


Anomaly-free $U(1)^m$ extensions of the Standard Model

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We construct anomaly-free $U(1)_1 \times U(1)_2 \times \dots \times U(1)_m$ gauge extensions of the Standard Model. To perform this construction, we put together anomaly-free $U(1)$ extensions of one and two families of fermions. The availability of free parameters that enter linearly in the equations for the fermion charges and the large number of different classes of extensions may help other model builders interested in their use to solve problems of particle physics.

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I. INTRODUCTION

Particle physics have problems that can be solved using gauged $U(1)^m \equiv U(1)_1 \times U(1)_2 \times \dots \times U(1)_m$ extensions of the Standard Model (SM) featuring multiple Z' bosons [1]. For $m = 1$, some examples are dark matter [2–8], neutrino masses [9–15], and the anomalous magnetic moment of the muon [16–18]. However, in order to be well behaved at high energies, a gauge symmetry must be free of anomalies [19–22]. General solutions to the anomaly cancellation equations are not always useful because free parameters may be trapped in complicated polynomial expressions. However, in composite gauge theories, there is a subset of fermions that independently cancel anomalies, and one can use these different subsets to add free parameters that are easily available because they enter linearly in the equations for fermion charges.

In this paper, we will construct composite anomaly-free $U(1)^m$ gauge extensions of the SM. In Secs. II and III, we will settle notation and define concepts, in Secs. IV and V we will make the construction using one-family and two-families anomaly-free $U(1)$ extensions as building blocks in each case, and in Sec. VI we will present other extensions that can be made with additional charged Weyl fermions.

II. ANOMALY EQUATIONS

Let $[z_{ij}^\ell]$ denote the charges of the SM fermions under an $U(1)^m = U(1)_1 \times U(1)_2 \times \dots \times U(1)_m$ gauge symmetry. The index $i \in \{1, 2, 3\}$ refers to the three families, $\ell \in \{1, 2, \dots, m\}$ to the $U(1)_\ell$ subgroup and $j \in \{1, 2, 3, 4, 5, 6\} \mapsto \{N, Q, D, L, U, E\}$ to the fermion

fields which are in the following representations of the SM gauge group $G_{\text{SM}} \equiv SU(3) \times SU(2) \times U(1)_Y$:

| Fermion fields | SU(3) | SU(2) | $U(1)_Y$ |
|----------------|--------------------|----------|----------|
| N | 1 | 1 | 0 |
| Q | 3 | 2 | 1 |
| D | $\bar{\mathbf{3}}$ | 1 | 2 |
| L | 1 | 2 | −3 |
| U | $\bar{\mathbf{3}}$ | 1 | −4 |
| E | 1 | 1 | 6 |

We assumed that right-handed neutrinos exist to generalize our results and to simplify their exposition. This assumption does no harm, as we can assign zero charges under the extended $U(1)$'s to them so that effectively they do not exist. Following the conventions of previous work [23,24], fermions are taken as left-handed fields which can be done via charge conjugation. We also assume that $U(1)$ charges are always given for a normalization of the gauge couplings where all charges are integers with no common divisor and the biggest charge, in absolute value, is positive. Often, we will use the notation $[z_{ij}^\ell]$ with the i or ℓ indices suppressed. If the i index is suppressed, we will be considering a single family of the SM; if the ℓ index is suppressed, the $U(1)$ group being considered will be clear in the context. In this paper, m is always a given natural number.

With this notation and convention, the hypercharges $[y_{ij}]$ of the SM fermions, referred from here on with the letter y , are $y_{i1} = 0$, $y_{i2} = 1$, $y_{i3} = 2$, $y_{i4} = -3$, $y_{i5} = -4$, and $y_{i6} = 6$ for all $i \in \{1, 2, 3\}$. We can also present it as a 3-tuple of $[y_j] \equiv [0, 1, 2, -3, -4, 6]$ or in matrix form

$$[y_{ij}] = \begin{bmatrix} y_j \\ y_j \\ y_j \end{bmatrix} = \begin{bmatrix} N & Q & D & L & U & E \\ 0 & 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 2 & -3 & -4 & 6 \end{bmatrix} \begin{matrix} \text{1st} \\ \text{2nd} \\ \text{3rd} \end{matrix} \cdot \quad (1)$$

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Without adding new fermions charged under hypercharge, a $U(1)$ gauge extension of the SM, $G_{\text{SM}} \times U(1)$, is *anomaly free* if the charges $[z_{ij}]$ associated with the additional $U(1)$ subgroup satisfy the following system of diophantine equations:

$$0 = \sum_{i,j} n_j z_{ij}^3, \quad (2a)$$

$$0 = \sum_{i,j} n_j z_{ij} y_{ij}^2, \quad (2b)$$

$$0 = \sum_{i,j} n_j z_{ij}^2 y_{ij}, \quad (2c)$$

$$0 = \sum_{i,j} \delta_j^{\text{grav}} n_j z_{ij}, \quad (2d)$$

$$0 = \sum_{i,j} \delta_j^{\text{su}(2)} n_j z_{ij}, \quad (2e)$$

$$0 = \sum_{i,j} \delta_j^{\text{su}(3)} n_j z_{ij}, \quad (2f)$$

where n_j is the number of Weyl fermions in the j multiplet, and δ_j^s is equal to 1 if multiplet j is charged under s and 0 otherwise, with $s \in \{\text{grav}, \text{su}(2), \text{su}(3)\}$. Using vector notation, we have $[n_j] = [1, 6, 3, 2, 3, 1]$, $[\delta_j^{\text{grav}}] = [1, 1, 1, 1, 1, 1]$, $[\delta_j^{\text{su}(2)}] = [0, 1, 0, 1, 0, 0]$, and $[\delta_j^{\text{su}(3)}] = [0, 1, 1, 0, 1, 0]$. More generally, a $U(1)^m = U(1)_1 \times U(1)_2 \times \dots \times U(1)_m$ gauge extension of the SM, $G_{\text{SM}} \times U(1)^m$, is anomaly free if the charges $[z_{ij}^\ell]$, $[z_{ij}^{\ell'}]$, and $[z_{ij}^{\ell''}]$ associated with the subgroups $U(1)_{\ell}$, $U(1)_{\ell'}$, and $U(1)_{\ell''}$ satisfy (2a)–(2f) and

$$0 = \sum_{i,j} n_j z_{ij}^\ell z_{ij}^{\ell'} z_{ij}^{\ell''} \quad (3)$$

for all $\ell, \ell', \ell'' \in \{1, 2, \dots, m\}$. These definitions are sufficient because an $G_{\text{SM}} \times U(1)^m$ gauge theory with no $SU(2)$ anomaly [25,26] does not have other global anomalies [27–30]. In this paper, we refer to *one-family* and *two-families* anomaly-free extensions; these are defined in the same way as a generic anomaly-free $U(1)^m$ extension but for a vector $[z_j]$ and a 2-tuple of vectors.

Recently, an atlas with solutions for the anomaly equations (2a)–(2f) was constructed [31]. Then a general solution was found for the first two equations [23], and for this solution, it was given a geometric interpretation [32]. While the present work was being written, the full system of equations was solved using this geometric method [33]. Other aspects of this system of equations including (3) were also explored [24,34,35].

Generalizing a notion presented in [24], we say that a gauge theory is *composite* if there is a nonempty proper

subset of fermions whose charges independently satisfy the anomaly equations. In particular, the SM is composite because each family of fermions is independently anomaly free. In fact, the only nonempty proper subsets of fermion fields of the SM whose hypercharge independently satisfies all anomaly equations are those that make a complete family. Therefore, without adding new Weyl fermions charged under hypercharge, all composite $U(1)^m$ extensions of the SM are made with $U(1)$ extensions that are one-family or two-families anomaly free. This implication is easy to prove by contradiction. First, assume that there is a $U(1)^m$ extension that does not satisfy the condition, then there would be a nonempty proper subset of fermions of the SM such that anomaly equations cancel independently which is not one or two families, but there is none.

III. FAMILY PERMUTATION

Before presenting the one-family and two-families anomaly-free extensions that we will use as building blocks, we will formalize a notion that will help in the exposition of our results. Given a collection of charges $[z_{ij}]$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) \in S_3^6$ where each $\sigma_j \in S_3$ is a permutation of $\{1, 2, 3\}$, we define the *family permutation* of $[z_{ij}]$ by σ as the new collection,

$$\sigma[z_{ij}] \equiv [z'_{ij}] = [z_{\sigma_j(i)j}]. \quad (4)$$

In words, the family permutation of $[z_{ij}]$ permutes the charges of each multiplet among the families in an independent way. With this definition, the family universality of the hypercharge can be stated precisely as

$$[y_{ij}] = \sigma[y_{ij}] \quad (5)$$

for all $\sigma \in S_3^6$. This symmetry suggests redundancy in the way fermions are partitioned into three families.

Concerning $U(1)$ extensions of the SM, we have the following proposition: the family permutation of an anomaly-free $U(1)$ extension is also anomaly free. To be precise, if $[z_{ij}]$ satisfies the anomaly equations (2a)–(2f), then $\sigma[z_{ij}]$ also satisfies them for all $\sigma \in S_3^6$. With the notation introduced, it is straightforward to verify this proposition. For example, if the charges $[z_{ij}]$ satisfy the anomaly equation (2b), then the charges $[z'_{ij}] = \sigma[z_{ij}]$ satisfy the same equation since

$$\sum_{i,j} g_j z'_{ij} y_{ij}^2 = \sum_{i,j} g_j z_{\sigma_j(i)j} y_{\sigma_j(i)j}^2 = 0, \quad (6)$$

where I used the fact that the hypercharge is family universal (5). More generally, if we have an anomaly-free $U(1)^m$ extension of the SM with charges $[z_{ij}^\ell]$, then the extension associated to $\sigma[z_{ij}^\ell]$ is also anomaly free.

If all $U(1)$ subgroups of $U(1)^m$ are family universal, the proposition presented above is of no interest since $\sigma[z_{ij}^\ell] = [z_{ij}^\ell]$ for all $\sigma \in S_3^6$ and $\ell \in \{1, 2, \dots, m\}$. However, if there is a $U(1)_k$ subgroup that is not family universal for some $1 \leq k \leq m$, then in general $\sigma[z_{ij}^k] \neq [z_{ij}^k]$, and we can obtain different extensions by doing family permutations. To be exact, for each anomaly-free $U(1)^m$ extension, there is a total of $|S_3^6| = 6^6 = 46656$ other anomaly-free extensions.

IV. ONE-FAMILY ANOMALY-FREE EXTENSIONS

The one-family anomaly-free $U(1)$ extensions of the SM are

$$[f_j(\vec{z})] = [z_1, z_2, 2z_2 - z_1, -3z_2, z_1 - 4z_2, 6z_2 - z_1] \quad (7)$$

for all $\vec{z} = (z_1, z_2) \in \mathbb{Z}^2$. Recall our convention for the j indices, namely, $j \in \{1, 2, 3, 4, 5, 6\} \mapsto \{N, Q, D, L, U, E\}$, and see [31] for details on how to derive this result. When $z_1 = 0$, the charges are proportional to one family of hypercharge; when $\vec{z} = (1, 0)$, they are proportional to the charges associated to $U(1)_{T3R}$ and to $U(1)_{B-L}$ when $\vec{z} = (3, 1)$.

A surprising fact is that a $U(1)^m$ gauge theory with the same fermionic content as one family of the SM and with charges given by (7) with different parameters for each $U(1)$ component is anomaly free. Formally, the charges $[z_j^\ell] = [f_j(\vec{z}^\ell)]$, $[z_j^{\ell'}] = [f_j(\vec{z}^{\ell'})]$, and $[z_j^{\ell''}] = [f_j(\vec{z}^{\ell''})]$ associated with the $U(1)_{\ell}$, $U(1)_{\ell'}$, and $U(1)_{\ell''}$ subgroups satisfy

$$\sum_j n_j z_j^\ell z_j^{\ell'} z_j^{\ell''} = \sum_j n_j f_j(\vec{z}^\ell) f_j(\vec{z}^{\ell'}) f_j(\vec{z}^{\ell''}) = 0 \quad (8)$$

for any $\vec{z}^\ell, \vec{z}^{\ell'}, \vec{z}^{\ell''} \in \mathbb{Z}^2$, and all $\ell, \ell', \ell'' \in \{1, 2, \dots, m\}$. This does not seem to be trivial because (7) is found as a solution to (2a)–(2f) and nothing else.

A consequence of this fact is that a $U(1)^m$ extension of the SM with the charges of each subset of fermions that make a complete family under each $U(1)$ given by a multiple of (7) satisfies all anomaly equations. To be more precise, given $m \in \mathbb{N}$ and $\sigma \in S_3^6$, a $U(1)^m = U(1)_1 \times U(1)_2 \times \dots \times U(1)_m$ extension with the charges associated with the $U(1)_\ell$ subgroup is given by

$$[z_{ij}^\ell] \equiv \sigma[k_i^\ell f_j(\vec{z}_i^\ell)] = \sigma \begin{bmatrix} k_1^\ell f_j(\vec{z}_1^\ell) \\ k_2^\ell f_j(\vec{z}_2^\ell) \\ k_3^\ell f_j(\vec{z}_3^\ell) \end{bmatrix}, \quad (9)$$

with $\vec{z}_1^\ell, \vec{z}_2^\ell, \vec{z}_3^\ell \in \mathbb{Z}^2$ and $k_1^\ell, k_2^\ell, k_3^\ell \in \mathbb{Z}$ for all $\ell \in \{1, 2, \dots, m\}$, is anomaly free. Note that σ does not depend on ℓ , because otherwise the anomaly equation (3) would not be satisfied in general. These extensions are composite

because anomalies cancel for subsets of fermions that make a complete family; for $\sigma = id$, the subsets are the actual families. Finally, these extensions have free parameters that enter linearly in the formulas for the fermion charges as stated in the Introduction.

V. TWO-FAMILIES ANOMALY-FREE EXTENSIONS

The two-families anomaly-free $U(1)$ extensions of the SM are

$$\begin{bmatrix} h_{1j}(\vec{z}, \vec{x}) \\ h_{2j}(\vec{z}, \vec{x}) \end{bmatrix} = \begin{bmatrix} f_j(\vec{z}) + g_j(\vec{x}) \\ f_j(\vec{z}) - g_j(\vec{x}) \end{bmatrix}, \quad (10)$$

with $[f_j(\vec{z})]$ defined as in (7) and $[g_j(\vec{x})]$ parametrized by four integer parameters depending on \vec{z} . If $z_1 = 0$, right-handed neutrinos will form a vectorlike pair with arbitrary charge, and

$$g_Q(\vec{x}) = -(x_1^2 - x_2^2 - x_3^2 + 2x_4^2), \quad (11a)$$

$$g_D(\vec{x}) = 2x_1 x_2, \quad (11b)$$

$$g_L(\vec{x}) = -(x_1^2 + x_2^2 + x_3^2 - 2x_4^2), \quad (11c)$$

$$g_U(\vec{x}) = 2x_1 x_4, \quad (11d)$$

$$g_E(\vec{x}) = 2x_1 x_3 \quad (11e)$$

for any $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$. If $z_1 \neq 0$, then

$$g_N(\vec{x}) = 2x_1(x_2^2 + 3x_3^2 - 3x_4^2), \quad (12a)$$

$$g_Q(\vec{x}) = -x_1^2 + x_2^4 + 9(x_3^2 - x_4^2)^2 - 2x_2^2(x_3^2 + x_4^2), \quad (12b)$$

$$g_D(\vec{x}) = 4x_1 x_2 x_3, \quad (12c)$$

$$g_L(\vec{x}) = -x_1^2 - x_2^4 - 9(x_3^2 - x_4^2)^2 + 2x_2^2(x_3^2 + x_4^2), \quad (12d)$$

$$g_U(\vec{x}) = 4x_1 x_2 x_4, \quad (12e)$$

$$g_E(\vec{x}) = 2x_1(x_2^2 - 3x_3^2 + 3x_4^2) \quad (12f)$$

for any $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$. Again, recall our convention for the j indices, $j \in \{1, 2, 3, 4, 5, 6\} \mapsto \{N, Q, D, L, U, E\}$, and see [31] for details on how to derive these results. The first family of solutions is proportional to the charges associated with $U(1)_{L\mu-L\tau}$ for $z_2 = x_3 = x_4 = 0$ and $x_1 = -x_2$.

Unfortunately, a $U(1)^2$ gauge theory with the same fermionic content as two families of the SM and charges given by (10) with different parameters for each $U(1)$ is not in general anomaly free. This means that we cannot use them, without other restrictions, to build $U(1)^m$ extensions

We regard these structured collections of charges as matrices with null entries and denote them by $[z_{kl}]$. The order in which the elements of each row are organized does not have any meaning. These structured collections of charges are all made with the hypercharges of the 45 charged Weyl fermions of the SM, but the k and l indices do not denote family and fermion field as before. These are partitions of the SM hypercharges such that for each k we have

$$0 = \sum_l z_{kl}, \quad (17a)$$

$$0 = \sum_l z_{kl}^3. \quad (17b)$$

These partitions can be used to construct different classes of composite anomaly-free extensions of the SM by finding $U(1)$ extensions to the different rows as we did in the previous sections for the family partition. But these extensions will not respect the full gauge structure of the SM because Weyl fermions are not built-in multiplets. However, we can solve this problem by adding pairs of Weyl fermions which are vectorlike under $U(1)_Y$ although chiral under $U(1)_Y \times U(1)$. We will illustrate this construction for (16) which is the partition that needs a smaller number of additional Weyl fermions. We will consider the simplest $U(1)$ extension, which is the repetition of a multiple of hypercharge in the new $U(1)$ which turns the cross anomaly equations into the cubic.

First, we add to the Standard Model four Weyl fermions $\psi_1, \psi_2, \psi_3,$ and ψ_4 which are singlets of $SU(2) \times SU(3)$ and have hypercharges, respectively, equal to 4, -4, 2, -2. Starting with $\{6, -4, -4, -4, -4, 2, 2, 2, 2, 2\}$ of (16), we first append the pair $\{2, -2\}$ to it. Now, we gather three -4's in one U multiplet and the six 2's in two D 's. Then, we attribute 6 to E , -4 to ψ_2 , and -2 to ψ_4 . For concreteness, the charges of the Standard Model fermions under this new $U(1)$ can be

$$[v_{ij}] = \begin{array}{cccccc} N & Q & D & L & U & E \\ \left[\begin{array}{cccccc} 0 & 0 & 2 & 0 & -4 & 6 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] & \begin{array}{l} \text{1st} \\ \text{2nd} \\ \text{3rd} \end{array} \end{array}, \quad (18)$$

and the additional chiral set with four Weyl fermions can have $U(1)_Y \times U(1)$ charges equal to

| Fermions fields | $U(1)_Y$ | $U(1)$ |
|-----------------|----------|--------|
| ψ_1 | 4 | 0 |
| ψ_2 | -4 | -4 |
| ψ_3 | 2 | 0 |
| ψ_4 | -2 | -2 |

This composite extension satisfies all anomaly equations associated with $U(1)_Y \times U(1)$ by construction and somewhat surprisingly also satisfies the (2e) and (2f) anomaly cancellation equations associated with $SU(3)$ and $SU(2)$,

$$\sum_{i,j} g_j \delta_j^{su(2)} u_{ij} = 0, \quad (19)$$

$$\sum_{i,j} g_j \delta_j^{su(3)} u_{ij} = 3 \times (-4) + 2 \times 3 \times 2 = 0. \quad (20)$$

Note that the additional Weyl fermions are vectorlike under $U(1)_Y$ but chiral under $U(1)_Y \times U(1)$.

We can repeat the same construction with $\{6, -4, -4, -3, -3, -3, -3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$ of (16a). First, we append $\{4, -4\}$ to it and gather the three -4's in one U multiplet, the six -3's in two L 's, and the twelve 1's in two Q 's. Then, we attribute the 6 to E , 4 to ψ_1 , and 2 to ψ_3 . Again, for concreteness, we can have

$$[u_{ij}] = \begin{array}{cccccc} N & Q & D & L & U & E \\ \left[\begin{array}{cccccc} z_1 & 1 & 0 & -3 & 0 & 0 \\ -z_1 & 1 & 0 & -3 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] & \begin{array}{l} \text{1st} \\ \text{2nd} \\ \text{3rd} \end{array} \end{array}, \quad (21)$$

and

| Fermions fields | $U(1)_Y$ | $U(1)$ |
|-----------------|----------|--------|
| ψ_1 | 4 | 4 |
| ψ_2 | -4 | 0 |
| ψ_3 | 2 | 2 |
| ψ_4 | -2 | 0 |

Note that we added opposite charges $z_1, -z_1 \in \mathbb{Z}$ to the pair of right-handed neutrinos which do not spoil anomaly cancellations. As for the other extension (18), the anomaly equations are satisfied for this one, in particular

$$\sum_{i,j} g_j \delta_j^{su(2)} u_{ij} = -2 \times 2 \times 3 + 2 \times 1 \times 6 = 0, \quad (22)$$

$$\sum_{i,j} g_j \delta_j^{su(3)} u_{ij} = -3 \times 4 - 2 \times 3 + 6 \times 1 = 0. \quad (23)$$

These two composite anomaly-free extensions are particular cases of the extensions with the charges of the SM fermions under the additional $U(1)$ given by

$$[u_{ij}] = \sigma(k_1[u_{ij}] + k_2[v_{ij}] + k_3[\delta_{i3}y_j]), \quad (24)$$

with $z_1, k_1, k_2, k_3 \in \mathbb{Z}$, $\sigma \in S_3^6$, and δ_{i3} the usual Kronecker delta. We also need to add a chiral set with four Weyl fermions that have $U(1)_Y \times U(1)$ charges given by

| Fermions fields | $U(1)_Y$ | $U(1)$ |
|-----------------|----------|---------|
| ψ_1 | 4 | $4k_1$ |
| ψ_2 | -4 | $-4k_2$ |
| ψ_3 | 2 | $2k_1$ |
| ψ_4 | -2 | $-2k_2$ |

Note that these extensions are in a sense the simplest possible with additional charged chiral fermions because they are made of multiples of hypercharge which turns the cross anomaly equations into the cubic that are satisfied by construction. This implies that we can build $U(1)^m$ extensions from them as we did in (9) and (15). A different approach would be to search other $U(1)$ extensions to the rows of (16a). The same construction can be done with the other partitions (16b)–(16d), but for them one needs to add more Weyl fermions.

VII. CONCLUSIONS

In this paper, we built composite anomaly-free $U(1)^m$ extensions of the SM. The availability of free parameters in (9) and (15) that enter linearly in the formulas for the fermion charges and the large number of different classes of extensions may help other model builders interested in their use to solve problems of particle physics.

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