

Matter-gravity coupling for fuzzy geometry and the Landau-Hall problemV. P. Nair^{*}*Physics Department, City College of the CUNY, New York, New York 10031, USA*

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We consider a set of physical degrees of freedom coupled to a finite-dimensional Hilbert space, which may be taken as modeling a fuzzy space or as the lowest Landau level of a Landau-Hall problem. These may be viewed as matter fields on a fuzzy space. Sequentially generalizing to arbitrary backgrounds, we argue that the effective action is given by the Chern-Simons form associated with the Dirac index density (with gauge and gravitational fields), with an Abelian gauge field shifted by the Poincaré-Cartan form for matter dynamics. The result is an action for matter fields where the Lagrangian is integrated with a density which is a specific polynomial in the curvatures.

DOI: [10.1103/PhysRevD.102.105008](https://doi.org/10.1103/PhysRevD.102.105008)**I. INTRODUCTION**

The energy levels of a charged particle in a magnetic field, the so-called Landau levels, have long been a useful structure to analyze many questions of physical interest. The quantum Hall effect is perhaps the most direct example of the use of these Landau levels [1]. In this context, a number of variants, including different topologies and different geometries (as characterized by metrics and spin connection) [2,3], as well as higher dimensions [4–7] have also been explored. The Landau levels have also provided a useful analytical tool for discussing effective actions, pair production by both Abelian and nonabelian gauge fields, etc., [8]. Another important reason for research interest in this area has to do with noncommutative geometry [9]. The set of degenerate states of a fixed Landau level can be used as a model for a noncommutative manifold, with operators on these degenerate states providing observables for the noncommutative space. The existence of symbols and star products corresponding to such operators renders the continuum or commutative space approximation to such noncommutative spaces easily tractable. It is worth emphasizing that noncommutative geometry has been a recurrent paradigm for many approaches to quantum gravity, both intrinsically as an idea in its own right [9] and as special cases in string theory [10]. Needless to say, there has been an enormous amount of recent research along these lines.

Offset against this large body of literature, it is interesting that there are still many open questions of physical

relevance. If we consider the lowest Landau level (LLL) as a model for a noncommutative space, we can construct fields living on such spaces. What are the characteristics of such a field theory? This is the key question we analyze here. The construction of noncommutative field theories has a long history in its own right. Most of this work has been at the level of promoting products of fields and their spatial (or spacetime) derivatives to star products, but using standard Lagrangians [11]. We are considering the construction of the action starting from operators on the Hilbert space (i.e., LLL) modeling the noncommutative space. The resulting action will be different in many features, especially in its relation to the background geometry. We have argued elsewhere that the LLL analysis can be used for understanding gravity on noncommutative spaces [12]. The present work may be viewed as extending such ideas to include matter couplings to gravity. Phrased another way, we ask the question whether there are particular peculiarities for matter-gravity coupling which we can extract from analyzing fields acting on the LLL.

Since we are modeling the noncommutative space by the LLL, there is a possibility of some confusion about the roles of the fields we are discussing. It is useful to clarify this at the outset. We will consider degrees of freedom which eventually lead to a set of fields we shall refer to as “matter fields”, designated by ϕ . But there will be a set of fermion fields defining the LLL itself, i.e., the noncommutative geometry. These latter ones, which we designate by ψ , ψ^\dagger , are what we shall refer to as the “spatial fields”. The question of interest for us is how the dynamics for the ϕ -fields is affected by the background geometry for ψ , ψ^\dagger . This is not simply a matter of writing a coupled action for both sets of fields and analyzing it, as we would normally do for interacting field theories, because, for us, the ϕ -fields do not exist outside of the LLL. This is the distinctive feature of our analysis.

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While the noncommutative geometry-gravity angle is the natural setting of the problem, it may also be viewed as a much more standard physical problem, of interest within the quantum Hall setting. If a set of fields ϕ are coupled to charged fermions (described by ψ, ψ^\dagger) and if these fermions are confined to the LLL, what is the theory of the fields ϕ within the LLL? How does this field theory respond to changes in the background fields, metric, and spin connection? Clearly, this is a natural next step to the many analyses which have been done for the pure electron system with arbitrary background gauge fields and metrics [2,3].

The organization of the paper and overall flow of logic may be summarized as follows. We start with the dynamics of a physical system whose observables are matrices acting on the states of the LLL (or the Hilbert space modeling the spatial geometry). Not surprisingly, this leads to a Hamiltonian or Lagrangian with star products for the fields and their derivatives. We will consider the required mathematical framework for the two-dimensional case in Sec. II and the more general higher dimensional cases in Sec. III. Complex Kähler geometry will play a crucial role in defining the star products. The path integral for the dynamics of the physical system under consideration, which we designate the matter fields, we will argue, is defined by a Chern-Simons action (related to the Dolbeault index density) with a shift of the (Abelian) gauge potential by the Poincaré-Cartan form. So far, the results will still be tied to the complex geometry of the background. Next, in Sec. IV, we want to generalize this to more general gauge and gravitational backgrounds. Toward this, we argue that there is a scaling of coordinates under which, if we restrict to low energy physics, one can ignore higher terms in the star products, thus giving an approximation not tied to the complex geometry. The resulting version can then be embedded in a more general geometry and the effective action constructed in terms of the Chern-Simons form associated with the Dirac index density. Again, the prescription for matter fields is to shift the Abelian gauge field in the Chern-Simons form for the Dirac index by the Poincaré-Cartan form for matter fields. Explicit formulas for the effective action in $2+1$ and $4+1$ dimensions are given. Finally, we give an action for a set of fermion fields, to be viewed as the fields which eventually define the spatial manifold, which leads to the prescription for the matter couplings we have obtained. The paper concludes with a short summary/discussion.

Explicit derivations of the effective action with perturbations to the background geometry and gauge fields for the LLL, in the absence of what we have referred to as matter fields, were given in [2,3,6] in $2+1$ dimensions and in [6,7] for higher dimensions. Also, a different way of constructing an effective action for the Landau-Hall problem for all odd spacetime dimensions, using the Dolbeault index theorem, was given in [13]. The present work may be

considered as an extension of these works to include matter couplings, and also to accommodate more general, not necessarily complex, geometries. An interesting feature of the emergent matter-gravity coupling is that the action is given by integrating the matter Lagrangian with a density which is a specific polynomial involving powers of the curvature. It is interesting to note that such couplings for matter and gravity have been the subject of recent research motivated by issues with dark matter [14].

II. MATTER FIELDS AND GRAVITY AND THE LLL IN TWO SPATIAL DIMENSIONS

We start by considering a physical system characterized by a set of operators which are the relevant dynamical variables. Among the operators, we assume there is a mutually commuting set which we denote by $\{q_\lambda\}$, where λ is an index labeling the distinct operators. Since we are aiming for a field theory eventually, we take the eigenvalues of the q 's to form a continuous set. The states of the physical system can be taken as the vectors $|q\rangle$ in a Hilbert space \mathcal{H} . Any nontrivial dynamics should allow for altering the state of the system, so there must be operators which do not commute with the q 's. We can take them to be a set of conjugate variables $\{p_\lambda\}$. Taking the Hamiltonian to be a function of these variables $\{q_\lambda, p_\lambda\}$, time evolution of the system by an infinitesimal amount ϵ is described by the transformation kernel

$$\langle q' | e^{-iH\epsilon} | q \rangle = \int [dp] \exp [ip_\lambda(q'_\lambda - q_\lambda) - H(p, q)\epsilon]. \quad (1)$$

It is also possible to carry out the integration over the p 's and write this in terms of the action.

The key point for us is that we want to interpret this as a field theory in the language of noncommutative geometry. The variables $\{q_\lambda\}$ should describe a field operator on some manifold \mathcal{M} in a suitable large N limit. For this, consider an N -dimensional Hilbert space \mathcal{H}_N . This is not the Hilbert space \mathcal{H} of the physical system we are considering, but the sequence of \mathcal{H}_N 's will model the noncommutative version of \mathcal{M} . Observables on \mathcal{H}_N correspond to $N \times N$ matrices. Thus, we want to identify q_λ as the mode amplitudes for a matrix \hat{q} , with matrix elements \hat{q}_{ij} , expanded as

$$\hat{q}_{ij} = \sum_\lambda q_\lambda (T_\lambda)_{ij}, \quad i, j = 1, 2, \dots, N, \quad (2)$$

where $\{T_\lambda\}$ form a basis for $N \times N$ matrices. We can take this to be an orthonormal basis obeying $\text{Tr}(T_\lambda T_{\lambda'}) = \delta_{\lambda\lambda'}$. In the large N limit, the algebra of the $N \times N$ matrices should become the algebra of functions on \mathcal{M} , with T_λ corresponding to a complete set of mode functions. There are two ways to pass from $\{q_\lambda\}, \{p_\lambda\}$ to functions on \mathcal{M} . If \mathcal{M} is a compact Kähler manifold, which is mostly the case we will be considering, we can take a suitable multiple of the

Kähler form as a symplectic structure and carry out quantization to construct \mathcal{H}_N . In this case, there will be a set of orthonormal “wave functions” u_i which are holomorphic. Strictly speaking, these are sections of a suitable power of the canonical line bundle on \mathcal{M} . The set $\{u_i\}$ can also be viewed as coherent states on \mathcal{M} obtained via standard coherent state quantization. Given this structure, there is a function ϕ on \mathcal{M} such that the matrix elements \hat{q}_{ij} in (2) can be obtained as

$$\hat{q}_{ij} = \int d\mu u_i^* \phi u_j. \quad (3)$$

Here $d\mu$ denotes the volume element for \mathcal{M} . The function ϕ is the contravariant symbol for \hat{q}_{ij} and the prescription (3) is the Berezin-Toeplitz quantization of ϕ .

We will be sequentially generalizing the results from the Kähler manifold to more general backgrounds, so a few comments on $d\mu$ may be appropriate here. On a general complex manifold with local complex coordinates $z^a, \bar{z}^{\bar{a}}$, the metric is of the form $ds^2 = g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}$ and $d\mu = \det g [dz d\bar{z}]$. (If we use real coordinates, this is identical to the more familiar $d\mu = \sqrt{\det g} dx$.) For a Kähler space, we have the additional condition $g_{a\bar{a}} = \partial_a \partial_{\bar{a}} K$ for a Kähler potential K . For Kähler manifolds which are of the coset type, say of the form G/H for a Lie group G and subgroup H (which we consider later), one can still construct $d\mu$ as explained here, but it is also possible to use the Haar measure in formulas such as (3), since the integrand will be invariant under H . The volume of H factors out and can be absorbed into a normalization factor.

Returning to the main line of reasoning, if \hat{A}, \hat{B} are $N \times N$ matrices, then the function which gives the operator or matrix product $(\hat{A}\hat{B})_{ij}$ via (3) is the star product of the functions A and B corresponding to the individual matrices; i.e.,

$$(\hat{A}\hat{B})_{ij} = \int d\mu u_i^* (A * B) u_j. \quad (4)$$

The trace of a matrix \hat{A} can be written as

$$\text{Tr}(\hat{A}) = \sum_i \hat{A}_{ii} = \int d\mu \left[\sum_i u_i^* u_i \right] A = \int d\mu \rho A. \quad (5)$$

We see that $\rho = \sum_i u_i^* u_i$ defines a density to be used in the integration.

Using these formulas, we can convert terms in H (and the action) into integrals over the star products of various contravariant symbols. Thus,

$$\sum_{\lambda} p_{\lambda} p_{\lambda} = \sum_{\lambda, \lambda'} \text{Tr}(p_{\lambda} T_{\lambda})(p_{\lambda'} T_{\lambda'}) = \text{Tr}(\hat{\Pi} \hat{\Pi}) = \int d\mu \rho \Pi * \Pi, \quad (6)$$

where $\hat{\Pi}_{ij} = \sum_{\lambda} p_{\lambda} (T_{\lambda})_{ij}$. As an example, consider choosing a Hamiltonian of the form

$$H = \frac{1}{2} \text{Tr}[\hat{\Pi} \hat{\Pi} + \beta_1 [T_{\alpha}, \hat{q}][T_{\alpha}, \hat{q}] + m_0^2 \hat{q} \hat{q}] + g_0 \text{Tr}(\hat{q}^4), \quad (7)$$

where β_1, m_0 , and g_0 are arbitrary constants. The last two may be identified as the bare mass and bare coupling constant. The commutator $[T_{\alpha}, \hat{q}]$ stands for the matrix version of the derivative, T_{α} being a specific set of matrices. Since we have not specified exactly how the commutators translate to derivatives and since we may have to do some scaling of spatial coordinates, we must allow for an arbitrary coefficient β_1 for the $[T_{\alpha}, \hat{q}]^2$ term. Expression (7) leads to the field theory Hamiltonian

$$H(\Pi, \phi) = \int d\mu \rho \left[\frac{1}{2} (\Pi * \Pi + \alpha_1 (\nabla_{\alpha} \phi) * (\nabla_{\alpha} \phi) + m_0^2 \phi * \phi) + g_0 \phi * \phi * \phi * \phi \right]. \quad (8)$$

Here α_1 is the version of β_1 once we make the translation of the commutator $[T_{\alpha}, \hat{q}]$ to a derivative of the field. If star products are approximated by ordinary products, which may be reasonable as $N \rightarrow \infty$, then we get a familiar form of the Hamiltonian density integrated with $d\mu \rho$ as the volume element.

Returning to the transformation kernel in (1), we first use the product of a sequence of such kernels and integrate over the q 's to obtain the Hamiltonian path integral in the usual way,

$$\begin{aligned} Z &= \mathcal{N} \int [Dp Dq] \exp \left(i \int dt [p_{\lambda} \dot{q}_{\lambda} - H(p, q)] \right) \\ &= \int [Dp Dq] \exp \left(- \int \mathcal{A}(p, q) \right) \\ \mathcal{A} &= -i [p_{\lambda} \dot{q}_{\lambda} - H(p, q)] dt. \end{aligned} \quad (9)$$

Here \mathcal{A} is the Poincaré-Cartan form for the system under consideration and \mathcal{N} is a normalization factor. (We define \mathcal{A} to be anti-Hermitian to agree with the convention used later for the gauge fields.) Written in terms of symbols, this expression for the path integral reads

$$Z = \mathcal{N} \int [D\Pi D\phi] \exp \left(- \int d\mu \rho \mathcal{A} \right), \quad (10)$$

where we now have the symbol for \mathcal{A} , also written as \mathcal{A} , in the exponent. Rewriting this in terms of the individual symbols for p and q would require the star products. Thus, we can also write

$$Z = \mathcal{N} \int [D\Pi D\phi] \exp \left(i \int dt d\mu \rho [\Pi * \dot{\phi} - H(\Pi, \phi)] \right). \quad (11)$$

The second way of passing from matrices to functions is via the covariant symbol. Here we start from the matrix elements of an operator \hat{A}_{ij} and form a function (A) defined by

$$(A) = \sum_{ij} \mathcal{D}_i \hat{A}_{ij} \mathcal{D}_j^*, \quad \mathcal{D}_i = \frac{u_i}{\sqrt{N}}. \quad (12)$$

Notice that the covariant symbol in the above equation defines a function on \mathcal{M} given the matrix elements \hat{A}_{ij} , while the contravariant symbol is a function on \mathcal{M} which leads to the matrix elements via (3). In this sense, they are converses of each other, but the symbols are not identical in general. By appropriately using the completeness properties of the \mathcal{D} 's, one can again pass from a Hamiltonian as in (7) to the form (8), with Π, ϕ replaced by the covariant symbols $(\Pi), (\phi)$, and the star product should also be the one pertaining to the covariant symbols. While this method has been used in a number of applications (e.g., see [6,7]), for what follows, we shall mostly use the contravariant symbols, although we will give a more explicit formula for the covariant symbol for $\mathcal{M} = S^2$ later.

The passage from a matrix expression to functions (with star products) as in (8) has been well known for many years. But our aim here is to go beyond that and consider the situation where there are perturbations to the background gauge fields and the spin connection on \mathcal{M} , these being the data needed for constructing \mathcal{H}_N and u_i .

Second, \mathcal{H}_N and u_i are obtained as the LLL and the corresponding set of wave functions for a Landau-Hall problem on \mathcal{M} . So, we can apply the analysis to various fields coupled to the electrons which fill the LLL. (We are taking the fields to refer to observables restricted to the LLL only. Fields which have an existence outside of the Hall system will have additional terms in the Hamiltonian and the action.)

There is another reason why the LLL setting is useful. Given the N -dimensional vector space \mathcal{H}_N and matrices as linear transformations of \mathcal{H}_N , we need the u_i to define symbols and star products. This choice is not unique. Hence, the large N limit we obtain can be different for different choices. For example, the continuum limit may correspond to the symplectic structure $n\Omega_K$ (where Ω_K is the Kähler form) or a perturbation of it in the form $n\Omega_K + d(\delta\alpha)$ since both will lead to the same number of states, at least for large n . This is equivalent to different choices of the background gauge fields. Is there an optimal choice? This would require a criterion selecting a particular background (of gauge fields and geometry) and so it would be the key principle for gravity on noncommutative spaces [12].

The placement of the problem in the LLL context gives a simple calculational scheme to analyze such questions.

III. MATTER FIELDS AND GRAVITY AND THE LLL IN HIGHER DIMENSIONS

We start with the framework for the Landau levels and the setup of the $\nu = 1$ state. For this, we consider fields ψ, ψ^\dagger which represent the electron or the charged fermions. They are subject to a $U(1)$ background gauge field, i.e., the magnetic field, and we will consider the fully filled LLL for these, i.e., the $\nu = 1$ Hall state. From the point of view of noncommutative geometry, the LLL will define the Hilbert space \mathcal{H}_N which serves as the model for the noncommutative version of \mathcal{M} . Thus, the fields ψ, ψ^\dagger define the noncommutative spatial geometry. For this reason, and to avoid confusion with the fields ϕ introduced previously, we will refer to ψ, ψ^\dagger as the spatial fields. The set of fields ϕ will be referred to as matter fields.

Toward setting up the Landau levels and the $\nu = 1$ state, initially we will consider the spatial manifold to be S^2 , so that spacetime is $S^2 \times \mathbb{R}$ [4,15]. The background magnetic field which leads to the Landau levels is thus a uniform magnetic field on S^2 , corresponding to a magnetic monopole at the center if we consider the S^2 as embedded in three dimensions. The Hamiltonian for the ψ, ψ^\dagger fields has the form

$$H = \int d\mu(g) \psi^\dagger \left[\frac{R_+ R_-}{2mr^2} \right] \psi. \quad (13)$$

We will view S^2 as $SU(2)/U(1)$, so that we can use an $SU(2)$ group element g to coordinatize the spatial manifold, modulo the $U(1)$ identification. On this group element, viewed as a 2×2 matrix, one can define left and right translation operators via

$$L_a g = t_a g, \quad R_a g = g t_a, \quad (14)$$

where t_a are a basis for the generators of $SU(2)$ in the 2×2 matrix representation. They may be taken as $t_a = \frac{1}{2} \sigma_a$, where σ_a are the Pauli matrices. The operators $R_\pm = R_1 \pm iR_2$ appearing in the Hamiltonian (13) are thus translation operators on S^2 . Also $d\mu(g)$ denotes the Haar measure on $SU(2)$ with the normalization $\int d\mu = 1$. The volume on S^2 differs from the $SU(2)$ volume by the $U(1)$ factor. Since we will be considering integrands which are invariant under the $U(1)$ action, integration over this extra $U(1)$ is immaterial and we will use the full $SU(2)$ volume. r is a scale parameter, which may be viewed as the radius of S^2 .

The translation operators R_\pm can be identified as covariant derivatives, so that having a nonzero background magnetic field B is equivalent to the requirement that the fermion fields obey $[R_+, R_-] \psi = 2R_3 \psi = -n\psi = -2Br^2 \psi$. Here n is an integer in accordance with the Dirac quantization condition. The eigenmodes of $R_+ R_-$ take the form

$$U_A^{(q)} = \sqrt{2q+n+1} \mathcal{D}_{A, -\frac{n}{2}}^{(\frac{q}{2})}(g), \quad (15)$$

where $\mathcal{D}_{A,B}^{(j)}(g)$ are the representatives of g in the spin- j representation with A, B labeling states within the representation. They take values $1, 2, \dots, (2j+1)$. Further, q is a semipositive integer, with $q=0$ corresponding to the lowest Landau level. The fermion field ψ has the mode expansion

$$\psi = \sum_i a_i u_i + \sum_{q \neq 0} a_A^{(q)} U_A^{(q)}, \quad (16)$$

where we have separated out the LLL, with $u_i = \sqrt{n+1} \mathcal{D}_{i, -\frac{n}{2}}^{(\frac{n}{2})}$. In terms of the creation and annihilation operators a_i, a_i^\dagger , which obey the standard fermion algebra, the completely filled LLL state is given by

$$|w\rangle = a_1^\dagger a_2^\dagger \cdots a_N^\dagger |0\rangle, \quad (17)$$

with $N = n+1$.

There are N single particle states corresponding to the LLL. These are characterized by the wave functions u_i in (16). They form the basis for a one-particle Hilbert space \mathcal{H}_N , which models fuzzy S^2 . They can also be constructed directly without embedding them in the larger framework of Landau levels. In terms of the group element g , the Kähler forms are given by

$$\alpha_K = i \text{Tr}(t_3 g^{-1} dg), \quad \Omega_K = d\alpha_K = -i \text{Tr}(t_3 g^{-1} dg g^{-1} dg). \quad (18)$$

These define the canonical line bundle for $S^2 \sim \mathbb{C}P^1$. The n th power of the canonical line bundle has the curvature $\Omega = n\Omega_K$ and u_i are sections of this line bundle. They are holomorphic since they obey

$$R_- u_i = \sqrt{N} R_- \mathcal{D}_{i, -\frac{n}{2}}^{(\frac{n}{2})} = 0. \quad (19)$$

These are the coherent states obtained by straightforward quantization of $(S^2, n\Omega_K)$ with the holomorphic polarization.

Observables restricted to the LLL are $N \times N$ matrices, which, as mentioned in Sec. II, can be expanded in terms of the basis $\{T_\lambda\}$. Since we are considering S^2 , such a basis is provided by the matrix analogs of the spherical harmonics. These are given by

$$\{T_\lambda\} = \left\{ \frac{1}{\sqrt{N}}, \frac{T_a}{\sqrt{3j(j+1)(2j+1)}}, \dots \right\}, \quad (20)$$

with $j = \frac{n}{2}$. Thus, we have a series of tensor operators T_λ with $SU(2)$ angular momentum $l = 0, 1, \dots, n$. The series naturally terminates at $l = n$ for $N \times N$ matrices. We have chosen the normalization condition $\text{Tr}(T_\lambda T_{\lambda'}) = \delta_{\lambda\lambda'}$.

The symbols corresponding to these matrices become the usual spherical harmonics as $n \rightarrow \infty$. In this way, the space of functions on the LLL lead to the commutative algebra of functions on S^2 as $n \rightarrow \infty$, in accordance with the expected structure for fuzzy S^2 .

For this example of the fuzzy version of S^2 , we can specify the covariant symbol for \hat{A} more explicitly as

$$(\hat{A}) = \sum_{ik} \mathcal{D}_{i, -\frac{n}{2}}^{(\frac{n}{2})}(g) A_{ik} \mathcal{D}_{k, -\frac{n}{2}}^{(\frac{n}{2})*}(g). \quad (21)$$

This is clearly a function on S^2 . Notice that the normalized wave functions are $u_i = \sqrt{N} \mathcal{D}_{i, -\frac{n}{2}}^{(\frac{n}{2})}(g)$, in agreement with (12).

We also have an explicit formula for $d\mu\rho$. Notice that the integral of $d\mu\rho$ is N , the number of states, or the dimension of the LLL. Since they are the kernel of the antiholomorphic derivative as in (19), they are given by the integral of the Dolbeault index density. The appropriate formula for two dimensions is

$$\mathcal{I}_{\text{Dolb}} = i \left(\frac{F}{2\pi} + \frac{R}{4\pi} \right). \quad (22)$$

The $U(1)$ background gauge field we have chosen is

$$F = -in\Omega_K, \quad (23)$$

where the normalization is specified as $\int \Omega_K / (2\pi) = 1$. The curvature of S^2 is given by $R = -i2\Omega_K$ and it is easily verified that $\mathcal{I}_{\text{Dolb}}$ integrates to $N = n+1$. We may thus expect that, even at the level of the density, before integration, $d\mu\rho$ can be identified as the two-form $\mathcal{I}_{\text{Dolb}}$,

$$d\mu\rho = \mathcal{I}_{\text{Dolb}} = i \left(\frac{F}{2\pi} + \frac{R}{4\pi} \right). \quad (24)$$

This is confirmed by several independent arguments. The simplest way is to note that $\sum_i u_i^* u_i$ is the number density of the fermions in the fully filled LLL. This is essentially the charge density and so it is related to an effective action for the background fields as

$$\delta S_{\text{eff}} = \int d\mu(g) \rho (i\delta A_0), \quad (25)$$

where A_0 is the time component of the background gauge potential. (We take this to be anti-Hermitian to agree with the convention for the other fields.) It is well known that the effective action is of the form $-\int A(F+R)/(4\pi)$, as calculated by a number of authors, even allowing for variations from the fixed background values of $F = -in\Omega_K$, $R = -i2\Omega_K$ [3,4]. The result (24) is then straightforward. It is also easy to understand this intuitively,

at least for the $U(1)$ background. A change of the field by δA is equivalent to the change of symplectic structure as $\Omega \rightarrow \Omega + d(i\delta A)$. The volume element of phase space is then $iF = \Omega + d(i\delta A)$, and hence it is the appropriate density for the number of states in the semiclassical approximation. To recapitulate, the advantage of writing $d\mu\rho$ in terms of the index density is that it applies even with perturbations to the gauge field or the geometry, so long as we remain within the same topological class.

We can now combine this with the path integral from (10). The exponent in the path integral is the symbol for the Poincaré-Cartan form \mathcal{A} of (9). Consider the effective action in terms of the gauge field A and the spin connection ω ,

$$S_{\text{eff}}(A, \omega) = -\frac{1}{4\pi} \int (AdA + AR), \quad (26)$$

with $R = d\omega$. The path integral is then given by

$$Z = \mathcal{N} \int [D\Pi D\phi] \exp(iS_{\text{eff}}(A + \mathcal{A}, \omega)). \quad (27)$$

Strictly speaking, we should use $S_{\text{eff}}(A + \mathcal{A}, \omega) - S_{\text{eff}}(A, \omega)$, but the extra factor $e^{iS_{\text{eff}}(A, \omega)}$ is a constant as far as the integration over the matter fields is concerned and can be absorbed in the normalization factor for now. In fact, there is good reason to keep $S_{\text{eff}}(A, \omega)$ in (27), it will be relevant for the dynamics of gravity itself.

Turning to higher dimensions, consider as an example, $\mathbb{C}\mathbb{P}^2 \times \mathbb{R}$. The fuzzy version of $\mathbb{C}\mathbb{P}^2$ can be modeled by a Hilbert space \mathcal{H}_N which can be identified as the LLL of a Landau-Hall problem on $\mathbb{C}\mathbb{P}^2 \sim (SU(3)/U(2))$. One can choose a constant $U(2)$ background for the gauge field, proportional to the curvatures of $\mathbb{C}\mathbb{P}^2$. The wave functions are coherent states or the holomorphic sections of a suitable line or vector bundle of the form $\sqrt{N}\mathcal{D}_{k,w}^{(r)}(g) = \sqrt{N}\langle r, k|g|r, w\rangle$ which is the matrix representative of an $SU(3)$ element g in the representation labeled as r . The state $|r, w\rangle$ has to be chosen to ensure that the commutators of right translation operators on $\mathbb{C}\mathbb{P}^2$ reproduce the chosen background field strengths. The Dolbeault index density takes the form

$$\mathcal{I}_{\text{Dolb}} = \frac{1}{2} \left(\frac{iF}{2\pi} + \frac{iR^0}{2\pi} \right)^2 - \frac{1}{12} \left[\left(\frac{iR^0}{2\pi} \right)^2 + \frac{1}{2} \text{Tr} \left(\frac{i\bar{R}}{2\pi} \frac{i\bar{R}}{2\pi} \right) \right], \quad (28)$$

where $R^0 = d\omega^0$ and $\bar{R} = -it_a R^a$ are the $U(1)$ and $SU(2)$ curvatures, $t_a = \frac{1}{2}\sigma_a$. The path integral for matter fields takes the same form as (27), namely, as the integral of $\exp(iS_{\text{eff}}(A + \mathcal{A}, \omega))$, with S_{eff} given by [13]

$$S_{\text{eff}}(A, \omega) = \frac{i^3}{(2\pi)^2} \int \left\{ \frac{1}{3!} (A + \omega^0) [d(A + \omega^0)]^2 - \frac{1}{12} (A + \omega^0) \left[(d\omega^0)^2 + \frac{1}{2} \text{Tr}(\bar{R} \wedge \bar{R}) \right] \right\}. \quad (29)$$

The Poincaré-Cartan form should also be defined with the star products appropriate to fuzzy $\mathbb{C}\mathbb{P}^2$.

What has emerged from the arguments presented in this section is a simple prescription on how to couple the matter fields to the spatial fields or the gravitational background at the level of the effective action, namely, as in (27). The $U(1)$ gauge field in $S_{\text{eff}}(A, \omega)$ is shifted by the Poincaré-Cartan form \mathcal{A} (for matter dynamics) as $A \rightarrow A + \mathcal{A}$ and the functional integration is done over Π, ϕ . This is the key result of the analysis. For the rest of this paper, we will explore possible generalizations.

IV. GENERALIZING THE BACKGROUND GEOMETRY

First, we want to consider (27) in the context of three-dimensional gravity, starting at the level of the effective action. It has been known for a long time that the gravitational action in $(2+1)$ dimensions can be written as the integral of the difference of two Chern-Simons terms. Since this requires the consideration of more general backgrounds, we first consider a simplification of the matter field dynamics. From what has been discussed before, the integral of the Poincaré-Cartan form for a scalar field has the structure

$$- \int d\mu\rho [dt\Pi * \dot{\phi} - dtH] = \frac{1}{4\pi} \int (2F + R) [dt\Pi * \dot{\phi} - dtH], \quad (30)$$

where H is as given in (8). So far, we have used dimensionless coordinates, normalizing the volume of S^2 to 1. We restore the normal assignment of dimensions by the scaling

$$dx \rightarrow \frac{dx}{al}, \quad (31)$$

where l has the dimensions of length and a is a constant, to be fixed shortly. On a background such as $S^2 \sim \mathbb{C}\mathbb{P}^1$ with $F = -in\Omega_K$, and the Kähler form is normalized such that $\int \Omega_K / (2\pi) = 1$, we get

$$d\mu\rho = n i \frac{dzd\bar{z}}{(1 + \bar{z}z)^2} + R\text{-term} \rightarrow \frac{n}{a^2 l^2} i \frac{d^2x}{(1 + x^2/(a^2 l^2))^2}. \quad (32)$$

We choose $a^2 = n$ now and also define $r = al$ as the radius of the sphere. Then,

$$-\int d\mu\rho[dt\Pi*\dot{\phi}-dtH]\rightarrow\int dt\frac{d^2x}{(1+x^2/r^2)^2}\left[\frac{\Pi*\dot{\phi}}{l^2}-\frac{H}{l^2}\right]. \quad (33)$$

We now introduce a further scaling of the fields by writing $\Pi = l\tilde{\Pi}$, $\phi = l\tilde{\phi}$. The first term on the right-hand side of (33) becomes $\int dV\tilde{\Pi}\dot{\tilde{\phi}}$, where dV denotes the volume element. The integral of the Hamiltonian becomes

$$\int dtH\rightarrow\int dV\left[\frac{1}{2}(\tilde{\Pi}*\tilde{\Pi}+\alpha_1a^2l^2(\nabla_y\tilde{\phi})*(\nabla_y\tilde{\phi})+m_0^2\tilde{\phi}*\tilde{\phi})+\lambda_0l^2\tilde{\phi}*\tilde{\phi}*\tilde{\phi}*\tilde{\phi}\right]. \quad (34)$$

We can now choose $\alpha_1a^2l^2 = 1$ to set the spatial gradient term to the usual form. (This is equivalent to choosing a speed of light.) Further, we have to identify the m_0 and λ_0l^2 as the new bare mass and bare coupling constant. The Hamiltonian then takes the standard form, but with star products.

Let us now consider the higher terms in the star product involving gradients of the fields. The first corrections are of the form

$$\frac{R_+fR_-h}{n}\sim l^2\frac{R_+\tilde{f}R_-\tilde{h}}{n}, \quad (35)$$

where f, h could be Π or ϕ . The derivatives, as written, are dimensionless. After the scaling of coordinates as in (31), this takes the form

$$l^2\frac{R_+\tilde{f}R_-\tilde{h}}{n}\sim(\nabla_y\tilde{f})(\nabla_y\tilde{h})\frac{a^2l^4}{n}\sim\frac{(\nabla_y\tilde{f})(\nabla_y\tilde{h})}{M^4}, \quad (36)$$

where $M = l^{-1}$. Notice that, so far, the scale of M is not fixed by anything. So there is some freedom in choosing this. Since the number of states is n (at large n) and the volume of the spatial universe is $a^2l^2 = nl^2$, we see that l or M^{-1} determines the size of one elemental state for the spatial manifold.¹ The corrections from the star products are therefore negligible in a regime of energies low compared to M when the magnitudes of $\nabla\Pi/M^2$ and $\nabla\phi/M^2$ are small. In this limit, we can replace the Poincaré-Cartan form by

$$\mathcal{A} = -i[\Pi\dot{\phi} - dtH]dt, \quad (37)$$

where star products are neglected in the expression for H as well. This is a significant simplification which is helpful for

¹If we interpret this within a gravity theory, the Planck scale is a natural choice for this. But M could be somewhat smaller or larger, although in any realistic sense, M cannot be too low, since it determines the limit of resolution for points of the spatial manifold itself.

generalization for the following reason. The star product is specific to a particular background. Although it can be generalized to some extent, it is tied to having holomorphicity for the wave functions used to construct the operators from the contravariant symbols.² This is an obstruction to the framing of the present problem within the context of general gravitational backgrounds. However, for the simplified version in (37), it can be done if we are interested in low energy dynamics for the matter fields where the star products are not important.

On a general gravitational background (which does not necessarily have a complex structure), we cannot use the Dolbeault index density, rather we shall consider the Dirac index density. Our aim is to show that the Chern-Simons form associated with the Dirac index density will reduce to the effective actions (26) and (29) when a particular choice of background is made. Therefore, the Chern-Simons forms serve as the effective action to be used in (27) for a general gauge and gravitational background.

Toward showing this result, for the 2 + 1-dimensional case, we start with the Dirac index density in four dimensions which is given by

$$\begin{aligned} \mathcal{I}_{\text{Dirac}} &= -\frac{\dim V}{24}p_1 - \frac{1}{2}\text{Tr}\left(\frac{F}{2\pi}\frac{F}{2\pi}\right) \\ &= \left[\frac{\dim V}{48}\text{Tr}\left(\frac{R}{2\pi}\frac{R}{2\pi}\right) - \frac{1}{2}\text{Tr}\left(\frac{F}{2\pi}\frac{F}{2\pi}\right)\right]. \end{aligned} \quad (38)$$

In the second line of this equation, the curvatures are written in terms of the vector representation of $SO(4)$ so that $\text{Tr}(RR) = R^{ab}R^{ba}$, $a, b = 1, 2, 3, 4$; p_1 in the first line is the Pontrjagin class given by $p_1 = R^{ab}R^{ab}/(8\pi^2)$. Also, $\dim V$ is the dimension of the vector bundle or the dimension of F 's viewed as matrices. For an Abelian background field, which is our case, $\dim V = 1$. The Chern-Simons term corresponding to (38) is given by³

$$S_{\text{eff}} = \int \left[\frac{1}{96\pi}\text{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right) - \frac{1}{4\pi}AdA \right]. \quad (39)$$

We want to argue that (39) is the effective action (or at least part of it) for our problem on a general gauge and gravitational background. Toward this, we will now show that this does lead to (27) if we take the spacetime manifold to be $S^2 \times \mathbb{R}$. In this case, the spin connection has only the nonzero component $\omega^{ij} = ie^{ij}\omega$, defined by the zero torsion condition $de^i + \omega^{ij}e^j = 0$, where $e^i = (e^1, e^2)$ are the frame fields for S^2 . ($e^3 = dt$ will be the third frame field, for \mathbb{R} .) The action (39) now reduces to

²It is possible to define a star product for any Poisson manifold [16]. But it is not clear how to use it in the present context.

³Our normalization is $d(C.S.) = 2\pi \times (\text{Index density})$.

$$S_{\text{eff}} = \int \left[\frac{1}{48\pi} \omega d\omega - \frac{1}{4\pi} A dA \right]. \quad (40)$$

To compare this with the effective action for $S^2 \times \mathbb{R}$ as in (26), (27), two changes are needed. Here we are discussing spinors, while (26), (27) apply to scalars where we could use the Dolbeault index density. Spinors transform non-trivially as $\psi \rightarrow e^{i\sigma_3\varphi/2}\psi$ under local spatial rotations while scalars do not respond to rotations. So, the factor $e^{i\sigma_3\varphi/2}$ must be canceled out to get a proper comparison with the Dolbeault index density. This can be done by the shift $A \rightarrow A + \frac{1}{2}\omega$ in (40). (In other words, we can view the case of scalars as this particular choice of background fields for the spinors.) And second, we must make the replacement $A \rightarrow a + \mathcal{A}$ to include the matter coupling. With these changes, the action becomes

$$S_{\text{eff}} \left(A + \mathcal{A} + \frac{1}{2}\omega, \omega \right) = -\frac{1}{4\pi} \int \left[A dA + A R + \frac{1}{6} \omega d\omega \right] - \int \mathcal{A} \left(\frac{F}{2\pi} + \frac{R}{4\pi} \right). \quad (41)$$

The first set of terms agrees with the action obtained in [13], and the second set of terms agrees with the present discussion in (25)–(27). Thus, we have shown that the general action (39) reduces to (41) so that our earlier results for $S^2 \times \mathbb{R}$ can be viewed as a special choice of background fields. Returning to the general case, we see that the action for describing the path integral for matter fields takes the form of (27) with

$$S_{\text{eff}} = \int \left[\frac{1}{96\pi} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right) - \frac{1}{4\pi} (A + \mathcal{A}) d(A + \mathcal{A}) \right] = \int \left[\frac{1}{96\pi} \text{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right) - \frac{1}{4\pi} A dA \right] - \frac{1}{2\pi} \int A dA. \quad (42)$$

Turning to the 4 + 1-dimensional case, the Dirac index density in six dimensions is given by

$$\mathcal{I}_{\text{Dirac}} = \dim V \frac{iF}{2\pi} \left[\frac{1}{3!} \left(\frac{iF}{2\pi} \right)^2 - \frac{1}{24} p_1 \right] + \frac{1}{2} \left(\frac{iF}{2\pi} \right) \text{Tr} \left(\frac{i\bar{F}}{2\pi} \frac{i\bar{F}}{2\pi} \right). \quad (43)$$

Here F is the $U(1)$ field strength, \bar{F} is the $SU(2)$ background field, and $\dim V$ is the dimension of the representation used for \bar{F} . As before, p_1 is the Pontrjagin class. The Chern-Simons form or effective action corresponding to (43) is given by

$$S_{\text{eff}}(A, \omega) = i^3 \dim V \int \left[\frac{1}{3!(2\pi)^2} A F F + \frac{1}{24} A p_1 \right] + \frac{i^3}{8\pi^2} A \text{Tr}(\bar{F} \bar{F}). \quad (44)$$

Again, our first task will be to show that this will lead to the effective action (29) when we make a special choice of background fields. We consider the case of zero non-Abelian background, i.e., $\bar{F} = 0$, $\dim V = 1$. In reducing (44) to spatial states defined by the Landau-Hall problem for scalars on a background geometry of the form $\mathbb{C}\mathbb{P}^2 \times \mathbb{R}$, there are two requirements on the $U(1)$ field. First of all, since $\mathbb{C}\mathbb{P}^2$ does not admit a spin structure, the use of the Dirac index density is problematic. One can use a spin^c structure, which means that we should make a shift of the $U(1)$ as $A \rightarrow A + \frac{1}{2}\omega^0$. (This is equivalent to choosing a $U(1)$ charge of the form $n + \frac{1}{2}$ where n is an integer.) This shift reduces the problem to spinors on $\mathbb{C}\mathbb{P}^2$ (with a spin^c structure). (For a discussion of the Dirac index for $\mathbb{C}\mathbb{P}^2$, relevant for our analysis, see [17].)

To get to scalars, so that we can compare with the Dolbeault index, we need a further shift by $\frac{1}{2}\omega^0$ to compensate for the transformation of spinors under rotations. Thus, in total, we should use $A \rightarrow A + \omega^0$. Further, in terms of the $U(1)$ and $SU(2)$ curvatures, p_1 reduces to

$$\frac{p_1}{24} = \frac{1}{(2\pi)^2} \left[-\frac{1}{12} d\omega^0 d\omega^0 - \frac{1}{24} \text{Tr}(\bar{R} \bar{R}) \right]. \quad (45)$$

With $A \rightarrow A + \omega^0$ and this formula for p_1 , the effective action (44) becomes

$$S_{\text{eff}} = \frac{i^3}{(2\pi)^2} \int \left[\frac{1}{3!} (A + \omega^0) (dA + d\omega^0)^2 - \frac{1}{12} (A + \omega^0) \left[d\omega^0 d\omega^0 + \frac{1}{2} \text{Tr}(\bar{R} \bar{R}) \right] \right]. \quad (46)$$

This agrees with the effective action obtained in [13] using the Dolbeault index theorem and quoted in (29). The charge density which is the variation of S_{eff} with respect to A_0 gives the correct $d\mu\rho$ for this case, and so the prescription (27) for fuzzy $\mathbb{C}\mathbb{P}^2$ is recovered for the particular choice of background.

Having shown that (44) does indeed reproduce the results for $\mathbb{C}\mathbb{P}^2 \times \mathbb{R}$ (for the appropriate choice of the geometric quantities), we can take it as the form of the action for general, not necessarily complex Kähler, backgrounds. The shift by the Poincaré-Cartan form produces the matter part of the action

$$S_{\text{matter}} = \frac{1}{32\pi^2} \int (iA) \left[\dim V \left(F_{\mu\nu} F_{\alpha\beta} + \frac{1}{24} R_{\mu\nu}^{ab} R_{\alpha\beta}^{ab} \right) + \text{Tr}(t_a t_b) \bar{F}_{\mu\nu}^a \bar{F}_{\alpha\beta}^b \right] dx^\mu \cdots dx^\beta, \quad (47)$$

where we used the real field components defined by

$$F = (-i) \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu, \quad \bar{F} = (-it_a) \frac{1}{2} F_{\mu\nu}^a dx^\mu dx^\nu. \quad (48)$$

The key emerging feature is that the Lagrangian for matter fields is multiplied by a specific polynomial involving powers of the curvature. The term involving the $U(1)$ field, namely, $F_{\mu\nu} F_{\alpha\beta}$ is the dominant one at large n , but there are curvature-dependent subdominant terms. It is curious to note that such couplings of matter fields to gravity have been extensively investigated recently, partly motivated by their potential to explain observations related to dark matter. For recent reviews on the subject of curvature-matter couplings, see [14].

Finally, we can ask how to formulate the coupling of matter fields directly at the level of the fermion fields defining the spatial geometry. The relevant action is the Dirac action with nontrivial gauge and gravitational fields. The gauge group should have a $U(1)$ component. For example, in $4 + 1$ dimensions, we consider the action

$$S = \int \bar{\psi}(i\gamma \cdot D)\psi. \quad (49)$$

If a specific representation of the Dirac matrices is needed, we will use

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (50)$$

In (49), we are using a four-component spinor ψ which would correspond to one chiral component of an eight-spinor in six dimensions. This means that there will be a parity anomaly for this theory. The Hamiltonian corresponding to (49) is

$$H = \int \psi^\dagger [-i\gamma^0 \gamma^\mu D_\mu] \psi, \quad \mu = 1, 2, 3, 4. \quad (51)$$

The nonzero positive and negative eigenvalues of the four-dimensional Dirac operator $-i\gamma^0 \gamma^\mu D_\mu$ are paired, with the corresponding eigenfunctions ψ_n and $\gamma^0 \psi_n$. The zero modes are not paired and their number is what is given by the Dirac index in four dimensions. In defining the vacuum state and calculating the charge, the key question is whether the zero modes are to be considered as part of the Dirac sea, hence filled, or as part of the unoccupied positive

energy states. The charge conjugation transformation for the spinor ψ is defined by

$$\psi = C\phi^*, \quad C^{-1}\gamma^a C = \gamma^{a*}, \quad C = \gamma^2 \gamma^4, \quad (52)$$

where ϕ is the charge conjugate of ψ . The C -odd definition of the charge can be evaluated on the vacuum as

$$Q|0_\mp\rangle = \frac{1}{2} \int [\psi^\dagger \psi - \phi^\dagger \phi] |0_\mp\rangle = \mp \frac{1}{2} N |0_\mp\rangle, \quad (53)$$

where the upper sign corresponds to the zero modes being unoccupied, the lower to the case when they are occupied, and N is the number of zero modes given by the Dirac index. This result shows that the effective action for (49) should have a Chern-Simons term with a level number of $\mp \frac{1}{2}$. This leads to an inconsistency. (This is the well-known parity anomaly, spelt out for $4 + 1$ dimensions here.) A consistent theory requires using the Dirac action (49) with a Chern-Simons term with level number $\pm \frac{1}{2}$ added. Taking the first sign in (53), the resulting vacuum will have zero charge and will lead to an effective action equal to the Chern-Simons term in (29) for the fully occupied $\nu = 1$ state, i.e., for the state where all the zero modes are occupied. Thus, our results for the coupling of matter fields to the fermions characterizing the spatial manifold are summarized by the action

$$S = \int \bar{\psi}(i\gamma \cdot D - m)\psi + \frac{1}{2} S_{CS}(A + \mathcal{A}, \omega) \\ \gamma \cdot D = \gamma^a e_a^{-1\mu} \left(\partial_\mu + A_\mu + \mathcal{A}_\mu + \frac{1}{8i} \omega_\mu^{bc} [\gamma_b, \gamma_c] \right). \quad (54)$$

The mass m is a small positive number whose role is to shift the energies upward. The zero modes of the Hamiltonian (51) will thus have small positive energies, leaving them unoccupied and making the choice of the vacuum state as $|0_- \rangle$ in (53). (This is the only reason for m ; it can be taken to be infinitesimally small.) And S_{CS} in (54) is the Chern-Simons action of (44) with the shift $A \rightarrow A + \mathcal{A}$,

$$S_{CS}(A + \mathcal{A}, \omega) = \left[i^3 \dim V \int \left[\frac{1}{3!(2\pi)^2} A F F + \frac{1}{24} A p_1 \right] + \frac{i^3}{8\pi^2} A \text{Tr}(\bar{F} \bar{F}) \right]_{A \rightarrow A + \mathcal{A}}. \quad (55)$$

The effective action, for the state with the zero modes fully occupied, obtained from (54) will be S_{CS} as in (55).

The formulation of the problem as in (54) is the main result of this paper, in its most general form.

V. DISCUSSION

Our analysis started with a finite-, say N , dimensional Hilbert space of states which could be identified as the

lowest Landau level of a Landau-Hall problem or as the Hilbert space modeling a fuzzy space. Observables on such a space are $N \times N$ matrices. We considered the path integral for the dynamics of such observables, specifically something which approaches a continuum field theory as $N \rightarrow \infty$. The action which defines such a path integral is given by a Chern-Simons form which includes a $U(1)$ gauge field A which is shifted as $A \rightarrow A + \mathcal{A}$ by the (star product version of the) Poincaré-Cartan form \mathcal{A} for the matter fields. We then extended this to more general backgrounds arguing that the Dirac index density can be used to construct the relevant Chern-Simons form. As far as matter fields are concerned, the end result is an action of the form $\int \rho \mathcal{L}$ where \mathcal{L} is the Lagrangian and ρ is a density which is a polynomial in the gauge fields and the curvature as determined by the index density. It was argued in [13] that the effective action for background gauge fields and gravity for the Landau-Hall system is given by a Chern-Simons form associated with the Dolbeault index density. The present work incorporates matter couplings in such a framework and further extends it to more general geometries.

As mentioned after (27), $S_{\text{eff}}(A + \mathcal{A}, \omega)$ has a term $S_{\text{eff}}(A, \omega)$ which is not related to matter couplings. We retained it in (27) expecting that such terms can be absorbed into the gravitational part of the action. Regarding such purely gravitational terms, we note that one can define a class of gravity theories on odd-dimensional spacetimes with an action which is the difference of two Chern-Simons forms. We have argued elsewhere for the natural emergence of such a structure with an interpretation in the framework of thermofield dynamics [12]. The inclusion of matter couplings as discussed here within such a structure would be an interesting next step, which we propose to pursue in a later publication.

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