Comment on "Brans-Dicke scalar field cosmological model in Lyra's geometry"

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(Received 21 November 2019; accepted 16 September 2020; published 4 November 2020)

In the present study, we comment on Brans-Dicke scalar field cosmological model in Lyra's geometry [Maurya and Zia, Phys. Rev. D **100**, 023503 (2019)]. In this comment, we investigate that there is no acceleration in the model proposed by the authors of Phys. Rev. D **100**, 023503 (2019). Therefore, despite the claims to the contrary the Brans-Dicke scalar field cosmological model in Lyra's geometry with high Brans-Dicke (BD) coupling parameter ω and constant β cannot produce late time acceleration in the universe.

DOI: 10.1103/PhysRevD.102.108301

I. INTRODUCTION

Today, the general theory of relativity (GR) is credited the most successful theory to describe the late time acceleration with inclusion of some type of exotic/dark energy in Einstein's field equation. In 1915, Einstein formulated GR and tried to give a satisfactory description to Mach's principle. But soon he realized that his theory does not follow Mach's principle. In the last century, some theories of gravitation had been postulated to follow Mach's principle. In 1961, Brans and Dicke [1] had proposed a scalar-tensor relativistic theory of gravitation in which Mach's principle is validated. With validation of Mach's principle, the proposed Brans-Dicke theory also describes the inflation era but it requires extra dark matter candidates to explain the galactic velocity profiles [2]. The concept of geometrizing gravitation in the form of GR gives a clue to researchers to think about the geometrizing electromagnetic field also. Weyl [3] had proposed a geometrized theory of electromagnetism and gravitation which is based the on nonintegrability of length transfer. In 1951, Lyra [4] had proposed geometrized theory without nonintegrability condition. In subsequent investigations, several authors [5–8] have constructed cosmological model in the framework of Lyra's geometry. It is shown from these investigations that the scalar-tensor treatment based on Lyra's geometry predicts the same effects, as GR, under observational limits.

II. THEORETICAL MODEL AND BASIC EQUATIONS

The Bianchi I space-time is read as [9]

$$ds^2 = -c^2dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{1}$$

where A, B, and C are directional scale factors and they are functions of t only. In Ref. [9], c^2 is missing from Eq. (16) but appeared in subsequent equations.

The Einstein's Brans-Dicke field equations in Lyra's manifold is read [9] as

$$G_{ij} + \frac{3}{2}\psi_i\psi_j - \frac{3}{4}g_{ij}\psi_k\psi^k = -\frac{8\pi T_{ij}}{\phi c^4} - \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) - \frac{1}{\phi}(\phi_{,i,j} - g_{ij}\Box\phi) \tag{2}$$

$$\Box\phi = \phi^{;i}_{,i} = \frac{8\pi T}{(3+2\omega)c^2} \tag{3}$$

where G_{ij}, ψ^i, ω, and ϕ are Einstein's curvature tensor, displacement vector field of Lyra's geometry, Brans-Dicke coupling constant and scalar field respectively. Also the timelike constant displacement vector is read as $\psi_i = (\beta, 0, 0, 0)$. In Eq. (2), T_{ij} denotes the energy-momentum tensor of perfect fluid.

As given in Ref. [9], the gravitational field equations for Bianchi I space-time are read as

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3}{4}\beta^2 = \frac{8\pi\rho}{\phi c^2} - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{4}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{3}{4}\beta^2 = -\frac{8\pi p}{\phi c^2} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\ddot{\phi}}{\phi} \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{3}{4}\beta^2 = -\frac{8\pi p}{\phi c^2} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) + \frac{\ddot{\phi}}{\phi} \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -\frac{8\pi p}{\phi c^2} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{\ddot{\phi}}{\phi} \tag{7}$$

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$$\frac{\ddot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=\frac{8\pi(\rho-3p)}{(2\omega+3)\phi c^2}. \tag{8}$$

Note that Eqs. (4)–(8) are same as the Eqs. (17)–(20) in Maurya and Zia [9]. It is worthwhile to note that the field equations (17)–(20) in Maurya and Zia [9] are wrong (see Refs. [1,10–13]). The correct field equations are as follows.

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}\dot{C}}{AC}-\frac{3}{4}\beta^2=\frac{8\pi\rho}{\phi c^2}+\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) \tag{9}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{\phi}}{\phi} \tag{10}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)-\frac{\ddot{\phi}}{\phi} \tag{11}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{\ddot{\phi}}{\phi}. \tag{12}$$

Accordingly Eqs. (27)–(30) are not correct in Refs. [9]. The procedure of solving the above equations is described below:

Subtracting Eqs. (10) from (11), (11) from (12), and (12) from (10), we obtain the following system of equations

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{C}}{AC}-\frac{\dot{B}\dot{C}}{BC}+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)\frac{\dot{\phi}}{\phi}=0 \tag{13}$$

$$\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{A}\dot{C}}{AC}+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{\dot{\phi}}{\phi}=0 \tag{14}$$

$$\frac{\ddot{C}}{C}-\frac{\ddot{A}}{A}+\frac{\dot{B}\dot{C}}{BC}-\frac{\dot{A}\dot{B}}{AB}+\left(\frac{\dot{C}}{C}-\frac{\dot{A}}{A}\right)\frac{\dot{\phi}}{\phi}=0. \tag{15}$$

The Eqs. (13)–(15) are the system of three equations with four unknown variables A, B, C, and ϕ. So, one cannot solve these equations in general. To obtain the explicit solution of above equations, we have to assume the following relation among the directional scale factors as

$$B=AD\ \&\ C=\frac{A}{D} \tag{16}$$

where $D=D(t)$ measures the anisotropy in the universe.

Equations (14) and (16) lead to

$$\frac{\ddot{D}}{D}-\frac{\dot{D}^2}{D^2}+\frac{\dot{D}}{D}\left(3\frac{\dot{A}}{A}+\frac{\dot{\phi}}{\phi}\right)=0 \tag{17}$$

After the integration of equation (19), we obtain

$$D=\exp\left[\int\frac{k}{A^3\phi}dt\right] \tag{18}$$

Now, the average scale factor is computed as

$$a^3=ABC=A^3\Rightarrow a=A \tag{19}$$

In light of Eq. (16), Eqs. (9) and (10) take the following form

$$3\frac{\dot{A}^2}{A^2}-\frac{\dot{D}^2}{D^2}-\frac{3}{4}\beta^2=\frac{8\pi\rho}{\phi c^2}+\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-3\frac{\dot{\phi}\dot{A}}{\phi A} \tag{20}$$

$$2\frac{\ddot{A}}{A}+\frac{\dot{A}^2}{A^2}+\frac{\dot{D}^2}{D^2}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi c^2}-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-2\frac{\dot{\phi}\dot{A}}{\phi A}-\frac{\ddot{\phi}}{\phi} \tag{21}$$

where $\frac{\dot{D}}{D}=\frac{k}{A^3\phi}$ with k as the arbitrary constant.

The deceleration parameter q and Hubble's parameter H are defined as

$$q=-\frac{a\ddot{a}}{\dot{a}^2}=-\frac{A\ddot{A}}{\dot{A}^2} \tag{22}$$

$$H=\frac{\dot{a}}{a}=\frac{\dot{A}}{A}. \tag{23}$$

Differentiating Eq. (23), we obtain

$$\dot{H}=\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{A^2}=\frac{\ddot{A}}{A}-H^2. \tag{24}$$

Now, deceleration parameter in terms of H is given by

$$q=-1-\frac{\dot{H}}{H^2}. \tag{25}$$

Also, we define the matter energy density parameter Ω_m, anisotropy parameter Ω_σ, β parameter Ω_β, and q_ϕ as [9]

$$\Omega_m=\frac{8\pi\rho}{3c^2H^2\phi},\qquad \Omega_\sigma=\frac{k^2}{3H^2A^6\phi^2},$$
$$\Omega_\beta=\frac{\beta^2}{4H^2},\qquad q_\phi=-\frac{\ddot{\phi}}{\phi H^2}. \tag{26}$$

Using Eqs. (23)–(26) and after some algebra, Eqs. (20), (21), and (8) lead to

$$\Omega_m+\Omega_\sigma+\Omega_\beta=1-\frac{\omega}{6}\Psi^2+\Psi \tag{27}$$

$$2q=1+3(\gamma\Omega_m+\Omega_\beta+\Omega_\sigma)+\frac{\omega}{2}\Psi^2+2\Psi-q_\phi \tag{28}$$

$$-q_\phi+3\Psi=\frac{3(1-3\gamma)}{2\omega+3}\Omega_m \tag{29}$$

where $\Psi=\frac{\dot{\phi}}{\phi H}$.

Solving Eqs. (27), (28) and (29), we obtain

$$q = 2 + \Psi - \frac{3[(1-\gamma)\omega + 1]}{2\omega + 3}\Omega_m \tag{30}$$

Also, it is easy to find that the main equations of the model in standard BD cosmology by introducing two effective parameters as

$$\rho_{\rm eff} = \rho + \frac{3\phi c^2}{32\pi}\beta^2$$

and

$$p_{\rm eff} = p + \frac{3\phi c^2}{32\pi}\beta^2.$$

Thus, Eqs. (20) and (21) are recast as

$$3\frac{\dot{A}^2}{A^2} - \frac{\dot{D}^2}{D^2} = \frac{8\pi}{\phi c^2}\rho_{\rm eff} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - 3\frac{\dot{A}\dot{\phi}}{A\phi} \tag{31}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\dot{D}^2}{D^2} = -\frac{8\pi}{\phi c^2}p_{\rm eff} - \frac{\omega\dot{\phi}^2}{2\phi^2} - 2\frac{\dot{A}\dot{\phi}}{A\phi} - \frac{\ddot{\phi}}{\phi}. \tag{32}$$

For analysis of model, it is convenient to consider the effective equation of state parameter ($\omega_{\rm eff}$) as

$$\omega_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{p + \frac{3\phi c^2}{32\pi}\beta^2}{\rho + \frac{3\phi c^2}{32\pi}\beta^2}.$$

In absence of matter, i.e., $\rho = p = 0$, the effective equation of state parameter is equal to $\omega_{\rm eff} = +1$. That is why the displacement vector cannot play the role of a cosmological constant in Brans-Dicke theory for which $\omega_{\rm eff} = -1$ is required. It is worthwhile to note that in Ref. [9], $\omega_{\rm eff} \neq -1$.

III. DISCUSSION

It is well known that for accelerating cosmological model $q < 0$. In the derived model, the fluid under consideration is a perfect fluid therefore $p \geq 0$. Also $\Psi = \frac{\dot{\phi}}{\phi H}$ is positive.

Thus, from Eq. (30), for accelerating cosmological model, we have

$$2 + \Psi - \frac{3[(1-\gamma)\omega + 1]}{2\omega + 3}\Omega_m < 0$$
$$\Rightarrow 2 + \Psi < \frac{3[(1-\gamma)\omega + 1]}{2\omega + 3}\Omega_m. \tag{33}$$

From Eq. (33), we conclude that in the derived model acceleration is only possible when $\frac{3[(1-\gamma)\omega+1]}{2\omega+3}\Omega_m > 2$ because $2 + \Psi$ is always greater than 2. The plot of numerically computed values of $\frac{3[(1-\gamma)\omega+1]}{2\omega+3}\Omega_m$ versus $0 \leq \gamma < 1/3$ for some particular values of ω is shown in Fig. 1. We choose γ in the range $0 \leq \gamma < 1/3$ for numerical result and analysis of model because the authors of Ref. [9] have taken this range of γ in describing the late acceleration of the universe. We observe that for $\omega = 49590$, the derived model does not validate Eq. (33) because the value of $\frac{3[(1-\gamma)\omega+1]}{2\omega+3}\Omega_m$ is less than 2 (see Fig. 1). From the numerical result plotted in Fig. 1, we also observe that for small negative values of BD coupling parameter (i. e. $\omega = -1.55$ and $\omega = -1.60$) the late time acceleration is possible in the derived model. Therefore, in spite of mathematical errors in Maurya and Zia [9], the late time acceleration is not possible with large BD coupling parameter ($\omega = 49590$). This result is in favor of investigations presented in Refs. [12,14]. Recently Akarsu *et al.* [15] have investigated some particular negative range of ω that lead acceleration in massive Brans-Dicke gravity. The variation of Ψ versus q and $0 \leq \gamma < 1/3$ for $\omega = 49590$ and $\omega = -1.55$ is depicted in Fig. 2. Since $\dot{\phi} > 0$ therefore $\Psi = \frac{\dot{\phi}}{\phi H}$ is always greater

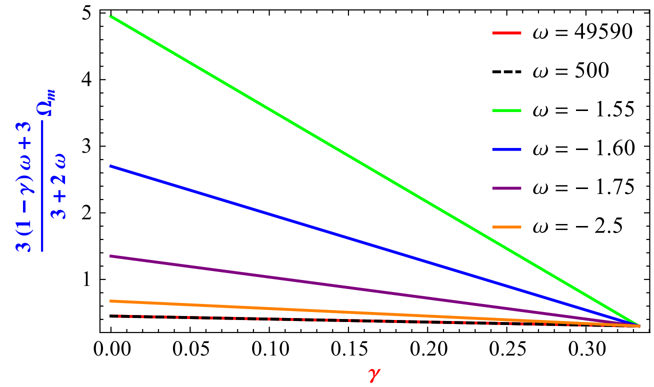

FIG. 1. The plot of $\frac{3(1-\gamma)(\omega+1)}{2\omega+3}$ versus γ for different numerical values of ω.

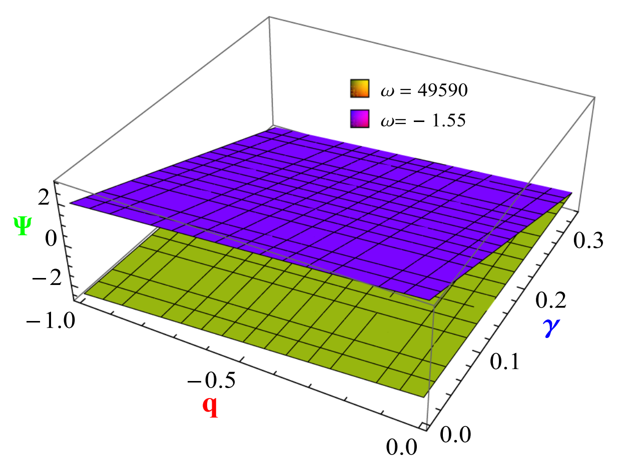

FIG. 2. The plot of Ψ versus γ and q for $\omega = 49590$ and $\omega = -1.55$.

than zero in Brans-Dicke theory of gravitation (see Ref. [10]). From Fig. 2, we observe that for $\omega = 49590$, in Maurya and Zia [9], positive Ψ and negative q do not exist simultaneously while $\omega = -1.55$ validates this condition.

Therefore, in the derived model, the late time acceleration is produced due to amalgamation of BD theory with small negative BD coupling parameter (adverse of the author's finding in Ref. [9]). In the literature, BD theory is invoked to fulfill the requirement of Mach's principle [1,10,16–18]. In Sen and Sen [12], the authors have investigated that a perfect fluid cannot support acceleration but a fluid with dissipative pressure can drive late time acceleration of the current universe. The present cosmic acceleration without resorting to a cosmological constant or quintessence matter have been investigated in BD theory but then Brans-Dicke coupling constant asymptotically acquires a small negative value for an accelerating universe at late time [14] while in Ref. [19], the authors have obtained solution for accelerating universe with ϕ^2 potential for large BD coupling constant without considering positive energy condition for matter and scalar field both. In the targeted paper [9], the authors have not clearly argued that the late time acceleration is due to scalar field. They focused on gauge function of Lyra's geometry and hypothetically assumed that $\beta =$ constant behaves like cosmological constant Λ. Further they argued that it may be a suitable candidate of dark energy and removes the cosmological constant problems while the investigations in Lyra's geometry clearly established the fact that time varying displacement vector $\beta(t)$ have the similar nature as $\Lambda(t)$, i.e., $\beta(t)$ and $\Lambda(t)$ both are decreasing functions of time [20]. It is important to note that this decreasing behavior of $\beta(t)$ with time does not contribute late time acceleration in the universe [21]. But in Maurya and Zia [9], it is conveniently assumed that constant β play the role of cosmological constant and leads the late time acceleration of the universe without giving concrete mathematical expression or exact physical reason behind it.

The continuity equation in Lyra's geometry is read as

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + 3\left[\chi(\rho + p) + \frac{3}{2}\beta^2\right]H = 0 \tag{34}$$

where $\chi = \frac{8\pi}{c^4}$.

The Eq. (22) of the targeted paper is entirely different from Eq. (34) of this comment. It seems that the authors have assumed only the general relativity case in energy conservation law. For $\beta =$ constant, Eq. (34) is given by

$$\chi\dot{\rho} + 3\left[\chi(\rho + p) + \frac{3}{2}\beta^2\right]H = 0. \tag{35}$$

Thus, the energy conservation law given in Ref. [9] is not correct in the context of Lyra's geometry which in turn implies that the matter energy density as given in Eq. (53) of thetargeted paper, may have different expression. In addition, we observe that for constant displacement vector, i.e., $\beta =$ constant, there is a constant contribution to Eqs. (4)–(7). Therefore, despite the claims to the contrary made by the authors, the model cannot be consistent with observations. Some important applications of Lyra's geometry with the time varying displacement vector are given in the Refs. [21–26].

IV. CONCLUSION

In this comment, we have shown that the field equations derived in Ref. [9], are not contributing late time acceleration with constant β and large ω but late time acceleration in the model may be the feature of universe due to Brans-Dicke scalar field that have smal negative value of ω. However, the actual physics of such acceleration with large BD coupling constant is not elaborated in Maurya and Zia [9]. It is convenient to assume $\beta =$ constant but this constant displacement vector does not contribute the late time acceleration of the universe with large ω. We have also corrected the field equations and subsequent equations which were wrong in Maurya and Zia [9]. It is worthwhile to note that we neither avoid the coexistence of BD scalar field with Lyra's geometry nor decline the similarities between time varying displacement vector $\beta(t)$ and $\Lambda(t)$ as both $\Lambda(t)$ and $\beta(t)$ are decreasing functions of time. As a final comment, we note that in spite of the good possibility of scalar field cosmological model in Lyra's geometry to provide a theoretical foundation for relativistic cosmology, the experimental point is yet to be considered and still the theory needs a fair trial.

ACKNOWLEDGMENTS

The author is grateful to the editor and referee for illuminating suggestions that have significantly improved our work in terms of research quality and presentation. We also acknowledge G. K. Goswami for a careful reading of manuscript prior to submission.

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