Magnetic field effect on the pion superfluid

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Magnetic field effect on the pion superfluid phase transition is investigated in the frame of a Pauli–Villars regularized Nambu–Jona-Lasinio model. Instead of directly dealing with a charged pion condensate, we apply Goldstone's theorem (massless Goldstone boson π^+) to determine the onset of the pion superfluid phase and obtain the phase diagram in the magnetic field, temperature, isospin, and baryon chemical potential space. In a weak magnetic field, it is analytically proven that the critical isospin chemical potential of the pion superfluid phase transition is equal to the mass of π^+ meson in the magnetic field. The pion superfluid phase is retarded to higher isospin chemical potential and can survive at a higher temperature and higher baryon chemical potential under the external magnetic field.

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The study of quantum chromodynamics (QCD) at finite isospin density and the corresponding pion superfluid phase attracts much attention due to its relation to the investigation of compact stars, isospin asymmetric nuclear matter, and heavy-ion collisions at intermediate energies. On the numerical side, while there are not yet precise lattice results at finite baryon density due to the Fermion sign problem, it is in principle no problem to do lattice simulation at finite isospin density [1–3]. On the analytical side, effective models such as the Nambu–Jona-Lasinio model (NJL), linear sigma model, and the chiral perturbation theory have been widely used to investigate pion superfluid phase structure [4–26]. There are two equivalent criteria for the critical point of pion superfluid phase transition, the nonvanishing charged pion condensate and the massless π^+ meson, which correspond to the spontaneous breaking of isospin symmetry and the Goldstone boson, respectively, guaranteed by Goldstone's theorem [27,28]. With vanishing temperature, the critical isospin chemical potential μ_I^c is the pion mass in vacuum m_π. When $\mu_I > m_\pi$, the u quark and $\bar{d}$ quark form coherent pairs and condensate, and the system enters the pion superfluid phase [1–26]. At hadron level, in the normal phase $(\mu_I < m_\pi)$ without a charged pion condensate, different pion modes explicitly show the mass splitting according to their isospin with $m_{\pi^\pm} = m_\pi \mp \mu_I$ and $m_{\pi^0} = m_\pi$. As $\mu_I = \mu_I^c = m_\pi$, the excitation of π^+ meson is free with zero momentum, which indicates the onset of the pion superfluid phase [2,12,15–17,24]. Inside the pion superfluid phase $(\mu_I \geq m_\pi)$, π^+ meson remains massless as the Goldstone mode [2,12,15–17,24].

Recently, the magnetic properties of QCD matter have become important. For instance, a certain class of neutron stars (magnetars) exhibits intense magnetic fields of strengths up to $10^{14–15}$ Gauss at the star surface, and the field is expected to become stronger towards the star center, about 10^{18} Gauss [29,30]. However, the magnetic field effect on the pion superfluid is still an open question. The difficulty lies in the fact that the pion superfluid is a phase with charged pion condensates. It breaks both the isospin symmetry in the flavor space and the translational invariance in the coordinate space, and thus the Fourier transformation between coordinate and momentum spaces is not as simple as for a neutral condensate or without a magnetic field. Lattice quantum chromodynamics (LQCD) simulations exhibit a sign problem at finite isospin chemical potential and the magnetic field. By using a Taylor expansion in the magnetic field, it is reported that at vanishing temperature, the onset of the pion condensate shifts to a larger isospin chemical potential under the magnetic fields [31], which is qualitatively consistent with the enhancement of the charged pion mass with growing magnetic fields [32]. In the study of effective models, people also focus on the charged pion condensate, but the interaction between the charged pion condensate and the magnetic field is simply neglected in Refs. [33,34] or, taken into account by the Ginzburg–Landau approach, assuming a tiny condensate in Ref. [35].

In this paper, we will study the pion superfluid phase transition at finite magnetic fields, temperature, isospin, and baryon chemical potential in the frame of a Pauli–Villars regularized NJL model, which is inspired by the Bardeen-Cooper-Shrieffer theory and describes remarkably well the quark pairing mechanisms and hadron mass

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spectra [36–41]. Instead of directly dealing with charged pion condensate, we investigate the magnetic field effect on pion superfluid through its Goldstone mode π^+, determining the critical point of pion superfluid phase transition by the massless π^+ meson. Seriously taking into account the breaking of translational invariance for charged particles, the pion propagators, in terms of quark bubbles, are analytically derived, and pion masses are solved. At the weak magnetic field, vanishing temperature, and vanishing baryon chemical potential, we analytically prove that the critical isospin chemical potential of the pion superfluid phase transition is equal to the π^+ mass in the magnetic field, the same as the vanishing magnetic field case [2,12,15–17,24]. Under the external magnetic field, the pion superfluid phase is shifted to a higher isospin chemical potential and can survive at a higher temperature and a higher baryon chemical potential.

The two-flavor NJL model is defined through the Lagrangian density in terms of quark fields ψ [36–41]

$$\mathcal{L} = \bar{\psi}(i\gamma_\nu D^\nu - m_0 + \gamma_0\mu)\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2]. \tag{1}$$

Here the covariant derivative $D_\nu = \partial_\nu + iQA_\nu$ couples quarks with electric charge $Q = \mathrm{diag}(Q_u, Q_d) = \mathrm{diag}(2e/3, -e/3)$ to the external magnetic field $\mathbf{B} = (0, 0, B)$ in z-direction through the potential $A_\nu = (0, 0, Bx_1, 0)$. The quark chemical potential $\mu = \mathrm{diag}(\mu_u, \mu_d) = \mathrm{diag}(\mu_B/3 + \mu_I/2, \mu_B/3 - \mu_I/2)$ is a matrix in the flavor space, with μ_u and μ_d being the u- and d-quark chemical potentials and μ_B and μ_I being the baryon and isospin chemical potentials. Note that G is the coupling constant in scalar and pseudoscalar channels. At finite isospin chemical potential and the magnetic field, the isospin symmetry $SU(2)_I$ is broken down to $U(1)_I$ symmetry, and the chiral symmetry $SU(2)_A$ is broken down to $U(1)_A$ symmetry. With the spontaneous breaking of chiral $U(1)_A$ symmetry and isospin $U(1)_I$ symmetry, the Goldstone mode reads π^0 meson and π^+ meson, respectively. Note that m_0 is the current quark mass characterizing the explicit chiral symmetry breaking.

Corresponding to the symmetries and their spontaneous breaking, we have two order parameters, the neutral chiral condensate $\langle\bar{\psi}\psi\rangle$ for chiral restoration phase transition and the charged pion condensate $\langle\bar{\psi}\gamma_5\tau^1\psi\rangle$ for pion superfluid phase transition. Under magnetic fields, the charged pion condensate breaks both the isospin symmetry in the flavor space and the translational invariance in the coordinate space, and thus the Fourier transformation between coordinate and momentum spaces is not as simple as for the neutral condensate or without a magnetic field. In our current work, to avoid the complication and difficulty of dealing with charged pion condensates under a magnetic field, we will start from the normal phase only with neutral chiral condensates and determine the critical point of pion superfluid phase transition by the appearance of the Goldstone boson, massless π^+ meson. Physically, it is equivalent to define the phase transition by the order parameter (charged pion condensate) and Goldstone mode (massless π^+ meson), as guaranteed by Goldstone's theorem [2,12,27,28].

In mean field approximation, the chiral condensate $\langle\bar{\psi}\psi\rangle$ or the dynamical quark mass $m_q = m_0 - 2G\langle\bar{\psi}\psi\rangle$ is controlled by the gap equation [42–48],

$$m_0 = m_q(1 - 2GJ_1), \tag{2}$$

$$J_1 = 3\sum_{f,n}\alpha_n \frac{|Q_fB|}{2\pi}\int\frac{dp_3}{2\pi}\frac{1}{E_f} \times [1 - f(E_f + \mu_f) - f(E_f - \mu_f)], \tag{3}$$

with the summation over all flavors and Landau energy levels, spin factor $\alpha_n = 2 - \delta_{n0}$, quark energy $E_f = \sqrt{p_3^2 + 2n|Q_fB| + m_q^2}$, and the Fermi–Dirac distribution function $f(x) = 1/(e^{x/T} + 1)$.

As quantum fluctuations above the mean field, mesons are constructed through quark bubble summations in the frame of random phase approximation [37–41]. Taking into account the interaction between charged mesons and magnetic fields, and generalizing our derivations in Ref. [48] to a finite quark chemical potential case, the meson propagator D_M can be expressed in terms of the meson polarization function Π_M with conserved Ritus momentum $\bar{k}$ [49–51],

$$D_M(\bar{k}) = \frac{G}{1 - G\Pi_M(\bar{k})}. \tag{4}$$

The meson pole mass m_M is defined through the pole of the propagator at zero momentum,

$$1 - G\Pi_M(k_0 = m_M) = 0. \tag{5}$$

Based on Goldstone's theorem for the spontaneous breaking of isospin symmetry, massless Goldstone mode π^+ exists in the pion superfluid phase. Therefore, the critical isospin chemical potential $\mu_I^{c\pi}$ for the pion superfluid can be identified by the condition

$$m_{\pi^+}(B, T, \mu_B, \mu_I^{c\pi}) = 0. \tag{6}$$

For the π^+ meson, we have

$$\Pi_{\pi^+}(k_0) = J_1 + J_2(k_0), \tag{7}$$

$$J_2(k_0) = \sum_{n,n'} \int \frac{dp_3}{2\pi} \frac{j_{n,n'}(k_0)}{4E_n E_{n'}} \times \left[\frac{f(-E_{n'}-\mu_u) - f(E_n - \mu_d)}{k_0 + \mu_I + E_{n'} + E_n} + \frac{f(E_{n'}-\mu_u) - f(-E_n-\mu_d)}{k_0+\mu_I - E_{n'} - E_n} \right], \tag{8}$$

$$j_{n,n'}(k_0) = [(k_0+\mu_I)^2/2 - n'|Q_u B| - n|Q_d B|] j^+_{n,n'} - 2\sqrt{n'|Q_u B| n |Q_d B|} j^-_{n,n'}, \tag{9}$$

with the u-quark energy $E_{n'} = \sqrt{p_3^2 + 2n'|Q_u B| + m_q^2}$ and d-quark energy $E_n = \sqrt{p_3^2 + 2n|Q_d B| + m_q^2}$. The detailed derivations of Eqs. (4), (7), (8), and (9) are written in the Appendix. Note that the lowest Landau-level term with $n = n' = 0$ does not contribute to the polarization function with $j^\pm_{0,0} = 0$ because the spins of u and $\bar{d}$ quarks at the lowest Landau level are aligned parallel to the magnetic field, but π^+ meson has spin zero. This leads to the heavy π^+ mass in the magnetic field [48] and thus delays the pion superfluid in the magnetic field (see the discussions of Fig. 1).

Because of the four-fermion interaction, the NJL model is not a renormalizable theory and needs regularization. Although the above analytical derivations do not depend on the regularization, in the numerical calculations we should choose a regularization scheme to obtain finite results for momentum integrals. The magnetic field does not cause extra ultraviolet divergence but introduces discrete Landau levels and anisotropy in momentum space. The usually used hard/soft momentum cutoff regularization schemes do not work well in the magnetic field since the momentum cutoff, together with the discrete Landau levels, will cause some nonphysical results [52–58], such as the oscillations of the chiral condensate, critical temperature, critical density, tachyonic pion mass, and the breaking of the law of causality for the Goldstone mode. In this work, we take into account the gauge invariant Pauli–Villars regularization scheme [47,48], where the quark momentum runs formally from zero to infinity, and the nonphysical results are cured [56–58]. The three parameters in the Pauli–Villars regularized NJL model, namely the current quark mass $m_0 = 5$ MeV, the coupling constant $G = 3.44\ \mathrm{GeV}^{-2}$, and the Pauli–Villars mass parameter $\Lambda = 1127$ MeV are fixed by fitting the chiral condensate $\langle\bar{\psi}\psi\rangle = -(250\ \mathrm{MeV})^3$, pion mass $m_\pi = 134$ MeV, and pion decay constant $f_\pi = 93$ MeV in a vacuum with $T = \mu_B = \mu_I = 0$ and $B = 0$.

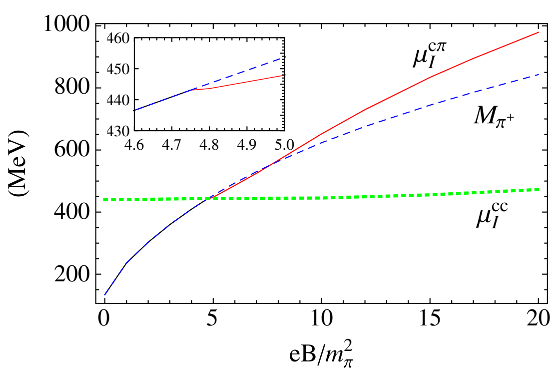

FIG. 1. Critical isospin chemical potential $\mu_I^{c\pi}$ (black and red solid lines) for the pion superfluid phase transition and μ_I^{cc} (green dotted line) for the chiral restoration phase transition as a function of the magnetic field at $T = \mu_B = 0$. Note that π^+ mass in the magnetic field $M_{\pi^+} = m_{\pi^+}(B, T = \mu_B = \mu_I = 0)$ is plotted in a blue dashed line for reference.

In Fig. 1, we plot the critical isospin chemical potential $\mu_I^{c\pi}$ (black and red solid lines) for the pion superfluid phase transition as a function of the magnetic field at $T = \mu_B = 0$, which is determined by the condition of the massless Goldstone boson $m_{\pi^+}(B, T = \mu_B = 0, \mu_I^{c\pi}) = 0$. Note that $\mu_I^{c\pi}$ increases with the magnetic field, which is qualitatively consistent with the conclusion of LQCD [31] and model calculations [35], and this means that the magnetic field delays/disfavors the pion superfluid phase transition at finite isospin chemical potential. Physically, it can be understood in this way. Locating the two constituent quarks at the lowest Landau level is forbidden for charged pions due to its zero spin. According to the quark energy $E_f = \sqrt{p_3^2 + 2n|Q_f B| + m_q^2}$, different electric charges of u and d quarks indicate a different effective quark mass $\sqrt{2n|Q_f B| + m_q^2}$ with a finite magnetic field and zero momentum $p_3 = 0$. This mass difference plays the role of an effective Fermi surface mismatch when u quark and $\bar{d}$ quark form cooper pairs. The larger the magnetic field (mass difference) is, the more difficult to form pion superfluid becomes, and this leads to the increasing $\mu_I^{c\pi}$ in the magnetic field.

Critical isospin chemical potential μ_I^{cc} for chiral restoration phase transition (see green dotted line in Fig. 1) is determined by the dynamical quark mass. At finite magnetic fields, chiral restoration is a first-order phase transition, and the quark mass jumps from a large value to a small value. It is noticeable that μ_I^{cc} and $\mu_I^{c\pi}$ are different from each other, except for one point at $eB = 4.75 m_\pi^2$ with $\mu_I^{cc} > \mu_I^{c\pi}$ at $eB < 4.75 m_\pi^2$ and $\mu_I^{cc} < \mu_I^{c\pi}$ at $eB > 4.75 m_\pi^2$.

The critical isospin chemical potential $\mu_I^{c\pi}$ is separated into two parts, denoted by the connecting point of the red and black solid lines at $eB = 4.75 m_\pi^2$ in Fig. 1. For $eB < 4.75 m_\pi^2$, we observe that the critical isospin chemical potential is equal to the π^+ mass in the magnetic field with $\mu_I^{c\pi} = M_{\pi^+} = m_{\pi^+}(B, T = \mu_B = \mu_I = 0)$, as shown by the overlap between the black solid line and blue dashed line in Fig. 1. This conclusion can be analytically proven, similar to the case without a magnetic field [12]. At $T = 0$, the Fermi–Dirac distribution $f(x)$ becomes a Heaviside step

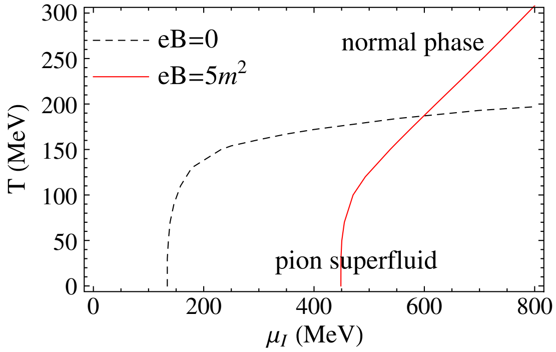

FIG. 2. Pion superfluid phase diagram in the $\mu_I - T$ plane with $\mu_B = 0$ and a fixed magnetic field. The black dashed line is for $eB/m_\pi^2 = 0$, and the red solid line is for $eB/m_\pi^2 = 5$.

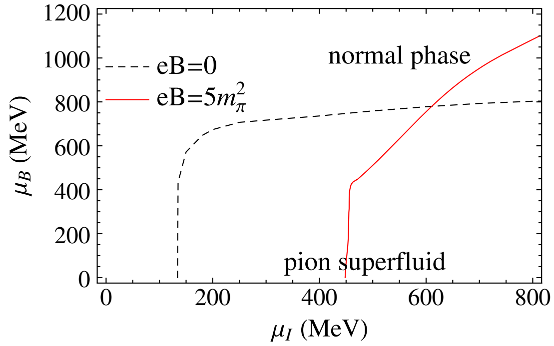

FIG. 3. Pion superfluid phase diagram in the $\mu_I - \mu_B$ plane with $T = 0$ and fixed magnetic field. The black dashed line is for $eB/m_\pi^2 = 0$, and the red solid line is for $eB/m_\pi^2 = 5$.

function $\theta(-x)$. With a fixed magnetic field, we solve a constant quark mass $m_q(B, T = \mu_B = 0, \mu_I) = m_q(B, T = \mu_B = \mu_I = 0)$ from gap Eq. (2) before the chiral restoration happens. And by straightforward comparison of gap Eq. (2) and pole Eq. (5), a linearly decreasing π^+ mass is obtained $m_{\pi^+}(B, T = \mu_B = 0, \mu_I) = M_{\pi^+} - \mu_I$. Applying Goldstone's theorem, the critical isospin chemical potential $\mu_I^{c\pi}$ for the pion superfluid is determined by the condition $m_{\pi^+}(B, T = \mu_B = 0, \mu_I^{c\pi}) = 0$. Therefore, we solve $\mu_I^{c\pi} = M_{\pi^+}$. At $eB = 4.75 m_\pi^2$, both the pion superfluid phase transition and the chiral restoration phase transition happen at the same critical isospin chemical potential $\mu_I^{c\pi} = \mu_I^{cc}$. Since chiral restoration is a first-order phase transition, associated with the quark mass jump, it leads to the discontinuous $\mu_I^{c\pi}$ for the pion superfluid phase transition, as shown by the different slopes of the black and red lines around $eB = 4.75 m_\pi^2$. For $eB > 4.75 m_\pi^2$, no such analytical derivations are available, and we should rely on the numerical calculations. The critical isospin chemical potential $\mu_I^{c\pi}$ is deviated from M_{π^+}, although they both increase in magnetic fields. With stronger magnetic fields, the deviation becomes larger.

We now turn on the temperature effect and depict the pion superfluid phase diagram in the $\mu_I - T$ plane with $\mu_B = 0$, the fixed magnetic field $eB/m_\pi^2 = 0$ (black dashed line), and $eB/m_\pi^2 = 5$ (red solid line) in Fig. 2. The phase transition line determined by the massless π^+ meson divides the $\mu_I - T$ plane into two regions. The pion superfluid phase is located in the high isospin chemical and low temperature region, and the quarks are in the normal phase for the low isospin chemical potential and/or high temperature region. With increasing temperature, the quark thermal motion becomes strong. It prohibits the quark pairing and leads to the phase transition from the pion superfluid phase to the normal phase. The critical temperature increases with isospin chemical potential. Compared to the vanishing magnetic field case, the pion superfluid phase is retarded to higher isospin chemical potential, and it survives in higher temperature under a finite magnetic field.

Figure 3 is the phase diagram in the $\mu_I - \mu_B$ plane with $T = 0$ and the fixed magnetic field. The black dashed line is for $eB/m_\pi^2 = 0$, and the red solid line is for $eB/m_\pi^2 = 5$. The pion superfluid phase is located in the high isospin chemical potential and low baryon chemical potential region. In the low isospin chemical potential and/or high baryon chemical potential region, quarks are in the normal phase. At zero baryon chemical potential, the u quark and $\bar{d}$ quark form coherent pairs and condensate on a uniform Fermi surface, as $\mu_I > \mu_I^{c\pi}$. When the baryon chemical potential is switched on, there appears to be a Fermi surface mismatch between the u quark and $\bar{d}$ quark, and it causes the phase transition from the pion superfluid phase to the normal phase. The critical baryon chemical potential increases with isospin chemical potential. With a stronger magnetic field, the pion superfluid phase happens at higher isospin chemical potential and survives at higher baryon chemical potential. It should be mentioned that when the baryon chemical potential is large enough, the quark system will enter the color superconductor phase [59–67]. Nonzero isospin chemical potential tends to destroy the color condensate, and thus in a large isospin chemical potential case, it is safe to neglect the color superconductor.

The magnetic field effect on the pion superfluid phase transition is studied in the frame of a Pauli–Villars regularized NJL model. Instead of directly dealing with the charged pion condensate, we apply Goldstone's theorem (massless Goldstone boson π^+) to determine the onset of the pion superfluid phase. Seriously taking into account the breaking of translational invariance, the charged pion propagator is constructed at the finite magnetic field, temperature, and chemical potential, and the π^+ mass and pion superfluid phase diagram are obtained. At weak magnetic field, vanishing temperature, and vanishing baryon chemical potential, it is analytically proven that the critical isospin chemical potential $\mu_I^{c\pi}$ is equal to the π^+ mass in the magnetic field, $\mu_I^{c\pi} = M_{\pi^+}$. Under the external magnetic field, the pion superfluid phase is retarded to higher isospin chemical potential and can survive at higher temperatures and higher baryon chemical potentials.

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APPENDIX: DERIVATIONS OF CHARGED MESON PROPAGATOR IN A MAGNETIC FIELD

As quantum fluctuations above the mean field, mesons are constructed through quark bubble summation in the frame of RPA [37–41]. Namely, the quark interaction via a meson exchange is effectively described by using the Dyson–Schwinger equation,

$$\mathcal{D}_M(x,z) = 2G\delta(x-z) + \int d^4y\, 2G\Pi_M(x,y)\mathcal{D}_M(y,z), \tag{A1}$$

where $\mathcal{D}_M(x,y)$ represents the meson propagator from x to y, and the corresponding meson polarization function is the quark bubble

$$\Pi_M(x,y) = i\mathrm{Tr}[\Gamma_M^* S(x,y)\Gamma_M S(y,x)] \tag{A2}$$

with the meson vertex

$$\Gamma_M = \begin{cases} 1 & M=\sigma \\ i\tau_+\gamma_5 & M=\pi_+ \\ i\tau_-\gamma_5 & M=\pi_- \\ i\tau_3\gamma_5 & M=\pi_0, \end{cases} \qquad \Gamma_M^* = \begin{cases} 1 & M=\sigma \\ i\tau_-\gamma_5 & M=\pi_+ \\ i\tau_+\gamma_5 & M=\pi_- \\ i\tau_3\gamma_5 & M=\pi_0. \end{cases} \tag{A3}$$

The quark propagator matrix in flavor space $S = \mathrm{diag}(S_u, S_d)$ is at mean field level, and the trace is taken in spin, color, and flavor spaces.

There are two equivalent ways to treat the quark and meson propagators in the magnetic field, the Schwinger scheme [44–46] and the Ritus scheme [49–51]. In the following, we derive the meson propagator with the Ritus method and comment on its advantage in the end.

In the Ritus scheme, one can well define the Fourier-like transformation for the particle propagator from the conserved Ritus momentum space to the coordinate space [49–51]. The quark propagator with flavor f in the coordinate space can be written as

$$\begin{aligned} S_f(x,y) &= \sum_n \int \frac{d^3\tilde{p}}{(2\pi)^3} e^{-i\tilde{p}\cdot(x-y)} P_n(x_1,p_2) \\ &\quad \times D_f(\bar{p}) P_n(y_1,p_2), \\ P_n(x_1,p_2) &= \frac{1}{2}[g_n^{s_f}(x_1,p_2) + I_n g_{n-1}^{s_f}(x_1,p_2)] \\ &\quad + \frac{is_f}{2}[g_n^{s_f}(x_1,p_2) - I_n g_{n-1}^{s_f}(x_1,p_2)]\gamma_1\gamma_2, \\ D_f^{-1}(\bar{p}) &= \gamma\cdot\bar{p} - m_q, \end{aligned} \tag{A4}$$

where $\tilde{p} = (p_0, 0, p_2, p_3)$ is the Fourier transformed momentum, $\bar{p} = (p_0+\mu_f, 0, -s_f\sqrt{2n|Q_fB|}, p_3)$ is the conserved Ritus momentum with n describing the quark Landau level in the magnetic field, μ_f is the quark chemical potential, $s_f = \mathrm{sgn}(Q_fB)$ is the quark sign factor, the magnetic field dependent function $g_n^{s_f}(x_1,p_2) = \phi_n(x_1 - s_f p_2/|Q_fB|)$ is controlled by the Hermite polynomial $H_n(\zeta)$ via $\phi_n(\zeta) = (2^n n!\sqrt{\pi}|Q_fB|^{-1/2})^{-1/2} e^{-\zeta^2|Q_fB|/2} H_n \times (\zeta/|Q_fB|^{-1/2})$, and $I_n = 1-\delta_{n0}$ is governed by the Landau energy level.

For charged mesons $M = \pi_\pm$ with spin zero, the Fourier transformation is extended to

$$\begin{aligned} \mathcal{D}_M(\bar{k}) &= \int d^4x d^4y F_k^*(x)\mathcal{D}_M(x,y)F_k(y), \\ \Pi_M(\bar{k}) &= \int d^4x d^4y F_k^*(x)\Pi_M(x,y)F_k(y), \end{aligned} \tag{A5}$$

where $F_k(x) = e^{-i\tilde{k}\cdot x} g_l^{s_M}(x_1,k_2)$ is the solution of the Klein–Gordon equation with index l describing the meson Landau level in the magnetic field, the Fourier transformed momentum $\tilde{k} = (k_0, 0, k_2, k_3)$, and the meson sign factor $s_M = \mathrm{sgn}(Q_MB)$, and $\bar{k} = (k_0, 0, -s_M\sqrt{(2l+1)|Q_MB|}, k_3)$ is the conserved four-dimensional Ritus momentum.

Taking into account the complete and orthogonal conditions

$$\sum_l \int \frac{d^3\tilde{k}}{(2\pi)^3} F_k(x)F_k^*(y) = \delta^{(4)}(x-y), \tag{A6}$$

$$\int d^4x F_k(x)F_{k'}^*(x) = (2\pi)^4\delta^{(3)}(\tilde{k}-\tilde{k}')\delta_{ll'}, \tag{A7}$$

and the Dyson–Schwinger Eq. (A1), the meson propagator in momentum space can be simplified as

$$\mathcal{D}_M(\bar{k}) = \frac{2G}{1-2G\Pi_M(\bar{k})}. \tag{A8}$$

Taking the mean field quark propagator (A4) and the definition (A2) for the quark bubble, we have the π_+ polarization function

$$\Pi_{\pi_+}(x,y) = 2i^3 N_c \mathrm{Tr}_D[\gamma_5 S_u(x,y)\gamma_5 S_d(y,x)]$$
$$= 2i^3 N_c \sum_{n,n'} \int \frac{d^3\tilde{p} d^3\tilde{q}}{(2\pi)^6} e^{-i(\tilde{p}-\tilde{q})(x-y)}$$
$$\times \mathrm{Tr}_D\left[P_n(x_1,q_2)P_{n'}(x_1,p_2)\frac{-\gamma\cdot\bar{p}+m_q}{\bar{p}^2-m_q^2} + P_n(y_1,q_2)P_{n'}(y_1,p_2)\frac{\gamma\cdot\bar{q}+m_q}{\bar{q}^2-m_q^2}\right] \tag{A9}$$

in coordinate space and

$$\Pi_{\pi_+}(\bar{k}) = 8i^3 N_c \int d^4x d^4y \sum_{n,n'} \int \frac{d^3\tilde{p} d^3\tilde{q}}{(2\pi)^6} e^{-i(\tilde{p}-\tilde{q}-\tilde{k})(x-y)}$$
$$\times g_l^+(x_1,k_2) g_l^+(y_1,k_2)$$
$$\times \left[A_+(x_1,y_1,k_2)\frac{m_q^2-\bar{p}_0\bar{q}_0+\bar{p}_3\bar{q}_3}{(\bar{p}^2-m_q^2)(\bar{q}^2-m_q^2)} + A_-(x_1,y_1,k_2)\frac{\bar{p}_2\bar{q}_2}{(\bar{p}^2-m_q^2)(\bar{q}^2-m_q^2)}\right] \tag{A10}$$

in Ritus-momentum space with

$$A_\pm(x_1,y_1,k_2) = \alpha_+(x_1,k_2)\alpha_+(y_1,k_2) \pm \alpha_+ \leftrightarrow \alpha_-,$$
$$\alpha_\pm(z,k_2) = \frac{[I_n g_{n-1}^{s_d}(z,q_2) g_{n'}^{s_u}(z,p_2) \pm u \leftrightarrow d]}{2}. \tag{A11}$$

Doing the integrations over x_0, x_2, x_3 and y_0, y_2, y_3, we obtain the momentum conservation relation $\tilde{p} = \tilde{q} + \tilde{k}$. At finite temperature, the integration over the particle energy is replaced by the Fermion/Boson Matsubara frequency summation in the imaginary time formalism of finite temperature field theory. After a straightforward derivation, we have the π_+ polarization function at the pole

$$\Pi_{\pi^+}(k_0) = J_1 + J_2(k_0), \tag{A12}$$

$$J_2(k_0) = \sum_{n,n'} \int \frac{dp_3}{2\pi} \frac{j_{n,n'}(k_0)}{4E_n E_{n'}} \left[\frac{f(-E_{n'}-\mu_u)-f(E_n-\mu_d)}{k_0+\mu_I+E_{n'}+E_n} + \frac{f(E_{n'}-\mu_u)-f(-E_n-\mu_d)}{k_0+\mu_I-E_{n'}-E_n}\right],$$
$$j_{n,n'}(k_0) = [(k_0+\mu_I)^2/2 - n'|Q_uB| - n|Q_dB|] j^+_{n,n'} - 2\sqrt{n'|Q_uB| n|Q_dB|}\, j^-_{n,n'}, \tag{A13}$$

$$j^\pm_{n,n'} = 2N_c \int dx_1 dy_1 dq_2/2\pi A_\pm(x_1,y_1,k_2), \tag{A14}$$

with the quark energies $E_{n'} = \sqrt{p_3^2 + 2n'|Q_uB| + m_q^2}$ and $E_n = \sqrt{q_3^2 + 2n|Q_dB| + m_q^2}$ and the Fermi–Dirac distribution function $f(E) = 1/(e^{E/T}+1)$.

In the end, let us briefly compare the Ritus and Schwinger schemes in constructing mesons under the external magnetic field. The difference is the treatment of the breaking of translational symmetry by the magnetic field [48]. A conserved momentum, called Ritus momentum, is introduced in the Ritus scheme and a Schwinger phase is embedded in the particle's propagator in the Schwinger scheme. For neutral mesons, the Ritus momentum is reduced to the normal momentum and the Schwinger phase disappears automatically; the two schemes are both convenient and give the same analytic formula. For charged mesons, we still obtain an algebraic equation for the meson propagator in terms of the conserved Ritus momentum, while the nontrivial Schwinger phase leads to an integral equation for the meson propagator, in which it is more difficult to extract the meson properties.

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