

Four-loop singularities of the massless fermion propagator in quenched three-dimensional QED

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We calculate the three- and four-loop corrections to the massless fermion propagator in three-dimensional quenched quantum electrodynamics with four-component fermions. The three-loop correction is finite and gauge invariant but the four-loop one has singularities except in the Feynman gauge where it is also finite. Our results explicitly show that, up to four loops, gauge-dependent terms are completely determined by lower order ones in agreement with the Landau-Khalatnikov-Fradkin transformation.

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I. INTRODUCTION

Three-dimensional quantum electrodynamics (QED₃) is an archetypal gauge field theory model of strongly interacting relativistic planar fermions. In Euclidean space, it is described by the action

$$S = \int d^3x \left[\frac{1}{4} F_{\mu\nu}^2 + \bar{\Psi}_i \gamma_\mu D_\mu \Psi_i \right], \quad (1)$$

where $D_\mu = \partial_\mu - ieA_\mu$, $i = 1, 2, \dots, n_f$ with n_f the number of four-component massless Dirac fermion flavors, Euclidean gamma matrices satisfy $\gamma_\mu^\dagger = \gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$ and the gauge coupling constant e^2 has dimension of mass.

For the past 40 years, this super-renormalizable model served as a toy model for exploring several key problems in quantum field theory such as infrared (IR) singularities in low-dimensional massless particle theories [1–5] (see recent progress in [6,7]) and dynamical symmetry breaking and fermion mass generation [8–18] (see recent progress in [19–23]). In the last 30 years, considerable interest in QED₃ also came from its applications to condensed matter physics systems with relativisticlike gapless quasiparticle

excitations at low-energies such as high- T_c superconductors [24–26], planar antiferromagnets [27], and graphene [28] (for graphene, see reviews in Ref. [29]).

A slight simplification of QED₃ takes place in the so-called quenched limit in which closed fermion loops are set to zero (this amounts to take $n_f = 0$). This limit arose as a useful approximation in the study of the lattice representation of four-dimensional quantum chromodynamics (QCD₄), see [30], where it was shown that a reasonable estimate of the hadron spectrum could be obtained by eliminating all internal quark loops. The quenched limit, and other approximation schemes such as the ladder (rainbow) approximation, were used in QED₄ for a long time to try solving nonperturbatively a more manageable truncated set of Schwinger-Dyson equations (see Refs. [31–33] and references therein to earlier papers). The quenched approximation in QED₄ is now still in use in order to include QED effects in lattice QCD calculations (see the recent Ref. [34] and discussions therein).

In a recent paper [7], we studied the gauge covariance of the massless fermion propagator of quenched QED₃ in a linear covariant gauge in dimensional regularization (following Refs. [33,35,36]). At this point, let us note that, as in the four-dimensional case, both the fermion propagator and vertex function of QED₃ possess the important property of being covariant under the Landau-Khalatnikov-Fradkin (LKF) transformations [37,38]. These transformations, which have a simple form in coordinate space, allow one to compute Green's functions in an arbitrary covariant gauge provided they are known in a particular gauge (for applications of the LKF transformations to QED₃, see the papers [39] and the review in Ref. [40]).

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The analysis of the LKF transformation of the massless fermion propagator of quenched QED₃ carried out in [7], led us to reconnect with the long-standing issue of IR singularities in QED₃. In particular, we concluded that, exactly in three dimensions ($d = 3$), all of the odd perturbative coefficients of the massless fermion propagator, starting from the third order one, have to vanish in any gauge in order for quenched QED₃ to be free of (IR) singularities. It turns out that there is a widespread opinion that quenched perturbation theory is IR finite [1] and recent lattice studies of quenched QED₃ seem to confirm it [6]. Usually, the presence of IR divergences is related to fermion loops. This is because, for dimensional reasons, higher-order diagrams contain higher powers of momentum in the denominator. For example, the two-loop fermion self-energy diagram with vacuum polarization gives rise to a logarithmic divergence [1] which corresponds to a $1/\epsilon$ pole in dimensional regularization in $d = 3 - 2\epsilon$ [2]. However, for the same dimensional reasons, higher-order quenched diagrams can lead to IR divergences, too. Indeed, it is easy to see that, at four loops, e.g., the diagram with an insertion of three one-loop fermion self-energies into a fermion line is logarithmically divergent in gauges different from the Landau gauge (this was for the first time mentioned in Ref. [3]). The question then is whether IR divergences of separate diagrams cancel in their sum or not. The LKF transformation by itself is unable to provide explicit values for the coefficients in a given gauge. Order by order calculations are therefore required (at least in a given gauge) in order to analyze the IR finiteness of QED₃ in accordance with our statement above.

In the present paper, we undertake this task and explicitly calculate the fermion propagator of quenched QED₃ at three and four loops in an arbitrary linear covariant gauge and in dimensional regularization in $d = 3 - 2\epsilon$. We find that the three-loop correction is finite and gauge invariant. Accordingly, the four-loop one has singularities. Our exact results show that, up to four loops, gauge-dependent terms are completely determined by lower order ones in perfect agreement with properties of the LKF transformation following the study [7].

The paper is organized as follows. In Sec. II, we specify our notations, provide some details of the calculations and present the results for the three and four-loop corrections to the fermion propagator. In Sec. III, we briefly recall the LKF transformation for the fermion propagator in momentum space and check its consistency with our four-loop perturbative results. Some predictions beyond four-loops are also presented. In Sec. IV, a representative sample of the computed diagrams is presented by focusing on the Landau gauge. The results are summarized and discussed in Sec. V.

II. FERMION PROPAGATOR: THREE- AND FOUR-LOOP COEFFICIENTS

A. Notations

In the following, we shall consider a Euclidean space of dimension $d = 3 - 2\epsilon$. The general form of the fermion propagator $S_F(p, \xi)$ in some gauge ξ reads:

$$S_F(p, \xi) = \frac{i}{\not{p}} P(p, \xi), \quad (2)$$

where the tensorial structure, e.g., the factor \not{p} containing Dirac γ matrices, has been extracted and $P(p, \xi)$ is a scalar function of $p = \sqrt{p^2}$.

It is convenient to first express $P(p, \xi)$ as

$$P(p, \xi) = \frac{1}{1 - \sigma(p, \xi)}, \quad (3)$$

where the one-particle-irreducible part, $\sigma(p, \xi)$, can be represented as

$$\sigma(p, \xi) = \sum_{m=1}^{\infty} \sigma_m(\xi) \left(\frac{\alpha}{2\sqrt{\pi}p} \right)^m \left(\frac{\bar{\mu}^2}{p^2} \right)^{m\epsilon}. \quad (4)$$

Here, $\sigma_m(\xi)$ are the coefficients of the loop expansion of the fermion self-energy, $\alpha = e^2/(4\pi)$ is the dimensionful coupling constant and $\bar{\mu}$ is the $\overline{\text{MS}}$ scale.

Following our previous paper [7], the fermion propagator can be equivalently represented as

$$P(p, \xi) = \sum_{m=0}^{\infty} a_m(\xi) \left(\frac{\alpha}{2\sqrt{\pi}p} \right)^m \left(\frac{\bar{\mu}^2}{p^2} \right)^{m\epsilon}, \quad (5)$$

where $a_m(\xi)$ are now the coefficients of the loop expansion of $P(p, \xi)$. As will be seen in Sec. III, the form (5) is convenient to study the properties of the propagator under the LKF transformation.

In both Eqs. (4) and (5), the expansion has been written in terms of the dimensionless ratio α/p with an additional conventional factor of $1/(2\sqrt{\pi})$. Its exact form is coming from the consideration of four-dimensional QED in [35] (see also Ref. [7] and discussions therein). Up to four loops, the coefficients $a_m(\xi)$ and $\sigma_m(\xi)$ are related to each other as

$$\begin{aligned} a_1 &= \sigma_1, & a_2 &= \sigma_2 + \sigma_1^2, & a_3 &= \sigma_3 + 2\sigma_2\sigma_1 + \sigma_1^3, \\ a_4 &= \sigma_4 + 2\sigma_3\sigma_1 + \sigma_2^2 + 3\sigma_2\sigma_1^2 + \sigma_1^4. \end{aligned} \quad (6)$$

B. Computational details

In quenched QED at one, two, three, and four loops we encountered 1, 2, 10, and 74 fermion self-energy diagrams, respectively. Let us note that the two-loop diagrams of

QED₃ were considered earlier in [1,2]. These papers mainly focused on the IR divergent two-loop diagram (with a fermion loop insertion) which is absent in the quenched case. In [2], higher order diagrams were considered but still with fermion loop insertions. To the best of our knowledge, the two-loop quenched QED₃ fermion propagator was calculated in [39]. Moreover, all two-loop diagrams of QED₃ were computed in [41] (see also the last paper in [29]) and their ε expansion provided near $d = 3$. The fact that the first singularities in the fermion propagator of QED₃ without vacuum polarization arise at four loops was mentioned for the first time in Ref. [3]. However, three- and four-loop corrections to the quenched QED₃ fermion propagator have not been previously computed. As will be shown in the next subsections, the three-loop correction is finite but IR singular diagrams do appear at four loops in the quenched case (in agreement with [3]) and there are 42 of them, the sum of which will be analyzed in the following.

In order to compute all of these diagrams and extract from them the unrenormalized fermion self-energy of QED₃ up to four loops, we first considered the corresponding results for the unrenormalized QCD quark propagator. The exact expression for the latter, written in terms of a set of master integrals and valid for arbitrary space-time dimension d and arbitrary gauge-fixing parameter ξ , is available up to four loops from [42] and also shipped with the FORCER package [43] designed for the reduction of four-loop massless propagator-type integrals. The fermion propagator of QED _{d} is obtained from this QCD _{d} result upon performing the following substitutions:

$$\begin{aligned} C_A &= d_A^{abcd} d_A^{abcd} = d_A^{abcd} d_F^{abcd} = 0, \\ C_F &= d_F^{abcd} d_F^{abcd} = T_F = 1. \end{aligned} \quad (7)$$

After that, the quenched limit of QED _{d} is obtained by setting $n_f = 0$ which discards all diagrams with closed fermion loops.

The main remaining task was then to compute all required propagator-type master integrals in an ε expansion around $d = 3$ ($\varepsilon = (3 - d)/2$). This could be achieved with the help of the dimensional recurrence and analyticity (DRA) method [44], which expresses the integrals in the form of fast convergent sums. The latter are then evaluated with high-precision numerical values. This in turn allows us to reconstruct the analytic expression of master integrals (in any space-time dimension) with the help of the PSLQ algorithm [45] once an adequate basis of transcendental constants is defined.

We note that near $d = 4$ ($\varepsilon = (4 - d)/2$), such calculations yield the expansions of all needed masters [46]. The results are well-known and available in input form for the SUMMERTIME package [47] with the package itself and also from [48].

The case $d = 3 - 2\varepsilon$ is less well known and was considered in the paper [47] from which the ε expansion of most of the needed master integrals for the current calculation is available. The successful reconstructions of [47] around $d = 3$, were carried out using a basis of transcendental constants consisting only of multiple zeta values (MZV) and alternating MZVs. As remarked already in [47], such a basis is too restrictive to enable the representation of all of the masters and some of them were left unreconstructed.

In our work we successfully reconstructed all the needed integrals and found agreement with results of [46] using a basis consisting of MZV and alternating MZVs. On top of that, we encountered one of the constants left unknown in [47]. By a careful analysis of the representation of one such integrals with known closed form expressions in the form of the ${}_3F_2$ functions [41],¹ we found that elements of its ε expansion belong to the set of generalized polylogarithms (GPLs) with fourth-root of unity alphabet. Extending our PSLQ basis to include the full set of GPLs with fourth root of unity arguments we successfully reconstructed its analytical value

$$\text{Sunset} = \text{Sunset} \cdot \left(-8\varepsilon \left(\mathcal{C}\pi^2 + 24\text{Cl}_4\left(\frac{\pi}{2}\right) \right) + O(\varepsilon^2) \right), \quad (8)$$

where we factored out the four-loop sunset integral to follow the normalization prescriptions of [47]. In Eq. (8), $\mathcal{C} = \text{Cl}_2(\pi/2)$ is Catalan's constant and $\text{Cl}_n(\theta)$ is Clausen's function which, for even weight, can be expressed through the classical polylogarithm as $\text{Cl}_{2k}(\theta) = \text{Im Li}_{2k}(e^{i\theta})$. As can be understood from the above result, the required extension of the basis of transcendental constants includes polylogarithms with fourth-root of unity argument, in the present case Clausen's function (see, for example, Ref. [50], where GPLs with second, fourth, and sixth root of unity arguments appear).

C. Results for the fermion self-energy (4)

We now present our results for $\sigma_m(\xi)$, which can be represented as

$$\sigma_m(\xi) = \sigma_m(0) + \xi \tilde{\sigma}_m(\xi), \quad (9)$$

where we have explicitly separated the part independent from ξ which corresponds to the full result in the Landau gauge.

¹The results of [41] were obtained based on the general approach of Ref. [49], where a class of more complicated diagrams with three arbitrary indices was studied and the corresponding results were expressed in terms of combinations of ${}_3F_2$ -hypergeometric functions of unit argument.

Considering the first two orders of the ε expansion, we have for the coefficients $\sigma_m(0)$

$$\sigma_1(0) = 0, \quad (10a)$$

$$\sigma_2(0) = \pi \left[\frac{3\pi^2}{4} - 7 - ((1 - 3l_2)\pi^2 + 12)\varepsilon \right], \quad (10b)$$

$$\sigma_3(0) = \pi^{5/2} \left[\frac{43\pi^2}{4} - 105 + \varepsilon \left\{ 2(185 - 105l_2 + 137\zeta_3) - \frac{\pi^2}{6}(451 - 171l_2) \right\} \right], \quad (10c)$$

$$\sigma_4(0) = \pi^2 \left[\left(\frac{43}{6}\pi^2 - 70 \right) \frac{1}{\varepsilon} + \bar{\sigma}_4 + \frac{5954}{3} + \frac{173}{18}\pi^2 - \frac{513}{10}\pi^4 \right], \quad (10d)$$

where $\bar{\sigma}_4$ contains the most complicated part

$$\bar{\sigma}_4 = 209l_2^4 + 5016a_4 + 4264\text{Cl}_4(\pi/2) + \left(\frac{533}{3}C - 930l_2 \right) \pi^2 + \frac{2078}{3}\zeta_3, \quad (11)$$

and

$$l_2 = \ln 2, \quad a_4 = \text{Li}_4(1/2), \quad \zeta_n = \text{Li}_n(1), \quad (12)$$

where Li_n are polylogarithms.

With the same accuracy, we have for the coefficients $\tilde{\sigma}_m(\xi)$

$$\tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}(1 - 2(1 - l_2)\varepsilon), \quad (13a)$$

$$\tilde{\sigma}_2(\xi) = \pi\xi \left[1 - \frac{\pi^2}{4} - (4 - (1 - l_2)\pi^2)\varepsilon \right], \quad (13b)$$

$$\tilde{\sigma}_3(\xi) = \pi^{5/2} \left[\frac{3\pi^2}{4} - 7 + \left(1 - \frac{\pi^2}{8} \right) \xi^2 + \varepsilon \left\{ -40 - 14l_2 + \frac{\pi^2}{2}(4 + 9l_2) + \left(2l_2 - 4 + \frac{3\pi^2}{4}(1 - l_2) \right) \xi^2 \right\} \right], \quad (13c)$$

$$\tilde{\sigma}_4(\xi) = \pi^2 \left[\left(70 - \frac{43\pi^2}{6} \right) \frac{1}{\varepsilon} + \frac{520}{3} - \frac{\pi^2}{9}(881 + 42l_2) + \frac{129\pi^4}{27} - \frac{548}{3}\zeta_3 + \xi \left(28 - \frac{33\pi^2}{4} + \frac{9\pi^4}{16} \right) + \xi^3 \left(-\frac{4}{3} + \frac{3\pi^2}{4} - \frac{\pi^4}{16} \right) \right]. \quad (13d)$$

We would like to note that the finite parts ($\varepsilon = 0$) of the coefficients $\sigma_1(\xi)$ and $\sigma_2(\xi)$ coincide with the corresponding ones in Ref. [41].

Moreover, from Eqs. (10d) and (13d), we notice that

$$\sigma_4(\xi) = \pi^2 \left(\frac{43}{6}\pi^2 - 70 \right) \frac{(1 - \xi)}{\varepsilon} + O(\varepsilon^0), \quad (14)$$

i.e., the total four-loop contribution is finite in the Feynman gauge.

D. Results for the fermion propagator (5)

As in the case of $\sigma_m(\xi)$ in (9), it is convenient to present the results for $a_m(\xi)$ in the form

$$a_m(\xi) = a_m(0) + \xi \tilde{a}_m(\xi), \quad (15)$$

where we have also explicitly separated the part independent from ξ which corresponds to the full result in the Landau gauge.

Since $\sigma_1(\xi) \sim \xi$, we see from (6) that $a_i(0) = \sigma_i(0)$ for $i \leq 3$ and thus $a_i(0)$ with $i \leq 3$ can straightforwardly be read off from Eqs. (10a), (10b), and (10c). In agreement with (6), we have for $a_4(0)$

$$\begin{aligned} a_4(0) &= \sigma_4(0) + \pi^2 \left(\frac{3\pi^2}{4} - 7 \right)^2, \\ &= \pi^2 \left[\left(\frac{43}{6}\pi^2 - 70 \right) \frac{1}{\varepsilon} + \bar{\sigma}_4 + \frac{6101}{3} - \frac{8}{9}\pi^2 - \frac{4059}{80}\pi^4 \right], \end{aligned} \quad (16)$$

where $\bar{\sigma}_4$ was defined in Eq. (11).

With the same accuracy, we have for the coefficients $\tilde{a}_m(\xi)$

$$\tilde{a}_1(\xi) = \tilde{\sigma}_1(\xi) = -\frac{\pi^{3/2}}{2}(1 - 2(1 - l_2)\varepsilon), \quad (17a)$$

$$\tilde{a}_2(\xi) = \pi\xi(1 - 4\varepsilon), \quad (17b)$$

$$\tilde{a}_3(\xi) = \pi^{5/2}\varepsilon \left(\frac{43\pi^2}{4} - 105 + 2\xi^2 \right), \quad (17c)$$

$$\begin{aligned} \tilde{a}_4(\xi) &= \frac{\pi^2}{3} \left[\left(210 - \frac{43\pi^2}{2} \right) \frac{1}{\varepsilon} + 520 + \frac{2\pi^2}{3}(32 - 21l_2) \right. \\ &\quad \left. - 548\zeta_3 + 6\xi \left(7 - \frac{3\pi^2}{4} \right) - \xi^3 \right]. \end{aligned} \quad (17d)$$

We would like to note that the finite parts ($\varepsilon = 0$) of the coefficients $a_1(\xi)$ and $a_2(\xi)$ coincide with the corresponding ones in Ref. [39] (see also Ref. [7] and discussions therein).

From the above results, we see that the coefficients $\tilde{a}_m(\xi)$ ($m = 2, 3, 4$) have simpler forms than the corresponding coefficients $\tilde{\sigma}_m(\xi)$. Moreover, as in the case of $\sigma_4(\xi)$, we notice from Eqs. (16) and (17d) that

$$a_4(\xi) = \sigma_4(\xi) + O(\varepsilon^0),$$

$$= \frac{2\pi^2}{3} \left(\frac{43\pi^2}{4} - 105 \right) (1 - \xi) \frac{1}{\varepsilon} + O(\varepsilon^0), \quad (18)$$

i.e., the total four-loop contribution is finite in the Feynman gauge.

III. LKF TRANSFORMATION

A. Comparison with the perturbative results up to four loops

It is convenient to introduce the x -space representation $S_F(x, \xi)$ of the fermion propagator as

$$S_F(x, \xi) = \not{x} X(x, \xi), \quad (19)$$

which is related by the Fourier transform to $S_F(p, \xi)$ in (2). The LKF transformation expresses the covariance of the fermion propagator under a gauge transformation. It can be derived by standard arguments, see, e.g., [37,38], and its general form can be written as (see Refs. [35,36]):

$$S_F(x, \xi) = S_F(x, \eta) e^{D(x)}, \quad (20a)$$

$$D(x) = e^2 \Delta \mu^{2\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iqx}}{q^4}, \quad \Delta = \xi - \eta, \quad (20b)$$

in $d = 3 - 2\varepsilon$. The calculation [33] yields:

$$D(x) = -\frac{\alpha \Delta}{2\pi\mu} \frac{\Gamma(1/2 - \varepsilon)}{1 + 2\varepsilon} (\pi\mu^2 x^2)^{1/2 + \varepsilon}. \quad (21)$$

The LKF transformation (20) relates [7] the coefficients $a_k(\xi)$ and $a_m(\xi)$ in (5) as

$$a_k(\xi) = \sum_{m=0}^k (-2\Delta)^{k-m} a_m(\eta) \Phi(m, k, \varepsilon) \phi(k - m, \varepsilon), \quad (22)$$

where

$$\Phi(m, k, \varepsilon) = \frac{\Gamma(3/2 - m/2 - (m+1)\varepsilon) \Gamma(1 + k/2 + k\varepsilon)}{\Gamma(1 + m/2 + m\varepsilon) \Gamma(3/2 - k/2 - (k+1)\varepsilon)}, \quad (23)$$

and

$$\phi(l, \varepsilon) = \frac{\Gamma^l(1/2 - \varepsilon)}{l!(1 + 2\varepsilon)^l \Gamma^l(1 + \varepsilon)}. \quad (24)$$

Consider $a_m(\xi)$ with $m \leq 4$. Keeping only the first two orders of the ε expansion, we have:

$$a_0(\xi) = a_0(\eta) = 1, \quad (25a)$$

$$a_1(\xi) = a_1(\eta) - \frac{\pi}{2} \delta(1 + 2\varepsilon(l_2 - 1)) a_0(\eta), \quad (25b)$$

$$a_2(\xi) = a_2(\eta) - \frac{4}{\pi} \delta(1 - 2\varepsilon(l_2 + 1)) a_1(\eta)$$

$$+ \delta^2(1 - 4\varepsilon) a_0(\eta), \quad (25c)$$

$$a_3(\xi) = a_3(\eta) + 6\pi\varepsilon\delta a_2(\eta) - 12\varepsilon\delta^2 a_1(\eta)$$

$$+ 2\pi\varepsilon\delta^3 a_0(\eta), \quad (25d)$$

$$a_4(\xi) = a_4(\eta) - \frac{2\delta}{3\pi\varepsilon} (1 + 2\varepsilon(3 - l_2)) a_3(\eta) - 2\delta^2 a_2(\eta)$$

$$+ \frac{8\delta^3}{3\pi} a_1(\eta) - \frac{\delta^4}{3} a_0(\eta), \quad (25e)$$

where $\delta = \sqrt{\pi}\Delta$.

Setting $\eta = 0$, i.e., choosing the initial gauge as the Landau gauge, we can see that our results for $\tilde{a}_m(\xi)$ are completely determined by $a_l(\xi)$, ($l < m$), i.e., by the coefficients of lower orders in agreement with the properties of the LKF transformation.

Moreover, the results of Eqs. (25) are in full agreement with the perturbative results presented in Sec. II D.

B. Beyond four loops

As can be seen from (18), the singularity of the four-loop coefficient $a_4(\xi)$ is $\sim(1 - \xi)$, i.e., the fermion propagator including up to four-loop corrections is finite in the Feynman gauge. This intriguing fact calls for a closer examination of higher order contributions and, as a first try, we will proceed by using the LKF transformation.

We therefore consider $a_5(\xi)$ and $a_6(\xi)$. From Eq. (22), we have:

$$a_5(\xi) = a_5(\eta) + \frac{45}{2} \pi\varepsilon\delta a_4(\eta) - \frac{15}{2} \delta^2 a_3(\eta) - 15\pi\varepsilon\delta^3 a_2(\eta)$$

$$+ 15\varepsilon\delta^4 a_1(\eta) - \frac{3}{2} \pi\varepsilon\delta^5 a_0(\eta), \quad (26a)$$

$$a_6(\xi) = a_6(\eta) + \frac{4\delta}{5\pi\varepsilon} a_5(\eta) - 9\delta^2 a_4(\eta) + \frac{2\delta^3}{\pi\varepsilon} a_3(\eta)$$

$$+ 3\delta^4 a_2(\eta) - \frac{12\delta^5}{5\pi} a_1(\eta) + \frac{\delta^6}{5} a_0(\eta). \quad (26b)$$

We may then take the η gauge as the Feynman gauge and consider $a_5(\xi)$ and $a_6(\xi)$ with accuracies $O(\varepsilon)$ and $O(\varepsilon^0)$, respectively. This yields

$$a_5(\xi) = a_5(1) - \frac{15}{2} \pi(\xi - 1)^2 a_3 + O(\varepsilon), \quad (27a)$$

$$a_6(\xi) = a_6(1) + \frac{4(\xi-1)}{5\sqrt{\pi}\varepsilon} a_5(1) + \frac{2\sqrt{\pi}(\xi-1)^3}{\varepsilon} a_3 + O(\varepsilon^0), \quad (27b)$$

where we took into account the fact that the finite part of a_3 is gauge-independent.

From these results, we see that the LKF transformation gives information about the ξ dependence of $a_5(\xi)$ and $a_6(\xi)$, as expected. Some singularities may still be hidden in $a_6(1)$ and further understanding of the singular structure of $a_6(\xi)$ requires explicit five- and six-loop computations (at least in a specific gauge).

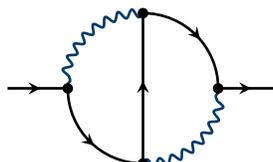
IV. DIAGRAMS IN THE LANDAU GAUGE

As we discussed in Sec. II B, there is a total of 87 diagrams to compute in order to derive the fermion propagator of quenched QED₃ up to four-loops with an arbitrary gauge-fixing parameter ξ . The results presented in Secs. II C and II D, were obtained by computing all of these diagrams.

In order to provide the interested reader with a representative sample of the graphs, we focus in this section on the Landau gauge. The reason is that it is the gauge where there is the least number of diagrams as most of them vanish in the limit $\xi = 0$. Moreover, as discussed in Sec. III, it is enough to compute the fermion propagator in this gauge as the LKF transformation allows us to reconstruct the full ξ dependence of the propagator.

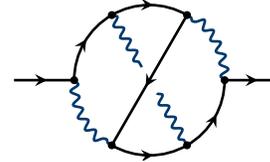
In the Landau gauge, there is no one-loop contribution and there are 1, 4, and 30 diagrams at two, three, and four loops, respectively; so there is a total of 35 diagrams. Focusing on the leading order contribution to the ε -expansion of these diagrams, the two- and three-loop contributions will be considered with an accuracy $O(\varepsilon)$ and the four-loop contributions with an accuracy $O(\varepsilon^0)$. So, amongst the 30 four-loop diagrams only the eight divergent ones need to be considered (the other 22 diagrams are finite). Moreover, taking into account the fact that mirror conjugate graphs take the same value, we are left with only three distinct graphs at three loops and four distinct graphs at four loops. Hence a total of eight distinct diagrams contribute to the Landau gauge quenched QED₃ fermion propagator up to four loops.

For the sake of clarity, we explicitly display the distinct graphs together with their values. The single diagram contributing at two-loop level is given by

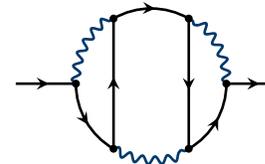


$$= -\frac{\pi(28 - 3\pi^2)}{4} + O(\varepsilon). \quad (28)$$

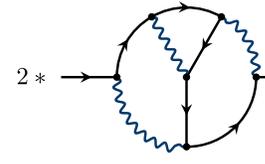
At three loops, the two benz diagrams in (29c) are mirror conjugate to each other and therefore share the same value. Hence, the three distinct three-loop graphs read:



$$= \frac{3\pi^{5/2}(10 - \pi^2)}{2} + O(\varepsilon), \quad (29a)$$

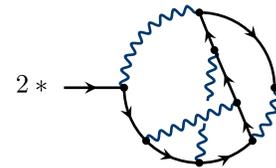


$$= -\frac{\pi^{5/2}(480 - 49\pi^2)}{12} + O(\varepsilon), \quad (29b)$$

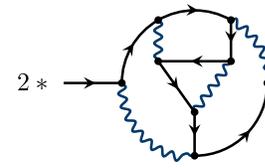


$$2 * \quad = -\frac{\pi^{5/2}(480 - 49\pi^2)}{6} + O(\varepsilon). \quad (29c)$$

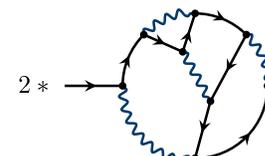
Similarly, the four-loop diagrams are grouped in pairs of mirror conjugate graphs and the four leading contributions are given by



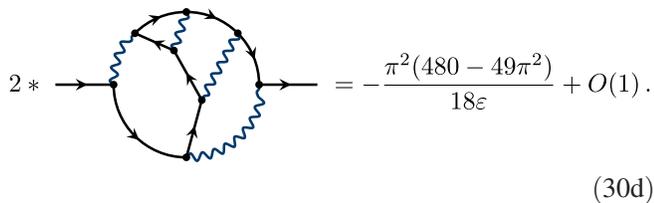
$$2 * \quad = \frac{\pi^2(10 - \pi^2)}{\varepsilon} + O(1), \quad (30a)$$



$$2 * \quad = -\frac{\pi^2(480 - 49\pi^2)}{18\varepsilon} + O(1), \quad (30b)$$



$$2 * \quad = -\frac{\pi^2(480 - 49\pi^2)}{18\varepsilon} + O(1), \quad (30c)$$



$$2 * \rightarrow \text{Diagram} = -\frac{\pi^2(480 - 49\pi^2)}{18\epsilon} + O(1). \quad (30d)$$

Summing all of the above contributions order by order in the loop expansion yields the coefficients $\sigma_i(0)$ ($i = 1-4$) in agreement with Eqs. (10) at the leading order of the ϵ expansion. With the accuracy used, these coefficients are equal to the coefficients $a_i(0)$ ($i = 1-4$). Substituting them in Eqs. (25) with $\eta = 0$ allows us to reconstruct the gauge-dependent part of $a_i(\xi)$ ($i = 1-4$) in agreement with Eqs. (17) at the leading order of the ϵ expansion.

V. SUMMARY AND CONCLUSION

In the present paper, we have examined the perturbative structure of the massless fermion propagator of quenched QED₃ up to four loops.

Our study was motivated by our recent publication [7] where the gauge covariance of the fermion propagator of quenched QED₃ was studied using the LKF transformation in dimensional regularization ($d = 3 - 2\epsilon$). This nonperturbative transformation revealed an interesting parity effect, whereby the contributions of odd orders, starting from the third one, to even orders are accompanied by singularities taking the form of poles, ϵ^{-1} , in dimensional regularization. In turn, even orders produce contributions to odd ones, starting from the third order, which are $\sim \epsilon$.

Following arguments in favor of the IR (and ultraviolet) perturbative finiteness of massless quenched QED₃ [1,6] and therefore assuming the existence of a finite limit as

$\epsilon \rightarrow 0$, we concluded in Ref. [7] that, exactly in $d = 3$, all odd coefficients $a_{2r+1}(\xi)$ in perturbation theory, except a_1 , should be exactly zero in any gauge.

This statement needed a check since analytical expressions for the fermion self-energy diagrams were known only at two-loop order. This is what we have done in the present paper by computing the three- and four-loop corrections to the massless fermion propagator, i.e., the coefficients $a_3(\xi)$ and $a_4(\xi)$, directly in the framework of perturbation theory (see Sec. IV for some details on the computed diagrams in the Landau gauge). We found that $a_3(\xi)$ is finite and gauge-independent when $\epsilon \rightarrow 0$. The coefficient $a_4(\xi)$ is, on the other hand, singular which violates the status of IR perturbative finiteness of massless quenched QED₃. The obtained singularity is such that all of its gauge-fixing dependent terms are entirely determined by lower order contributions in agreement with the properties of the LKF transformation.

In closing, let us note that the four-loop singularities were found to contribute to the coefficient $a_4(\xi)$ with a factor $\sim(1 - \xi)$, and, thus, $a_4(\xi)$ is finite in the Feynman gauge. The reason for this intriguing effect is not clear at present and its elucidation requires additional research.

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