Magnetic corrections to the boson self-coupling and boson-fermion coupling in the linear sigma model with quarks

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We compute the magnetic-field-induced modifications to the boson self-coupling and the boson-fermion coupling, in the static limit, using an effective model of QCD, the linear sigma model with quarks. The former is computed for arbitrary field strengths as well as using the strong field approximation. The latter is obtained in the strong field limit. The arbitrary field result for the boson self-coupling depends on the ultraviolet renormalization scale, and this dependence cannot be removed by a simple vacuum subtraction. Using the strong field result as a guide, we find the appropriate choice for this scale and discuss the physical implications. The boson-fermion coupling depends on the Schwinger phase, and we show how this phase can be treated consistently in such a way that the magnetic-field-induced vertex modification is both gauge invariant and can be written with an explicit factor corresponding to energy-momentum conservation for the external particles. Both couplings show a modest decrease with the field strength.

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I. INTRODUCTION

The effects of magnetic fields on the properties of strongly interacting matter have gathered a great deal of interest over the last several years. The main driving motivation is the lattice QCD (LQCD) discovery of the inverse magnetic catalysis (IMC) phenomenon [1], whereby for temperatures above that of chiral restoration, the quark-antiquark condensate decreases and the chiral restoration temperature itself also decreases, as a function of the field intensity. The origin of IMC has been intensively studied; see, for example, Refs. [2–15].

In addition, much effort has also been devoted to studying the basic properties of magnetized hadronic degrees of freedom. The subject is important—e.g., for systems such as cold neutron stars and heavy-ion collisions. As is well known, the nuclear equation of state is affected by baryon and meson masses and couplings, which motivates studies aimed to understand how these parameters change in the presence of electromagnetic fields [16–28]. Different effective QCD models [29–56] and LQCD simulations [57–63] as well as holographic QCD models [64–67] have been used to describe the behavior of light meson masses. More recently, efforts have also been carried out to describe the behavior of light baryons in the presence of magnetic fields [68–71]. In particular, the recent LQCD results for the magnetic-field-driven modifications of neutral and charged mesons show that the neutral pion mass monotonically decreases, whereas the mass of the charged pions monotonically increases, both as functions of the field intensity [57,72]. The former cannot be fully reproduced by calculations within effective models that do not consider accounting for magnetic field modifications of the couplings [73,74].

When the linear sigma model with quarks (LSMq) is used as an effective QCD model, it has been shown that the IMC can be reasonably well described when temperature, as well as magnetic field corrections, are incorporated into self-energies and couplings [6,8]. The decreasing of the neutral pion mass with the magnetic field strength can also be understood when, in the weak field limit, the meson self-coupling is dressed to include magnetic field effects [75]. In order to find out whether or not the behavior of the pion masses can be described over a wider range of magnetic

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field intensities in the LSMq, it is important to compute the magnetic-field-induced corrections to the interaction vertices.

In this work, we address this question and compute the one-loop magnetic field corrections to the boson self-coupling and the boson-fermion coupling in the LSMq. In doing so, we address some important details involving the effects introduced by the renormalization scale, as well as those introduced by the Schwinger phase in calculations involving three particles propagating within loops in the presence of magnetic fields. The work is organized as follows: In Sec. II, we introduce the LSMq, writing the Lagrangian in terms of the charged pion degrees of freedom and including an explicit symmetry-breaking term. In Sec. III, we recall the way the magnetic field effects are introduced for the propagators of charged bosons and fermions. In Sec. IV, we compute the modification to the boson self-coupling in the presence of a magnetic field. We show that the modification depends on the renormalization scale, and that for this to match the result obtained in the strong field limit, one needs to resort to a suitable choice for this scale. In Sec. V, we compute the magnetic-field-induced modification to the boson-fermion coupling and discuss in detail the effect of the Schwinger phase. We show that this leads to a plausible result respecting energy-momentum conservation for the external particles when these are described as plane waves, and thus when we neglect propagation over large space-time intervals. Finally, we summarize and provide an outlook of our results in Sec. VI, leaving for the appendixes the details of the calculation of the boson self-coupling and the boson-fermion coupling.

II. THE LSMq

The LSMq is an effective theory that captures the approximate chiral symmetry of QCD. It describes the interactions among small-mass mesons and quarks. We work with a Lagrangian invariant under $SU(2)_L \times SU(2)_R$ chiral transformations

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{a^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - ig\gamma^5 \bar{\psi}\,\vec{\tau}\cdot\vec{\pi}\psi - g\bar{\psi}\psi\sigma, \tag{1}$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices,

$$\psi_{L,R} = \begin{pmatrix} u \\ d \end{pmatrix}_{L,R}, \tag{2}$$

is a $SU(2)_{L,R}$ doublet, σ is a real scalar field, and $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ is a triplet of real scalar fields. π_3 corresponds to the neutral pion, whereas the charged ones are represented by the combinations

$$\pi_- = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2), \qquad \pi_+ = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2). \tag{3}$$

λ is the boson's self-coupling, and g is the fermion-boson coupling. $a^2 > 0$ is the mass parameter. Equation (1) can be written in terms of the charged and neutral-pion degrees of freedom as

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \pi_0)^2] + \partial_\mu \pi_- \partial^\mu \pi_+ + \frac{a^2}{2}(\sigma^2 + \pi_0^2) \\ & + a^2 \pi_- \pi_+ - \frac{\lambda}{4}(\sigma^4 + 4\sigma^2 \pi_- \pi_+ + 2\sigma^2 \pi_0 + 4\pi_-^2 \pi_+^2 \\ & + 4\pi_- \pi_+ \pi_0^2 + \pi_0^4) + i\bar{\psi}\partial\!\!\!/\psi - g\bar{\psi}\psi\sigma - ig\gamma^5 \bar{\psi}(\tau_+ \pi_+ \\ & + \tau_- \pi_- + \tau_3 \pi_0)\psi, \end{aligned} \tag{4}$$

where we introduce the combination of Pauli matrices

$$\tau_+ = \frac{1}{\sqrt{2}}(\tau_1 + i\tau_2), \qquad \tau_- = \frac{1}{\sqrt{2}}(\tau_1 - i\tau_2). \tag{5}$$

After chiral symmetry is spontaneously broken, the field σ acquires a nonvanishing vacuum expectation value $\sigma \to \sigma + v$, which breaks the $SU(2)_L \times SU(2)_R$ symmetry down to $SU(2)_{L+R}$, resulting in the Lagrangian

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \pi_0 \partial^\mu \pi_0 + \partial_\mu \pi_- \partial^\mu \pi_+ \\ & - \frac{1}{2}m_\sigma^2 \sigma^2 - \frac{1}{2}m_\pi^2 \pi_0^2 - m_\pi^2 \pi_- \pi_+ + i\bar{\psi}\partial\!\!\!/\psi \\ & - m_f \bar{\psi}\psi + \mathcal{L}_{\text{int}} - V_{\text{tree}}, \end{aligned} \tag{6}$$

where the interaction Lagrangian is defined as

$$\begin{aligned}\mathcal{L}_{\text{int}} = {}& -\frac{\lambda}{4}\sigma^4 - \lambda v \sigma^3 - \lambda v^3 \sigma - \lambda \sigma^2 \pi_- \pi_+ - 2\lambda v \sigma \pi_- \pi_+ \\ & - \frac{\lambda}{2}\sigma^2 \pi_0^2 - \lambda v \sigma \pi_0^2 - \lambda \pi_-^2 \pi_+^2 - \lambda \pi_- \pi_+ \pi_0^2 - \frac{\lambda}{4}\pi_0^4 \\ & + a^2 v \sigma - g\bar{\psi}\psi\sigma - ig\gamma^5 \bar{\psi}(\tau_+ \pi_+ + \tau_- \pi_- + \tau_3 \pi_0)\psi, \end{aligned} \tag{7}$$

and the tree-level potential can be expressed as

$$V_{\text{tree}} = -\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4. \tag{8}$$

As can be seen from Eqs. (6), (7), and (8), there are new terms which depend on v. In particular, the fields develop dynamic masses given by

$$m_\sigma^2 = 3\lambda v^2 - a^2, \qquad m_\pi^2 = \lambda v^2 - a^2, \qquad m_f = gv. \tag{9}$$

The tree-level potential develops a minimum, called the vacuum expectation value of the σ field, namely

$$v_0 = \sqrt{\frac{a^2}{\lambda}}. \tag{10}$$

Notice that when $v = v_0$, the linear term in σ vanishes and the pions become massless. However, the σ and quark fields remain massive.

In order to include a finite vacuum pion mass, one adds an explicit symmetry-breaking term in the Lagrangian of Eq. (6) such that

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{m_\pi^2}{2} v(\sigma + v). \tag{11}$$

This term modifies the tree-level potential. In particular, the minimum is shifted such that

$$v_0 \to v_0' = \sqrt{\frac{a^2 + m_\pi^2}{\lambda}}. \tag{12}$$

Correspondingly, the expressions for the masses, evaluated at the minimum obtained after the explicit breaking of the symmetry, are given by

$$\begin{aligned} m_f(v_0') &= g\sqrt{\frac{a^2 + m_\pi^2}{\lambda}}, \\ m_\sigma^2(v_0') &= 2a^2 + 3m_\pi^2, \\ m_\pi^2(v_0') &= m_\pi^2. \end{aligned} \tag{13}$$

Furthermore, from Eq. (9), we can get an expression for the parameter a, which is given by

$$a = \sqrt{\frac{m_\sigma^2 - 3m_\pi^2}{2}}. \tag{14}$$

Setting $m_\pi = 140$ MeV and $m_\sigma = 400$–600 MeV, we get $a = 225$–390 MeV.

We conclude this section by listing the Feynman rules deduced from the Lagrangian density in Eq. (7). After accounting for the number of permutations for a set of equivalent lines and a factor of i coming from the action, these are displayed in Figs. 1 and 2. Figure 1 shows the vertices arising in the meson sector, and Fig. 2 shows the quark-meson vertices. Dashed lines represent the neutral and charged pions, and double lines represent the σ, whereas thin solid lines represent the d quark and thick solid lines represent the u quark.

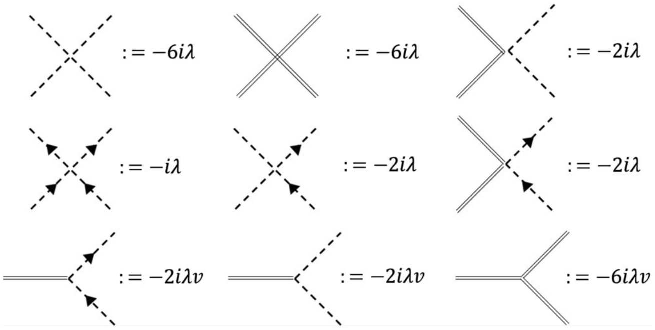


FIG. 1. Meson interactions in the LSMq. Dashed lines are used to represent the neutral and charged pions, whereas double lines represent the σ.

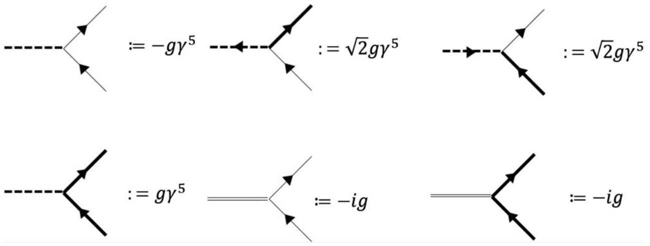


FIG. 2. Quark-meson interactions in the LSMq. Dashed lines represent the neutral and charged pions, whereas the double lines represent the σ. Solid lines represent the quarks. Thin solid lines represent the d quark, and thick solid lines represent the u quark.

III. MAGNETIC-FIELD-DEPENDENT BOSON AND FERMION PROPAGATORS

In order to consider the propagation of the charged modes within a magnetized background, we make the minimal substitution

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu, \tag{15}$$

where q is the particle's electric charge and A_μ is the vector potential. Choosing the magnetic field to point in the direction of the $\hat{z}$ axis, namely $\vec{B} = B\hat{z}$, and working in an arbitrary gauge, we have

$$A^\mu(x) = \frac{1}{2} x_\nu F^{\nu\mu} + \partial^\mu \Lambda(x), \tag{16}$$

where Λ is a well-behaved function which describes a gauge transformation from the symmetric gauge to an arbitrary gauge.

Notice that the ordinary derivative becomes the covariant derivative only for particles with a nonvanishing electric charge. As a consequence, the propagation of charged bosons and fermions is described by propagators in the presence of a constant magnetic field. Using Schwinger's proper time representation, the fermion propagator can be written as

$$S(x, x') = e^{i\Phi(x,x')} S(x - x'), \tag{17}$$

where $\Phi(x, x')$ is the Schwinger phase given by

$$\Phi(x, x') = q \int_x^{x'} d\xi_\mu \left[A^\mu(\xi) + \frac{1}{2} F^{\mu\nu}(\xi - x')_\nu \right] \tag{18}$$

and represents the translationally and gauge noninvariant part of the propagator in the presence of a magnetic background. Using Eq. (16) with Eq. (18), the Schwinger phase can be computed using the expression

$$\Phi(x,x') = q\left[\frac{1}{2}x^{\mu}F_{\mu\nu}x'^{\nu} + \Lambda(x') - \Lambda(x)\right]. \tag{19}$$

The translationally and gauge-invariant part of the propagator is provided by $S(x-x')$, which can be expressed in terms of its Fourier transform as

$$S(x-x') = \int \frac{d^4p}{(2\pi)^4} S(p) e^{-ip\cdot(x-x')}, \tag{20}$$

where

$$\begin{aligned} S(p) = {}& \int_0^{\infty} \frac{ds}{\cos(|q_fB|s)} e^{is\left(p_{\parallel}^2 - p_{\perp}^2\frac{\tan(|q_fB|s)}{|q_fB|s} - m_f^2 + i\epsilon\right)} \\ & \times \Bigg[(\cos(|q_fB|s) + \gamma_1\gamma_2 \sin(|q_fB|s)\mathrm{sign}(q_fB)) \\ & \times (m_f + \not{p}_{\parallel}) - \frac{\not{p}_{\perp}}{\cos(|q_fB|s)}\Bigg]. \end{aligned} \tag{21}$$

In a similar fashion, for a charged scalar field we have

$$\begin{aligned} D(x,x') &= e^{i\Phi(x,x')}D(x-x'), \\ D(x-x') &= \int \frac{d^4p}{(2\pi)^4} D(p) e^{-ip\cdot(x-x')}, \end{aligned} \tag{22}$$

with

$$D(p) = \int_0^{\infty} \frac{ds}{\cos(|q_bB|s)} e^{is(p_{\parallel}^2 - p_{\perp}^2\frac{\tan(|q_bB|s)}{|q_bB|s} - m_b^2 + i\epsilon)}, \tag{23}$$

where the boson and fermion masses and electric charges are m_b, q_b and m_f, q_f, respectively.

The propagators in Eqs. (21) and (23) can also be expanded as a sum over Landau levels. In this last representation, the expressions for the charged scalar and fermion propagators are given by

$$iD(p) = 2ie^{-\frac{p_{\perp}^2}{|q_bB|}} \sum_{n=0}^{\infty} \frac{(-1)^n L_n^0(\frac{2p_{\perp}^2}{|q_bB|})}{p_{\parallel}^2 - m_b^2 - (2n+1)|q_bB| + i\epsilon}, \tag{24}$$

$$iS(p) = ie^{-\frac{p_{\perp}^2}{|q_fB|}} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(p)}{p_{\parallel}^2 - m_f^2 - 2n|q_fB| + i\epsilon}, \tag{25}$$

respectively, where

$$\begin{aligned} D_n(p) = {}& 2(\not{p}_{\parallel} + m_f)\mathcal{O}^{+} L_n^0\left(\frac{2p_{\perp}^2}{|q_fB|}\right) \\ & - 2(\not{p}_{\parallel} + m_f)\mathcal{O}^{-} L_{n-1}^0\left(\frac{2p_{\perp}^2}{|q_fB|}\right) \\ & + 4\not{p}_{\perp} L_{n-1}^1\left(\frac{2p_{\perp}^2}{|q_fB|}\right), \end{aligned} \tag{26}$$

and $L_n^m(x)$ are the generalized Laguerre polynomials. In Eq. (26), the operators $\mathcal{O}^{\pm}$ are defined as

$$\mathcal{O}^{\pm} = \frac{1}{2}(1 \pm i\gamma_1\gamma_2\mathrm{sign}(qB)). \tag{27}$$

We now proceed to use the interaction vertices and the magnetic-field-dependent propagators to find the one-loop corrections to the boson self-coupling and boson-fermion coupling in the presence of a magnetic field.

IV. MAGNETIC CORRECTIONS TO THE BOSON SELF-COUPLING

The magnetic-field-induced corrections to the boson self-coupling λ can be obtained at one-loop order from the Feynman diagram depicted in Fig. 3, where the loop pions are the charged ones. In our approximation, the external particles are taken as plane waves—that is, the states they represent do not experience the effects of the magnetic field. The only particles affected by the magnetic background are the charged loop particles. With this approach, we intend to capture the distinction between the modification of the interaction, that in a perturbative approach is a short-distance effect, from the asymptotic propagation of the external particles, which corresponds to a long-distance effect. Therefore, since the correction we look for is, in this sense, independent of whether the external bosons are charged or neutral, the electric charge of the external particles is irrelevant. Thus, the correction we look for is written as

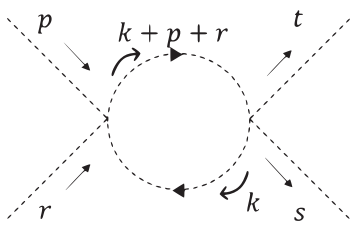


FIG. 3. Feynman diagram representing the magnetic correction to the boson self-coupling at one loop. The loop particles are considered as electrically charged, whereas the external ones can be either charged or neutral.

$$-i6\lambda\Gamma^B_\lambda = \int \frac{d^4k}{(2\pi)^4}(-2i\lambda)iD_{\pi^-}(k)(-2i\lambda) \times iD_{\pi^-}(k+p+r) + \mathrm{CC}, \tag{28}$$

where CC denotes the charge conjugate term, and the subindex in the boson propagator indicates the propagating species. Notice that since the loop involves the same propagating particle, the Schwinger phase vanishes.

According to the explicit computation shown in Appendix A and in the *static* limit $p_0, r_0 \to 0$, $\vec{p} = \vec{r} = \vec{0}$, we obtain

$$\Gamma^B_\lambda = -\frac{\lambda}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \psi^0\left(\frac{|q_bB| + m_\pi^2}{2|q_bB|}\right) + \ln\left(\frac{\mu^2}{2|q_bB|}\right)\right], \tag{29}$$

where ψ^0 is the digamma function and $|q_bB| = |eB|$. In the modified minimal subtraction scheme $\overline{\mathrm{MS}}$, the first three terms in Eq. (29) are associated with the corresponding vertex counterterm. Therefore, the finite magnetic correction to the boson self-coupling is given by

$$\Gamma^B_\lambda = -\frac{\lambda}{12\pi^2}\left[\ln\left(\frac{\mu^2}{2|q_bB|}\right) - \psi^0\left(\frac{|q_bB| + m_\pi^2}{2|q_bB|}\right)\right]. \tag{30}$$

Notice that the result in Eq. (30) depends on the ultraviolet renormalization scale μ. In order to gain some insight on the appropriate choice of this scale, we can compare this result with the one obtained in the strong field limit, where, as a good approximation, one can consider just the lowest Landau level (LLL) contribution, $n = 0$, for the charged boson propagators of Eq. (24), namely

$$iD^{LLL}(p) = \frac{2ie^{-\frac{p_\perp^2}{|q_bB|}}}{p_\parallel^2 - m_b^2 - |q_bB| + i\epsilon}. \tag{31}$$

Therefore, using Eq. (31) with Eq. (28), and working also in the static limit, the magnetic correction to the boson self-coupling in the LLL is given by

$$\Gamma^{LLL}_\lambda = -\frac{\lambda}{6\pi^2}\frac{|q_bB|}{|q_bB| + m_\pi^2}, \tag{32}$$

which is independent of μ. On the other hand, in the absence of a magnetic field, it is easy to show that the one-loop correction to the boson self-coupling is given by

$$\Gamma_\lambda = -\frac{\lambda}{12\pi^2}\ln\left(\frac{\mu^2}{m_\pi^2}\right). \tag{33}$$

Notice that in order to obtain the limits when $|q_bB| \to 0$ in Eq. (33), and when $|q_bB| \to \infty$ in Eq. (32), from the arbitrary field strength result of Eq. (30), it is necessary that μ depends on $|q_bB|$. In fact, the match is obtained when μ^2 is explicitly chosen as $\mu^2 = m_\pi^2 + 2|q_bB|$, for which the arbitrary field strength result becomes

$$\Gamma^B_\lambda = -\frac{\lambda}{12\pi^2}\left[\ln\left(\frac{m_\pi^2 + 2|q_bB|}{2|q_bB|}\right) - \psi^0\left(\frac{|q_bB| + m_\pi^2}{2|q_bB|}\right)\right]. \tag{34}$$

With this choice, the result reproduces the behavior of the coupling in both extreme limiting values of $|q_bB|$, and it is also compatible with the behavior of the coupling found in Ref. [75] for the weak field case. This behavior is shown in Fig. 4, where we plot the effective magnetic-field-dependent boson self-coupling $\lambda^{\mathrm{eff}} = \lambda(1 + \Gamma^B_\lambda)$ as a function of the field strength. In contrast, when μ is taken at a fixed value, the arbitrary field result does not match the LLL case. We interpret this result as signaling that when the field strength is the largest energy scale, μ needs to be taken also as this large scale, since otherwise the computation is not consistent when the strength of the magnetic field surpasses a given fixed scale. At the same time, when the field strength vanishes, the only remaining energy scale is the pion mass, and μ needs to be taken solely as this energy scale. Furthermore, notice that $2|q_bB|$ corresponds to the square of the energy gap between Landau levels, and thus that in order for μ to correspond to the

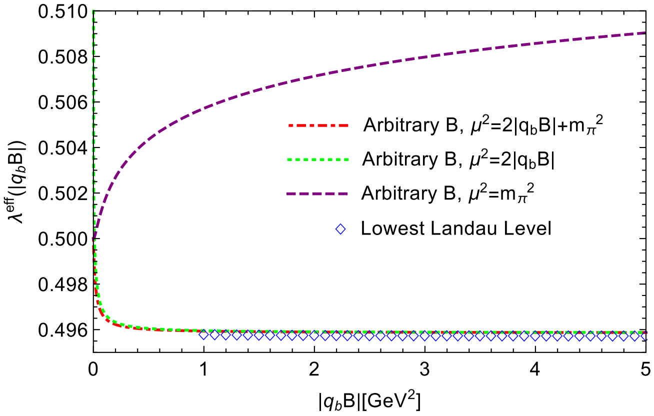


FIG. 4. Comparison of the magnetic field dependence of the effective boson self-coupling $\lambda^{\mathrm{eff}} = \lambda(1 + \Gamma^B_\lambda)$ in the arbitrary field approach and the strong field limit, both computed in the static limit. For the calculation, we use $\lambda = 0.5$ and $m_\pi = 0.140$ GeV. Shown are the cases where for the arbitrary field intensity calculation, the ultraviolet renormalization scale μ^2 is taken as $m_\pi^2 + 2|q_bB|$ (red dashed line), $\mu^2 = 2|q_bB|$ (green dotted line), and a fixed value $\mu^2 = m_\pi^2$ (purple dashed line). Notice that, although the choice $\mu^2 = 2|q_bB|$ does give a good description of the LLL result for large field strengths, when $|q_bB| = 0$ the effective coupling diverges, which signals that this choice is not appropriate. For the rest of the cases, the self-coupling relative change from the vacuum value is rather small, of order 0.8%.

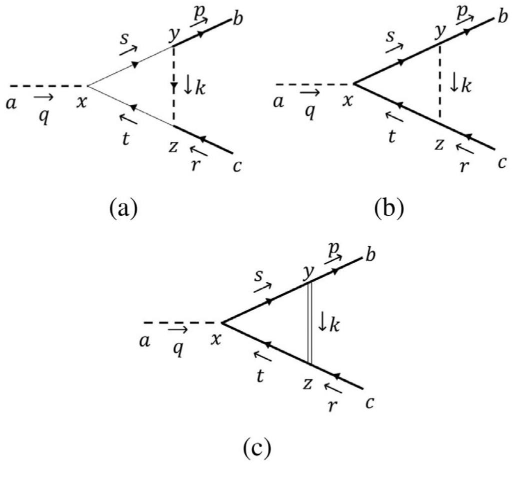
largest energy scale, it is important that for large values of the field strength, μ^2 is taken as the square of this energy gap. In contrast, as also shown in Fig. 4, the usual prescription [34,76], whereby one just subtracts the vacuum correction, represented by the purple dashed line computed with $\mu^2 = m_\pi^2$, behaves opposite to what is expected from the result obtained using the LLL propagator. Since the latter provides a reliable approximation for large field strengths, we conclude that a simple vacuum subtraction prescription leads to a nonreliable limit for large values of the field strength.

V. MAGNETIC CORRECTIONS TO THE BOSON-FERMION COUPLING

The magnetic corrections to the coupling constant g at one-loop level can be obtained from the sum of the three Feynman diagrams depicted in Fig. 5. Since the correction can be obtained from the sum of the allowed Feynman diagrams coupling one boson and two quarks, here we consider the magnetic correction to the boson-fermion coupling for the choice of external particles shown in Fig. 5. Also, as discussed in the previous section, since the use of propagators in the LLL approximation provides a reliable description in the case of the strong field limit, we hereby restrict ourselves to this case using the LLL approximation, Eq. (31), for the boson propagator and the fermion propagator also in the LLL, given by

$$iS^{LLL}(k) = 2ie^{\frac{-k_\perp^2}{|q_f B|}} \frac{\not{k}_\parallel + m_f}{k_\parallel^2 - m_f^2 + i\epsilon} \mathcal{O}^{\pm}. \tag{35}$$

We start by computing the contribution from the diagram in Fig. 5(a). We first compute the quantity $I^B_{1,g}$, which is given explicitly by



(a) (b)

(c)

FIG. 5. Feynman diagrams that contribute to the boson-fermion coupling at one-loop order. The diagrams show the case with a neutral pion and two u quarks as the external particles.

$$\begin{aligned} I^B_{1,g} = \int d^4x d^4y d^4x \int \frac{d^4s}{(2\pi)^4} \frac{d^4t}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} e^{i\Phi_{1,l}} e^{-ip\cdot y} \\ \times (\sqrt{2}g\gamma^5) e^{-is\cdot(x-y)} iS_d(s)(-g\gamma^5) e^{iq\cdot x} e^{-it\cdot(z-x)} \\ \times iS_d(t)(\sqrt{2}g\gamma^5) e^{-ik\cdot(y-z)} iD_{\pi^-}(k) e^{ir\cdot z} + \mathrm{CC}. \end{aligned} \tag{36}$$

The information from the Schwinger phases is contained in the function $\Phi_{1,l}(x,y,z)$. This function depends on the space-time points located at the vertices. For the calculation to have a solid physical meaning, this phase should be a gauge-invariant quantity. We proceed to show this fact explicitly.

Notice that the *total* Schwinger phase $\Phi_{1,t}$ associated with the Feynman diagram in Fig. 5(a) contains not only the information of the space-time points at the interaction vertices x, y, z, but also the information coming from the *external* space-time points a, b, c. Therefore $\Phi_{1,t}$ is given explicitly by

$$\begin{aligned} \Phi_{1,t} = \Phi_d(x,y) + \Phi_{\pi^-}(y,z) + \Phi_d(z,x) + \Phi_u(y,b) \\ + \Phi_u(c,z). \end{aligned} \tag{37}$$

Using Eq. (19) with Eq. (37), we have

$$\begin{aligned} \Phi_{1,t} = -\frac{1}{2} q_d F_{\mu\nu}(y^\mu x^\nu + x^\mu z^\nu) - \frac{1}{2} q_{\pi^-} F_{\mu\nu} z^\mu y^\nu \\ - \frac{1}{2} q_u F_{\mu\nu}(b^\mu y^\nu + z^\mu c^\nu) + q_u[\Lambda(b) - \Lambda(c)]. \end{aligned} \tag{38}$$

Notice that terms depending on Λ evaluated at the internal space-time points add up to zero. Therefore, the integration over the configuration space becomes independent of the gauge choice. However, this would not be the case were we just to consider the phase factors associated with the particles within the loop, since the result of the integration would then become gauge dependent. This observation is essential, since otherwise one faces a nonconservation of electric charge at each vertex when just considering the phases within the loop. On the other hand, Eq. (38) contains a mixing between the phases associated with loop particles, $\Phi_{1,l}$, and the phases from external particles, Φ_{ext}, where the last term is associated with the external charged lines in the diagram and can be written as

$$\Phi_{\mathrm{ext}} = -\frac{1}{2} q_u F_{\mu\nu}(b^\mu x^\nu + x^\mu c^\nu) + q_u[\Lambda(b) - \Lambda(c)]. \tag{39}$$

In order to separate these contributions, we write

$$\Phi_{1,t} = \Phi_{1,l} + \Phi_{\mathrm{ext}}. \tag{40}$$

We resort to considering that the external particles can be described as plane waves. Physically, this means that we consider the propagation of the external particles during short distances and times. In this manner, we neglect long-distance effects introduced when the magnetic field acts over the external particles. Therefore, we can take $y^\mu \approx b^\mu$ and $z^\mu \approx c^\mu$ such that

$$\Phi_{1,t} = -\frac{1}{2} q_u F_{\mu\nu}(b^\mu x^\nu + x^\mu c^\nu) + q_u[\Lambda(b) - \Lambda(c)], \\ -\frac{1}{2} q_{\pi^-} F_{\mu\nu}(y^\mu x^\nu + z^\mu y^\nu + x^\mu z^\nu). \tag{41}$$

Using this approximation, we can separate the phase factors coming from external and internal loop particles. Thus, for the computation of the magnetic field correction for the coupling g, we need only to account for the last term in Eq. (41), whereas the first and second terms in Eq. (41) are associated with the external phase given by Eq. (39). Therefore, we have

$$\Phi_{1,l} = -\frac{1}{2} q_{\pi^-} F_{\mu\nu}(y^\mu x^\nu + z^\mu y^\nu + x^\mu z^\nu). \tag{42}$$

It is important to note that the contribution from the Schwinger phase is gauge invariant. Using the fact that $F_{21} = -F_{12} = |B|$ and $q_{\pi^-} = -|e|$, we get

$$\Phi_{1,l} = \frac{1}{2}|eB|\varepsilon_{ij}(x_i y_j + y_i z_j + z_i x_j), \qquad i, j = 1, 2, \tag{43}$$

where ε_{ij} is the Levi-Civita symbol. Having identified the Schwinger phase contribution, we can perform the integration over coordinates. Upon doing so, we obtain the energy-momentum conservation for the external particles, and can write

$$I^B_{1,g} = (2\pi)^4 \delta^{(4)}(p - r - q) g\gamma^5 \Gamma^B_{1,g}, \tag{44}$$

where $g\gamma^5\Gamma_{1,g}$ is identified as the contribution to the magnetic field correction to the vertex, given explicitly by

$$g\gamma^5\Gamma^B_{1,g} = \int \frac{d^2 s_\perp d^2 t_\perp}{\pi^2 |eB|^2} \frac{d^4 k}{(2\pi)^4} (\sqrt{2} g\gamma^5) i S_d(k_\parallel + p_\parallel, s_\perp) \\ \times (-g\gamma^5) i S_d(k_\parallel + r_\parallel, t_\perp)(\sqrt{2} g\gamma^5) i D_{\pi^-}(k_\parallel, k_\perp) \\ \times e^{i\frac{2}{|eB|}\varepsilon_{ij}(s-q-t)_i (s-p-k)_j} + \mathrm{CC}. \tag{45}$$

Following the procedure explicitly shown in Appendix B and the static limit $p_0 = r_0 = m_f$ and $\vec{p} = \vec{r} = \vec{0}$, we get

$$\Gamma^{LLL}_{1,g} = \frac{g^2 |eB|}{16\pi^2 m_f^2} \int_0^1 du \frac{u}{u^2 + \alpha(1-u)} \\ \times \left[1 + \frac{(2-u)u}{u^2 + \alpha(1-u)}\right], \tag{46}$$

where $\alpha = (m_\pi^2 + |eB|)/m_f^2$.

Next, we compute the contribution from the Feynman diagram depicted in Fig. 5(b). This contribution can be obtained from the function $I^B_{2,g}$, which is given by

$$I^B_{2,g} = \int d^4x d^4y d^4z \int \frac{d^4 s}{(2\pi)^4} \frac{d^4 t}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} e^{i\Phi_{2,t}} e^{-ip\cdot y} \\ \times (g\gamma^5) e^{-is\cdot(x-y)} i S_u(s)(g\gamma^5) e^{iq\cdot x} e^{-it\cdot(z-x)} \\ \times i S_u(t)(g\gamma^5) e^{-ik\cdot(y-z)} i D_{\pi^0}(k) e^{ir\cdot z} + \mathrm{CC}. \tag{47}$$

In a similar fashion, we first compute the Schwinger phase associated with the whole diagram in Fig. 5(b)—namely,

$$\Phi_{2,t} = \Phi_u(x, y) + \Phi_u(z, x) + \Phi_u(y, b) + \Phi_u(c, z). \tag{48}$$

Using Eq. (19) with Eq. (48), we get

$$\Phi_{2,t} = -\frac{1}{2} F_{\mu\nu} q_u (y^\mu x^\nu + b^\mu y^\nu + x^\mu z^\nu + z^\mu c^\nu) \\ + q_u[\Lambda(b) - \Lambda(c)]. \tag{49}$$

Once again, terms that depend on Λ, evaluated at internal points, vanish. On the other hand, the Schwinger phase associated with the tree-level diagram is given by Eq. (39). Adding and subtracting the first term from this equation and Eq. (49), we have

$$\Phi_{2,t} = -\frac{1}{2} F_{\mu\nu} q_u (y^\mu x^\nu + b^\mu y^\nu + x^\mu z^\nu + z^\mu c^\nu) \\ + q_u[\Lambda(b) - \Lambda(c)] - \frac{1}{2} q_u F_{\mu\nu}(b^\mu x^\nu + x^\mu c^\nu) \\ + \frac{1}{2} q_u F_{\mu\nu}(b^\mu x^\nu + x^\mu c^\nu). \tag{50}$$

Assuming that $y^\mu \approx b^\mu$ and $z^\mu \approx c^\mu$ (short space-time interval propagation after the interaction), we can write

$$\Phi_{2,t} = -\frac{1}{2} q_u F_{\mu\nu}(b^\mu x^\nu + x^\mu c^\nu) + q_u[\Lambda(b) - \Lambda(c)]. \tag{51}$$

This result coincides with Eq. (39). Therefore, we can conclude that the Schwinger phase associated with the loop particles vanishes:

$$\Phi_{2,l} = 0. \tag{52}$$

Upon integration over configuration space, we can identify the contribution to the magnetic correction from this diagram, $g\gamma^5\Gamma_{2,g}$, as

$$I^B_{2,g} = (2\pi)^4 \delta^{(4)}(p - r - q) g\gamma^5 \Gamma_{2,g}. \tag{53}$$

Again, notice that by using this approximation, we recover the energy-momentum conservation for the external particles, whereas the magnetic correction is associated with the loop and can be expressed as

$$g\gamma^5\Gamma^B_{2,g} = \int \frac{d^4k}{(2\pi)^4}(g\gamma^5)iS_u(k+p)(g\gamma^5)iS_u(k+r) \times (g\gamma^5)iD_{\pi^0}(k) + \mathrm{CC}. \tag{54}$$

The computation of this quantity is explicitly performed in Appendix B in the strong field limit and can be expressed as

$$\Gamma^{LLL}_{2,g} = -\frac{g^2}{2\pi^2 m_f^2}\int_0^1 du \int_0^\infty dk_\perp k_\perp e^{-\frac{3k_\perp^2}{|eB|}} \times \frac{u}{u^2+\beta(1-u)}\left[1+\frac{(2-u)u}{u^2+\beta(1-u)}\right], \tag{55}$$

where $\beta = (k_\perp^2 + m_\pi^2)/m_f^2$.

The diagram in Fig. 5(c) can be computed from the quantity $I^B_{3,g}$, given explicitly by

$$I^B_{3,g} = \int d^4x d^4y d^4z \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{i\Phi_{3,l}} e^{-ip\cdot y} \times (-ig)e^{-is\cdot(x-y)} iS_u(s)(g\gamma^5)e^{iq\cdot x}e^{-it\cdot(z-x)} \times iS_u(t)(-ig)e^{-ik\cdot(y-z)} iD_\sigma(k)e^{ir\cdot z} + \mathrm{CC}. \tag{56}$$

In a similar fashion, one can compute the Schwinger phase from this loop, $\Phi_{3,l}(x,y,z)$. It is easy to see that this phase satisfies $\Phi_{3,t} = \Phi_{2,t}$, and therefore, the internal Schwinger phase vanishes when considering short-range propagation of the external particles—namely,

$$\Phi_{3,l} = 0. \tag{57}$$

After performing the integration over the configuration space, we obtain the relation between $I^B_{3,g}$ and the contribution to the magnetic correction to the boson-fermion coupling, $g\gamma^5\Gamma_{3,g}$, given by

$$I^B_{3,g} = (2\pi)^4\delta^{(4)}(p-r-q)g\gamma^5\Gamma_{3,g}, \tag{58}$$

with

$$g\gamma^5\Gamma^B_{3,g} = \int \frac{d^4k}{(2\pi)^4}(-ig)iS_u(k+p)(g\gamma^5)iS_u(k+r) \times (-ig)iD_\sigma(k) + \mathrm{CC}. \tag{59}$$

Once again, using the LLL propagators and following the explicit procedure shown in Appendix B, we get

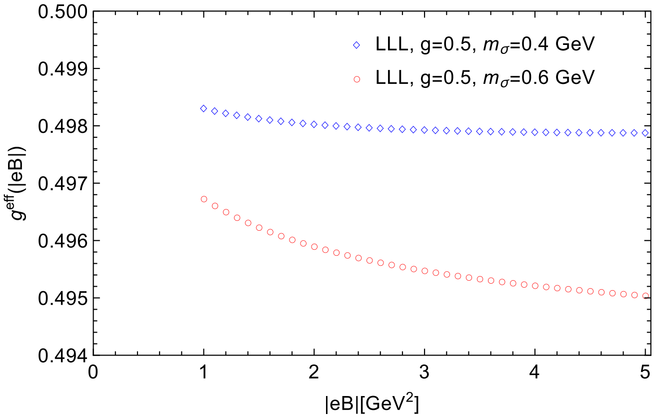


FIG. 6. Magnetic field dependence of the effective boson-fermion coupling $g^{\mathrm{eff}} = g(1+\Gamma^B_g)$ in the static limit and the strong field approximation. For the calculation we used $g = 0.5$, $m_\pi = 0.140$ GeV, $m_f = 0.3$ GeV, and the two values $m_\sigma = 0.4$, 0.6 GeV. In both cases, g^{eff} monotonically decreases in an interval $|eB| = 1$–5 GeV^2. Notice that the relative change with regard to the vacuum is of order 0.5% and 1%, respectively.

$$\Gamma^{LLL}_{3,g} = \frac{g^2}{2\pi^2 m_f^2}\int_0^1 du \int_0^\infty dk_\perp k_\perp e^{-\frac{3k_\perp^2}{|eB|}} \times \frac{u}{u^2+\gamma(1-u)}\left[1+\frac{(2-u)u}{u^2+\gamma(1-u)}\right], \tag{60}$$

where $\gamma = (k_\perp^2 + m_\sigma^2)/m_f^2$.

The total magnetic correction to the boson-fermion coupling in the strong field limit is given by the sum of the three contributions—namely,

$$\Gamma^{LLL}_g = \Gamma^{LLL}_{1,g} + \Gamma^{LLL}_{2,g} + \Gamma^{LLL}_{3,g}. \tag{61}$$

The effective boson-fermion coupling, g^{eff}, is thus given by

$$g^{\mathrm{eff}} = g(1+\Gamma^{LLL}_g). \tag{62}$$

Figure 6 shows the behavior of the boson-fermion coupling as a function of the field strength. For the calculation we set $m_\pi = 0.140$ GeV, $m_f = 0.3$ GeV and $m_\sigma = 0.4$, 0.6 GeV. Notice that the coupling decreases monotonically over a large range of the field strength. However, the relative change is rather small.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have computed the magnetic-field-induced corrections to the boson self-coupling and to the boson-fermion coupling in the LSMq, in the static limit. For the former, we have performed the computation for an arbitrary field strength, as well as in the strong field approximation. For the latter, we worked in the strong field limit. We have shown that the full magnetic field corrections for the boson self-coupling depend on the ultraviolet renormalization scale, and that this dependence cannot be removed by the usual vacuum subtraction. The reason for this behavior is that for a fixed ultraviolet renormalization scale, the calculation is not valid any longer when the field strength surpasses that fixed value, thus becoming the largest energy scale. Taking as a guide the result in the strong field limit, we have found the appropriate choice for the renormalization scale that produces the expected behavior for the two extreme limits—namely, when the field vanishes, or when this becomes very large.

For the calculation of the effective boson-fermion coupling, we have shown that when considering that the external charged particles propagate only during short space-time intervals, the effects coming from the Schwinger phase become gauge invariant, and that the usual energy-momentum conservation can be factored out from the vertex function.

Recall that vertex corrections are functions of arbitrary values of the external particle's momenta. In this work, we have considered the static limit approximation whereby the vertices are computed for vanishing external momenta. We emphasize that this approximation goes in hand with the short-distance approximation whereby the motion of these particles is considered to happen during very short times, so as to also neglect the magnetic field effects that would otherwise build up during large times, deviating the motion of external charged particles from free propagation. Within these approximations, the effective boson self-coupling and boson-fermion coupling show a modest monotonic decrease over a large interval of magnetic field strengths.

The results of this work can now be used to find the corrections to the mass of neutral and charged pions introduced by magnetic field effects. Other possible scenarios of physical interest where the findings of this work can have a potential impact include the properties of the nuclear equation of state within dense and compact astrophysical objects, such as the cores of neutron stars, which are affected by magnetic-field-dependent baryon and meson masses and couplings, and the shear and bulk viscosity in quark-meson matter. The first of these topics is currently being actively pursued and will be soon reported elsewhere.

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APPENDIX A: MAGNETIC CORRECTIONS TO THE BOSON SELF-COUPLING

To compute the magnetic correction to the boson self-coupling, we start from the Landau level representation of the charged boson propagator in Eq. (24) and use it in the expression for the magnetic correction to λ given by

$$-i6\lambda\Gamma_\lambda^B = \int \frac{d^4k}{(2\pi)^4}(-2i\lambda)iD_{\pi^-}^B(k)(-2i\lambda) \times iD_{\pi^-}^B(k+p+r) + \text{CC}. \tag{A1}$$

Performing a Wick rotation in k and $s = r + p$, such that $k_0 \to ik_4$ and $s_0 \to is_4$, then

$$k_\|^2 \to -k_{E\|}^2, (k+s)_\|^2 \to -(k+s)_{E\|}^2, d^4k \to id^4k_E. \tag{A2}$$

We now introduce two Schwinger parameters, x_1, x_2, $d^2x = dx_1dx_2$, such that the magnetic correction can be written as

$$\Gamma_\lambda^B = -\frac{16}{3}\lambda\int\frac{d^4k_E}{(2\pi)^4}\int d^2x\sum_{n,m=0}^{\infty} r_1^n r_2^m L_n^0(s_1)L_m^0(s_2) \times e^{-\frac{k_\perp^2}{|q_bB|}-\frac{(k+s)_\perp^2}{|q_bB|}-x_1[\alpha(k_{E\|})+|q_bB|]-x_2[\beta(k_{E\|})+|q_bB|]}, \tag{A3}$$

where

$$\begin{aligned} s_1 = 2k_\perp^2/|q_bB|, s_2 &= 2(k+s)_\perp^2/|q_bB|, \\ \alpha(k_{E\|}) &= k_{E\|}^2 + m_\pi^2 - i\epsilon, \\ \beta(k_{E\|}) &= (k+s)_{E\|}^2 + m_\pi^2 - i\epsilon, \end{aligned} \tag{A4}$$

and $r_i = -e^{-2|q_bB|x_i}$, $i = 1, 2$. Using the generating function of Laguerre polynomials

$$\sum_{n=0}^{\infty} r_i^n L_n^0(s_i) = \frac{1}{1-r_i}e^{-\frac{r_i}{1-r_i}s_i}, \tag{A5}$$

we obtain

$$\Gamma_\lambda^B = -\frac{16}{3}\lambda\int d^2x\frac{e^{-(x_1+x_2)|q_bB|}}{(1-r_1)(1-r_2)}I(x_1,x_2)J(x_1,x_2), \tag{A6}$$

where we define

$$
\begin{aligned}
I(x_1,x_2) &= \int \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{k_\perp^2}{|q_bB|}(1-2\eta(x_1))-\frac{(k+s)_\perp^2}{|q_bB|}(1-2\eta(x_2))},\\
J(x_1,x_2) &= \mu^{4-d}\int \frac{d^{d-2}k_{E\parallel}}{(2\pi)^{d-2}} e^{-x_1\alpha(k_{E\parallel})-x_2\beta(k_{E\parallel})},\\
\eta(x_i) &= \frac{1}{e^{2|q_bB|x_i}+1},
\end{aligned}
\tag{A7}
$$

with $i = 1, 2$ and $\epsilon \to 0$. To carry out the integrals, we use dimensional regularization, namely

$$
\int \frac{d^4k_E}{(2\pi)^4} \to \mu^{4-d}\int \frac{d^{d-2}k_{E\parallel}}{(2\pi)^{d-2}}\int \frac{d^2k_\perp}{(2\pi)^2}. \tag{A8}
$$

First, to find $I(x_1,x_2)$, we consider the change of variables

$$
q_\perp = k_\perp + \frac{1-2\eta(x_2)}{2(1-\eta(x_1)-\eta(x_2))}s_\perp, \tag{A9}
$$

and the identity

$$
1-2\eta(x_i) = \tanh(|q_bB|x_i). \tag{A10}
$$

Completing the square, we have

$$
I(x_1,x_2) = \frac{|q_bB|/4\pi}{\tanh(|q_bB|x_1)+\tanh(|q_bB|x_2)}\exp\left[-\frac{\tanh(|q_bB|x_1)\tanh(|q_bB|x_2)}{|q_bB|(\tanh(|q_bB|x_1)+\tanh(|q_bB|x_2))}s_\perp^2\right]. \tag{A11}
$$

Next, $J(x_1,x_2)$ can be found using the change of variables

$$
q_{E\parallel} = k_{E\parallel} + \frac{x_2}{x_1+x_2}s_{E\parallel}. \tag{A12}
$$

Carrying out the integral and using $d = 4-2\varepsilon$, we obtain

$$
J(x_1,x_2) = \mu^{2\varepsilon}\left(\frac{1}{4\pi(x_1+x_2)}\right)^{1-\varepsilon} e^{-\frac{x_1x_2}{x_1+x_2}s_{E\parallel}^2-(x_1+x_2)m_\pi^2}. \tag{A13}
$$

Using the identities

$$
\frac{e^{-x_i|q_bB|}}{1-r_i} = \frac{1}{2\cosh(|q_bB|x_i)}, \tag{A14}
$$

$$
\frac{1}{\sinh(|q_bB|(x_1+x_2))} = \frac{1}{\tanh(|q_bB|x_1)+\tanh(|q_bB|x_2)}\frac{1}{\cosh(|q_bB|x_1)\cosh(|q_bB|x_2)} \tag{A15}
$$

together with Eqs. (A11) and (A13), we get

$$
\begin{aligned}
\Gamma_\lambda^B = {}& -\frac{\lambda}{12\pi^2}\int d^2x \frac{(4\pi\mu^2)^\varepsilon}{(x_1+x_2)^{1-\varepsilon}}\frac{|q_bB|}{\sinh(|q_bB|(x_1+x_2))}\exp\left[-\frac{\tanh(|q_bB|x_1)\tanh(|q_bB|x_2)}{|q_bB|(\tanh(|q_bB|x_1)+\tanh(|q_bB|x_2))}s_\perp^2\right]\\
&\times\exp\left[-\frac{x_1x_2}{x_1+x_2}s_{E\parallel}^2-(x_1+x_2)m_\pi^2\right].
\end{aligned}
\tag{A16}
$$

We perform the change of variables

$$
x_1 = s(1-y), \qquad x_2 = sy, \qquad dx_1dx_2 = sdsdy. \tag{A17}
$$

These variables have the domains $0 < y < 1$ and $s > 0$. Substituting these new variables, we obtain

$$
\begin{aligned}
\Gamma_\lambda^B = {}& -\frac{\lambda}{12\pi^2}\int_0^\infty ds\int_0^1 dy(4\pi\mu^2 s)^\varepsilon\frac{|q_bB|}{\sinh(|q_bB|s)}\exp\left[-\frac{\tanh(|q_bB|s(1-y))\tanh(|q_bB|sy)}{|q_bB|(\tanh(|q_bB|s(1-y))+\tanh(|q_bB|sy))}s_\perp^2\right]\\
&\times\exp\left[-sy(1-y)s_{E\parallel}^2-sm_\pi^2\right].
\end{aligned}
\tag{A18}
$$

Equation (A18) is the general expression for the magnetic correction to the boson self-coupling. Notice that this expression contains a divergence that should be regularized. Considering the static limit in Eq. (A18), which implies that $s_{E\parallel}^2 \to 0$ and $s_\perp^2 = 0$, the general magnetic correction reduces to

$$\Gamma_\lambda^B = -\frac{\lambda}{12\pi^2}\int_0^\infty ds (4\pi\mu^2 s)^\varepsilon \frac{|q_bB|}{\sinh(|q_bB|s)} e^{-sm_\pi^2}. \tag{A19}$$

Notice that in this limit, both integrals can be solved analytically:

$$\begin{aligned}\int_0^\infty ds \frac{s^\varepsilon e^{-sm_\pi^2}}{\sinh(|q_bB|s)} = \frac{1}{|q_bB|}\left(\frac{1}{2|q_bB|}\right)^\varepsilon \Gamma(\varepsilon+1)\\ \times \zeta\left(\varepsilon+1, \frac{|q_bB|+m_\pi^2}{2|q_bB|}\right),\end{aligned} \tag{A20}$$

where ζ is the Hurwitz zeta function. Considering an expansion for $\varepsilon \to 0$, we have

$$\begin{aligned}\left(\frac{4\pi\mu^2}{2|q_bB|}\right)^\varepsilon \approx 1 + \varepsilon \ln\left(\frac{4\pi\mu^2}{2|q_bB|}\right),\\ \Gamma(\varepsilon+1) \approx 1 - \varepsilon\gamma_E,\\ \zeta\left(\varepsilon+1, \frac{|q_bB|+m_\pi^2}{2|q_bB|}\right) \approx \frac{1}{\varepsilon} - \psi^0\left(\frac{|q_bB|+m_\pi^2}{2|q_bB|}\right),\end{aligned} \tag{A21}$$

where ψ^0 is the digamma function. Therefore, we finally obtain

$$\begin{aligned}\Gamma_\lambda^B = -\frac{\lambda}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \psi^0\left(\frac{|q_bB|+m_\pi^2}{2|q_bB|}\right)\right.\\ \left. + \ln\left(\frac{\mu^2}{2|q_bB|}\right)\right],\end{aligned} \tag{A22}$$

where $|q_bB| = |eB|$.

APPENDIX B: MAGNETIC CORRECTIONS TO THE BOSON-FERMION COUPLING IN THE STRONG FIELD LIMIT

We start writing the contribution from the diagram in Fig. 5(a), which can be obtained from the expression

$$\begin{aligned}I_{1,g}^B = \int d^4x d^4y d^4x \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{i\Phi_{1,l}} e^{-ip\cdot y}\\ \times (\sqrt{2}g\gamma^5) e^{-is\cdot(x-y)} iS_d(s)(-g\gamma^5) e^{iq\cdot x} e^{-it\cdot(z-x)}\\ \times iS_d(t)(\sqrt{2}g\gamma^5) e^{-ik\cdot(y-z)} iD_{\pi^-}(k) e^{ir\cdot z} + \mathrm{CC},\end{aligned} \tag{B1}$$

where the Schwinger phase contribution is finite and is given by

$$\Phi_{1,l} = \frac{1}{2}|eB|\varepsilon_{ij}(x_iy_j + y_iz_j + z_ix_j), \qquad i,j = 1,2. \tag{B2}$$

The integration over configuration space can be performed using the factorization between parallel and perpendicular components. Recall that for four-vectors a_μ and b_μ,

$$a_\mu b^\mu = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 = a_\parallel \cdot b_\parallel - a_\perp \cdot b_\perp. \tag{B3}$$

Thus, integrating over configuration space and taking into account Eq. (B3) to include the Schwinger phase contribution, we obtain

$$\begin{aligned}I_{1,g}^B = \delta^{(2)}(p-q-r)_\perp \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{4}{|eB|^2}(2\pi)^{10}\\ \times \delta^{(2)}(s-q-t)_\parallel \delta^{(2)}(p-s+k)_\parallel \delta^{(2)}(t-k-r)_\parallel\\ \times (\sqrt{2}g\gamma^5) iS_d(s)(-g\gamma^5) iS_d(t)(\sqrt{2}g\gamma^5) iD_{\pi^-}(k)\\ \times e^{i\frac{2}{|eB|}\varepsilon_{ij}(s-q-t)_i(s-p-k)_j} + \mathrm{CC}.\end{aligned} \tag{B4}$$

We first integrate over $d^2s_\parallel$ and $d^2t_\parallel$, using the Dirac delta distributions to get

$$\begin{aligned}I_{1,g}^B = (2\pi)^4\delta^{(4)}(p-q-r)\int \frac{d^2s_\perp d^2t_\perp}{\pi^2|eB|^2}\frac{d^4k}{(2\pi)^4}(\sqrt{2}g\gamma^5)\\ \times iS_d(k_\parallel + p_\parallel, s_\perp)(-g\gamma^5) iS_d(k_\parallel + r_\parallel, t_\perp)(\sqrt{2}g\gamma^5)\\ \times iD_{\pi^-}(k_\parallel, k_\perp) e^{i\frac{2}{|eB|}\varepsilon_{ij}(s-q-t)_i(s-p-k)_j} + \mathrm{CC}.\end{aligned} \tag{B5}$$

Notice that with this procedure, we can identify the Dirac delta distribution for energy-momentum conservation in Eq. (B5) such that

$$I_{1,g}^B = (2\pi)^4\delta^{(4)}(p-r-q) g\gamma^5\Gamma_{1,g}^B. \tag{B6}$$

The contribution to the magnetic correction to the boson-fermion coupling, $g\gamma^5\Gamma_{1,g}^B$, is thus given by

$$\begin{aligned}g\gamma^5\Gamma_{1,g}^B = \int \frac{d^2s_\perp d^2t_\perp}{\pi^2|eB|^2}\frac{d^4k}{(2\pi)^4}(\sqrt{2}g\gamma^5) iS_d(k_\parallel + p_\parallel, s_\perp)\\ \times (-g\gamma^5) iS_d(k_\parallel + r_\parallel, t_\perp)(\sqrt{2}g\gamma^5) iD_{\pi^-}(k_\parallel, k_\perp)\\ \times e^{i\frac{2}{|eB|}\varepsilon_{ij}(s-q-t)_i(s-p-k)_j} + \mathrm{CC}.\end{aligned} \tag{B7}$$

Equation (B7) is general enough and could be computed using either the complete propagators or approximations to them. In this work, we consider the propagators in the strong field limit. Substituting Eqs. (31) and (35) and adding the charge conjugate contribution, we have

$$\Gamma^{LLL}_{1,g} = \frac{16ig^2}{\pi^2|eB|^2}\int d^2s_\perp d^2t_\perp \frac{d^4k}{(2\pi)^4} e^{-\frac{s_\perp^2}{|q_dB|}-\frac{t_\perp^2}{|q_dB|}-\frac{k_\perp^2}{|q_\pi - B|}} \times \frac{\mathcal{N}_1}{A_1B_1C_1} e^{i\frac{2}{|eB|}\varepsilon_{ij}(s-q-t)_i(s-p-k)_j}, \tag{B8}$$

where we have defined for convenience the quantities

$$\begin{aligned} \mathcal{N}_1 &= (\slashed{k}_\parallel + \slashed{p}_\parallel + m_d)(m_d - \slashed{k}_\parallel - \slashed{r}_\parallel), \\ A_1 &= (k_\parallel + p_\parallel)^2 - m_d^2 + i\epsilon, \\ B_1 &= (k_\parallel + r_\parallel)^2 - m_d^2 + i\epsilon, \\ C_1 &= k_\parallel^2 - m_\pi^2 - |eB| + i\epsilon. \end{aligned} \tag{B9}$$

We also resort to work in the static limit, setting the perpendicular coordinates of external momenta to zero. On doing so, we can integrate over the perpendicular coordinates relative to the magnetic field. The result is given by

$$\Gamma^{LLL}_{1,g} = \frac{ig^2|eB|}{4\pi}\int \frac{d^2k_\parallel}{(2\pi)^2}\frac{\mathcal{N}_1}{A_1B_1C_1}, \tag{B10}$$

introducing the Feynman parametrization

$$\frac{1}{A_1B_1C_1} = \int_0^1 dx \int_0^{1-x} \frac{2dy}{(A_1x + B_1y + C_1(1-x-y))^3}. \tag{B11}$$

The denominator of Eq. (B11) can be expressed as

$$A_1x + B_1y + C_1(1-x-y) = (k_\parallel + xp_\parallel + yr_\parallel)^2 - \Delta + i\epsilon, \tag{B12}$$

where

$$\Delta = (xp_\parallel + yr_\parallel)^2 - xp_\parallel^2 + xm_d^2 - yr_\parallel^2 + ym_d^2 + (1-x-y)(m_\pi^2 + |eB|). \tag{B13}$$

On the other hand, it is useful to consider the change of variables as $k_\parallel = l_\parallel - xp_\parallel - yr_\parallel$, $dk_\parallel = dl_\parallel$. Then the numerator, $\mathcal{N}$, can be written as

$$\mathcal{N}_1 = -l_\parallel^2 - 2xyp_\parallel\cdot r_\parallel + m_d\slashed{p}_\parallel - m_d\slashed{r}_\parallel - x(x-1)p_\parallel^2 - y(y-1)r_\parallel^2 - (1-x-y)\slashed{p}_\parallel\slashed{r}_\parallel + m_d^2, \tag{B14}$$

where we have already discarded linear terms of $l_\parallel$. At this point, we can use the Dirac equation for outgoing states, assuming that they are not affected by the external magnetic field. This means that the spinors satisfy the Dirac equation in vacuum:

$$\bar{u}(p_\parallel)\slashed{p}_\parallel = \bar{u}(p_\parallel)m_u, \qquad \slashed{r}_\parallel u(r_\parallel) = m_u u(r_\parallel). \tag{B15}$$

Here, it is worth noting that in this computation, we assume that the values of the quark masses remain fixed to just their vacuum values, $m_d = m_u = m_f$. Then, taking the static limit, $p_3 = r_3 = 0$ and $p_0 = r_0 = m_f$, we get

$$\bar{u}(p_\parallel)\mathcal{N}_1 u(r_\parallel) = \bar{u}(p_\parallel)(-l_\parallel^2 + 2m_f^2(x+y) - m_f^2(x+y)^2)u(r_\parallel). \tag{B16}$$

Thus, once we consider $\bar{u}(p_\parallel)\Gamma^{LLL}_{1,g}u(r_\parallel)$ and use Eq. (B16), we get

$$\Gamma^{LLL}_{1,g} = \frac{ig^2|eB|}{2\pi}\int_0^1 dx\int_0^{1-x} dy\int\frac{d^2l_\parallel}{(2\pi)^2}\left[\frac{-l_\parallel^2}{(l_\parallel^2 - \Delta + i\epsilon)^3} + \frac{2m_f^2(x+y) - m_f^2(x+y)^2}{(l_\parallel^2 - \Delta + i\epsilon)^3}\right], \tag{B17}$$

where with the above assumptions, Δ is simplified to become

$$\Delta = m_f^2(x+y)^2 + (1-(x+y))(m_\pi^2 + |eB|). \tag{B18}$$

In order to integrate over $d^2l_\parallel$, we consider the following equations:

$$\mu^{4-d}\int\frac{d^{d-2}l_\parallel}{(2\pi)^{d-2}}\frac{1}{(l_\parallel^2 - \Delta)^3} = -\frac{i}{4\pi}\frac{1}{2}\frac{1}{\Delta^2} + \mathcal{O}(\varepsilon), \tag{B19}$$

$$\mu^{4-d}\int\frac{d^{d-2}l_\parallel}{(2\pi)^{d-2}}\frac{l_\parallel^2}{(l_\parallel^2 - \Delta)^3} = \frac{i}{4\pi}\frac{1}{2}\frac{1}{\Delta} + \mathcal{O}(\varepsilon). \tag{B20}$$

According to Eqs. (B19) and (B20), we get

$$\Gamma^{LLL}_{1,g} = \frac{g^2|eB|}{16\pi^2m_f^2}\int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{(x+y)^2 + \alpha(1-(x+y))} + \frac{2(x+y) - (x+y)^2}{((x+y)^2 + \alpha(1-(x+y)))^2}\right], \tag{B21}$$

where $\alpha = (m_\pi^2 + |eB|)/m_f^2$. With the purpose of finding the integral over Feynman parameters, consider the following linear transformation:

$$u = x + y, \qquad v = 1 - x. \tag{B22}$$

The Jacobian satisfies $\det(J) = 1$, and the region of integration becomes $u \in [0,1]$ and $v \in [1-u, 1]$. Thus,

$$\Gamma^{LLL}_{1,g} = \frac{g^2|eB|}{16\pi^2m_f^2}\int_0^1 du\int_{1-u}^1 dv\frac{1}{u^2 + \alpha(1-u)} \times\left[1 + \frac{(2-u)u}{u^2 + \alpha(1-u)}\right]. \tag{B23}$$

Performing the integration over dv, we get the final expression for this contribution:

$$\Gamma_{1,g}^{LLL} = \frac{g^2|eB|}{16\pi^2 m_f^2}\int_0^1 du \frac{u}{u^2+\alpha(1-u)} \times \left[1+\frac{(2-u)u}{u^2+\alpha(1-u)}\right]. \tag{B24}$$

We now proceed with $I_{2,g}$, which is given by

$$I_{2,g}^B = \int d^4x d^4y d^4z \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{i\Phi_{2,l}} e^{-ip\cdot y} \times (g\gamma^5) e^{-is\cdot(x-y)} iS_u(s)(g\gamma^5) e^{iq\cdot x} e^{-it\cdot(z-x)} \times iS_u(t)(g\gamma^5) e^{-ik\cdot(y-z)} iD_{\pi^0}(k) e^{ir\cdot z} + \mathrm{CC}. \tag{B25}$$

Performing the integration over configuration space, we have

$$I_{2,g}^B = \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^{12}\delta^{(4)}(s-t-q) \times \delta^{(4)}(p-s+k)\delta^{(4)}(t-k-r)(g\gamma^5) iS_u(s) \times (g\gamma^5) iS_u(t)(g\gamma^5) iD_{\pi^0}(k) + \mathrm{CC}. \tag{B26}$$

Integrating over d^4s and d^4t, we obtain

$$I_{2,g}^B = (2\pi)^4\delta^{(4)}(p-r-q)\int \frac{d^4k}{(2\pi)^4}(g\gamma^5) iS_u(k+p) \times (g\gamma^5) iS_u(k+r)(g\gamma^5) iD_{\pi^0}(k) + \mathrm{CC}. \tag{B27}$$

At this point, we can identify the contribution to the magnetic correction from this diagram, $g\gamma^5\Gamma_{2,g}^B$, which can be expressed as

$$I_{2,g}^B = (2\pi)^4\delta^{(4)}(p-r-q) g\gamma^5\Gamma_{2,g}^B, \tag{B28}$$

where

$$g\gamma^5\Gamma_{2,g}^B = \int \frac{d^4k}{(2\pi)^4}(g\gamma^5) iS_u(k+p)(g\gamma^5) iS_u(k+r) \times (g\gamma^5) iD_{\pi^0}(k) + \mathrm{CC}. \tag{B29}$$

Using Eqs. (31) and (35) to account for the strong field limit, we have

$$\Gamma_{2,g}^{LLL} = -4ig^2 \int \frac{d^4k}{(2\pi)^4} e^{-\frac{(k+p)_\perp^2}{|q_uB|}-\frac{(k+r)_\perp^2}{|q_uB|}} \frac{\mathcal{N}_2}{A_2B_2C_2}, \tag{B30}$$

where we define

$$\begin{aligned} \mathcal{N}_2 &= (\not{k}_\parallel + \not{p}_\parallel + m_u)(m_u - \not{k}_\parallel + \not{r}_\parallel), \\ A_2 &= (k_\parallel + p_\parallel)^2 - m_u^2 + i\epsilon, \\ B_2 &= (k_\parallel + r_\parallel)^2 - m_u^2 + i\epsilon, \\ C_2 &= k^2 - m_\pi^2 + i\epsilon. \end{aligned} \tag{B31}$$

We now introduce a Feynman parametrization in the same fashion of Eq. (B11). The denominator can be written as

$$A_2x + B_2y + C_2(1-x-y) = (k_\parallel + xp_\parallel + yr_\parallel)^2 - \Delta_\perp + i\epsilon, \tag{B32}$$

where

$$\Delta_\perp = (xp_\parallel + yr_\parallel)^2 - xp_\parallel^2 + (x+y)m_u^2 - yr_\parallel^2 + (1-x-y)(m_\pi^2 + k_\perp^2). \tag{B33}$$

Let us consider the change of variable $k_\parallel = l_\parallel - xp_\parallel - yr_\parallel$, $dk_\parallel = dl_\parallel$; then, in terms of these variables, the numerator $\mathcal{N}_2$ can be written as

$$\mathcal{N}_2 = -l_\parallel^2 - 2xy p_\parallel\cdot r_\parallel + m_u\not{p}_\parallel - m_u\not{r}_\parallel - x(x-1)p_\parallel^2 - y(y-1)r_\parallel^2 - (1-x-y)\not{p}_\parallel\not{r}_\parallel + m_u^2, \tag{B34}$$

where we have already discarded linear terms in $l_\parallel$. We now use the Dirac equation for outgoing states once we set $p_i = r_i = 0$, $i = 1, 2$ and assume that these states are not affected by the external magnetic field, according to Eq. (B15). Finally, setting $p_3 = r_3 = 0$ and $p_0 = r_0 = m_u$, we get

$$\bar{u}(p_\parallel)\mathcal{N}_2 u(r_\parallel) = \bar{u}(p_\parallel)(-l_\parallel^2 + 2m_u^2(x+y) - m_u^2(x+y)^2)u(r_\parallel). \tag{B35}$$

Thus, once we have considered $\bar{u}(p_\parallel)\Gamma_{2,g}^{LLL}u(r_\parallel)$, we have

$$\Gamma_{2,g}^{LLL} = -8ig^2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^2k_\perp d^2l_\parallel}{(2\pi)^4} e^{-\frac{2k_\perp^2}{|q_uB|}} \times \left[\frac{-l_\parallel^2}{(l_\parallel^2 - \Delta_\perp + i\epsilon)^3} + \frac{2m_u^2(x+y) - m_u^2(x+y)^2}{(l_\parallel^2 - \Delta_\perp + i\epsilon)^3}\right], \tag{B36}$$

where $\Delta_\perp$ is simplified according to the previous assumptions to become

$$\Delta_\perp = m_u^2(x+y)^2 + (1-(x+y))(k_\perp^2 + m_\pi^2). \tag{B37}$$

The integral over $d^2l_\parallel$ is found to be

$$\Gamma_{2,g}^{LLL} = -\frac{g^2}{\pi}\int_0^1 dx \int_0^{1-x} dy \int \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{2k_\perp^2}{|q_uB|}} \times\left[\frac{1}{\Delta_\perp} + \frac{2m_u^2(x+y) - m_u^2(x+y)^2}{\Delta_\perp^2}\right]. \quad \text{(B38)}$$

Using the change of variables given in Eq. (B22), the integral over dv can be performed to get

$$\Gamma_{2,g}^{LLL} = -\frac{g^2}{\pi m_u^2}\int_0^1 du \int \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{2k_\perp^2}{|q_uB|}} \times \frac{u}{u^2+\beta(1-u)}\left[1+\frac{(2-u)u}{u^2+\beta(1-u)}\right], \quad \text{(B39)}$$

where $\beta = (k_\perp^2 + m_\pi^2)/m_u^2$. We write the integration using polar coordinates:

$$d^2k_\perp = dk_1dk_2 = k_\perp dk_\perp d\theta, \quad \text{(B40)}$$

where $k_\perp = \sqrt{k_1^2+k_2^2}$ and $\theta \in [0, 2\pi]$. Performing the integral over $d\theta$ and substituting $|q_uB| = 2|eB|/3$ and $m_u = m_f$, we have

$$\Gamma_{2,g}^{LLL} = -\frac{g^2}{2\pi^2 m_f^2}\int_0^1 du \int_0^\infty dk_\perp k_\perp e^{-\frac{3k_\perp^2}{|eB|}} \times \frac{u}{u^2+\beta(1-u)}\left[1+\frac{(2-u)u}{u^2+\beta(1-u)}\right]. \quad \text{(B41)}$$

Finally, $I_{3,g}^B$ can be written as

$$I_{3,g}^B = \int d^4xd^4yd^4z \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{i\Phi_{3,l}} e^{-ip\cdot y} \times (-ig)e^{-is\cdot(x-y)} iS_u(s)(g\gamma^5)e^{iq\cdot x}e^{-it\cdot(z-x)} \times iS_u(t)(-ig)e^{-ik\cdot(y-z)} iD_\sigma(k)e^{ir\cdot z} + \text{CC}. \quad \text{(B42)}$$

After integration over configuration space, we get

$$I_{3,g}^B = \int \frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} (2\pi)^{12}\delta^{(4)}(s-t-q) \times \delta^{(4)}(p-s+k)\delta^{(4)}(t-k-r)(-ig)iS_u(s) \times (g\gamma^5)iS_u(t)(-ig)iD_\sigma(k) + \text{CC}. \quad \text{(B43)}$$

Integrating over d^4s and d^4t, we have

$$I_{3,g}^B = (2\pi)^4\delta^{(4)}(p-q-r)\int \frac{d^4k}{(2\pi)^4}(-ig)iS_u(k+p) \times (g\gamma^5)iS_u(k+r)(-ig)iD_\sigma(k) + \text{CC}, \quad \text{(B44)}$$

from which we can identify the contribution to the magnetic correction according to the expression

$$I_{3,g}^B = (2\pi)^4\delta^{(4)}(p-r-q)g\gamma^5\Gamma_{3,g}, \quad \text{(B45)}$$

where

$$g\gamma^5\Gamma_{3,g}^B = \int \frac{d^4k}{(2\pi)^4}(-ig)iS_u(k+p)(g\gamma^5)iS_u(k+r) \times (-ig)iD_\sigma(k) + \text{CC}. \quad \text{(B46)}$$

We now use the propagators for the charged particles in the LLL. After simplifying and adding the contribution from the charge conjugate diagram, we get

$$\Gamma_{3,g}^{LLL} = 4ig^2\int \frac{d^4k}{(2\pi)^4} e^{-\frac{(k+p)_\perp^2}{|q_uB|}-\frac{(k+r)_\perp^2}{|q_uB|}} \frac{\mathcal{N}_3}{A_3B_3C_3}, \quad \text{(B47)}$$

where we define

$$\begin{aligned} \mathcal{N}_3 &= (m_u - \not{k}_\parallel - \not{p}_\parallel)(\not{k}_\parallel + \not{r}_\parallel + m_u), \\ A_3 &= (k_\parallel + p_\parallel)^2 - m_u^2 + i\epsilon, \\ B_3 &= (k_\parallel + r_\parallel)^2 - m_u^2 + i\epsilon, \\ C_3 &= k^2 - m_\sigma^2 + i\epsilon. \end{aligned} \quad \text{(B48)}$$

The denominator can be written as

$$A_3x + B_3y + C_3(1-x-y) = (k_\parallel + xp_\parallel + yr_\parallel)^2 - \Delta_\perp + i\epsilon, \quad \text{(B49)}$$

where

$$\Delta_\perp = (xp_\parallel + yr_\parallel)^2 - xp_\parallel^2 + (x+y)m_u^2 - yr_\parallel^2 + (1-x-y)(m_\sigma^2 + k_\perp^2). \quad \text{(B50)}$$

Using the change of variable $k_\parallel = l_\parallel - xp_\parallel - yr_\parallel$, $dk_\parallel = dl_\parallel$, the numerator, $\mathcal{N}_3$, can be written as

$$\mathcal{N}_3 = -l_\parallel^2 - 2xyp_\parallel\cdot r_\parallel - m_u\not{p}_\parallel + m_u\not{r}_\parallel - x(x-1)p_\parallel^2 - y(y-1)r_\parallel^2 - (1-x-y)\not{p}_\parallel\not{r}_\parallel + m_u^2, \quad \text{(B51)}$$

where we have neglected linear terms of $l_\parallel$. We proceed as for the previous cases. We use Eq. (B15) and work in the static limit, $\vec{p} = \vec{r} = \vec{0}$ and $p_0 = r_0 = m_u$, to obtain

$$\begin{aligned}&\bar{u}(p_\parallel)\mathcal{N}_3 u(r_\parallel)\\&\quad=\bar{u}(p_\parallel)(-l_\parallel^2+2m_u^2(x+y)-m_u^2(x+y)^2)u(r_\parallel).\end{aligned}\tag{B52}$$

Thus, the integral can be written as

$$\begin{aligned}\Gamma_{3,g}^{LLL}=8ig^2\int\frac{d^2k_\perp}{(2\pi)^2}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^2l_\parallel}{(2\pi)^2}e^{-\frac{2k_\perp^2}{|q_uB|}}\\\times\left[\frac{-l_\parallel^2}{(l_\parallel^2-\Delta_\perp+i\epsilon)^3}+\frac{2m_u^2(x+y)-m_u^2(x+y)^2}{(l_\parallel^2-\Delta_\perp+i\epsilon)^3}\right],\end{aligned}\tag{B53}$$

where

$$\Delta_\perp=m_u^2(x+y)^2+(1-(x+y))(k_\perp^2+m_\sigma^2).\tag{B54}$$

Now, we can perform the integration over $d^2l_\parallel$ to get

$$\begin{aligned}\Gamma_{3,g}^{LLL}=\frac{g^2}{\pi}\int\frac{d^2k_\perp}{(2\pi)^2}\int_0^1 dx\int_0^{1-x}dy e^{-\frac{2k_\perp^2}{|q_uB|}}\\\times\left[\frac{1}{\Delta_\perp}+\frac{2m_u^2(x+y)-m_u^2(x+y)^2}{\Delta_\perp^2}\right].\end{aligned}\tag{B55}$$

The last expression can be simplified if we consider the change of variables given by Eq. (B22). After integration over dv, we have

$$\begin{aligned}\Gamma_{3,g}^{LLL}=\frac{g^2}{\pi m_u^2}\int_0^1 du\int\frac{d^2k_\perp}{(2\pi)^2}e^{-\frac{2k_\perp^2}{|q_uB|}}\\\times\frac{u}{u^2+\gamma(1-u)}\left[1+\frac{(2-u)u}{u^2+\gamma(1-u)}\right],\end{aligned}\tag{B56}$$

where $\gamma=(k_\perp^2+m_\sigma^2)/m_u^2$. We can now perform another integration after switching to polar coordinates according to Eq. (B40). Performing the integration for $d\theta$ and substituting $|q_uB|=2|eB|/3$ and $m_u=m_f$, we have the final result

$$\begin{aligned}\Gamma_{3,g}^{LLL}=\frac{g^2}{2\pi^2 m_f^2}\int_0^1 du\int_0^\infty dk_\perp k_\perp e^{-\frac{3k_\perp^2}{|eB|}}\\\times\frac{u}{u^2+\gamma(1-u)}\left[1+\frac{(2-u)u}{u^2+\gamma(1-u)}\right].\end{aligned}\tag{B57}$$

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