite set of allowed HWs in SMMs. We focus on the R sector, where $r - s$ is odd and the maximal level degeneracy $P_R(\ell)$ is obtained from the generating function $\sum_{\ell=0}^\infty q^\ell P_R(\ell) = \prod_{n=1}^\infty (1 + q^n) / (1 - q^n) =: 1/\varphi_R(q)$.

Unitarity and irreducibility require that negative-norm states should be absent from the Verma module and null (zero-norm) states should be systematically removed. A large class of linearly independent null states consists of states obtained from the repeated action of lowering operators L_{-m}, G_{-r} ($m > 0, r \geq 0$) onto a singular vector $|\chi\rangle$, i.e., a descendant state satisfying the HW condition $L_n|\chi\rangle = 0$ for $n > 0$ and $G_r|\chi\rangle = 0$ for $r > 0$. Such states form a null Verma submodule [27]. A singular vector is itself necessarily a null state, but the converse is not generally true. For all Virasoro and most super-Virasoro HWs, the first null state dictated by the Kac determinant (at level $\ell = rs/2$) happens to also be a singular vector; the same Kac determinant can thus be used again to find

another singular vector in the resulting null Verma submodule, and such a procedure repeats, leading to a nested structure of null Verma submodules which comprises all null states [28]. Based on this nested structure, Kiritsis derived character formulas for NS and R irreducible Verma modules with HWs *not* equal to $c/24$ [29]. The $c/24$ HW only appears in the R sector of the even- m series of SMMs, and we focus on those for the rest of the paper.

The crucial concurrence between first null state and singular vector is however not warranted by the Kac determinant and breaks down for the module with HW $h_\star := h_{\frac{m}{2}, \frac{m}{2}+1}(c) = c/24$ [30–32]. As for other R modules, the HW states come in degenerate pairs $|h_\star^\pm\rangle$ of opposite fermion parity where $|h_\star^-\rangle = G_0|h_\star^+\rangle$, but contrary to other R modules the odd-parity state $|h_\star^-\rangle$ is annihilated by G_0 since $G_0^2 = L_0 - \frac{c}{24}$ (and is thus also null). As a result, the linear system corresponding to the singular-vector condition is in general overdetermined. For example, when $m = 2$, one has $\langle h_\star^+ | L_1 L_{-1} | h_\star^+ \rangle = 0$ but $G_1 L_{-1} | h_\star^+ \rangle = \frac{3}{2} | h_\star^+ \rangle \neq 0$, i.e., the first null state $L_{-1} | h_\star^+ \rangle$ built from the even-parity HW state $| h_\star^+ \rangle$ is not singular. Analogously for $m = 4$, one can prove that no linear combinations of level-3 descendants of $| h_\star^+ \rangle$ are singular.

IV. CHARACTER OF THE $c/24$ MODULE

To resolve this issue, we first propose the study of an auxiliary module \tilde{V}_\star , defined by contrast with the original module $V_\star := V(c, h_\star)$ as

$$V_\star : G_0 | h_\star^+ \rangle = |\text{null}\rangle \rightarrow \tilde{V}_\star : G_0 | \tilde{h}_\star^+ \rangle = 0. \quad (1)$$

The entire class of null states built solely on $| h_\star^- \rangle$ is effectively subsumed into a single representative zero vector so that $| \tilde{h}_\star^- \rangle$ does not appear and the Verma module is halved, which greatly simplifies the construction of irreducible HW representations. The Kac determinant for \tilde{V}_\star remains formally unchanged: $\det \tilde{V}_{\star,0}^+ = 1$, $\det \tilde{V}_{\star,0}^- = 0$, $\det \tilde{V}_{\star, \ell > 0}^\pm = \prod_{r,s \geq 1} (h - h_{r,s}(c))^{P_R(\ell - \frac{rs}{2})}$. Thus irreducible modules constructed from V_\star and \tilde{V}_\star , albeit fundamentally different, necessarily possess the same character.

We first investigate the structure of the reducible auxiliary modules \tilde{V}_\star^\pm built entirely upon the HW state $| \tilde{h}_\star^+ \rangle$. According to the Friedan-Qiu-Shenker (FQS) prescription [5], a set of linearly independent vectors spanning the level- ℓ degenerate subspace is given by $G_{-m_1} G_{-m_2} \cdots L_{-n_1} L_{-n_2} \cdots | \tilde{h}_\star^+ \rangle$, where $0 < m_1 < m_2 < \cdots$, $0 < n_1 \leq n_2 \leq \cdots$, and $\sum_i m_i + \sum_i n_i = \ell$. Imposing $G_0 | \tilde{h}_\star^+ \rangle = 0$ leads to two major differences in the structure of V_\star^\pm and \tilde{V}_\star^\pm : (i) In contrast to V_\star^\pm , singular vectors $| \tilde{\chi}_\star \rangle$ are restored in \tilde{V}_\star^\pm and first appear at levels dictated by the Kac determinant. However, generically $| \tilde{\chi}_\star \rangle$ is not annihilated by G_0 although $| \tilde{h}_\star^+ \rangle$ is. (ii) At a given level and for a fixed fermion parity, the set of null descendant states built upon

the two degenerate singular vectors $| \tilde{\chi}_\star \rangle$ and $G_0 | \tilde{\chi}_\star \rangle$ by the FQS prescription is in general linearly dependent; thus null states in \tilde{V}_\star^\pm do not form Verma submodules. As will be argued, the linear-dependence relations among null states evoked in (ii) are organized into an infinite hierarchy that plays a role analogous to that of the nested embedding of null Verma submodules for $h \neq c/24$ HWs and allows us to compute the irreducible character of the original V_\star^\pm modules.

The zeroth echelon in this hierarchy corresponds to the first singular vector $| \tilde{\chi}_\star \rangle$ of a given fermion parity, which appears at level $\ell_0 := \frac{m}{4} (\frac{m}{2} + 1)$ according to the Kac determinant and can be written as a linear combination of level- ℓ_0 descendants of $| \tilde{h}_\star^+ \rangle$,

$$| \tilde{\chi}_\star \rangle = \hat{\mathcal{L}}_0 [f_1^{(0)}, \dots, f_{\frac{1}{2}P_R(\Delta\ell_0)}^{(0)}] | \tilde{h}_\star^+ \rangle, \quad (2)$$

where $f_1^{(0)}, \dots, f_{\frac{1}{2}P_R(\Delta\ell_0)}^{(0)}$ are the coefficients of this linear combination, and we define the k th-echelon ($k \geq 0$) generalized lowering operator,

$$\begin{aligned} \hat{\mathcal{L}}_k [f_1^{(k)}, \dots, f_{\frac{1}{2}P_R(\Delta\ell_k)}^{(k)}, g_1^{(k)}, \dots, g_{\frac{1}{2}P_R(\Delta\ell_k)}^{(k)}] \\ := f_1^{(k)} L_{-1}^{\Delta\ell_k} + \cdots + f_{\frac{1}{2}P_R(\Delta\ell_k)}^{(k)} G_{-1} G_{1-\Delta\ell_k} \\ + g_1^{(k)} G_{-1} L_{-1}^{\Delta\ell_k-1} G_0 + \cdots + g_{\frac{1}{2}P_R(\Delta\ell_k)}^{(k)} G_{-\Delta\ell_k} G_0, \end{aligned} \quad (3)$$

with $\Delta\ell_k := \ell_k - \ell_{k-1}$ and $\ell_k := (1+k)^2 \ell_0$. Equation (3) is the most general fermion-parity-preserving operator that raises the level of a state by $\Delta\ell_k$, and ℓ_k for $k \geq 1$ is the level at which higher-level singular vectors appear according to the Kac determinant. The first $\frac{1}{2}P_R(\Delta\ell_k)$ terms in Eq. (3) involve bosonic generators, while the remaining terms involve fermionic generators times the zero-mode operator G_0 . From Eq. (1), the latter vanish when acting on $| \tilde{h}_\star^+ \rangle$ and are thus excluded from Eq. (2).

The first echelon in the hierarchy corresponds to the linear dependence of null states built upon $| \tilde{\chi}_\star \rangle$ and $G_0 | \tilde{\chi}_\star \rangle$, which first appears at level ℓ_1 and can be expressed as

$$\hat{\mathcal{L}}_1 | \tilde{\chi}_\star \rangle = 0. \quad (4)$$

The set of $f_i^{(1)}$ and $g_i^{(1)}$ coefficients implicit in Eq. (4) is uniquely determined by $f_i^{(0)}$ up to an overall multiplicative constant. The second echelon expresses the fact that linear-dependence relations generated from (4) become themselves linearly dependent at higher levels. In general, the k th-echelon consists of linear-dependence relations among the linear-dependence relations of the $(k-1)$ th-echelon and can be summarized compactly as

$$\hat{\mathcal{L}}_k \hat{\mathcal{L}}_{k-1} | \ell_{k-2} \rangle = 0, \quad k \geq 2, \quad (5)$$

where the coefficients $f_i^{(k)}$, $g_i^{(k)}$, again omitted for simplicity, are uniquely determined from $f_i^{(k-1)}$, $g_i^{(k-1)}$ up to an overall multiplicative constant, and $|\ell_{k-2}\rangle$ denotes an arbitrary state at level ℓ_{k-2} . In [33], we substantiate this hierarchy conjecture with explicit calculations for $m = 2$; a calculation for $m = 4$ with a computer algebra system yields analogous results.

The hierarchy of linear-dependence relations embodied in Eqs. (4) and (5) can now be used to calculate the character of the original irreducible $c/24$ module $\mathcal{M}_{\frac{m}{2}, \frac{m}{2}+1}^\pm$: the number of linearly independent null states to be removed from \tilde{V}_\star^\pm at each level is reduced by one whenever a linear-dependence relation occurs. Implementing this procedure recursively, one obtains [33]

$$\begin{aligned} \chi_{\text{R}}(\mathcal{M}_{\frac{m}{2}, \frac{m}{2}+1}^\pm) &= \frac{1}{2\varphi_{\text{R}}(q)} \sum_{n \in \mathbb{Z}} (q^{\frac{1}{2}m(m+2)n^2} \\ &\quad - q^{\frac{1}{2}m(m+2)(n+\frac{1}{2})^2}) \pm \frac{1}{2}. \end{aligned} \quad (6)$$

V. MODULAR-INVARIANT TORUS PARTITION FUNCTION

Since the full SCFT is nonlocal, we restrict ourselves to the Gliozzi-Scherk-Olive (GSO)-projected spin model [5] with even fermion parity. On a torus, the required invariance of the partition function under modular transformations constrains which HW representations appear in the theory and how often [23]. A modular-invariant contribution to the partition function for even m was found to be [34–36]

$$\begin{aligned} Z^+ &= \sum_{(r,s) \in \Delta_{\text{NS}}} (|\chi_{\text{NS}}(\mathcal{M}_{r,s})|^2 + |\chi_{\widetilde{\text{NS}}}(\mathcal{M}_{r,s})|^2) \\ &\quad + |\chi_{\text{R}}^{c/24}(q)|^2 + 2 \sum_{(r,s) \in \Delta_{\text{R}}} |\chi_{\text{R}}(\mathcal{M}_{r,s}^+)|^2, \end{aligned} \quad (7)$$

where Δ_{NS} (Δ_{R}) denotes the set of independent HWs in the NS (R) sector, $\chi_{\widetilde{\text{NS}}}$ corresponds to the NS character twisted by the insertion of the fermion-parity operator, and the primed sum over Δ_{R} means that the $c/24$ HW ($r = \frac{m}{2}$, $s = \frac{m}{2} + 1$) is excluded. The asymmetry in Eq. (7) between the latter and other R HWs comes from the fact that the $c/24$ HW occupies the self-symmetric point of the Kac table [34–36]. The function $\chi_{\text{R}}^{c/24}(q) := \varphi_{\text{R}}(q)^{-1} \sum_{n \in \mathbb{Z}} (q^{\frac{1}{2}m(m+2)n^2} - q^{\frac{1}{2}m(m+2)(n+\frac{1}{2})^2})$, whose form is highly constrained by modular invariance, should be the character of the $c/24$ module but disagrees with the result (6) of an explicit evaluation in representation theory. To resolve this paradox, we introduce an additional pair of R HW ground states $|w_\star^\pm\rangle$, obeying

$$\begin{aligned} L_0|w_\star^\pm\rangle &= \frac{c}{24}|w_\star^\pm\rangle, & \langle w_\star^-|w_\star^- \rangle &= 1, \\ |w_\star^+\rangle &= G_0|w_\star^-\rangle = |\text{null}\rangle, \end{aligned} \quad (8)$$

and giving rise to a second irreducible $c/24$ module built on the non-null HW state $|w_\star^-\rangle$. The level degeneracies of the modules built on $|h_\star^+\rangle$ and $|w_\star^-\rangle$ only differ at level zero, since the parity of the non-null HW state is opposite for both, and the sum of their characters inferred from Eq. (6) yields precisely $\chi_{\text{R}}^{c/24}(q)$. We thus interpret Eq. (7) as an off-diagonal modular invariant involving the product of holomorphic and antiholomorphic characters for two distinct $c/24$ modules, in sharp contrast to the diagonal form for m odd. The introduction of an additional R ground state also resolves the arbitrariness in the original definition $|h_\star^-\rangle = G_0|h_\star^+\rangle$ [5], which could have equally been chosen as $|h_\star^+\rangle = G_0|h_\star^-\rangle$.

VI. BOUNDARY SCFT AND A NEW CARDY STATE

We now explore the consequences of this symmetrization of the R ground-state manifold on boundary SCFT. Superconformally invariant boundary states, the Cardy states, can be expanded on a basis of Ishibashi states $|h_\gamma\rangle\rangle$, $\gamma = \pm 1$, which are constructed for each irreducible module and obey the gluing conditions $(L_n - \bar{L}_{-n})|h_\gamma\rangle\rangle = 0$ and $(G_r + i\gamma\bar{G}_{-r})|h_\gamma\rangle\rangle = 0$ [24]. For the $c/24$ HW, we now have two sets of Ishibashi states,

$$|h_{\star,\gamma}^{\text{R}}\rangle\rangle = \sum_{\mathbf{q}} |h_\star, \mathbf{q}\rangle \otimes U_\gamma |\overline{h_\star, \mathbf{q}}\rangle, \quad (9)$$

$$|w_{\star,\gamma}^{\text{R}}\rangle\rangle = \sum_{\mathbf{q}'} |w_\star, \mathbf{q}'\rangle \otimes U_\gamma |\overline{w_\star, \mathbf{q}'}\rangle, \quad (10)$$

where U_γ is an antiunitary operator that commutes with the holomorphic fermion-parity operator $(-1)^F$ and obeys $U_\gamma L_n U_\gamma^{-1} = L_n$, $U_\gamma G_r U_\gamma^{-1} = -i\gamma G_r (-1)^F$. The sums run over a complete set of states in each module, with \mathbf{q}, \mathbf{q}' a set of quantum numbers sufficient to label each state (holomorphic fermion parity, level, and other quantum numbers). An equally valid basis, which facilitates the construction of the Cardy states, is given by the bonding and antibonding combinations of (9) and (10),

$$|\frac{c}{24\pm}^{\text{R}}\rangle\rangle := |h_{\star,\pm}^{\text{R}}\rangle\rangle \pm |w_{\star,\pm}^{\text{R}}\rangle\rangle. \quad (11)$$

Cardy states [21] $|\alpha_\gamma\rangle\rangle, |\beta_\gamma\rangle\rangle$ are defined by the property that the partition function $Z_{\alpha,\beta,\gamma}$ of the (GSO-projected) SCFT on a cylinder of length L and circumference R can be evaluated in either the open-string or closed-string pictures. In the open-string picture, one has periodic time evolution along the R direction according to a Hamiltonian

$H_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}} = \frac{\pi}{L} (L_0 - \frac{c}{24})_{\alpha_\gamma, \beta_{\gamma'}}$ with BCs $\alpha_\gamma, \beta_{\gamma'}$ along the L direction,

$$Z_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}(q) = \frac{1}{2} \text{Tr}_{\{\text{NS}\}} [e^{-RH_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}}] + \frac{1}{2} \text{Tr}_{\{\text{NS}\}} [(-1)^F e^{-RH_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}}] \\ + \frac{1}{2} \text{Tr}_{\{\text{R}\}} [e^{-RH_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}}] + \frac{1}{2} \text{Tr}_{\{\text{R}\}} [(-1)^F e^{-RH_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}}], \quad (12)$$

where traces are over holomorphic states only and all four spin structures on the cylinder are considered separately. In the closed-string picture, one has a transition amplitude between Cardy states with finite time evolution along the L direction,

$$Z_{\alpha_\gamma, \beta_{\gamma'}}^{\text{closed}}(\tilde{q}) = \langle\langle \Theta \alpha_\gamma \| e^{-LH^{\text{closed}}} \| \beta_{\gamma'} \rangle\rangle, \quad (13)$$

with Hamiltonian $H^{\text{closed}} = \frac{2\pi}{R} (L_0 + \bar{L}_0 - \frac{c}{12})$, and Θ is the antiunitary CPT operator. The parameters $q = e^{-\pi R/L} = e^{2\pi i\tau}$ and $\tilde{q} = e^{-4\pi L/R} = e^{2\pi i\tilde{\tau}}$ are related by a modular S transformation $\tilde{\tau} = -1/\tau$. The open-string partition function is evaluated as

$$Z_{\alpha_\gamma, \beta_{\gamma'}}^{\text{open}}(q) = \frac{1}{2} \sum_{i \in \Delta_{\text{NS}}} (n_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\text{NS}}^i(q) + \tilde{n}_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\tilde{\text{NS}}}^i(q)) \\ + \frac{1}{2} (m_{\alpha_\gamma, \beta_{\gamma'}}^{h^+} + m_{\alpha_\gamma, \beta_{\gamma'}}^{w^-}) \chi_{\text{R}}^{c/24}(q) \\ + \sum_{i \in \Delta_{\text{R}}} m_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\text{R}}^i(q) + \frac{1}{2} (\tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{h^+} - \tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{w^-}) \\ = \frac{1}{2} \sum_{i \in \Delta_{\text{NS}}} (n_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\text{NS}}^i(q) + \tilde{n}_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\tilde{\text{NS}}}^i(q)) \\ + \sum_{i \in \Delta_{\text{R}}} m_{\alpha_\gamma, \beta_{\gamma'}}^i \chi_{\text{R}}^i(q) + \frac{1}{2} \tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{c/24}, \quad (14)$$

where the multiplicities $n_{\alpha_\gamma, \beta_{\gamma'}}^i, \tilde{n}_{\alpha_\gamma, \beta_{\gamma'}}^i, m_{\alpha_\gamma, \beta_{\gamma'}}^i, \tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^i \in \mathbb{Z}$ denote how many times the irreducible HW module i appears in the spectrum of the SCFT with BCs $\alpha_\gamma, \beta_{\gamma'}$ for a given choice of spin structure. In the last equality, we define $m_{\alpha_\gamma, \beta_{\gamma'}}^{h^+} + m_{\alpha_\gamma, \beta_{\gamma'}}^{w^-} =: 2m_{\alpha_\gamma, \beta_{\gamma'}}^{c/24}$ and $\tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{h^+} - \tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{w^-} =: \tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{c/24}$, establishing later that $m_{\alpha_\gamma, \beta_{\gamma'}}^{c/24}$ is integer. Note that the two independent $c/24$ modules in general contribute a non-trivial constant from the twisted sum in the R sector, which is at the origin of the new Cardy state to be discussed shortly. Conversely, expanding Cardy states on the basis of Ishibashi states as $\|\alpha_\gamma\rangle\rangle = \sum_{j \in \Delta_{\text{NS}}} B_{\alpha_\gamma}^{j, \text{NS}} |j_\gamma^{\text{NS}}\rangle\rangle + \sum_{j \in \Delta_{\text{R}}} B_{\alpha_\gamma}^{j, \text{R}} |j_\gamma^{\text{R}}\rangle\rangle$, and similarly for $\|\beta_{\gamma'}\rangle\rangle$, where $B_{\alpha_\gamma}^{j, \text{NS,R}} := \langle\langle j_\gamma^{\text{NS,R}} | \alpha_\gamma \rangle\rangle$ are the expansion coefficients, the closed-string

partition function (13) can be calculated and expressed in terms of these coefficients and the characters $\chi_{\text{NS}}^j(\tilde{q}), \chi_{\tilde{\text{NS}}}^j(\tilde{q}), \chi_{\text{R}}^j(\tilde{q})$ [33]. The aforementioned constant term in (14) is matched by a corresponding term in (13), which arises from $\langle\langle \Theta_{24\pm}^{c, \text{R}} | e^{-LH^{\text{closed}}} | \frac{c, \text{R}}{24\pm} \rangle\rangle = 2$. Equating Eqs. (14) and (13), and using the transformation properties of the characters under a modular S transformation [34–36], we find a set of four Cardy equations [33],

$$\sum_{i \in \Delta_{\text{NS}}} n_{\alpha_\gamma, \beta_{\gamma'}}^i S_{ij}^{[\text{NS}, \text{NS}]} = 2(\delta_{\gamma^+} \delta_{\gamma'^+} + \delta_{\gamma^-} \delta_{\gamma'^-}) B_{\alpha_\gamma}^{j, \text{NS}} B_{\beta_{\gamma'}}^{j, \text{NS}}, \quad (15)$$

$$\sum_{i \in \Delta_{\text{R}}} m_{\alpha_\gamma, \beta_{\gamma'}}^i \frac{1}{\lambda_i} S_{ij}^{[\text{R}, \tilde{\text{NS}}]} = (\delta_{\gamma^+} \delta_{\gamma'^-} + \delta_{\gamma^-} \delta_{\gamma'^+}) B_{\alpha_\gamma}^{j, \text{NS}} B_{\beta_{\gamma'}}^{j, \text{NS}}, \quad (16)$$

$$\sum_{i \in \Delta_{\text{NS}}} \tilde{n}_{\alpha_\gamma, \beta_{\gamma'}}^i S_{ij}^{[\tilde{\text{NS}}, \text{R}]} \lambda_j = 4(\delta_{\gamma^+} \delta_{\gamma'^+} + \delta_{\gamma^-} \delta_{\gamma'^-}) B_{\alpha_\gamma}^{j, \text{R}} B_{\beta_{\gamma'}}^{j, \text{R}}, \quad (17)$$

$$\tilde{m}_{\alpha_\gamma, \beta_{\gamma'}}^{c/24} = 4(\delta_{\gamma^+} \delta_{\gamma'^-} + \delta_{\gamma^-} \delta_{\gamma'^+}) B_{\alpha_\gamma}^{c, \text{R}} B_{\beta_{\gamma'}}^{c, \text{R}}, \quad (18)$$

where S is the modular S -matrix [22], j runs over HWs in the appropriate sector, and $\lambda_{c/24} = 1, \lambda_{j \neq c/24} = \sqrt{2}$.

Adopting the method in Ref. [21], the Cardy states can be obtained as solutions to the above equations. For HWs $k, l \in \Delta_{\text{NS}}$, we obtain

$$\|k_+^{\text{NS}}\rangle\rangle = \frac{1}{\sqrt{2}} \sum_{j \in \Delta_{\text{NS}}} \frac{S_{kj}^{[\text{NS}, \text{NS}]}}{\sqrt{S_{0j}^{[\text{NS}, \text{NS}]}}} |j_+^{\text{NS}}\rangle\rangle, \quad (19)$$

$$\|l_+^{\tilde{\text{NS}}}\rangle\rangle = \sum_{j \in \Delta_{\text{R}}} \frac{\sqrt{\lambda_j}}{2} \frac{S_{lj}^{[\tilde{\text{NS}}, \text{R}]}}{\sqrt{S_{0j}^{[\tilde{\text{NS}}, \text{R}]}}} |j_+^{\text{R}}\rangle\rangle, \quad (20)$$

while for HWs $d \in \Delta_{\text{R}}$, we obtain

$$\|d_+^{\text{R}}\rangle\rangle = \sum_{j \in \Delta_{\text{NS}}} \frac{\sqrt{2}}{\lambda_d} \frac{S_{dj}^{[\text{R}, \tilde{\text{NS}}]}}{\sqrt{S_{0j}^{[\text{NS}, \text{NS}]}}} |j_+^{\text{NS}}\rangle\rangle. \quad (21)$$

Finally, for even m , the existence of the self-symmetric $c/24$ HW yields an additional Cardy state,

$$\|\frac{c, \tilde{\text{R}}}{24-}\rangle\rangle = \frac{1}{2\sqrt{S_{0, c/24}^{[\tilde{\text{NS}}, \text{R}]}}} |\frac{c, \text{R}}{24-}\rangle\rangle. \quad (22)$$

VII. FUSION RULES

Identifying the multiplicities appearing in the open-string partition function (14) with the fusion coefficients of the SCFT [21], one obtains the complete set of fusion rules for the even- m series of SMMs,

$$[\frac{c}{24}^{\tilde{R}}] \times [\frac{c}{24}^{\tilde{R}}] = \sum_{i \in \Delta_{\text{NS}}} \tilde{n}_{\frac{c}{24}^{\tilde{R}}, \frac{c}{24}^{\tilde{R}}}^i [i^{\tilde{\text{NS}}}], \quad (23)$$

$$[l^{\tilde{\text{NS}}}] \times [\frac{c}{24}^{\tilde{R}}] = \tilde{m}_{l^{\tilde{\text{NS}}}, \frac{c}{24}^{\tilde{R}}}^{\frac{c}{24}}, \quad (24)$$

$$[l^{\tilde{\text{NS}}}] \times [l^{\tilde{\text{NS}}}] = \sum_{i \in \Delta_{\text{NS}}} \tilde{n}_{l^{\tilde{\text{NS}}}, l^{\tilde{\text{NS}}}}^i [i^{\tilde{\text{NS}}}], \quad (25)$$

$$[d^{\text{R}}] \times [d^{\text{R}}] = \sum_{i \in \Delta_{\text{NS}}} n_{d^{\text{R}}, d^{\text{R}}}^i [i^{\text{NS}}], \quad (26)$$

$$[k^{\text{NS}}] \times [d^{\text{R}}] = \sum_{i \in \Delta_{\text{R}}} m_{k^{\text{NS}}, d^{\text{R}}}^i [i^{\text{R}}], \quad (27)$$

$$[k^{\text{NS}}] \times [k^{\text{NS}}] = \sum_{i \in \Delta_{\text{NS}}} n_{k^{\text{NS}}, k^{\text{NS}}}^i [i^{\text{NS}}], \quad (28)$$

and the corresponding Verlinde formula,

$$\tilde{n}_{\frac{c}{24}^{\tilde{R}}, \frac{c}{24}^{\tilde{R}}}^i = \frac{(S_{\frac{c}{24}, i}^{\tilde{\text{NS}}, \text{R}})^{-1}}{S_{0, \frac{c}{24}}^{\tilde{\text{NS}}, \text{R}}}, \quad \tilde{m}_{l^{\tilde{\text{NS}}}, \frac{c}{24}^{\tilde{R}}}^{\frac{c}{24}} = \frac{S_{l^{\tilde{\text{NS}}}, \frac{c}{24}}^{\tilde{\text{NS}}, \text{R}}}{S_{0, \frac{c}{24}}^{\tilde{\text{NS}}, \text{R}}}, \quad (29)$$

$$\tilde{n}_{l^{\tilde{\text{NS}}}, l^{\tilde{\text{NS}}}}^i = \sum_{j \in \Delta_{\text{R}}} \frac{S_{l^{\tilde{\text{NS}}}, j}^{\tilde{\text{NS}}, \text{R}} S_{l^{\tilde{\text{NS}}}, j}^{\tilde{\text{NS}}, \text{R}} (S_{l^{\tilde{\text{NS}}}, j}^{\tilde{\text{NS}}, \text{R}})^{-1}}{S_{0, j}^{\tilde{\text{NS}}, \text{R}}}, \quad (30)$$

$$n_{d^{\text{R}}, d^{\text{R}}}^i = \sum_{j \in \Delta_{\text{NS}}} \frac{4 S_{d^{\text{R}}, j}^{\text{R}, \tilde{\text{NS}}} S_{d^{\text{R}}, j}^{\text{R}, \tilde{\text{NS}}} (S_{d^{\text{R}}, j}^{\text{NS}, \text{NS}})^{-1}}{\lambda_d \lambda_{d'} S_{0, j}^{\text{NS}, \text{NS}}}, \quad (31)$$

$$m_{k^{\text{NS}}, d^{\text{R}}}^i = \sum_{j \in \Delta_{\text{NS}}} \frac{\lambda_j S_{k^{\text{NS}}, j}^{\text{NS}, \text{NS}} S_{d^{\text{R}}, j}^{\text{R}, \tilde{\text{NS}}} (S_{d^{\text{R}}, j}^{\text{R}, \tilde{\text{NS}}})_{ji}^{-1}}{\lambda_d S_{0, j}^{\text{NS}, \text{NS}}}, \quad (32)$$

$$n_{k^{\text{NS}}, k^{\text{NS}}}^i = \sum_{j \in \Delta_{\text{NS}}} \frac{S_{k^{\text{NS}}, j}^{\text{NS}, \text{NS}} S_{k^{\text{NS}}, j}^{\text{NS}, \text{NS}} (S_{k^{\text{NS}}, j}^{\text{NS}, \text{NS}})^{-1}}{S_{0, j}^{\text{NS}, \text{NS}}}. \quad (33)$$

We check explicitly that all fusion coefficients are integers.

VIII. SUMMARY

We show that the standard representation-theoretic approach to the determination of irreducible characters in CFT fails for the R-sector $c/24$ HW in the even- m series of $\mathcal{N} = 1$ SMMs and conjecture an infinite hierarchy of linear-dependence relations to compute the character of this module. Modular invariance on the torus is restored with this character provided that the R ground-state manifold is augmented by a non-null parity-odd state, which in turn yields additional bulk fusion channels and superconformally invariant boundary states. One implication of this symmetrization of the ground-state manifold is the possibility of spontaneously broken supersymmetry and the associated Goldstino in even- m SMMs, for which numerical evidence has been presented in [37].

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