

Revisiting modular symmetry in magnetized torus and orbifold compactifications

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We study the modular symmetry in T^2 and orbifold compactifications with magnetic fluxes. There are $|M|$ zero modes on T^2 with the magnetic flux M . Their wave functions as well as massive modes behave as modular forms of weight $1/2$ and represent the double covering group of $\Gamma \equiv SL(2, \mathbb{Z})$, $\tilde{\Gamma} \equiv \widetilde{SL}(2, \mathbb{Z})$. Each wave function on T^2 with the magnetic flux M transforms under $\tilde{\Gamma}(2|M|)$, which is the normal subgroup of $\widetilde{SL}(2, \mathbb{Z})$. Then, $|M|$ zero modes are representations of the quotient group $\tilde{\Gamma}'_{2|M|} \equiv \tilde{\Gamma}/\tilde{\Gamma}(2|M|)$. We also study the modular symmetry on twisted and shifted orbifolds T^2/\mathbb{Z}_N . Wave functions are decomposed into smaller representations by eigenvalues of twist and shift. They provide us with reduction of reducible representations on T^2 .

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I. INTRODUCTION

The standard model (SM) is now well established. However, the origin of the flavor structure of quarks and leptons is still one of the mysteries of the SM. Various studies have been carried out to understand the flavor structure. One of the interesting approaches is to impose some non-Abelian discrete flavor symmetries [1–9] on the flavors of quarks and leptons. Various discrete symmetries such as S_N , A_N , $\Delta(3N^2)$, $\Delta(6N^2)$ are used. Then, these flavor symmetries are broken by vacuum expectation values (VEVs) of gauge singlet scalars, the so-called flavons in order to realize the masses and the mixing angles of quarks and leptons. However, a complicated vacuum alignment is required.

Superstring theory predicts six-dimensional (6D) compact space in addition to our four-dimensional (4D) space-time. Such a compact space may provide us with origins of non-Abelian discrete flavor symmetries. (See, e.g., [10,11].) In particular, the torus as well as orbifolds has the modular symmetry as geometrical symmetry. Zero modes transform under the modular symmetry. That is, the modular symmetry is a flavor symmetry in a sense. This transformation behavior has been studied in magnetized D-brane models [12–15] and heterotic orbifold models [16–20]. (See also [21–23].) However, the modular flavor symmetry is different from the conventional flavor

symmetries. Yukawa couplings as well as higher order couplings are not singlets, but transform under the modular symmetry.

Interestingly, the modular symmetry includes the finite modular groups Γ_N for $N = 2, 3, 4, 5$, which are isomorphic to S_3 , A_4 , S_4 , A_5 [24], respectively. Recently inspired by these aspects, a new bottom-up approach of flavor models has been studied extensively [25–41]. In those models, some finite modular groups are applied as the flavor symmetries. Furthermore, it is notable that the Yukawa couplings are functions of the modulus τ , which describes the shape of the compact space, and are assigned to modular forms, which transform nontrivially under the modular transformations. The flavor modular symmetry can be broken by the VEV of the modulus τ without flavons.

As mentioned above, the modular symmetry is quite important from both top-down and bottom-up approaches. That could become a bridge to connect high and low energy scales. Our purpose of this paper is to study the modular symmetry in more detail. We study how wave functions on T^2 with magnetic flux transform under the modular symmetry. Furthermore, we also study twisted and shifted orbifolds. Orbifold twist and shift decompose wave functions by their eigenvalues. That provides us with reduction of reducible representations. Also, it provides us with a new approach to construct three-generation models from the phenomenological viewpoint.

This paper is organized as follows. In Sec. II, we briefly review the modular symmetry on T^2 and modular forms. After reviewing T^2 with magnetic flux in Sec. III A, we study the modular symmetry on the magnetized T^2 in Sec. III B. We find that the wave functions on the magnetized T^2 are transformed under the modular transformations like

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modular forms of weight $1/2$ for $\tilde{\Gamma}(2|M|)$. We also study the modular symmetry on various magnetized T^2/\mathbb{Z}_N orbifolds in Secs. IV–VI. In Sec. IV, we study that on the T^2/\mathbb{Z}_N twisted orbifolds. In Sec. V, we study that on the T^2/\mathbb{Z}_N shifted orbifolds. Furthermore, in Sec. VI, we study that on the T^2/\mathbb{Z}_N twisted and shifted orbifolds. In those sections, we find that the modular symmetry remains on the T^2/\mathbb{Z}_2 twisted orbifold and the “full” shifted T^2/\mathbb{Z}_N orbifolds. In particular, the full T^2/\mathbb{Z}_2 shifted orbifold is consistent with the T^2/\mathbb{Z}_2 twisted orbifold. Section VII is the conclusion. Appendix A shows the extension for the generalized CP symmetry with the modular symmetry on the magnetized T^2 . Appendix B shows the detail calculation of discussion in Sec. III B. Appendix C shows examples of the magnetized T^2/\mathbb{Z}_2 twisted and shifted orbifold models.

II. MODULAR SYMMETRY AND MODULAR FORMS

In this section, we briefly review the modular symmetry on T^2 and the modular forms. (See e.g., [42–45]. See also [34].) First, we review the modular symmetry on T^2 . The torus T^2 can be constructed as division of the complex plane \mathbb{C} by a two-dimensional (2D) lattice Λ , i.e., $T^2 \simeq \mathbb{C}/\Lambda$. The lattice Λ is spanned by two lattice vectors e_k ($k = 1, 2$). We denote the complex coordinate of \mathbb{C} as u and that of the T^2 as $z \equiv u/e_1$. We also introduce the complex modulus parameter $\tau \equiv e_2/e_1$ ($\text{Im}\tau > 0$). However, there is some ambiguity in choice of the lattice vectors. The lattice spanned by the following lattice vectors e'_k ($k = 1, 2$),

$$\begin{aligned} \begin{pmatrix} e'_2 \\ e'_1 \end{pmatrix} &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}, \\ \gamma &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z}) \equiv \Gamma, \end{aligned} \quad (1)$$

is the same lattice spanned by the lattice vectors e_k ($k = 1, 2$). Under the above $SL(2, \mathbb{Z})$ transformation, the coordinate of T^2 and the modulus are transformed as

$$\gamma: z \equiv \frac{u}{e_1} \rightarrow z' \equiv \frac{u}{e'_1} = \frac{z}{c'\tau + d'}, \quad (2)$$

$$\gamma: \tau \equiv \frac{e_2}{e_1} \rightarrow \tau' \equiv \frac{e'_2}{e'_1} = \frac{a'\tau + b'}{c'\tau + d'}. \quad (3)$$

This is the modular transformation. The group $\Gamma \equiv SL(2, \mathbb{Z})$ is generated by two generators,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

They satisfy the following algebraic relations¹:

$$S^2 = -\mathbb{I}, \quad S^4 = (ST)^3 = \mathbb{I}. \quad (5)$$

Under S and T , the coordinate of T^2 and the modulus, (z, τ) , are transformed as

$$S: (z, \tau) \rightarrow \left(-\frac{z}{\tau}, -\frac{1}{\tau}\right), \quad T: (z, \tau) \rightarrow (z, \tau + 1). \quad (6)$$

Note that $-\mathbb{I}: (z, \tau) \rightarrow (-z, \tau)$.

Next, we review the modular forms. The principal congruence subgroup of level N , $\Gamma(N)$ is the normal subgroup of Γ defined by

$$\begin{aligned} \Gamma(N) &\equiv \left\{ h = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \right. \\ &\equiv \left. \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \end{aligned} \quad (7)$$

where we have $\Gamma(1) \simeq \Gamma$. The modular forms $f(\tau)$ of weight k for $\Gamma(N)$ are holomorphic functions of τ which transform as

$$f(h(\tau)) = (c''\tau + d'')^k f(\tau), \quad h = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \Gamma(N). \quad (8)$$

Here, k is an integer while k is even for $N = 1, 2$ because of $-\mathbb{I} \in \Gamma(N)$. The above modular forms of weight k for $\Gamma(N)$ transform as

$$f(\gamma(\tau)) = (c'\tau + d')^k \rho(\gamma) f(\tau), \quad \gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma, \quad (9)$$

under Γ transformation, where ρ is a unitary representation of the quotient group $\Gamma'_N \equiv \Gamma/\Gamma(N)$. Thus, the representation of Γ'_N , ρ , satisfies the following relations:

$$\begin{aligned} \rho(S)^4 &= [\rho(S)\rho(T)]^3 = \rho(T)^N = \mathbb{I}, \\ \rho(S)^2\rho(T) &= \rho(T)\rho(S)^2. \end{aligned} \quad (10)$$

Note that since the relation $(-1)^k \rho(-\mathbb{I}) = \mathbb{I}$ should be satisfied, it is required that $\rho(-\mathbb{I}) = \mathbb{I}$ [$\rho(-\mathbb{I}) = -\mathbb{I}$] when k is even [odd]. Consequently, when k is even, ρ becomes a representation of $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$, where we define $\bar{\Gamma} \equiv \Gamma/\{\pm\mathbb{I}\}$ and $\bar{\Gamma}(N) \equiv \Gamma(N)/\{\pm\mathbb{I}\}$ ($N = 1, 2$) while $\bar{\Gamma} \equiv \Gamma(N)$ ($N > 2$). As mentioned in Sec. I, Γ_N are

¹They satisfy $(ST^{-1})^3 = -\mathbb{I}$, $(ST^{-1})^6 = \mathbb{I}$.

isomorphic to $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$. Furthermore, we define the automorphy factor as

$$J_k(\gamma, \tau) \equiv (c'\tau + d')^k, \quad \gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma, \quad (11)$$

which satisfies

$$J_k(\gamma_1\gamma_2, \tau) = J_k(\gamma_1, \gamma_2(\tau))J_k(\gamma_2, \tau), \quad \gamma_1, \gamma_2 \in \Gamma. \quad (12)$$

In the next section, we study the modular symmetry for wave functions on the magnetized T^2 . In the following analysis, we extend the above discussion on modular forms.

III. MODULAR SYMMETRY IN THE MAGNETIZED T^2 MODEL

We consider ten-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, as an effective field theory of magnetized D-brane models of superstring theory, compactified on $T^2 \times T^2 \times T^2$ with nonvanishing magnetic fluxes. Magnetic fluxes induce degenerate zero modes corresponding to flavors of quarks and leptons as well as massive modes. In particular, we focus on one T^2 with a magnetic flux, and start with 6D theory. In this case, the wave function of the fermion on 6D space-time, $\lambda(X)$, is decomposed into the wave function on 4D space-time, $\psi_n^j(x)$, and the wave function on T^2 , $\psi_n^j(z)$ as follows:

$$\lambda(X) = \sum_n \sum_j \psi_n^j(x) \otimes \psi_n^j(z), \quad (13)$$

where we chose $\psi_n^j(z)$ as the eigenstate of the 2D Dirac operator $i\mathcal{D}_2$ as

$$i\mathcal{D}_2\psi_n^j(z) = m_n\psi_n^j(z). \quad (14)$$

Here, we denote the n th excited and j th degenerate wave function as ψ_n^j , and also we denote the coordinates of 6D space-time, 4D space-time, and T^2 as X , x , and z , respectively. Then, the 6D action for massless fermion $\lambda(X)$ is reduced to 4D action as

$$\begin{aligned} S &= \int d^6X \overline{\lambda(X)} i\mathcal{D}_6 \lambda(X) \\ &= \int d^4x \sum_{m,n} \sum_{j,k} \left(\int d^2z \overline{\psi_m^j(z)} \psi_n^k(z) \right) \\ &\quad \times \overline{\psi_m^j(x)} (i\mathcal{D}_4 + m_n) \psi_n^k(x). \end{aligned} \quad (15)$$

In this section, we study the modular symmetry for the wave functions on the magnetized T^2 .

A. Wave functions on magnetized T^2

First, we briefly review the wave functions on $T^2 \simeq \mathbb{C}/\Lambda$ with $U(1)$ magnetic flux² [46]. The metric on T^2 is given by

$$ds^2 = 2h_{\mu\nu} dz^\mu d\bar{z}^\nu, \quad h = |e_1|^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \quad (16)$$

The $U(1)$ magnetic flux on T^2 ,

$$F = \frac{i\pi M}{\text{Im}\tau} dz \wedge d\bar{z}, \quad (17)$$

should satisfy the quantization condition,

$$\int_{T^2} F = 2\pi M, \quad M \in \mathbb{Z}. \quad (18)$$

It is induced from the following vector potential:

$$\begin{aligned} A &= A_z dz + A_{\bar{z}} d\bar{z} \\ &= -\frac{i\pi M}{2\text{Im}\tau} (\bar{z} + \bar{\zeta}) dz + \frac{i\pi M}{2\text{Im}\tau} (z + \zeta) d\bar{z} \\ &= \frac{\pi M}{\text{Im}\tau} \text{Im}((\bar{z} + \bar{\zeta}) dz), \end{aligned} \quad (19)$$

where ζ is a Wilson line. The above vector potential satisfies the following boundary conditions:

$$A(z+1) = A(z) + d\left(\frac{\pi M}{\text{Im}\tau} \text{Im}z\right) = A(z) + d\chi_1(z), \quad (20)$$

$$A(z+\tau) = A(z) + d\left(\frac{\pi M}{\text{Im}\tau} \text{Im}\bar{\tau}z\right) = A(z) + d\chi_2(z), \quad (21)$$

which correspond to $U(1)$ gauge transformation.

Then, the wave functions on the T^2 with the above gauge background satisfy the following boundary conditions:

$$\psi(z+1) = e^{i\chi_1(z)} \psi(z) = e^{i\pi M \frac{\text{Im}(z+\bar{\zeta})}{\text{Im}\tau}} \psi(z), \quad (22)$$

$$\psi(z+\tau) = e^{i\chi_2(z)} \psi(z) = e^{i\pi M \frac{\text{Im}\bar{\tau}(z+\bar{\zeta})}{\text{Im}\tau}} \psi(z), \quad (23)$$

where we consider the unit $U(1)$ charge, $q = 1$. Note that the quantization condition $M \in \mathbb{Z}$ originates from the above boundary conditions. The zero-mode wave function of the two-dimensional spinor with charge $q = 1$,

$$\psi^M(z) = \begin{pmatrix} \psi_+^M(z) \\ \psi_-^M(z) \end{pmatrix}, \quad \psi_-^M(z) = \overline{\psi_+^M(z)}, \quad (24)$$

is obtained by solving the zero-mode Dirac equation,

²The following analysis in this paper can be applied for $U(N)$ magnetic fluxes.

$$i(\gamma^z D_z + \gamma^{\bar{z}} D_{\bar{z}})\psi^M(z) = 0, \quad (25)$$

where $\gamma^z, \gamma^{\bar{z}}$ are written by

$$\gamma^z = \frac{1}{e_1} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \gamma^{\bar{z}} = \frac{1}{\bar{e}_1} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad (26)$$

satisfying $\{\gamma^z, \gamma^{\bar{z}}\} = 2h^{z\bar{z}}$, and we denote the covariant derivative as $D_z = \partial_z - iA_z (D_{\bar{z}} = \partial_{\bar{z}} - iA_{\bar{z}})$. From Eq. (26), each component of Eq. (25) is described by

$$iD\psi_+^M(z) \equiv \frac{2i}{\bar{e}_1} \bar{D}_{\bar{z}}\psi_+^M(z) = 0, \quad (27)$$

$$-i\bar{D}\psi_-^M(z) \equiv \frac{2i}{e_1} D_z\psi_-^M(z) = 0. \quad (28)$$

When the magnetic flux M is positive, $\psi_+^M(z)$ and $\psi_-^M(z) = \overline{\psi_+^M(z)}$ have $|M|$ number of degenerate zero modes described by

$$\begin{aligned} \psi_0^{j,|M|}(z, \tau) &= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \sum_{l \in \mathbb{Z}} e^{i\pi|M|\tau(\frac{j}{|M|}+l)^2} \\ &\quad \times e^{2\pi i|M|(z+\zeta)(\frac{j}{|M|}+l)} \\ &= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \vartheta \left[\begin{matrix} j \\ 0 \end{matrix} \right] (|M|z, |M|\tau), \end{aligned} \quad (29)$$

with $\forall j \in \mathbb{Z}_{|M|} = \{0, 1, 2, \dots, |M| - 1\}$, where $\mathcal{A} = |e_1|^2 \text{Im}\tau$ is the area of T^2 and ϑ denotes the Jacobi theta function defined as

$$\vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu, \tau) = \sum_{l \in \mathbb{Z}} e^{\pi i(a+l)^2 \tau} e^{2\pi i(a+l)(\nu+b)}. \quad (30)$$

Similarly, when M is negative, $\psi_-^M(z)$ has $|M|$ degenerate zero modes, whose wave functions are the same as the above. Thus, we can realize a chiral theory.

Furthermore, the wave functions of the n th excited modes [47], whose squared masses are $m_n^2 = \frac{4\pi M}{\mathcal{A}} n$, can be described by

$$\begin{aligned} \psi_n^{j,|M|}(z, \tau) &= \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_0^{j,|M|}(z, \tau) \\ &= \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}}\right)^n \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \\ &\quad \times \sum_{l \in \mathbb{Z}} e^{i\pi|M|\tau(\frac{j}{|M|}+l)^2} e^{2\pi i|M|(z+\zeta)(\frac{j}{|M|}+l)} \\ &\quad \times H_n \left(\sqrt{2\pi|M|\text{Im}\tau} \left(\frac{\text{Im}(z+\zeta)}{\text{Im}\tau} + \frac{j}{|M|} + l \right) \right), \end{aligned} \quad (31)$$

where we use the creation and annihilation operators,

$$a^\dagger = \sqrt{\frac{\mathcal{A}}{4\pi|M|}} \bar{D}, \quad a = \sqrt{\frac{\mathcal{A}}{4\pi|M|}} D, \quad (32)$$

which satisfy $[a, a^\dagger] = 1$, and $H_n(x)$ is the Hermite function. We note that the wave functions in Eqs. (29) and (31) are normalized by

$$\int_{T^2} dz d\bar{z} \overline{\psi_m^{j,|M|}(z)} \psi_n^{k,|M|}(z) = (2\text{Im}\tau)^{-1/2} \delta_{j,k} \delta_{m,n}. \quad (33)$$

From Eqs. (33) and (15), we can obtain the following 4D kinetic terms:

$$S_K = \int d^4x \sum_{n=0}^{\infty} \sum_{j=0}^{|M|-1} \frac{\overline{\psi_n^{j,|M|}(x)} i \not{D}_4 \psi_n^{j,|M|}(x)}{(-i\tau + i\bar{\tau})^{1/2}}, \quad (34)$$

which means that the wave functions on the 4D space-time, $\psi_n^{j,|M|}(x)$, have modular weight $-k = -1/2$. Thus, the modular symmetry in the 4D low-energy effective field theory is determined by behaviors of wave functions on the magnetized T^2 . In the next section, we study the modular symmetry on the magnetized T^2 . Before ending this section, we also note that the wave functions in Eqs. (29) and (31) satisfy the following relation:

$$\psi_n^{j,|M|}(-z, \tau) = \psi_n^{|M|-j,|M|}(z, \tau). \quad (35)$$

B. Modular symmetry in the magnetized T^2 model

Here, we study how the fields on the magnetized T^2 are transformed under the modular transformation, Eq. (6).

The transformation of the Wilson line ζ is the same as the coordinate z , i.e., $\Gamma \ni \gamma: \zeta \rightarrow \zeta/(c'\tau + d')$. The fields F in Eq. (17) and A in Eq. (19) are modular invariant. The equations of motions for $\psi^M(z)$ with any excited modes, including zero modes, are also modular invariant. On the other hand, while the boundary conditions for $\psi^M(z)$ in Eqs. (22) and (23) are consistent with the S transformation, they are consistent with the T transformation only if M is even. In general, the boundary conditions are consistent with the modular transformation in Eqs. (2) and (3) only if M is even or both $a'b'$ and $c'd'$ are even, where a', b', c', d' are elements of $\gamma \in \Gamma_{1,2} \subset \Gamma$. (See Ref. [15].) Here, we focus on the models with $M = \text{even}$.

The wave functions of the n th excited modes in Eq. (31), including the zero modes in Eq. (29), are transformed as

$$\begin{aligned} S: \psi_n^{j,|M|}(z, \tau) &\rightarrow \psi_n^{j,|M|} \left(-\frac{z}{\tau}, -\frac{1}{\tau} \right) \\ &= (-\tau)^{1/2} \sum_{k=0}^{|M|-1} e^{i\pi/4} \frac{1}{\sqrt{|M|}} e^{2\pi i \frac{jk}{|M|}} \psi_n^{k,|M|}(z, \tau), \end{aligned} \quad (36)$$

$$T: \psi_n^{j,|M|}(z, \tau) \rightarrow \psi_n^{j,|M|}(z, \tau + 1) = e^{i\pi \frac{j^2}{|M|}} \psi_n^{j,|M|}(z, \tau), \quad (37)$$

under the modular transformation, Eq. (6). Note that the creation operator is modular invariant and commutative with the above coefficients. Thus, the wave functions transform like modular forms of weight $1/2$. It is consistent with Eq. (34). Modular forms of weight $1/2$ are relevant to the double covering group of $\Gamma = SL(2, \mathbb{Z})$, $\tilde{\Gamma} \equiv \widetilde{SL}(2, \mathbb{Z})$. (See e.g., [44,48–51].) The double covering group $\tilde{\Gamma} \equiv \widetilde{SL}(2, \mathbb{Z})$ is defined by

$$\tilde{\Gamma} \equiv \{[\gamma, \epsilon] | \gamma \in \Gamma, \epsilon \in \{\pm 1\}\}. \quad (38)$$

The multiplication of arbitrary two elements, $[\gamma_1, \epsilon_1]$, $[\gamma_2, \epsilon_2] \in \tilde{\Gamma}$, is defined by

$$[\gamma_1, \epsilon_1][\gamma_2, \epsilon_2] = [\gamma_1\gamma_2, A(\gamma_1, \gamma_2)\epsilon_1\epsilon_2], \quad (39)$$

where $A(\gamma_1, \gamma_2)$ is called Kubota's twisted 2-cocycle [50] for Γ , defined as follows. We first introduce Kubota's function $\chi: \Gamma \rightarrow \mathbb{Z}$, defined by

$$\chi(\gamma) = \begin{cases} c', & (c' \neq 0) \\ d', & (c' = 0). \end{cases} \quad (40)$$

We also introduce the Hirbert symbol, defined by

$$(a, b)_H = \begin{cases} -1, & (a < 0 \text{ and } b < 0) \\ 1, & (\text{otherwise}). \end{cases} \quad (41)$$

Then, $A(\gamma_1, \gamma_2)$ is defined by

$$A(\gamma_1, \gamma_2) = \left(\frac{\chi(\gamma_1\gamma_2)}{\chi(\gamma_1)}, \frac{\chi(\gamma_1\gamma_2)}{\chi(\gamma_2)} \right)_H. \quad (42)$$

Actually, it satisfies the following cocycle relation:

$$A(\gamma_1, \gamma_2)A(\gamma_1\gamma_2, \gamma_3) = A(\gamma_1, \gamma_2\gamma_3)A(\gamma_2, \gamma_3). \quad (43)$$

Here, we set

$$\mathbb{I} \equiv [\mathbb{I}, 1], \quad \tilde{S} \equiv [S, 1], \quad \tilde{T} \equiv [T, 1]. \quad (44)$$

They satisfy the following algebraic relations:

$$\begin{aligned} \tilde{S}^2 &= [-\mathbb{I}, 1] \equiv \tilde{Z}, & \tilde{S}^4 &= (\tilde{S}\tilde{T})^3 = [\mathbb{I}, -1] = \tilde{Z}^2, \\ \tilde{S}^8 &= (\tilde{S}\tilde{T})^6 = [\mathbb{I}, 1] = \mathbb{I} = \tilde{Z}^4. \end{aligned} \quad (45)$$

Note that inverses of \tilde{S} , \tilde{T} , \tilde{Z} are written by

$$\tilde{S}^{-1} = [S^{-1}, 1], \quad \tilde{T}^{-1} = [T^{-1}, 1], \quad \tilde{Z} = [-\mathbb{I}, -1]. \quad (46)$$

Hereafter, we often denote an element of $\tilde{\Gamma}$, $[\gamma, \epsilon]$, as $\tilde{\gamma}$, where $\tilde{\gamma}$ is, in general, independent of γ .

Due to the above extension by $\epsilon \in \{\pm 1\}$, the definition of the automorphy factor in Eq. (11) is also extended by $\epsilon \in \{\pm 1\}$ as follows:

$$\begin{aligned} \tilde{J}_{k/2}(\tilde{\gamma}, \tau) &\equiv \epsilon^k J_{k/2}(\gamma, \tau) = \epsilon^k (c'\tau + d')^{k/2}, \quad k \in \mathbb{Z}, \\ \tilde{\gamma} &= \left[\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \epsilon \right] \in \tilde{\Gamma}, \end{aligned} \quad (47)$$

where we take $(-1)^{k/2} = e^{-in\pi/2}$. From Eqs. (12) and (39), Eq. (47) satisfies the following relation:

$$\begin{aligned} \tilde{J}_{k/2}(\tilde{\gamma}_1\tilde{\gamma}_2, \tau) &= (A(\gamma_1, \gamma_2))^k \tilde{J}_{k/2}(\tilde{\gamma}_1, \tilde{\gamma}_2(\tau)) \tilde{J}_{k/2}(\tilde{\gamma}_2, \tau), \\ \tilde{\gamma}_1 &= [\gamma_1, \epsilon_1], \quad \tilde{\gamma}_2 = [\gamma_2, \epsilon_2] \in \tilde{\Gamma}, \end{aligned} \quad (48)$$

where the extension by $\epsilon \in \{\pm 1\}$ does not affect the modular transformation, i.e., $\tilde{\gamma}(z, \tau) = \gamma(z, \tau)$. It allows us to study modular forms of weight $k/2$, where k is integer. Considering the above extension, the wave functions on the magnetized T^2 transform as

$$\psi_n^{j,|M|}(\tilde{\gamma}(z, \tau)) = \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \sum_{k=0}^{|M|-1} \rho(\tilde{\gamma})_{jk} \psi_n^{k,|M|}(z, \tau), \quad \tilde{\gamma} \in \tilde{\Gamma}, \quad (49)$$

$$\rho(\tilde{S})_{jk} = e^{i\pi/4} \frac{1}{\sqrt{|M|}} e^{2\pi i \frac{jk}{|M|}}, \quad \rho(\tilde{T})_{jk} = e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \quad (50)$$

under the modular transformation. Note that $\tilde{Z}(z, \tau) = (-z, \tau)$ and $\tilde{Z}^2(z, \tau) = (z, \tau)$ require $\tilde{J}_{1/2}(\tilde{Z}, \tau)\rho(\tilde{Z}) = \delta_{|M|-j,k}$ and $\tilde{J}_{1/2}(\tilde{Z}^2, \tau)\rho(\tilde{Z}^2) = \delta_{j,k}$, respectively. Actually, we can check that the following relations,

$$\begin{aligned} \tilde{J}_{1/2}(\tilde{Z}, \tau) &= \tilde{J}_{1/2}(\tilde{S}^2, \tau) = -i, \\ \rho(\tilde{Z})_{jk} &= \rho(\tilde{S})_{jk}^2 = i\delta_{|M|-j,k}, \end{aligned} \quad (51)$$

$$\begin{aligned} \tilde{J}_{1/2}(\tilde{Z}^2, \tau) &= \tilde{J}_{1/2}(\tilde{S}^4, \tau) = \tilde{J}_{1/2}((\tilde{S}\tilde{T})^3, \tau) = -1, \\ \rho(\tilde{Z}^2)_{jk} &= \rho(\tilde{S})_{jk}^4 = [\rho(\tilde{S})\rho(\tilde{T})]_{jk}^3 = -\delta_{j,k}, \end{aligned} \quad (52)$$

$$\begin{aligned} \tilde{J}_{1/2}(\tilde{Z}^4, \tau) &= \tilde{J}_{1/2}(\tilde{S}^8, \tau) = \tilde{J}_{1/2}((\tilde{S}\tilde{T})^6, \tau) = 1, \\ \rho(\tilde{Z}^4)_{jk} &= \rho(\tilde{S})_{jk}^8 = [\rho(\tilde{S})\rho(\tilde{T})]_{jk}^6 = \delta_{j,k}, \end{aligned} \quad (53)$$

$$\tilde{J}_{1/2}(\tilde{T}^n, \tau) = 1, \quad \forall n \in \mathbb{Z}, \quad \rho(\tilde{T})_{jk}^{2|M|} = \delta_{j,k}, \quad (54)$$

$$\rho(\tilde{Z}^n)\rho(\tilde{T}) = \rho(\tilde{T})\rho(\tilde{Z})^n, \quad n = 1, 2, 3, \quad (55)$$

are satisfied.³ Therefore, the wave functions on the magnetized T^2 transform under the modular transformation like modular forms of weight $1/2$ for $\tilde{\Gamma}(2|M|)$,⁴ which is the normal subgroup of $\tilde{\Gamma}$, defined as

$$\tilde{\Gamma}(2|M|) \equiv \{[h, e] \in \tilde{\Gamma} | h \in \Gamma(2|M|), e = 1\}. \quad (57)$$

Then ρ is a unitary representation of the quotient group $\tilde{\Gamma}'_{2|M|} \equiv \tilde{\Gamma}/\tilde{\Gamma}(2|M|)$. That is, the group generated by ρ is homomorphic with $\tilde{\Gamma}'_{2|M|}$.

For example, when $M = 2$, the S and T transformations are represented as

$$\rho(\tilde{S}) = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho(\tilde{T}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (58)$$

They generate the group G^2 whose order is 96, and it is isomorphic to

$$G^2 \simeq T' \times Z_4. \quad (59)$$

When $M = 4$, the S and T transformations are represented as

$$\rho(\tilde{S}) = \frac{e^{i\pi/4}}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \quad (60)$$

$$\rho(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \end{pmatrix}.$$

They generate the group G^4 , whose order is 384, and it is isomorphic to

$$G^4 \simeq \Delta(48) \rtimes Z_8. \quad (61)$$

In Appendix A, we study the extension for the generalized CP symmetry with the modular symmetry on the magnetized T^2 .

So far, we have considered the Wilson line ζ , which transforms as $\zeta \rightarrow \zeta/(c'\tau + d')$. In that case, the modular transformation is restrictive due to the consistency with the

³The following relations are also satisfied:

$$[\rho(\tilde{S})\rho(\tilde{T})^{-1}]_{jk}^3 = i\delta_{|M|-j,k}, \quad [\rho(\tilde{S})\rho(\tilde{T})^{-1}]_{jk}^6 = -\delta_{j,k},$$

$$[\rho(\tilde{S})\rho(\tilde{T})^{-1}]_{jk}^{12} = \delta_{j,k}. \quad (56)$$

⁴According to Ref. [48], $\widetilde{SL}(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z})$ can be split on $\Gamma(2|M|)$ since $2|M| \in 4\mathbb{Z}$.

boundary conditions for $\psi^M(z)$. Before ending this section, we comment about another possibility. In particular, if a Wilson line ζ is also changed to $\zeta + 1$, which is the gauge transformation, simultaneously with the T transformation, the boundary conditions for $\psi^M(z)$ are consistent with the modular transformation even if M is odd, although the equations of motions for $\psi^M(z)$ are modified. In this case, the zero-mode wave function for j after the T transformation can be expanded by the all excited-mode wave functions for j before the T transformation as follows:

$$T: \psi_0^{j|M|}(z + \zeta, \tau) \rightarrow \psi_0^{j|M|}(z + \zeta + 1, \tau + 1)$$

$$= (-1)^j e^{i\pi \frac{j^2}{|M|}} e^{-\frac{\pi j M}{8\text{Im}\tau}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(i \sqrt{\frac{\pi|M|}{4\text{Im}\tau}} \right)^n$$

$$\times \psi_n^{j|M|}(z + \zeta, \tau), \quad (62)$$

where we use the following the generating function of the Hermite function:

$$e^{-y^2+2xy} = \sum_{n=0}^{\infty} H_n(x) \frac{y^n}{n!}. \quad (63)$$

The detail calculation is shown in Appendix B. Similarly, the n th excited-mode wave function for j after the T transformation can be also expanded by the all excited-mode wave functions for j before the T transformation.

In this section, we have discussed the modular symmetry on magnetized T^2 . In the following sections, we study the modular symmetry on various magnetized T^2/\mathbb{Z}_N orbifolds.

IV. MODULAR SYMMETRY IN MAGNETIZED T^2/\mathbb{Z}_N TWISTED ORBIFOLD MODELS

In this section, we study the modular symmetry of the wave functions on the magnetized T^2/\mathbb{Z}_N twisted orbifolds [12,52–54]. Here and hereafter, we often omit the KK index n , because each KK level satisfies the same relations in what follows. For simplicity, we do not introduce nonvanishing discrete Wilson lines, although we can discuss models with nonvanishing Wilson lines similarly. The T^2/\mathbb{Z}_N twisted orbifold can be obtained by further identifying the points on $T^2 \simeq \mathbb{C}/\Lambda$ which are rotated by $\alpha_N^k \equiv e^{2\pi i k/N}$, $\forall k \in \mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}$. That is the \mathbb{Z}_N twist, i.e., $(\alpha_N^k)^N = 1$. Hence, a lattice point, except for the origin, should move to another lattice point after any \mathbb{Z}_N twist. It allows only if $N = 2, 3, 4, 6$. Moreover, the modulus $\tau = e_2/e_1$ should be fixed to be $\tau = \alpha_N = e^{2\pi i/N}$ for $N = 3, 4, 6$, although any τ is allowed for $N = 2$. It means that only ST, S, ST^{-1} transformations of the modular transformations are consistent for $N = 3, 4, 6$, respectively, while there remains the full modular symmetry for $N = 2$.

The wave function on the magnetized T^2/\mathbb{Z}_N twisted orbifold, $\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z)$, must satisfy the following boundary condition:

$$\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(\alpha_N z) = \alpha_N^m \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z), \quad m \in \mathbb{Z}_N. \quad (64)$$

Hence, such wave functions can be written by linear combinations of wave functions on the magnetized T^2 as

$$\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z) = \mathcal{N}_N^t \sum_{k=0}^{N-1} (\alpha_N^m)^{-k} \psi_{T^2}^{j,|M|}(\alpha_N^k z), \quad (65)$$

where \mathcal{N}_N^t is the normalization factor determined by remaining Eq. (33). Furthermore, $\psi_{T^2}^{j,|M|}(\alpha_N^k z)$ satisfies the same equation of motion as $\psi_{T^2}^{j,|M|}(z)$. In addition, if $\psi_{T^2}^{j,|M|}(\alpha_N^k z)$ also satisfies the same boundary condition as $\psi_{T^2}^{j,|M|}(z)$, $\psi_{T^2}^{j,|M|}(\alpha_N^k z)$ can be expanded by the same excited mode of $\psi_{T^2}^{j,|M|}(z)$.

First, we consider the magnetized T^2/\mathbb{Z}_2 twisted orbifold. In this case, since the wave function $\psi_n^{j,|M|}(\alpha_2 z, \tau)$ satisfies the same boundary conditions, Eqs. (22) and (23), $\psi_n^{j,|M|}(\alpha_2 z, \tau)$ can be expressed by $\psi_n^{j,|M|}(z, \tau)$, i.e., Eq. (35). Therefore, the wave function on the magnetized T^2/\mathbb{Z}_2 twisted orbifold basis, $\psi_{T^2/\mathbb{Z}_2}^{j,|M|}(z)$, can be written by linear combinations of wave functions on the magnetized T^2 basis, $\psi_{T^2}^{j,|M|}(z)$, as

$$\psi_{T^2/\mathbb{Z}_2}^{j,|M|}(z, \tau) = \mathcal{N}_2^t \sum_{k=0}^{|M|-1} (\delta_{j,k} + (-1)^m \delta_{|M|-j,k}) \psi_{T^2}^{k,|M|}(z, \tau), \quad (66)$$

where the normalization factor \mathcal{N}_2^t is determined by $\mathcal{N}_2^t = 1, 1/2$, and $1/\sqrt{2}$ for $j = 0, |M|/2$, and the others, respectively. Note that there are no \mathbb{Z}_2 -odd modes, $\psi_{T^2/\mathbb{Z}_2}^{j,|M|}(z, \tau)$, for $j = 0, |M|/2$. When M is even, the numbers of \mathbb{Z}_2 -even ($m = 0$) and -odd ($m = 1$) modes are $(|M|/2 + 1)$ and $(|M|/2 - 1)$, respectively.⁵ On the T^2/\mathbb{Z}_2 twisted orbifold basis, Eq. (50) is deformed by

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2^0}(\tilde{S})_{jk} &= e^{i\pi/4} \frac{2}{\sqrt{|M|}} \cos(2\pi jk/|M|), \\ \rho_{T^2/\mathbb{Z}_2^0}(\tilde{T})_{jk} &= e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \end{aligned} \quad (67)$$

⁵When M is odd, the numbers of \mathbb{Z}_2 -even and \mathbb{Z}_2 -odd modes are $((|M| - 1)/2 + 1)$ and $((|M| - 1)/2)$, respectively.

$$\rho_{T^2/\mathbb{Z}_2^1}(\tilde{S})_{jk} = e^{i\pi/4} \frac{2i}{\sqrt{|M|}} \sin(2\pi jk/|M|),$$

$$\rho_{T^2/\mathbb{Z}_2^1}(\tilde{T})_{jk} = e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \quad (68)$$

where we need to multiply $\rho_{T^2/\mathbb{Z}_2^0}(\tilde{S})$ further by $1/\sqrt{2}$ when j or k is 0 or $|M|/2$. The above deformations induce deformation of the relation in Eq. (51) as

$$\rho_{T^2/\mathbb{Z}_2^m}(\tilde{Z})_{jk} = \rho_{T^2/\mathbb{Z}_2^0}(\tilde{S})_{jk}^2 = (-1)^m i \delta_{j,k}, \quad (69)$$

while the other relations are the same as the T^2 basis. Thus, the representations on the T^2/\mathbb{Z}_2 twisted orbifold basis satisfy the same algebraic relations as that on the T^2 basis, although the dimensions of the representations are different. For example, when $M = 4$, the wave functions on the T^2/\mathbb{Z}_2 twisted orbifold basis are expressed as

$$\begin{pmatrix} \psi_{T^2/\mathbb{Z}_2}^{0,4}(z, \tau) \\ \psi_{T^2/\mathbb{Z}_2}^{1,4}(z, \tau) \\ \psi_{T^2/\mathbb{Z}_2}^{2,4}(z, \tau) \end{pmatrix} = \begin{pmatrix} \psi_{T^2}^{0,4}(z, \tau) \\ \frac{1}{\sqrt{2}} (\psi_{T^2}^{1,4}(z, \tau) + \psi_{T^2}^{3,4}(z, \tau)) \\ \psi_{T^2}^{2,4}(z, \tau) \end{pmatrix}, \quad (70)$$

$$\psi_{T^2/\mathbb{Z}_2}^{1,4}(z, \tau) = \frac{1}{\sqrt{2}} (\psi_{T^2}^{1,4}(z, \tau) - \psi_{T^2}^{3,4}(z, \tau)). \quad (71)$$

The representations of the S and T transformations for \mathbb{Z}_2 -even modes are expressed as

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2^0}(\tilde{S}) &= \frac{e^{i\pi/4}}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_{T^2/\mathbb{Z}_2^0}(\tilde{T}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (72)$$

which are the generators of the group G_0^4 . The group G_0^4 has the order 384 and is isomorphic to

$$G_0^4 \simeq \Delta(48) \rtimes \mathbb{Z}_8, \quad (73)$$

which is the same as the group on T^2 in Eq. (61). The above wave functions in Eq. (70) correspond to a triplet under $G_0^4 \simeq \Delta(48) \rtimes \mathbb{Z}_8$. The representations of the S and T transformations for the \mathbb{Z}_2 -odd mode, on the other hand, are expressed as

$$\rho_{T^2/\mathbb{Z}_2^1}(\tilde{S}) = e^{3\pi i/4}, \quad \rho_{T^2/\mathbb{Z}_2^1}(\tilde{T}) = e^{i\pi/4}, \quad (74)$$

which are the generators of the group G_1^4 . The group G_1^4 is nothing but

$$G_1^4 \simeq Z_8, \quad (75)$$

which is a subgroup of $G_0^4 \simeq \Delta(48) \rtimes Z_8$. The above representation in Eq. (71) is a representation of this Z_8 symmetry and it also corresponds to a singlet under $G_0^4 \simeq \Delta(48) \rtimes Z_8$.

Thus, the T^2/Z_2 twisted orbifold is consistent with the modular symmetry. Furthermore, the wave functions on T^2 are decomposed into smaller representations by Z_2 eigenvalues, even and odd, that is, the T^2/Z_2 twisted orbifold basis. For smaller $|M|$, this basis of wave functions provide us with irreducible representations of $\tilde{\Gamma}_{2|M}^t \equiv \tilde{\Gamma}/\tilde{\Gamma}(2|M)$. For larger $|M|$, wave functions on the T^2/Z_2 twisted orbifold basis could be decomposed further. We will study it in Sec. VI.

Next, we comment about the other magnetized T^2/Z_N twisted orbifolds. In the case of T^2/Z_4 , since the wave function $\psi_n^{j,|M|}(\alpha_4^k z, \alpha_4)$ satisfies the same boundary condition as $\psi_n^{j,|M|}(z, \alpha_4)$, $\psi_n^{j,|M|}(\alpha_4^k z, \alpha_4)$ can be expanded by $\psi_n^{j,|M|}(z, \alpha_4)$. Actually, it can be done by considering the S transformation for $\psi_n^{j,|M|}(z, \alpha_4)$ [12]. Therefore, the wave function on the magnetized T^2/Z_4 twisted orbifold basis, $\psi_{T^2/Z_4}^{j,|M|}(z)$, can be expanded by linear combinations of wave functions on the magnetized T^2 basis, $\psi_{T^2}^{j,|M|}(z)$, as

$$\begin{aligned} \psi_{T^2/Z_4}^{j,|M|}(z, \alpha_4) &= \mathcal{N}_4^t \sum_{k=0}^{|M|-1} \left(\left(\delta_{j,k} + (-1)^m \delta_{|M|-j,k} \right) \right. \\ &\quad \left. + \frac{e^{-i\pi m/2}}{\sqrt{|M|}} \left(e^{2\pi i \frac{jk}{|M|}} + (-1)^m e^{-2\pi i \frac{jk}{|M|}} \right) \right) \\ &\quad \times \psi_{T^2}^{k,|M|}(z, \alpha_4). \end{aligned} \quad (76)$$

On the T^2/Z_4 twisted orbifold basis, the representation of S transformation is diagonalized as

$$\rho_{T^2/Z_4}(\tilde{S})_{jk} = e^{i\pi/4} (e^{i\pi/2})^m \delta_{j,k}. \quad (77)$$

That is the Z_8 symmetry.

In the case of T^2/Z_N for $N = 3, 6$, however, the wave function $\psi_n^{j,|M|}(\alpha_N^k z, \alpha_N)$ satisfies the same boundary condition as $\psi_n^{j,|M|}(z, \alpha_N)$ only if M is even. Thus, when M is even, $\psi_n^{j,|M|}(\alpha_N^k z, \alpha_N)$ can be expanded by $\psi_n^{j,|M|}(z, \alpha_N)$. Actually, it can be done by considering ST , ST^{-1} transformations for $\psi_n^{j,|M|}(z, \alpha_N)$; $N = 3, 6$, respectively [12]. Therefore, the wave function on the magnetized T^2/Z_N ; $N = 3, 6$ twisted orbifold base, $\psi_{T^2/Z_N}^{j,|M|}(z)$, can be expanded by linear combinations of wave functions on the magnetized T^2 basis, $\psi_{T^2}^{j,|M|}(z)$, as

$$\psi_{T^2/Z_3}^{j,|M|}(z, \alpha_3) = \mathcal{N}_3^t \sum_{k=0}^{|M|-1} \left(\delta_{j,k} - \frac{e^{2\pi i m/3}}{\sqrt{|M|}} \left(e^{-i\pi/12} e^{2\pi i \frac{jk}{|M|}} e^{i\pi \frac{k^2}{|M|}} + e^{-2\pi i m/3} e^{i\pi/12} e^{-i\pi \frac{k^2}{|M|}} e^{-2\pi i \frac{jk}{|M|}} \right) \right) \psi_{T^2}^{k,|M|}(z, \alpha_3), \quad (78)$$

$$\begin{aligned} \psi_{T^2/Z_6}^{j,|M|}(z, \alpha_6) &= \mathcal{N}_6^t \sum_{k=0}^{|M|-1} \left(\left(\delta_{j,k} + (-1)^m \delta_{|M|-j,k} \right) + \frac{e^{-i\pi m/3}}{\sqrt{|M|}} \left(\left(e^{i\pi/12} e^{2\pi i \frac{jk}{|M|}} e^{-i\pi \frac{k^2}{|M|}} + e^{-i\pi m/3} e^{-i\pi/12} e^{i\pi \frac{k^2}{|M|}} e^{2\pi i \frac{jk}{|M|}} \right) \right. \right. \\ &\quad \left. \left. - (-1)^m \left(e^{i\pi/12} e^{-2\pi i \frac{jk}{|M|}} e^{-i\pi \frac{k^2}{|M|}} + e^{-i\pi m/3} e^{-i\pi/12} e^{i\pi \frac{k^2}{|M|}} e^{-2\pi i \frac{jk}{|M|}} \right) \right) \right) \psi_{T^2}^{k,|M|}(z, \alpha_6). \end{aligned} \quad (79)$$

On the T^2/Z_3 twisted orbifold base, the representation of ST transformation is diagonalized as

$$\rho_{T^2/Z_3}(\tilde{S}\tilde{T})_{jk} = e^{i\pi/3} (e^{2\pi i/3})^m \delta_{j,k}. \quad (80)$$

That is the Z_6 symmetry. On the T^2/Z_6 twisted orbifold base, the representation of ST^{-1} transformation is diagonalized as

$$\rho_{T^2/Z_6}(\tilde{S}\tilde{T}^{-1})_{jk} = e^{i\pi/6} (e^{i\pi/3})^m \delta_{j,k}. \quad (81)$$

That is the Z_{12} symmetry. Thus, there remain Z_{2N} symmetries in $\rho(\gamma)$ on the magnetized Z_N twisted orbifolds

for $N = 3, 4, 6$. Remaining $\rho(\gamma)$ represent a spinor representation under Z_N twist. Obviously, $\rho(\gamma)$ on the T^2 and Z_2 bases also correspond to spinor representations under the 2D (discrete) rotation.

V. MODULAR SYMMETRY IN MAGNETIZED T^2/Z_N SHIFTED ORBIFOLD MODELS

In this section, we study the modular symmetry for the wave functions on the magnetized T^2/Z_N shifted orbifolds [55]. The T^2/Z_N shifted orbifold can be obtained by further identifying the points on $T^2 \simeq \mathbb{C}/\Lambda$ which are shifted by $ke_N^{(m,n)} \equiv k(m+n\tau)/N$, $\forall k, \exists m, \exists n \in Z_N = \{0, 1, 2, \dots, N-1\}$. Then, the wave functions on the T^2/Z_N shifted orbifold have to also satisfy the following boundary condition:

$$\begin{aligned}\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z + e_N^{(m,n)}) &= \alpha_N^\ell e^{i\chi_N^{(m,n)}(z)} \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z) \\ &= e^{2\pi i \ell/N} e^{i\pi |M| \left(\frac{\text{Im} \tilde{\epsilon}_N^{(m,n)}(z+\zeta)}{\text{Im} \tilde{\epsilon}_N} + \frac{mn}{N} \right)} \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z),\end{aligned}\quad (82)$$

with $\ell \in \mathbb{Z}_N$, which is consistent with the boundary condition for $z \rightarrow z + m + n\tau$, in addition to Eqs. (22) and (23). Furthermore, these boundary conditions constrain the magnetic flux M to be $M = Nt$, $t \in \mathbb{Z}$. The above wave function can be written by linear combinations of wave functions on the magnetized T^2 as

$$\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z) = \mathcal{N}_N^s \sum_{k=0}^{N-1} (\alpha_N^\ell)^{-k} e^{-ik\chi_N^{(m,n)}(z)} \psi_{T^2}^{j,|M|}(z + ke_N^{(m,n)}), \quad (83)$$

where \mathcal{N}_N^s is the normalization factor determined by remaining Eq. (33). Furthermore, since $e^{-ik\chi_N^{(m,n)}(z,\tau)} \psi_{T^2}^{j,|M|}(z + ke_N^{(m,n)})$ satisfies the same equation of motion as $\psi_{T^2}^{j,|M|}(z)$ and also satisfies the same boundary condition as $\psi_{T^2}^{j,|M|}(z)$, one can expand $e^{-ik\chi_N^{(m,n)}(z,\tau)} \psi_{T^2}^{j,|M|}(z + ke_N^{(m,n)})$ by the same excited mode of $\psi_{T^2}^{j,|M|}(z)$. Then, the wave function on the magnetized T^2/\mathbb{Z}_N shifted orbifold, $\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z)$, can be expanded by linear combinations of wave functions on the magnetized T^2 , $\psi_{T^2}^{j,|M|}(z)$, as

$$\begin{aligned}\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z, \tau) &= \mathcal{N}_N^s \sum_{k=0}^{N-1} e^{-2\pi i k(\ell - mj)/N} e^{-i\pi k(N-k)mn|t|/N} \\ &\quad \times \psi_{T^2}^{j+k|t|,|M|}(z, \tau),\end{aligned}\quad (84)$$

which can be obtained from Eqs. (29) and (31) directly. The normalization factor \mathcal{N}_N^s is determined as $\mathcal{N}_N^s = 1/\sqrt{N}$ for $n \neq 0$ or $\mathcal{N}_N^s = 1/N$ for $n = 0$.

We discuss the modular symmetry on the magnetized T^2/\mathbb{Z}_N shifted orbifolds. There is the modular symmetry on the T^2/\mathbb{Z}_N shifted orbifold only if the points on $T^2 \simeq \mathbb{C}/\Lambda$ which are shifted by $e_N^{(m,n)} = (m + n\tau)/N$; $\forall m, \forall n \in \mathbb{Z}_N$ are further identified. Hereafter, we call this T^2/\mathbb{Z}_N shifted orbifold the full T^2/\mathbb{Z}_N shifted orbifold. The full T^2/\mathbb{Z}_N shifted orbifold with magnetic flux M corresponds to $T'^2 \simeq \mathbb{C}/\Lambda'$, $\Lambda' \equiv \Lambda/N$ with magnetic flux M/N^2 . The boundary conditions for the wave functions on the magnetized full T^2/\mathbb{Z}_N shifted orbifold are written by

$$\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z + e_N^{(1,0)}) = \alpha_N^{\ell_1} e^{i\chi_N^{(1,0)}(z)} \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z), \quad (85)$$

$$\psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z + e_N^{(0,1)}) = \alpha_N^{\ell_2} e^{i\chi_N^{(0,1)}(z)} \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z). \quad (86)$$

The above boundary conditions are consistent with Eqs. (22), (23), and (82) for $\forall m, \forall n \in \mathbb{Z}_N$, where we denote ℓ in Eq. (82) as $\ell^{(m,n)}$, determined by $\ell^{(m,n)} \equiv m\ell_1 + n\ell_2 \pmod{N}$. From the above boundary conditions, we obtain $s \equiv M/N^2 \in \mathbb{Z}$. The eigenfunctions for $\forall e_N^{(m,n)}$ shifts which satisfy the above boundary conditions are expressed as

$$\begin{aligned}\Psi_{T^2/\mathbb{Z}_N}^{r,|s|}(z, \tau) &\equiv \psi_{T^2/\mathbb{Z}_N}^{j,|M|}(z, \tau) \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-2\pi i k \ell_2/N} \psi_{T^2}^{j+kN|s|,|M|}(z, \tau), \\ M &= N^2 s, \quad s \in \mathbb{Z}, \\ j &= Nr + \ell_1 \in \mathbb{Z}_{N|s|}, \\ r &\in \mathbb{Z}_{|s|}, \ell_1, \ell_2 \in \mathbb{Z}_N,\end{aligned}\quad (87)$$

where we note that $\ell_1 \equiv j \pmod{N}$. Furthermore, when we consider $s = \text{even}$, the boundary conditions, Eqs. (85) and (86), are consistent with the modular transformation. On this full T^2/\mathbb{Z}_N shifted orbifold basis, Eq. (50) is deformed as

$$\begin{aligned}\Psi_{T^2/\mathbb{Z}_N}^{r,|s|}(\tilde{\gamma}(z, \tau)) &= \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \sum_{r'=0}^{|s|-1} \sum_{\ell'_1, \ell'_2=0}^{N-1} \rho_{T^2/\mathbb{Z}_N}^{(\ell_1, \ell_2)}(\tilde{\gamma})_{rr', (\ell_1, \ell_2)(\ell'_1, \ell'_2)} \\ &\quad \times \Psi_{T^2/\mathbb{Z}_N}^{r',|s|}(\tilde{\gamma})_{r', (\ell'_1, \ell'_2)}(z, \tau),\end{aligned}\quad (88)$$

for $\tilde{\gamma} \in \tilde{\Gamma}$, where

$$\begin{aligned}\rho_{T^2/\mathbb{Z}_N}^{(\ell_1, \ell_2)}(\tilde{S})_{rr', (\ell_1, \ell_2)(\ell'_1, \ell'_2)} &= e^{i\pi/4} \frac{1}{\sqrt{|s|}} e^{2\pi i (\frac{\ell_1}{N} + r)(\frac{\ell'_1}{N} + r')/|s|} \delta_{\ell_2, \ell'_1} \delta_{N - \ell_1, \ell'_2},\end{aligned}\quad (89)$$

$$\rho_{T^2/\mathbb{Z}_N}^{(\ell_1, \ell_2)}(\tilde{T})_{rr', (\ell_1, \ell_2)(\ell'_1, \ell'_2)} = e^{i\pi(\frac{\ell_1}{N} + r)/|s|} \delta_{r, r'} \delta_{\ell_1, \ell'_1} \delta_{\ell_2 - \ell_1, \ell'_2}. \quad (90)$$

The above deformations induce the deformation of the relation in Eq. (51) as

$$\begin{aligned}\rho_{T^2/\mathbb{Z}_N}^{(\ell_1, \ell_2)}(\tilde{Z})_{rr', (\ell_1, \ell_2)(\ell'_1, \ell'_2)} &= \rho_{T^2/\mathbb{Z}_N}^{(\ell_1, \ell_2)}(\tilde{S})_{rr', (\ell_1, \ell_2)(\ell'_1, \ell'_2)}^2 \\ &= e^{2\pi i \ell_2/N} i \delta_{|s| - r - 1, r'} \delta_{N - \ell_1, \ell'_1} \delta_{N - \ell_2, \ell'_2}.\end{aligned}\quad (91)$$

We should modify several terms in the following particular case. Since $N - \ell_1 \equiv \ell_1 \pmod{N}$ is satisfied when $\ell_1 = 0$ or $N = 2$, $\delta_{|s|-r-1,r'}$ should be modified into $\delta_{|s|-r,r'}$ when $\ell_1 = 0$ or $\delta_{|s|-r-\ell_1,r'}$ when $N = 2$. Furthermore, when $r = 0$ in addition to $\ell_1 = 0$ or $N = 2$, $e^{2\pi i \ell_2 / N}$ does not appear even if $\ell_2 \neq 0$. Note that Eq. (91) leads to the following relation⁶:

$$\Psi_{T^2/\mathbb{Z}_N}^{r,|s|}(-z, \tau) = e^{2\pi i \ell_2 / N} \Psi_{T^2/\mathbb{Z}_N}^{|s|-r-1,|s|}(z, \tau). \quad (92)$$

The other relations except for Eq. (91) are the same as the T^2 basis, where we note that the representation of T^N transformation is diagonalized. However, the \mathbb{Z}_N -shift invariant modes on the full T^2/\mathbb{Z}_N shifted orbifold, i.e., $(\ell_1, \ell_2) = (0, 0)$, in particular, correspond to the modes on the $T^2/N \simeq T^2$ with magnetic flux $s = M/N^2 \in 2\mathbb{Z}$. In other words, the \mathbb{Z}_N -shift invariant modes behave like

modular forms for $\tilde{\Gamma}(2|M|/N^2)$, while the other modes correspond to modular forms for $\tilde{\Gamma}(2|M|)$.

VI. MODULAR SYMMETRY IN MAGNETIZED T^2/\mathbb{Z}_N TWISTED AND SHIFTED ORBIFOLD MODELS

In this section, we study the modular symmetry for the wave functions on the magnetized T^2/\mathbb{Z}_N twisted and shifted orbifolds. The modular symmetry remains on the T^2/\mathbb{Z}_2 twisted orbifold. In order for the T^2/\mathbb{Z}_2 twisted orbifold to be consistent with the full T^2/\mathbb{Z}_N shifted orbifold, the following condition should be also satisfied:

$$N - \ell_{1,2} \equiv \ell_{1,2} \pmod{N}, \quad (93)$$

for $\forall \ell_{1,2} \in \mathbb{Z}_N$. Therefore, the only full T^2/\mathbb{Z}_2 shifted orbifold is consistent with the T^2/\mathbb{Z}_2 twisted orbifold.⁷ The wave functions on the magnetized T^2/\mathbb{Z}_2 twisted and shifted orbifold are expressed as

$$\begin{aligned} \Psi_{T^2/\mathbb{Z}_2}^{r,|s|} &= \mathcal{N}_2^{st} (\Psi_{T^2/\mathbb{Z}_2}^{r,|s|} + (-1)^{m+\ell_2} \Psi_{T^2/\mathbb{Z}_2}^{|s|-r-\ell_1,|s|}) = \mathcal{N}_2^{st} (\psi_{T^2/\mathbb{Z}_2}^{2r+\ell_1,4|s|} + (-1)^{\ell_2} \psi_{T^2/\mathbb{Z}_2}^{2r+\ell_1+2|s|,4|s|}) \\ &= \mathcal{N}_2^{st} (\psi_{T^2}^{2r+\ell_1,4|s|} + (-1)^{\ell_2+m} \psi_{T^2}^{2(|s|-r-\ell_1)+\ell_1,4|s|} + (-1)^{\ell_2} \psi_{T^2}^{2(|s|+r)+\ell_1,4|s|} + (-1)^m \psi_{T^2}^{2(2|s|-r-\ell_1)+\ell_1,4|s|}), \\ s \in 2\mathbb{Z}, \quad r \in \mathbb{Z}_{\frac{|s|}{2}+1-\ell_1}, \quad m, \ell_1, \ell_2 \in \mathbb{Z}_2, \end{aligned} \quad (94)$$

where \mathcal{N}_2^{st} is the normalization factor determined by remaining Eq. (33). Note that $\ell^{(1,1)} \equiv \ell_1 + \ell_2 \pmod{2}$. The numbers of the degenerate modes for $(m; \ell_1, \ell_2) = (0; 0, 0), (1; 0, 0)$ are $(|M|/8 + 1), (|M|/8 - 1)$, respectively, while the numbers of the degenerate modes for the other $(m; \ell_1, \ell_2)$ are $|M|/8$, where $M \in 8\mathbb{Z}$. On this T^2/\mathbb{Z}_2 twisted and shifted orbifold basis, Eqs. (89) and (90) as well as Eq. (50) are deformed as

$$\rho_{T^2/\mathbb{Z}_2}^{(0;\ell_1,\ell_2)}(\tilde{S})_{rr',(\ell_1,\ell_2)(\ell'_1,\ell'_2)} = e^{i\pi/4} \frac{2}{\sqrt{|s|}} \cos(2\pi(\ell_1/2 + r)(\ell'_1/2 + r')/|s|) \delta_{\ell_2,\ell'_1} \delta_{\ell_1,\ell'_2}, \quad (95)$$

$$\rho_{T^2/\mathbb{Z}_2}^{(0;\ell_1,\ell_2)}(\tilde{T})_{rr',(\ell_1,\ell_2)(\ell'_1,\ell'_2)} = e^{i\pi(\frac{\ell_1}{N}+r)^2/|s|} \delta_{r,r'} \delta_{\ell_1,\ell'_1} \delta_{\ell_2-\ell_1,\ell'_2}, \quad (96)$$

$$\rho_{T^2/\mathbb{Z}_2}^{(1;\ell_1,\ell_2)}(\tilde{S})_{rr',(\ell_1,\ell_2)(\ell'_1,\ell'_2)} = e^{i\pi/4} \frac{2i}{\sqrt{|s|}} \sin(2\pi(\ell_1/2 + r)(\ell'_1/2 + r')/|s|) \delta_{\ell_2,\ell'_1} \delta_{\ell_1,\ell'_2}, \quad (97)$$

$$\rho_{T^2/\mathbb{Z}_2}^{(1;\ell_1,\ell_2)}(\tilde{T})_{rr',(\ell_1,\ell_2)(\ell'_1,\ell'_2)} = e^{i\pi(\frac{\ell_1}{N}+r)^2/|s|} \delta_{r,r'} \delta_{\ell_1,\ell'_1} \delta_{\ell_2-\ell_1,\ell'_2}. \quad (98)$$

⁶The following calculation is useful to confirm Eq. (92):

$$\begin{aligned} e^{-2\pi i k \ell_2 / N} \Psi_{T^2}^{(Nr+\ell_1)+kN|s|,N^2|s|}(-z) &= e^{-2\pi i k \ell_2 / N} \Psi_{T^2}^{N^2|s|-(Nr+\ell_1)+kN|s|,N^2|s|}(z) \\ &= e^{2\pi i \ell_2 / N} e^{-2\pi i (N-k-1)(N-\ell_2) / N} \Psi_{T^2}^{(N(|s|-r-1)+N-\ell_1)+(N-k-1)N|s|,N^2|s|}(z) \\ &= e^{2\pi i \ell_2 / N} e^{-2\pi i k' \ell'_2 / N} \Psi_{T^2}^{(Nr'+\ell'_1)+k'N|s|,N^2|s|}(z). \end{aligned}$$

⁷The other T^2/\mathbb{Z}_N twisted orbifolds with $N = 3, 4, 6$, on the other hand, are not consistent with any full T^2/\mathbb{Z}_N shifted orbifolds since they require $\ell_1 = \ell_2$.

They satisfy the same relations as the T^2/\mathbb{Z}_2 twisted orbifold basis. In particular, the \mathbb{Z}_2 -shift invariant modes, i.e., $(m; \ell_1, \ell_2) = (m; 0, 0)$, correspond to the modes on the $(T^2/N)/\mathbb{Z}_2 \simeq T^2/\mathbb{Z}_2$ twisted orbifold with magnetic flux $s = M/4 \in 2\mathbb{Z}$. For example, when $M = 8$ ($s = 2$), the \mathbb{Z}_2 -shift invariant wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold basis are expressed as

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{0,2} \\ \Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{1,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,8} + \psi_{T^2}^{4,8}) \\ \frac{1}{\sqrt{2}}(\psi_{T^2}^{2,8} + \psi_{T^2}^{6,8}) \end{pmatrix}, \quad (99)$$

and the S and T transformations for Eq. (99) are the same as Eq. (58). When $M = 16$ ($s = 4$), the \mathbb{Z}_2 -shift invariant wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold basis are expressed as

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{0,4} \\ \Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{1,4} \\ \Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{2,4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,16} + \psi_{T^2}^{8,16}) \\ \frac{1}{2}(\psi_{T^2}^{2,16} + \psi_{T^2}^{6,16} + \psi_{T^2}^{10,16} + \psi_{T^2}^{14,16}) \\ \frac{1}{\sqrt{2}}(\psi_{T^2}^{4,16} + \psi_{T^2}^{12,16}) \end{pmatrix}, \quad (100)$$

$$\Psi_{T^2/\mathbb{Z}_2^{(0;0,0)}}^{1,4} = \frac{1}{2}(\psi_{T^2}^{2,16} - \psi_{T^2}^{6,16} + \psi_{T^2}^{10,16} - \psi_{T^2}^{14,16}), \quad (101)$$

and the representations of the S and T transformations are the same as Eqs. (72) and (74). We express all wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold base for $M = 8, 16$ and the representations of the S and T transformations for them in Appendix C.

As a result, when $M = 0 \pmod{8}$, both the \mathbb{Z}_2 twist and the full \mathbb{Z}_2 shift are consistent with the modular symmetry. The wave functions can be decomposed into smaller representations by their eigenvalues. Thus, a combination between the \mathbb{Z}_2 twist and the full \mathbb{Z}_2 shift provides us with a reduction of reducible representations towards irreducible representations $\tilde{\Gamma}'_{2|M|} \equiv \tilde{\Gamma}/\tilde{\Gamma}(2|M|)$.

VII. CONCLUSION

We have studied the modular symmetry of wave functions on the magnetized $T^2 \simeq \mathbb{C}/\Lambda$. When the magnetic flux M is even, the wave functions behave as modular forms of weight $1/2$ and represent the double covering group of $\Gamma \equiv SL(2, \mathbb{Z})$, $\tilde{\Gamma} \equiv \widetilde{SL}(2, \mathbb{Z})$. Each wave function on T^2 with the magnetic flux M transforms under $\tilde{\Gamma}(2|M|)$. Then, $|M|$ zero modes as well as massive modes are representations of the quotient group $\tilde{\Gamma}'_{2|M|} \equiv \tilde{\Gamma}/\tilde{\Gamma}(2|M|)$.

If we change the Wilson line $\zeta \rightarrow \zeta + 1$ simultaneously with the T transformation of the modular transformations, T^2 with any magnetic flux M is consistent with the modular transformations. However, the zero-mode wave functions after the T transformation are expanded by the all excited-mode wave functions before the T transformation.

We have also studied the modular symmetry for the wave functions on various magnetized T^2/\mathbb{Z}_N orbifolds. The T^2/\mathbb{Z}_N twisted orbifold can be constructed for $N = 2, 3, 4, 6$. However, the modulus $\tau \equiv e_2/e_1$ is fixed as $\tau = e^{2\pi i/N}$ for $N = 3, 4, 6$ while any τ is allowed for $N = 2$. It means that the only ST, S, ST^{-1} transformations of the modular transformations remain for $N = 3, 4, 6$, respectively. They correspond to \mathbb{Z}_{2N} symmetries. On the other hand, there remains the full modular symmetry for $N = 2$. The representations of the modular transformations on the T^2/\mathbb{Z}_2 twisted orbifold basis satisfy the same algebraic relations as the representations on the T^2 basis. However, the representations on the T^2 basis are decomposed into smaller representations on the T^2/\mathbb{Z}_2 twisted orbifold basis.

In order for the T^2/\mathbb{Z}_N shifted orbifold to be consistent with the modular transformations, all \mathbb{Z}_N -shifted points should be identified, where we call it the full T^2/\mathbb{Z}_N shifted orbifold. The full T^2/\mathbb{Z}_N shifted orbifold with the magnetic flux M corresponds to $T^{2'} \simeq \mathbb{C}/\Lambda'$, $\Lambda' \equiv \Lambda/N$ with the magnetic flux $s \equiv M/N^2 \in 2\mathbb{Z}$. In particular, the \mathbb{Z}_N -shift invariant modes correspond to the modes on $T^2/N \simeq T^{2'}$. Therefore, the \mathbb{Z}_N -shift invariant modes behave like modular forms for $\tilde{\Gamma}(2|M|/N^2)$, while the other modes behave as modular forms for $\tilde{\Gamma}(2|M|)$.

Furthermore, the only full T^2/\mathbb{Z}_2 shifted orbifold is consistent with the T^2/\mathbb{Z}_2 twisted orbifold. On that T^2/\mathbb{Z}_2 twisted and shifted orbifold, the \mathbb{Z}_2 -shift invariant modes correspond to the modes on the $(T^2/N)/\mathbb{Z}_2$ twisted orbifold with the magnetic flux $s \equiv M/4 \in 2\mathbb{Z}$.

The wave functions on T^2 are decomposed into smaller representations by the \mathbb{Z}_2 twist and shift. They provide us with a reduction of representations towards irreducible representations. Also, the combination of the \mathbb{Z}_2 twist and shift provides us with a new approach to realize three generations from the phenomenological viewpoints.⁸ It is interesting to study three-generation models by a combination of \mathbb{Z}_2 twist and shift. We would study elsewhere.

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⁸See [56,57] for classifications of three-generation models by T^2/\mathbb{Z}_N twisted orbifolds.

APPENDIX A: EXTENSION FOR GENERALIZED CP SYMMETRY WITH THE MODULAR SYMMETRY ON THE MAGNETIZED T^2

Here, we study the extension for generalized CP symmetry with the modular transformations on the magnetized T^2 .⁹ The CP transformation for the modulus τ is defined as $CP: \tau \rightarrow -\bar{\tau}$, where it remains $\text{Im}(-\bar{\tau}) > 0$. It is derived from

$$CP: \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \rightarrow \begin{pmatrix} e_2^{CP} \\ e_1^{CP} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{e}_2 \\ \bar{e}_1 \end{pmatrix},$$

$$CP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A1})$$

$$CP: z \equiv \frac{u}{e_1} \rightarrow z^{CP} \equiv \frac{u^{CP}}{e_1^{CP}} = \frac{\bar{u}}{-\bar{e}_1} = -\bar{z}, \quad (\text{A2})$$

$$CP: \tau \equiv \frac{e_2}{e_1} \rightarrow \tau^{CP} \equiv \frac{e_2^{CP}}{e_1^{CP}} = \frac{\bar{e}_2}{-\bar{e}_1} = -\bar{\tau}. \quad (\text{A3})$$

The CP matrix in Eq. (A1) satisfies the following relations:

$$CP^2 = \mathbb{I}, \quad (CP)S(CP)^{-1} = S^{-1}, \quad (CP)T(CP)^{-1} = T^{-1}. \quad (\text{A4})$$

When we also consider the above CP transformation in addition to the modular transformations, the modular group $\Gamma = SL(2, \mathbb{Z})$ is extended to $\Gamma^* \equiv SL(2, \mathbb{Z}) \rtimes \mathbb{Z}_2^{CP} \simeq GL(2, \mathbb{Z})$. Under the extended modular transformation by $\gamma^* = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma^*$, (z, τ) transforms as

$$\gamma^*: (z, \tau) \rightarrow \begin{cases} \left(\frac{z}{c'\tau + d'}, \frac{a'\tau + b'}{c'\tau + d'} \right), & (\det \gamma^* = 1) \\ \left(\frac{\bar{z}}{c'\bar{\tau} + d'}, \frac{a'\bar{\tau} + b'}{c'\bar{\tau} + d'} \right), & (\det \gamma^* = -1), \end{cases} \quad (\text{A5})$$

where the above in Eq. (A5) is just modular transformation and the below in Eq. (A5) contains odd numbers of CP transformation. It leads to redefine the automorphy factor as

$$J_k(\gamma^*, \tau) \equiv \begin{cases} (c'\tau + d')^k, & (\det \gamma^* = 1) \\ (c'\bar{\tau} + d')^k, & (\det \gamma^* = -1), \end{cases}$$

$$\gamma^* = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma^*, \quad (\text{A6})$$

where it satisfies Eq. (12). Note that γ^* does not mean the complex conjugate of γ but an element of Γ^* .

⁹See [19,33] for the relation between the modular symmetry and CP symmetry. See also Ref. [58], and references therein, for CP in superstring theory.

In order to see how the wave functions on the magnetized T^2 transform under the extended modular transformation by $\gamma^* \in \Gamma^*$, furthermore, we consider the double covering group of $\Gamma^* \simeq GL(2, \mathbb{Z})$, $\tilde{\Gamma}^* \simeq \tilde{GL}(2, \mathbb{Z})$, similar to Eq. (38). Note that only Eq. (42) is redefined as

$$A(\gamma_1^*, \gamma_2^*) = (\det \gamma_1^*, \det \gamma_2^*)_H \left(\frac{\chi(\gamma_1^* \gamma_2^*)}{\chi(\gamma_1^*)}, \frac{\chi(\gamma_1^* \gamma_2^*)}{\chi(\gamma_2^*) \det \gamma_1^*} \right)_H. \quad (\text{A7})$$

(See Ref. [49].) In particular, we set

$$\tilde{CP} \equiv [CP, 1]. \quad (\text{A8})$$

Then, Eqs. (A8), (44), and (46) lead to the following relations:

$$\begin{aligned} (\tilde{CP})^2 &= [\mathbb{I}, -1] = \tilde{Z}^2, \\ (\tilde{CP})^4 &= [\mathbb{I}, 1] = \mathbb{I} = \tilde{Z}^4, \\ (\tilde{CP})^{-1} &= [(CP)^{-1}, -1], \\ (\tilde{CP})\tilde{S}(\tilde{CP})^{-1} &= [S^{-1}, 1] = \tilde{S}^{-1}, \\ (\tilde{CP})\tilde{T}(\tilde{CP})^{-1} &= [T^{-1}, 1] = \tilde{T}^{-1}, \end{aligned} \quad (\text{A9})$$

in addition to Eq. (45). The automorphy factor is the same as Eq. (47) and satisfies Eq. (48), where we should apply Eqs. (A6) and (A7).

Here, we study the CP transformation of the fields on the magnetized T^2 . In addition to Eqs. (A2) and (A3), it is also needed that the magnetic flux M is flipped as

$$CP: M \rightarrow -M. \quad (\text{A10})$$

In this case, any field in Sec. III A after the CP transformation corresponds to the complex conjugate of the field. In particular, the wave functions of the n th excited modes in Eq. (31), including the zero modes in Eq. (29), transform as

$$CP: \psi_n^{j,M}(z, \tau) \rightarrow \psi_n^{j,|M|}(-\bar{z}, -\bar{\tau}) = \overline{\psi_n^{j,|M|}(z, \tau)}, \quad (\text{A11})$$

under the CP transformation. Considering Eqs. (49) and (A11), we can obtain the following form:

$$\psi_n^{j,|M|}(\tilde{CP}(z, \tau)) = \tilde{J}_{1/2}(\tilde{CP}, \tau) \sum_{k=0}^{|M|-1} \rho(\tilde{CP})_{jk} \overline{\psi_n^{k,|M|}(z, \tau)}, \quad (\text{A12})$$

$$\tilde{J}_{1/2}(\tilde{CP}, \tau) = (-1)^{1/2} = e^{-i\pi/2} = -i, \quad \rho(\tilde{CP})_{jk} = i\delta_{j,k}. \quad (\text{A13})$$

We can also check the following relations:

$$\tilde{J}_{1/2}((\widetilde{CP})^{-1}, \tau) = -(-1)^{1/2} = -e^{-i\pi/2} = i, \quad \rho(\widetilde{CP})_{jk}^{-1} = -i\delta_{j,k}. \quad (\text{A14})$$

From Eqs. (A13), (A14), and (50), we can obtain the following relations:

$$\tilde{J}_{1/2}(\tilde{Z}^2, \tau) = \tilde{J}_{1/2}((\widetilde{CP})^2, \tau) = -1, \quad \rho(\tilde{Z})^2 = \rho(CP)^2 = -\delta_{j,k}, \quad (\text{A15})$$

$$\tilde{J}_{1/2}(\tilde{Z}^4, \tau) = \tilde{J}_{1/2}((\widetilde{CP})^4, \tau) = -1, \quad \rho(\tilde{Z})^4 = \rho(CP)^4 = \delta_{j,k}, \quad (\text{A16})$$

$$\tilde{J}_{1/2}((\widetilde{CP})\tilde{S}(\widetilde{CP})^{-1}, \tau) = \tilde{J}_{1/2}(\tilde{S}^{-1}, \tau), \quad [\rho(\widetilde{CP})\overline{\rho(\tilde{S})}\rho(\widetilde{CP})^{-1}]_{jk} = \rho(\tilde{S})_{jk}^{-1}, \quad (\text{A17})$$

$$\tilde{J}_{1/2}((\widetilde{CP})\tilde{T}(\widetilde{CP})^{-1}, \tau) = \tilde{J}_{1/2}(\tilde{T}^{-1}, \tau), \quad [\rho(\widetilde{CP})\overline{\rho(\tilde{T})}\rho(\widetilde{CP})^{-1}]_{jk} = \rho(\tilde{T})_{jk}^{-1}, \quad (\text{A18})$$

which are the representations of Eq. (A9). Then, ρ becomes the representation of $\tilde{\Gamma}'_{2|M|} \equiv \tilde{\Gamma}^*/\tilde{\Gamma}(2|M|)$.

APPENDIX B: MODULAR TRANSFORMATION WITH GAUGE TRANSFORMATION

Here, we derive Eq. (62),

$$\begin{aligned} T: \psi_0^{j|M|}(z + \zeta, \tau) &\rightarrow \psi_0^{j|M|}(z + \zeta + 1, \tau + 1) \\ &= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta+1)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \sum_{l \in \mathbf{Z}} e^{i\pi|M|(\tau+1)(\frac{j}{|M|}+l)^2} e^{2\pi i|M|(z+\zeta)(\frac{j}{|M|}+l)} \\ &= (-1)^2 e^{i\pi\frac{j^2}{|M|}} \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \sum_{l \in \mathbf{Z}} e^{i\pi|M|(\frac{\text{Im}(z+\zeta)}{\text{Im}\tau} + \frac{j}{|M|}+l)} e^{i\pi|M|\tau(\frac{j}{|M|}+l)^2} e^{2\pi i|M|(z+\zeta)(\frac{j}{|M|}+l)} \\ &= (-1)^2 e^{i\pi\frac{j^2}{|M|}} e^{-\frac{\pi|M|}{8\text{Im}\tau}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(i\sqrt{\frac{\pi|M|}{8\text{Im}\tau}}\right)^n \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} \\ &\quad \times e^{i\pi|M|(z+\zeta)\frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \sum_{l \in \mathbf{Z}} e^{i\pi|M|\tau(\frac{j}{|M|}+l)^2} e^{2\pi i|M|(z+\zeta)(\frac{j}{|M|}+l)} H_n\left(\sqrt{2\pi|M|\text{Im}\tau}\left(\frac{\text{Im}z}{\text{Im}\tau} + \frac{j}{|M|} + l\right)\right) \\ &= (-1)^j e^{i\pi\frac{j^2}{|M|}} e^{-\frac{\pi|M|}{8\text{Im}\tau}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(i\sqrt{\frac{\pi|M|}{4\text{Im}\tau}}\right)^n \psi_n^{j|M|}(z + \zeta, \tau), \end{aligned} \quad (\text{B1})$$

where we use the following the generating function of the Hermite function:

$$e^{-y^2+2xy} = \sum_{n=0}^{\infty} H_n(x) \frac{y^n}{n!}. \quad (\text{B2})$$

APPENDIX C: EXAMPLES OF THE T^2/\mathbb{Z}_2 TWISTED AND SHIFTED ORBIFOLD BASE

Here, we express examples of the wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold basis and the representations of the S and T transformations for them. In particular, we show them for $M = 8$ and 16.

When $M = 8$ ($s = 2$), the wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold basis are expressed as

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,2(0;0,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,2(0;0,0)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,8} + \psi_{T^2}^{4,8}) \\ \frac{1}{\sqrt{2}}(\psi_{T^2}^{2,8} + \psi_{T^2}^{6,8}) \end{pmatrix}, \quad (\text{C1})$$

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,2(0,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,2(0,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,2(0,1,1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,8} - \psi_{T^2}^{4,8}) \\ \frac{1}{2}(\psi_{T^2}^{1,8} + \psi_{T^2}^{3,8} + \psi_{T^2}^{5,8} + \psi_{T^2}^{7,8}) \\ \frac{1}{2}(\psi_{T^2}^{1,8} - \psi_{T^2}^{3,8} - \psi_{T^2}^{5,8} + \psi_{T^2}^{7,8}) \end{pmatrix}, \quad (\text{C2})$$

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,4(0,0,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(0,0,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{2,4(0,0,0)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,16} + \psi_{T^2}^{8,16}) \\ \frac{1}{2}(\psi_{T^2}^{2,16} + \psi_{T^2}^{6,16} + \psi_{T^2}^{10,16} + \psi_{T^2}^{14,16}) \\ \frac{1}{\sqrt{2}}(\psi_{T^2}^{4,16} + \psi_{T^2}^{12,16}) \end{pmatrix}, \quad (\text{C7})$$

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,2(1,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,2(0,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,2(0,1,1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{2,8} - \psi_{T^2}^{6,8}) \\ \frac{1}{2}(\psi_{T^2}^{1,8} - \psi_{T^2}^{3,8} + \psi_{T^2}^{5,8} - \psi_{T^2}^{7,8}) \\ \frac{1}{2}(\psi_{T^2}^{1,8} + \psi_{T^2}^{3,8} - \psi_{T^2}^{5,8} - \psi_{T^2}^{7,8}) \end{pmatrix}. \quad (\text{C3})$$

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,4(0,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(0,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,4(0,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(0,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,4(0,1,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(0,1,1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_{T^2}^{0,16} - \psi_{T^2}^{8,16}) \\ \frac{1}{2}(\psi_{T^2}^{2,16} - \psi_{T^2}^{6,16} - \psi_{T^2}^{10,16} + \psi_{T^2}^{14,16}) \\ \frac{1}{2}(\psi_{T^2}^{1,16} + \psi_{T^2}^{7,16} + \psi_{T^2}^{9,16} + \psi_{T^2}^{15,16}) \\ \frac{1}{2}(\psi_{T^2}^{3,16} + \psi_{T^2}^{5,16} + \psi_{T^2}^{11,16} + \psi_{T^2}^{13,16}) \\ \frac{1}{2}(\psi_{T^2}^{1,16} - \psi_{T^2}^{7,16} - \psi_{T^2}^{9,16} + \psi_{T^2}^{15,16}) \\ \frac{1}{2}(\psi_{T^2}^{3,16} - \psi_{T^2}^{5,16} - \psi_{T^2}^{11,16} + \psi_{T^2}^{13,16}) \end{pmatrix}, \quad (\text{C8})$$

The representations of the S and T transformations for Eq. (C1) are expressed as

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{S}) &= \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{T}) &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \end{aligned} \quad (\text{C4})$$

$$\Psi_{T^2/\mathbb{Z}_2}^{1,4(0,0,0)} = \frac{1}{2}(\psi_{T^2}^{2,16} - \psi_{T^2}^{6,16} + \psi_{T^2}^{10,16} - \psi_{T^2}^{14,16}), \quad (\text{C9})$$

which are the same as Eq. (58). The representations of the S and T transformations for Eqs. (C2) and (C3) are expressed as

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2}^{(0,\ell_1,\ell_2)}(\tilde{S}) &= e^{i\pi/4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \rho_{T^2/\mathbb{Z}_2}^{(0,\ell_1,\ell_2)}(\tilde{T}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{i\pi/8} \\ 0 & e^{i\pi/8} & 0 \end{pmatrix}, \end{aligned} \quad (\text{C5})$$

$$\begin{pmatrix} \Psi_{T^2/\mathbb{Z}_2}^{0,4(1,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(1,0,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,4(1,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(1,1,0)} \\ \Psi_{T^2/\mathbb{Z}_2}^{0,4(1,1,1)} \\ \Psi_{T^2/\mathbb{Z}_2}^{1,4(1,1,1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\psi_{T^2}^{2,16} + \psi_{T^2}^{6,16} - \psi_{T^2}^{10,16} - \psi_{T^2}^{14,16}) \\ \frac{1}{\sqrt{2}}(\psi_{T^2}^{4,16} - \psi_{T^2}^{12,16}) \\ \frac{1}{2}(\psi_{T^2}^{1,16} - \psi_{T^2}^{7,16} + \psi_{T^2}^{9,16} - \psi_{T^2}^{15,16}) \\ \frac{1}{2}(\psi_{T^2}^{3,16} - \psi_{T^2}^{5,16} + \psi_{T^2}^{11,16} - \psi_{T^2}^{13,16}) \\ \frac{1}{2}(\psi_{T^2}^{1,16} + \psi_{T^2}^{7,16} - \psi_{T^2}^{9,16} - \psi_{T^2}^{15,16}) \\ \frac{1}{2}(\psi_{T^2}^{3,16} + \psi_{T^2}^{5,16} - \psi_{T^2}^{11,16} - \psi_{T^2}^{13,16}) \end{pmatrix}. \quad (\text{C10})$$

and

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2}^{(1,\ell_1,\ell_2)}(\tilde{S}) &= e^{i\pi/4} i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \rho_{T^2/\mathbb{Z}_2}^{(1,\ell_1,\ell_2)}(\tilde{T}) &= \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & e^{i\pi/8} \\ 0 & e^{i\pi/8} & 0 \end{pmatrix}, \end{aligned} \quad (\text{C6})$$

The representations of the S and T transformations for Eqs. (C8) and (C10) are expressed as

$$\begin{aligned} \rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{S}) &= \frac{e^{i\pi/4}}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\ \rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{T}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (\text{C11})$$

respectively.

When $M = 16$ ($s = 4$), the wave functions on the T^2/\mathbb{Z}_2 twisted and shifted orbifold basis are expressed as

which are the same as Eq. (72), and

$$\rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{S}) = e^{3\pi i/4}, \quad \rho_{T^2/\mathbb{Z}_2}^{(0,0,0)}(\tilde{T}) = e^{i\pi/4}, \quad (\text{C12})$$

which are the same as Eq. (74), respectively. The representations of the S and T transformations for Eqs. (C8) and (C10) are expressed as

$$\rho_{T^2/\mathbb{Z}_2^{(0;\ell_1,\ell_2)}}(\tilde{S}) = e^{i\pi/4} \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\pi/8) & \sin(\pi/8) \\ 0 & 0 & 0 & 0 & \sin(\pi/8) & -\cos(\pi/8) \end{pmatrix},$$

$$\rho_{T^2/\mathbb{Z}_2^{(0;\ell_1,\ell_2)}}(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\pi/16} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\pi/16}i \\ 0 & 0 & e^{i\pi/16} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\pi/16}i & 0 & 0 \end{pmatrix}, \quad (\text{C13})$$

and

$$\rho_{T^2/\mathbb{Z}_2^{(1;\ell_1,\ell_2)}}(\tilde{S}) = e^{i\pi/4}i \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin(\pi/8) & \cos(\pi/8) \\ 0 & 0 & 0 & 0 & \cos(\pi/8) & -\sin(\pi/8) \end{pmatrix},$$

$$\rho_{T^2/\mathbb{Z}_2^{(1;\ell_1,\ell_2)}}(\tilde{T}) = \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\pi/16} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\pi/16}i \\ 0 & 0 & e^{i\pi/16} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\pi/16}i & 0 & 0 \end{pmatrix}, \quad (\text{C14})$$

respectively.

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