Reply to "Comment on 'Brans-Dicke scalar field cosmological model in Lyra's geometry' "

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The present study is a response to the raising comments on Brans-Dicke scalar field cosmological models in Lyra's geometry [A. K. Yadav, Phys. Rev. D **10**, 108301 (2020)] by Dr. Anil Kumar Yadav. We have corrected the manuscripts and remove the contradictions observed by the comment author. Now, the present cosmological model is a transit phase model with Brans-Dicke coupling constant $\omega < -1.33$ while the equation of state parameter (EoS) lies in the range $0 \leq \gamma < 0.33$. The Brans-Dicke scalar field is responsible candidate for the acceleration in expansion of the universe.

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I. INTRODUCTION

Recently we have investigated Brans-Dicke scalar field cosmological models in Lyra's geometry in a spatially homogeneous and anisotropic Bianchi Type-I space-time and published an research article in the form of Ref. [1]. We have found that the cosmological model [1] is an accelerating universe model with Brans-Dicke coupling constant $\omega > 40000$. But in Ref. [2] Yadav has found some sign errors in Eqs. (17)–(20) in his investigation and on the behalf of this he claimed that the model is not accelerated for $\omega > 40000$ while it is accelerated for small and negative value of ω. He has also commented on equation of energy conservation [Eq. (22)] for missing the gauge function β term from it. He has also, commented on the role of gauge function β because we have claimed that both scalar-field ϕ is responsible for signature-flipping deceleration parameter and the constant displacement vector behaves just like cosmological constant Λ-term.

In response to these comments, we have agreed with some comments on Ref. [1] by Yadav and this is a formal reply to those comments. We have corrected the sign errors and reinvestigated all the corresponding scenarios of the cosmological models and found that it is accelerating only for small negative values of Brans-Dicke coupling constant $\omega \approx -1.8$. We have also corrected the energy conservation equation and solved it for ρ with constant β and constant equation of state γ and have found an interesting equation of energy density ρ which clearly presents the direct contribution of gauge function β in the total energy density. In this reinvestigation we have skipped those part of Ref. [1] that is either not changed or necessary in this study.

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After the revision the outline of the paper is as follows: Sec. I is introductory in nature. In Sec. II the field equations in Lyra geometry with Brans-Dicke modifications are described. Section III deals with the cosmological solutions that have established relations among energy parameters Ω_m, Ω_σ and Ω_β. In Sec. IV, we obtained expressions for Hubble's constant, luminosity distance, and apparent magnitude in terms of redshift and scale factor. We have also estimated the present values of energy parameters and Hubble's constant. The deceleration parameter (DP), age of the universe, and certain physical properties of the universe are presented in Sec. V. Finally, conclusions are summarized in Sec. VI.

II. EINSTEIN'S BRANS-DICKE FIELD EQUATIONS IN LYRA'S GEOMETRY

Now, the Eqs. (17)–(20) in Ref. [1] are corrected as

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}\dot{C}}{AC}-\frac{3}{4}\beta^2=\frac{8\pi\rho}{\phi c^2}+\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) \tag{1}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{\phi}}{\phi} \tag{2}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)-\frac{\ddot{\phi}}{\phi} \tag{3}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{\ddot{\phi}}{\phi}. \tag{4}$$

Here the overdot denotes derivative with respect to time t.

III. COSMOLOGICAL SOLUTIONS OF THE FIELD EQUATIONS

The equation of energy conservation Eq. (22) in Ref. [1] is corrected as

$$8\pi\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+3H\left[8\pi(\rho+p)+\frac{3}{2}\beta^2\right]=0. \tag{5}$$

Here, we have considered β as a constant and hence, Eq. (22) read as

$$8\pi\dot{\rho}+3H\left[8\pi(\rho+p)+\frac{3}{2}\beta^2\right]=0. \tag{6}$$

The equation of state for the model is defined as

$$p=\gamma\rho, \tag{7}$$

where γ is EoS parameter of the fluid filled in the universe and the Hubble parameter H is given by $H=\frac{\dot{a}}{a}$, where a is the average scale factor.

There are two cases for the value of the EoS parameter γ in Eq. (7), first one is $\gamma=$ constant and second is γ time-dependent. Because of our study in commented paper, we have considered constant equation of state γ. Integrating (6), we get Eq. (8) in place of Eq. (24) in Ref. [1]:

$$\rho=\left[\rho_0+\frac{3\beta^2}{16\pi(1+\gamma)}\right]\left(\frac{A_0}{A}\right)^{3(1+\gamma)}-\frac{3\beta^2}{16\pi(1+\gamma)}. \tag{8}$$

Now, from the corrected field Eqs. (1)–(4), we have obtained the following equations in place of Eqs. (27)–(29) in Ref. [1]:

$$\frac{B}{A}=c_2\exp\left(\int\frac{c_1}{a^3\phi}dt\right) \tag{9}$$

$$\frac{C}{A}=c_4\exp\left(\int\frac{c_3}{a^3\phi}dt\right) \tag{10}$$

$$\frac{C}{B}=c_6\exp\left(\int\frac{c_5}{a^3\phi}dt\right). \tag{11}$$

And hence, Eq. (30) in Ref. [1] becomes

$$D=\exp\left(\int\frac{k}{(ABC)\phi}dt\right). \tag{12}$$

We get the following relations,

$$B=AD \quad \& \quad C=\frac{A}{D} \tag{13}$$

where $D=D(t)$ measures the anisotropy of the universe.

The average scale factor a is defined as $a=(ABC)^{\frac{1}{3}}$ and using Eq. (13), we obtain

$$a=(ABC)^{\frac{1}{3}}=A. \tag{14}$$

Therefore, Eqs. (33), (34) in Ref. [1], read as respectively

$$3\left(\frac{\dot{A}}{A}\right)^2-\left(\frac{\dot{D}}{D}\right)^2-\frac{3}{4}\beta^2=\frac{8\pi\rho}{\phi c^2}+\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-3\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}\right) \tag{15}$$

$$2\left(\frac{\ddot{A}}{A}\right)+\left(\frac{\dot{A}}{A}\right)^2+\left(\frac{\dot{D}}{D}\right)^2+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-2\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}\right)-\frac{\ddot{\phi}}{\phi} \tag{16}$$

where

$$\frac{\dot{D}}{D}=\frac{k}{A^3\phi} \tag{17}$$

and k is an arbitrary constant and Eq. (17) replaces the Eq. (38) in Ref. [1].

Also, Eq. (37) is read as Eq. (6) and its solution obtained as (8).

Now, we define the matter energy density parameter (Ω_m), curvature anisotropy parameter (Ω_σ) and β parameter Ω_β as [in place of Eq. (39) in Ref. [1]]:

$$\Omega_m=\frac{8\pi\rho}{3c^2H^2\phi}, \qquad \Omega_\sigma=\frac{k^2}{3H^2A^6\phi^2}, \qquad \Omega_\beta=\frac{\beta^2}{4H^2}. \tag{18}$$

Now, Eqs. (41), (42), and (44) in Ref. [1] becomes respectively as follows:

$$\Omega_m+\Omega_\sigma+\Omega_\beta=1-\frac{\omega}{6}\xi^2+\xi \tag{19}$$

$$\gamma\Omega_m+\Omega_\sigma+\Omega_\beta=\frac{2}{3}q-\frac{1}{3}-\frac{\omega}{6}\xi^2-\frac{2}{3}\xi+\frac{1}{3}q_\phi \tag{20}$$

and

$$q=2+\xi-\frac{3[(1-\gamma)\omega+1]}{2\omega+3}\Omega_m \tag{21}$$

where $\xi=\frac{\dot{\phi}}{\phi H}$.

Eqs. (45), (46), and (47) in Ref. [1] revised as:

$$\phi = \phi_0 \left(\frac{A}{A_0}\right)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} \tag{22}$$

$$\xi = \frac{1-3\gamma}{\omega-\omega\gamma+1}, \tag{23}$$

and

$$\Omega_m + \Omega_\sigma + \Omega_\beta = 1 + \frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}. \tag{24}$$

IV. EXPRESSIONS FOR HUBBLE'S CONSTANT, LUMINOSITY DISTANCE, APPARENT MAGNITUDE, ETC.

A. Hubble's constant

The energy conservation Eq. (6) is integrable for constant EoS parameter ($\gamma = \text{constant}$), giving rise to the following expression among matter density ρ, average scale factor $a(t) = A(t)$ and the redshift z of the universe [in place of Eq. (53) in Ref. [1]]:

$$\rho = \left[\rho_0 + \frac{3\beta^2}{16\pi(1+\gamma)}\right](1+z)^{3(1+\gamma)} - \frac{3\beta^2}{16\pi(1+\gamma)}. \tag{25}$$

Now, Eqs. (54) and (55) in Ref. [1], revised as respectively:

$$H = \frac{H_0}{\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}}\sqrt{\Omega'_{m0}\left(\frac{A_0}{A}\right)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}\left(\frac{A_0}{A}\right)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}\left(\frac{A_0}{A}\right)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}} \tag{26}$$

and

$$H = \frac{H_0}{\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}}\sqrt{\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(1+z)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}} \tag{27}$$

respectively. Where $\Omega_{m0} = \Omega'_{m0} - \Omega_{\phi0}$, $\Omega'_{m0} = \frac{8\pi}{3c^2H_0^2\phi_0}(\rho_0 + \frac{3\beta^2}{16\pi(1+\gamma)})$, $\Omega_{\phi0} = \frac{\beta^2}{2c^2(1+\gamma)H_0^2\phi_0}$ $\Omega_{\sigma0} = \frac{8\pi k^2}{3H_0^2A_0^6\phi_0^2}$ and $\Omega_{\beta0} = \frac{\beta^2}{4H_0^2}$.

B. Luminosity distance

Therefore, the Eq. (58) in Ref. [1] is revised as

$$D_L = \frac{c(1+z)\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(1+z)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}}} \tag{28}$$

C. Apparent magnitude

The expression for apparent magnitude as Eq. (60) in Ref. [1] is revised as

$$m = 16.08 + 5\log_{10}\left(\frac{(1+z)\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}}{0.026}\right) + 5\log_{10}\left(\int_0^z \frac{dz}{\sqrt{\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(1+z)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}}}\right). \tag{29}$$

TABLE I. Outcomes of the R^2 − test for the best fit curve of apparent magnitude $m(z)$ in Eq. (29). The values of coefficients Ω_m, Ω_σ, Ω_β, and ω are at 95% confidence of bounds.

Function	γ	ω	Ω'_{m0}	$\Omega_{\phi 0}$	$\Omega_{\sigma 0}$	$\Omega_{\beta 0}$	R^2	$RMSE$
$m(z)$	0	−1.777	0.4885	0.09992	2.159×10^{-12}	0.001204	0.9944	0.2359
$m(z)$	0.1	−1.717	0.3899	0.10000	2.221×10^{-14}	4.888×10^{-7}	0.9944	0.2358
$m(z)$	0.2	−1.656	0.3100	0.01000	2.221×10^{-14}	2.161×10^{-9}	0.9944	0.2358
$m(z)$	0.3	−1.524	0.2994	0.01000	2.256×10^{-14}	2.948×10^{-5}	0.9944	0.2364

D. Energy parameters at present

We consider 580 high red shift ($0.015 \leq z \leq 1.414$) SN Ia supernova data of observed apparent magnitudes along with their possible error from union 2.1 compilation [3]. In our present study, we have used a technique to estimate the present values of energy parameters Ω_{m0}, $\Omega_{\sigma 0}$, and $\Omega_{\beta 0}$ by comparing the theoretical and observed results with the help of R^2 formula.

$$R^2_{SN} = 1 - \frac{\sum_{i=1}^{580}[(m_i)_{ob} - (m_i)_{th}]^2}{\sum_{i=1}^{580}[(m_i)_{ob} - (m_i)_{mean}]^2}. \tag{30}$$

The ideal case $R^2 = 1$ occurs when the observed data and theoretical function $m(z)$ agree exactly. On the basis of maximum value of R^2, we get the best fit present values of Ω_m, Ω_σ, and Ω_β for the apparent magnitude $m(z)$ function as shown in Eq. (29) which is given in Table I. For this, coupling constant ω is taken as > -2 and the theoretical values are calculated from Eq. (29). We have found the best fit present values of Ω_m, Ω_σ, and Ω_β are $(\Omega_m)_0 = 0.3885$, $(\Omega_\sigma)_0 = 2.159\times 10^{-12}$, and $(\Omega_\beta)_0 = 0.001204$ for maximum $R^2 = 0.9944$ with root mean square error (RMSE) 0.2359, i.e., $m(z) \pm 0.2359$ and their R^2 values only 0.56% far from the best one.

E. Estimation of present values of Hubble's constant H_0

We consider a data set of the observed values of the Hubble parameter $H(z)$ versus the red shift z with possible error [4–7]. These data points were obtained by various researchers from time to time, by using different age approach.

In our model, Hubble's constant $H(z)$ versus redshift "z" relation Eq. (27) is reduced to

$$H^2 = (5.7640)H_0^2[0.3006(1+z)^{1.5714} - 0.01052(1+z)^{-1.4286} + 2.371\times 10^{-5}(1+z)^6 + 0.0003737] \tag{31}$$

Where we have taken $(\Omega_m)_0 = 0.2901$, $(\Omega_\sigma)_0 = 2.371\times 10^{-5}$, $(\Omega_\beta)_0 = 0.0003737$, and the coupling constant $\omega = -1.7$. The Hubble Space Telescope (HST) observations of Cepheid variables [8] provides present value of Hubble constant H_0 in the range $H_0 = 73.8 \pm 2.4$ km/s/Mpc. A large number of data sets of theoretical values of Hubble constant $H(z)$ versus z, corresponding to H_0 in the range $(60.45 \leq H_0 \leq 74.21)$ are obtained by using Eq. (31). It should be noted that the redshift z are taken from [4–7] and each data set will consist of 19 data points.

In order to get the best fit theoretical data set of Hubble's constant $H(z)$ versus z, we calculate R^2 − test by using following statistical formula:

$$R^2_{SN} = 1 - \frac{\sum_{i=1}^{19}[(H_i)_{ob} - (H_i)_{th}]^2}{\sum_{i=1}^{19}[(H_i)_{ob} - (H_i)_{mean}]^2} \tag{32}$$

Here the sums are taken over datasets of observed and theoretical values of Hubble's constants. The observed values are taken from [4–7] and theoretical values are calculated from Eq. (27). Using the above R^2-test, we have found the best fit function of $H(z)$ for the Eq. (27) which is mentioned in Table II.

From the Table II, one can see that the best fit value of Hubble constant H_0 is 50.02 for maximum $R^2 = 0.8875$ with root mean square error $RMSE = 16.7907$, i.e., $H_0 = 50.02 \pm 16.7907$ and their R^2 values only 11.25% far from the best one.

TABLE II. Outcomes of the R^2 − test for the best fit curve of Hubble parameter $H(z)$ in Eq. (27). The values of coefficients Ω_m, Ω_σ, Ω_β, and ω are at 95% confidence of bounds.

Function	γ	ω	Ω'_{m0}	$\Omega_{\phi 0}$	$\Omega_{\sigma 0}$	$\Omega_{\beta 0}$	H_0	R^2	$RMSE$
$H(z)$	0	−1.800	0.3555	0.09829	7.894×10^{-5}	3.671×10^{-7}	56.94	0.8849	16.3662
$H(z)$	0.1	−1.700	0.3006	0.01052	2.371×10^{-5}	0.0003737	50.02	0.8875	16.7907
$H(z)$	0.2	−1.613	0.2773	2.337×10^{-14}	4.357×10^{-14}	0.06744	40.77	0.8891	15.5200
$H(z)$	0.3	−1.550	0.2700	2.220×10^{-14}	2.220×10^{-14}	0.1821	40.00	0.8625	15.7799

V. ESTIMATION OF CERTAIN OTHER PHYSICAL PARAMETERS OF THE UNIVERSE

A. Age of the universe

By using the standard formula

$$t - t_0 = \int_{t_0}^{t} dt = \int_{A_0}^{A} \frac{dA}{AH}$$

we obtain the values of t in terms of scale factor and redshift respectively (Eqs. (33) and (34) replaced Eqs (74) and (75) in Ref. [1] respectively):

$$t_0 - t = \int_{A_0}^{A} \frac{\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}dA}{AH_0\sqrt{\Omega'_{m0}(\frac{A_0}{A})^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(\frac{A_0}{A})^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(\frac{A_0}{A})^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}}} \tag{33}$$

$$H_0(t_0 - t) = \int_{0}^{z} \frac{\sqrt{1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}}dz}{(1+z)\sqrt{\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(1+z)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}}} \tag{34}$$

For $\omega = -1.7$, $(\Omega_m)_0 = 0.2901$, $(\Omega_\sigma)_0 = 2.371 \times 10^{-5}$, and $(\Omega_\beta)_0 = 0.0003737$, Eq. (34) gives $t_0 \to 0.8729H_0^{-1}$ for high redshift. This means that the present age of the universe is $t_0 = 17.07^{+5.60}_{-3.47}$ Gyrs as per our model. From WMAP data, the empirical value of present age of universe is 13.73 ± 0.13 Gyrs which is closed to present age of universe, estimated by us in this paper.

Figure 1 shows the variation of time over redshift. At $z \to \infty$ the value of $H_0 t_0 = 2.51$. This provides present age of the universe. This also indicated the consistency with recent observations.

B. Deceleration parameter

From Eq. (21), we obtain the expressions for DP as [Eqs. (76), (77), and (78) in Ref. [1] is replaced by two Eqs. (35) and (36)]

$$q = 2 + \frac{1-3\gamma}{\omega-\omega\gamma+1} - \frac{3[(1-\gamma)\omega+1]}{2\omega+3}(\Omega'_m - \Omega_\phi) \tag{35}$$

or

$$q = 2 + \frac{1-3\gamma}{\omega-\omega\gamma+1} - \frac{\frac{3[(1-\gamma)\omega+1]}{2\omega+3}\left(\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}}\right)\left(1+\frac{(1-3\gamma)(5\omega-3\omega\gamma+6)}{6(\omega-\omega\gamma+1)^2}\right)}{\Omega'_{m0}(1+z)^{\frac{3\omega-3\omega\gamma^2+4}{\omega-\omega\gamma+1}} - \Omega_{\phi0}(1+z)^{\frac{1-3\gamma}{\omega-\omega\gamma+1}} + \Omega_{\sigma0}(1+z)^{6+\frac{(1-3\gamma)^2}{(\omega-\omega\gamma+1)^2}} + \Omega_{\beta0}} \tag{36}$$

At the present phase ($z = 0$) of the universe is accelerating $q \leq 0$, i.e., $\frac{\ddot{a}}{a} \geq 0$, so we must have

$$\Omega_{\beta0} \leq \left[\frac{[6(\omega-\omega\gamma+1)^2 + (1-3\gamma)(5\omega-3\omega\gamma+6)][3(1-\gamma)\omega+3]}{6(2\omega+3)(\omega-\omega\gamma+1)(2\omega-2\omega\gamma-3\gamma+3)} - 1\right]\Omega_{m0} - \Omega_{\sigma0} \tag{37}$$

For $\omega = -1.7$ and $\Omega_m = 0.2901$, $\Omega_\sigma = 2.371 \times 10^{-5}$, $0 \leq \gamma < \frac{1}{3}$ the maximum value of $\Omega_{\beta0}$ is given by $\Omega_{\beta0} \leq 0.0044$. Putting $z = 0$ in Eq. (36), the present value of deceleration parameter is obtained as

$$q_0 = -0.1248. \tag{38}$$

The Eq. (36) also provides

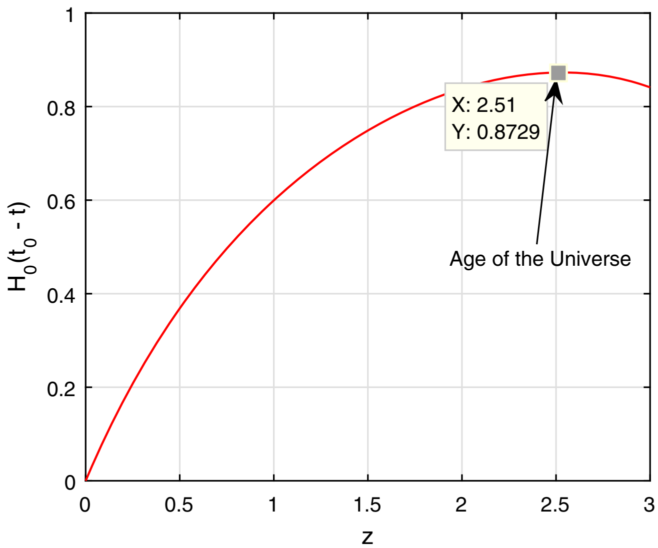

FIG. 1. Plot of $H_0(t_0 - t)$ versus redshift z.

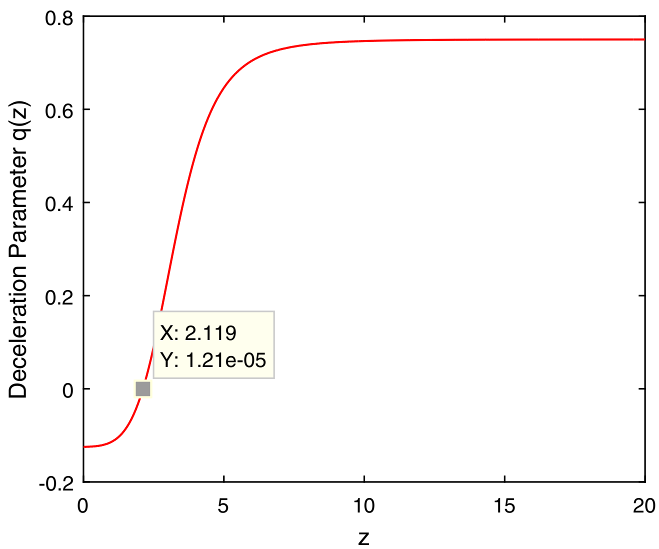

FIG. 2. Variation of deceleration parameter versus red shift.

$$z_c \approx 2.119 \quad \text{at} \quad q = 0. \tag{39}$$

Therefore, the universe attains to the accelerating phase when $z < z_c$.

Converting redshift into time from Eq. (39), the value of z_c is reduced to

$$z_c = 2.119 \sim 0.8508 H_0^{-1} \text{ Gyrs} \sim 14.61 \text{ Gyrs}. \tag{40}$$

TABLE III. Cosmological parameters at present for $0 \leq \gamma < \frac{1}{3}$.

Cosmological parameters	Values at present
BD coupling constant ω	-1.7
Matter energy parameter Ω_{m0}	0.2901
Gauge function β parameter $\Omega_{\beta 0}$	0.0003737
Anisotropic energy parameter $\Omega_{\sigma 0}$	2.371×10^{-5}
Hubble's constant H_0	50.02
Deceleration parameter q_0	-0.1248
Matter energy density ρ_{m0}	$0.5621 h_0^2 \times 10^{-29}$ gm/cm^3
Energy density $\rho_{\beta 0}$	$7.0256 h_0^2 \times 10^{-33}$ gm/cm^3
Anisotropic energy density $\rho_{\sigma 0}$	$4.4575 h_0^2 \times 10^{-34}$ gm/cm^3
Age of the Universe t_0	$17.07^{+5.60}_{-3.47}$ Gyrs

So, the acceleration must have begun in the past at 14.61 Gyrs. Figure 2 shows how deceleration parameter increases from negative to positive over redshift which means that in the past the universe was decelerating and at a instant $z_c \cong 2.119$, it became stationary there after it goes on accelerating.

VI. CONCLUSION

We summarize our results by presenting Table III which displays the values of cosmological parameters at present obtained by us.

In our reinvestigation we have found that our derived model is an accelerated universe model for the small negative values of $\omega \approx -1.7$. We have also found that the constant displacement vector behaves important contribution in early time of the universe and the scalar-field ϕ is the responsible candidate for present acceleration in the expansion of the universe. The behavior of β and ϕ in the history of the universe may be interesting and it requires more investigations for its viable characteristics. Thus, the model creates more interest in researchers to study the behavior of gauge function β and scalar field ϕ and their coupling in the formulation of the universe model.

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