# Analog Hawking effect: A master equation

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We consider further the problem of the analog Hawking radiation. We propose a fourth order ordinary differential equation, which allows us to discuss the problem of Hawking radiation in analog gravity in a unified way, encompassing fluids and dielectric media. In a suitable approximation, involving weak dispersive effects, Wentzel-Kramers-Brillouin solutions are obtained far from the horizon (turning point), and furthermore an equation governing the behavior near the horizon is derived, and a complete set of analytical solutions is obtained also near the horizon. The subluminal case of the original fluid model introduced by Corley and Jacobson and the case of dielectric media are discussed. We show that in this approximation scheme there is a mode which is not directly involved in the pair-creation process. Thermality is verified and a framework for calculating the gray-body factor is provided.

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## I. INTRODUCTION

The analog Hawking effect has been largely discussed in literature, and we are interested in focusing our attention on the analytical side of calculations in the presence of dispersion. As is well known, the problem is very difficult and requires techniques borrowed from asymptotic analysis, see e.g., the following (nonexhaustive) list of papers [1–21]. Even if the mathematics to be adopted is quite similar, still different systems seem to require different tools to be discussed, and what is done for fluids is not just the same as for dielectric media. Even if a strictly unified framework *a priori* is not mandatory, still it is interesting to point out that such a framework exists and allows one to draw common conclusions for the various physical situations at hand, and to realize an universality for the analogous Hawking effect (see e.g., [6]).

In this paper, we propose a fourth order ordinary differential equation as a master equation allowing one to deal with the analogous Hawking effect in condensed matter systems in a systematic way, in the approximation of weak dispersive effects. This is *per se* interesting, because (i) a single master equation is shown to be enough for describing different physical situations. In this paper we deal with the subluminal version of the fluid model introduced by Corley and Jacobson [2,22], and also with the case of dielectric media. In the companion paper [23] we discuss also the case of the analogous Hawking effect in Bose-Einstein condensates (BEC) (superluminal case), and in water.

As a second element of interest, (ii) a single approximation is done, allowing one to reduce the problem into a form which is analogous to the one described in a series of works [24–28]. To be more specific, we adopt the limit of weak dispersive effects in all models (for the previous literature, concerning analytical calculations, see e.g., [8,14].) Furthermore, (iii) a new kind of near-horizon expansion (expansion near the turning point) is adopted, allowing one to get a completeness of states also in that physical region; in particular we can take into account explicitly the $-s$ modes (also called $v$ modes; see Sec. IV B) which are neglected in other near-horizon expansions. Moreover, the near-horizon equation one obtains is universal, i.e., it has the same form for all the models we take into account, and this is at the root of the universality of the Hawking effect in analog gravity.

Next, (iv) the nature of the horizon (turning point) is clearly emerging, and the role of both $v/c$ in the fluid models, and of the horizon equation $n = c/v$ (phase horizon) in the dielectric case are enhanced. Connection formulas allow one to calculate the fundamental ratio $|J_x^+|^2/|J_x^-|^2$, where $J_x^\pm$ stays for the (conserved) current associated with the dispersive modes of wave numbers $k_\pm$ ($k_-$ is associated with negative norm) (see Secs. IV B, IV C). As is well known, this ratio qualifies thermality of the Hawking analog radiation. Last, but not least, (v) one may also provide a general rule for the computation of the gray-body factor, which is in agreement with the analysis carried out in [8,14] as far as the Corley model is concerned, and that

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is extended to the dielectric model to be discussed herein. As general assumptions, in agreement with the aforementioned previous literature, we consider the situation where dispersive effects are mild and the relevant background fields like $v(x)$, $c(x)$ in the fluid models and $n(x)$ in the dielectric models are asymptotically constant and bounded. In a remarkable correspondence with the standard black hole case, the gray-body factor is simply due to "scattering on a barrier," provided by the geometry of the Hawking modes created in the region of the horizon, and is not directly associated with the presence of the horizon itself. The fourth wave number mode, a short wave number mode distinct from the Hawking mode, is then actually decoupled at the horizon.

## II. THE MASTER EQUATION: A ORR–SOMMERFELD TYPE FOURTH ORDER EQUATION

We show that three significant cases of wave equations in dispersive analog gravity can be reconduced to the equation

$$\epsilon^2 \frac{d^4\Phi}{dx^4} \pm \left[p_3(x,\epsilon)\frac{d^2\Phi}{dx^2} + p_2(x,\epsilon)\frac{d\Phi}{dx} + p_1(x,\epsilon)\Phi\right] = 0, \tag{2.1}$$

where the upper sign occurs in the case of subluminal dispersion and the lower one in the case of superluminal dispersion. The latter case is considered in Nishimoto's works (see e.g., [25] and references therein). Furthermore,

$$p_i(x,\epsilon) = \sum_{n=0}^{\infty} p_{\mathrm{in}}(x)\epsilon^n \tag{2.2}$$

is assumed. As $\epsilon \to 0$ one finds the so-called reduced equation

$$p_{30}(x)\frac{d^2\Phi}{dx^2} + p_{20}(x)\frac{d\Phi}{dx} + p_{10}(x)\Phi = 0. \tag{2.3}$$

Solutions of

$$p_{30}(x) = 0 \tag{2.4}$$

define the turning points (TPs) of the equation, and in the analysis of the reduced equation the behavior of solutions in the neighborhood of the TPs is of utmost relevance for the scattering problem we mean to delve into. In the following, we limit ourselves to the case of a single TP, to be identified with $x = 0$ without loss of generality. In [25] it is assumed that the reduced equation displays a Fuchsian singularity at the TP (nothing actually prevents the general equation in itself to admit a regular behavior). One may then expect two kinds of solutions:

$$\Phi^{(1)} = 1 + \sum_{n=1}^{\infty} d_n x^n, \tag{2.5}$$

$$\Phi^{(2)} = x^{1-\lambda}\left(1 + \sum_{n=1}^{\infty} e_n x^n\right), \tag{2.6}$$

where $\lambda$ is related to a root of the so-called indicial equation associated with the reduced equation in the neighbourhood of the TP. This kind of solution appears to be useful in the Wentzel-Kramers-Brillouin (WKB) approximation, which in our scheme, differently from the hypotheses in [24,25], can be extended to hold also in the asymptotic region of unboundedly large values of $x$. It is worth mentioning that the first solution above is regular at the turning point. This is relevant also in the following sections.

The great advantage of referring to the above equation is that sophisticated analytical calculations carried out mostly by [25] are just available, where a considerable effort has to be exploited in order to keep under control the asymptotic formulas and the associated connection formulas.

## III. A SUMMARY OF THE APPROXIMATION METHOD NEAR THE TURNING POINT

We sketch for the sake of completeness the essentials of the approximation method near the TP as described in [25], of which we maintain the same notation. The starting point consists in rewriting Eq. (2.1) as the first order system

$$\epsilon\frac{dY}{dx} = P(x,\epsilon)Y, \tag{3.1}$$

where

$$Y = \begin{pmatrix} y \\ y' \\ y'' \\ \epsilon y^{(3)} \end{pmatrix}, \tag{3.2}$$

and

$$P(x,\epsilon) = \begin{pmatrix} 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \\ \mp p_1(x,\epsilon) & \mp p_2(x,\epsilon) & \mp p_3(x,\epsilon) & 0 \end{pmatrix}, \tag{3.3}$$

where, again, the upper sign is relative to the subluminal case. The "stretching and shearing transformations"

$$x - a = \epsilon^{2/3}s, \tag{3.4}$$

$$Y = \Omega(\epsilon)W, \tag{3.5}$$

$$\Omega(\epsilon) := \mathrm{diag}\{\epsilon^{4/3}, \epsilon^{2/3}, 1, \epsilon^{1/3}\}, \tag{3.6}$$

where $a$ is the turning point, allow one to obtain

$$\frac{dW}{ds} = A(s,\epsilon)W, \tag{3.7}$$

where

$$A(s,\epsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mp p_1(x(s),\epsilon)\epsilon^{2/3} & \mp p_2(x(s),\epsilon) & \mp p_3(x(s),\epsilon)\epsilon^{-2/3} & 0 \end{pmatrix}, \tag{3.8}$$

and $x(s) = a + \epsilon^{2/3}s$. The functions $p_i(x,\epsilon)$ $(i = 1, 2, 3)$ can be expanded in power series of $\epsilon$ with coefficients which are polynomials of $x - a$ in the neighborhood of the TP, and in turn the matrix $A$ can be expanded in power series of $\epsilon^{1/3}$ with polynomial coefficients of $s$. Solutions are constructed in the form

$$W(s,\epsilon) = \sum_{i=0}^{\infty} W_i(s)\epsilon^{i/3}; \tag{3.9}$$

at the lowest order, $W_0(s)$ must satisfy

$$\frac{dW_0}{ds} = A_0(s)W_0, \tag{3.10}$$

where

$$A_0(s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \mp p_{20}(a) & \mp p'_{30}(a)s & 0 \end{pmatrix}. \tag{3.11}$$

Equation (3.10) is equivalent to the fourth order differential equation

$$\frac{d^4w}{dz^4} \pm \left(z\frac{d^2w}{dz^2} + \lambda\frac{dw}{dz}\right) = 0, \tag{3.12}$$

where

$$z = (p'_{30}(a))^{1/3}s = (p'_{30}(a))^{1/3}\epsilon^{-2/3}(x-a), \tag{3.13}$$

and

$$\lambda = \frac{p_{20}(a)}{p'_{30}(a)}. \tag{3.14}$$

A further corroboration of Eq. (3.12) is contained in Appendix A. Solutions to Eq. (3.12) are found by means of Laplace integrals

$$w_j(z) = \frac{1}{2\pi i}\int_{C_j} dt t^{\lambda-2} \exp\left(zt \pm \frac{1}{3}t^3\right), \tag{3.15}$$

with a suitable choice for the paths $C_j$ in the complex $t$ plane. See e.g., [25,27] for the superluminal case, where solutions of (3.15) are also known as generalized Airy functions. See Fig. 1 on the left side for paths $C_j$ adopted in [25,27], with $j = 1, \ldots, 6$.

It is interesting to deduce the solutions above directly, in order to point out the subtleties in solving (3.12). We first deduce solutions (3.15), by means of the Laplace-transform formalism: by putting

$$w_j(z) = \frac{1}{2\pi i}\int_{C_j} dt \phi(t)\exp(zt), \tag{3.16}$$

we find

$$\frac{1}{2\pi i}\int_{C_j} dt(t^4 + zt^2 + \lambda t)\phi(t)\exp(zt) = 0, \tag{3.17}$$

and, as usual, thanks to an integration by parts

$$\begin{aligned} &\frac{1}{2\pi i}\int_{C_j} dt z t^2\phi(t)\exp(zt) \\ &= \frac{1}{2\pi i} t^2\phi(t)\exp(zt)|_{C_j} \\ &- \frac{1}{2\pi i}\int_{C_j} dt\left[\frac{d}{dt}(t^2\phi(t))\right]\exp(zt), \end{aligned} \tag{3.18}$$

where $t^2\phi(t)\exp(zt)|_{C_j}$ is the variation of $t^2\phi(t)$ along $C_j$. One obtains solutions by putting

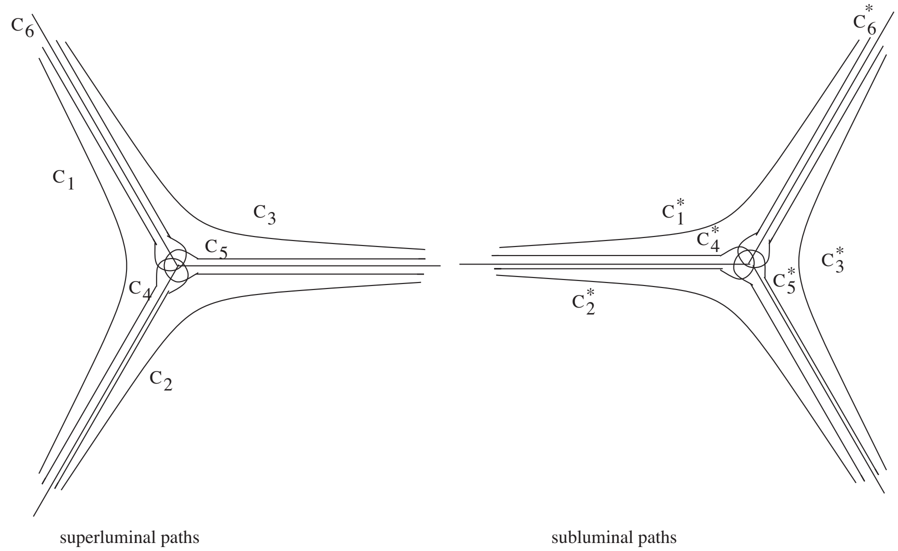


FIG. 1. Paths for the superluminal case (left side, see [25,27]), labeled with $C_j$, $j = 1, \ldots, 6$, and also for the subluminal case (obtained by a rotation of $-\pi/3$). An asterisk has been introduced for the paths in the latter case.

$$\frac{d}{dt}(t^2\phi(t)) = \pm\left(t^2 + \frac{\lambda}{t}\right)t^2\phi(t), \tag{3.19}$$

which provides us (3.15), and imposing also

$$t^\lambda \exp\left(zt \pm \frac{1}{3}t^3\right)\bigg|_{C_j} = 0. \tag{3.20}$$

The fourth solution, i.e., the constant solution, which is present in a trivial way as a solution of the original Eq. (3.12), seems to be quite "hidden" in the Laplace-transform formalism. Naively, it would seem that one could find it by a suitable choice of the path $C$ along which the complex integration is performed. For example, one might easily find the zero solution as a integral along any closed path nonintersecting the cut. Still, this reasoning is too naive. What is a bit difficult to realize is that the equation obtained by Laplace transform admits also a distributional solution: indeed, it can be rewritten as

$$t^2\frac{d\phi(t)}{dt} + (2t \mp (t^4 + \lambda t))\phi(t) = 0. \tag{3.21}$$

As a consequence, it is easy to show that

$$\phi(t) = \delta(t) \tag{3.22}$$

is a distributional solution, where $\delta(t)$ is the Dirac delta. By direct substitution, we get a first term which is $t^2\delta'(t)$, which is zero (cf. [29], Sec. 9.11, Eq. (4) ). At the same time, in a distributional sense, we get also $(2t \mp (t^4 + \lambda t))\delta(t) = 0$. Then, the constant solution arises in this framework as

$$w_C(z) = \int_C dt\delta(t)\exp(zt) = 1. \tag{3.23}$$

Note that we are allowed to put (cf. [8])

$$z = \mathrm{sign}(z)|z|, \tag{3.24}$$

as we are interested in real values of $z$ (and $x$).

Paths extending to infinity in the complex $t$ plane must be restricted to allowed regions. In the superluminal case ($-$ sign in front of the cubic term in the exponential), we have the same regions as for the Airy functions, with $\theta := \arg(t)$:

$$\theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right). \tag{3.25}$$

In the subluminal case ($+$ sign in front of the cubic term in the exponential), we find the complementary regions:

$$\theta \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right). \tag{3.26}$$

It is interesting to point out that one may select a basis of solutions. For example, for the superluminal case, we list the approximations of the solutions in the asymptotic region (large $z$) as determined in [25]:

$$w_1(z) = -\frac{e^{\lambda\pi i}}{2\sqrt{\pi}} z^{\frac{\lambda}{2}-\frac{5}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}(1 + O(z^{-\frac{3}{2}})), \qquad |\arg(z)| < \pi, \tag{3.27a}$$

$$w_2(z) = \frac{e^{-\lambda\pi i}}{2\sqrt{\pi}} z^{\frac{\lambda}{2}-\frac{5}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}(1 + O(z^{-\frac{3}{2}})), \qquad \frac{\pi}{3} < \arg(z) < \frac{7\pi}{3}, \tag{3.27b}$$

$$w_3(z)=\frac{i}{2\sqrt{\pi}}z^{\frac{\lambda}{2}-\frac{5}{4}}e^{\frac{2}{3}z^{\frac{3}{2}}}(1+O(z^{-\frac{3}{2}})),\quad -\frac{\pi}{3}<\arg(z)<\frac{5\pi}{3},\tag{3.27c}$$

$$w_4(z)=\frac{e^{\lambda\pi i}-e^{-\lambda\pi i}}{2\pi i}\Gamma(\lambda-1)z^{1-\lambda}(1+O(z^{-3})),\quad -\pi<\arg(z)<\frac{\pi}{3},\tag{3.27d}$$

$$w_5(z)=\frac{e^{\lambda\pi i}-e^{-\lambda\pi i}}{2\pi i}\Gamma(\lambda-1)z^{1-\lambda}(1+O(z^{-3})),\quad \frac{\pi}{3}<\arg(z)<\frac{5\pi}{3},\tag{3.27e}$$

$$w_6(z)=\frac{e^{\lambda\pi i}-e^{-\lambda\pi i}}{2\pi i}\Gamma(\lambda-1)z^{1-\lambda}(1+O(z^{-3})),\quad -\frac{\pi}{3}<\arg(z)<\pi.\tag{3.27f}$$

A basis of solutions is obtained by considering one of the following sets:

$$W_0^{(1)}\coloneqq\{1,w_6(z),w_3(z),w_1(z)\},\tag{3.28a}$$

$$W_0^{(2)}\coloneqq\{1,w_5(z),w_2(z),w_3(z)\},\tag{3.28b}$$

$$W_0^{(3)}\coloneqq\{1,w_4(z),w_1(z),w_2(z)\},\tag{3.28c}$$

for

$$\arg(z)\in\left(-\frac{\pi}{3},\pi\right),\tag{3.29a}$$

$$\arg(z)\in\left(\frac{\pi}{3},\frac{5\pi}{3}\right),\tag{3.29b}$$

$$\arg(z)\in\left(-\pi,\frac{\pi}{3}\right),\tag{3.29c}$$

respectively [27].

It is also easy to show that, both in the superluminal and in the subluminal case, by choosing suitably also the subluminal solutions, one finds

$$w_1(z)=\psi^{\lambda-1}w_3(\psi z)=\psi^{2(\lambda-1)}w_2(\psi^2 z),\tag{3.30a}$$

$$w_4(z)=\psi^{\lambda-1}w_6(\psi z)=\psi^{2(\lambda-1)}w_5(\psi^2 z),\tag{3.30b}$$

where

$$\psi\coloneqq e^{i\frac{2}{3}\pi}.\tag{3.31}$$

One may also notice that, by considering

$$\bar{w}_{\bar{j}}(z)\coloneqq(e^{-i\frac{\pi}{3}})^{\lambda-1}w_j(e^{-i\frac{\pi}{3}}z),\tag{3.32}$$

for $j=1,\ldots,6$, one may formally find basis sets also for the subluminal case. See Fig. 1, right side.

In the following sections, we shall exploit the aforementioned mathematical formalism in order to study two models for the analogous Hawking effect in condensed matter system. We shall consider two subluminal cases, represented by the Corley subluminal model and the Hopfield model for dielectric media. In the companion paper [23], we shall deal with the superluminal case represented by Bose–Einstein condensates, and also the further subluminal case represented by surface waves.

## IV. CORLEY MODEL: SUBLUMINAL CASE

We refer mainly to Corley in the subluminal case, which is considered in [2,8,14]. It represents the simplest model one can consider in this field, and, differently from e.g., BEC and water waves, to be discussed in the companion paper [23], it does not allow a variable speed of sound velocity $c(x)$. Furthermore, it cannot be related to the dielectric model which is discussed in the following section. As such, it is of limited physical interest, still we discuss it in our framework as a useful benchmark for our master equation and for our approximations. We shall not discuss the superluminal case for brevity. From the action

$$S=\frac{1}{2}\int d^2x\left[((\partial_t+v\partial_x)\phi)^2+\phi\frac{1}{k_0^2}\partial_x^4\phi\right]\tag{4.1}$$

displayed in [2,22] one obtains by separation of variables the fourth order ordinary differential equation

$$\frac{1}{k_0^2}\partial_x^4\varphi+\left(1-\frac{v^2(x)}{c^2}\right)\partial_x^2\varphi+2\frac{v(x)}{c^2}(i\omega-v'(x))\partial_x\varphi - i\frac{\omega}{c^2}(i\omega-v'(x))\varphi=0,\tag{4.2}$$

where $v(x)$ is the velocity field and $v'(x)$ stands for its first derivative with respect to $x$, and we have restored the (constant) sound velocity $c$. In order to reproduce the features of the master equation above, one must consider the following choice of the scale parameter: we assume as a significant physical scale, as in [2,22], the scale $k_0$ associated with the nonlinearity. By defining the dimensionless parameter[1]

$$\epsilon\coloneqq\frac{\omega}{ck_0},\tag{4.3}$$

[1]It might be questioned such a choice of expansion parameter, as other choices could appear as more natural, e.g., one could consider $\kappa$ [cf. (4.9)] in place of $\omega$. Still, it can be verified that in the error estimates like e.g., in (4.20) nothing substantial changes.

and the dimensionless coordinate $\xi = x\omega/c$, (4.2) becomes (with a small abuse of notation)

$$\epsilon^2\partial_\xi^4\varphi + \left(1-\frac{v^2(\xi)}{c^2}\right)\partial_\xi^2\varphi + 2\frac{v(\xi)}{c}\left(i-\frac{v'(\xi)}{c}\right)\partial_\xi\varphi + \left(1+i\frac{v'(\xi)}{c}\right)\varphi = 0. \tag{4.4}$$

Assuming that $k_0 \gg \omega/c$, we get $0 < \epsilon^2 \ll 1$. Moreover, we have

$$p_3(\xi,\epsilon) = 1-\frac{v^2(\xi)}{c^2} = p_{30}(\xi), \tag{4.5a}$$

$$p_2(\xi,\epsilon) = 2\frac{v(\xi)}{c}\left(i-\frac{v'(\xi)}{c}\right) = p_{20}(\xi), \tag{4.5b}$$

$$p_1(\xi,\epsilon) = 1+i\frac{v'(\xi)}{c} = p_{10}(\xi). \tag{4.5c}$$

There is no higher order contribution to the coefficients for this specific model (which is actually exceptional from this point of view). This is not true in the case of the other models we take into consideration herein and in the companion paper. We remark that the expansion parameter (4.3) defining our limit of weak dispersion is the same as in [8,14].

### A. The reduced equation

We notice that, in the limit $\epsilon \to 0$, one obtains the reduced equation, which we express in the original coordinates

$$\left(1-\frac{v^2(x)}{c^2}\right)\partial_x^2\varphi + 2\frac{v(x)}{c^2}(i\omega - v'(x))\partial_x\varphi - i\frac{\omega}{c^2}(i\omega - v'(x))\varphi = 0, \tag{4.6}$$

and, accordingly to [2], we assume $v(x) \le 0$, so that the TP coincides with the solution of

$$v(x) + c = 0. \tag{4.7}$$

In the neighborhood of the TP we have

$$v(x) \simeq -c + \kappa x, \tag{4.8}$$

where

$$\kappa \coloneqq v'(x=0). \tag{4.9}$$

The region where this approximation holds is called linear region henceforth. Notice that this is purposefully the same denomination as e.g., in [8].

The indicial equation for Eq. (4.6) provides a vanishing root $\alpha_1 = 0$ and a nonvanishing one $\alpha_2 = i\frac{\omega}{c\kappa}$, so that, being $\lambda = 1-\alpha_2$, one gets

$$\lambda = 1 - i\frac{\omega}{\kappa}, \tag{4.10}$$

which is not an integer number for any $\omega > 0$.

### B. WKB approximation

By now, we assume $x > 0$, i.e., $|v| < c$, which means that the external region is taken into account. We put

$$\varphi(\xi) = \exp\left(\frac{\theta(\xi)}{\epsilon}\right)\sum_{i=0}^{\infty}\epsilon^i y_i(\xi), \tag{4.11}$$

and refer e.g., to the presentation given in [30]. To the lowest order, we obtain

$$\theta'^4 + \left(1-\frac{v^2}{c^2}\right)\theta'^2 = 0, \tag{4.12}$$

whose solutions are $\theta' = 0$ (multiplicity two), and

$$\theta'_\pm = \pm i\sqrt{1-\frac{v^2}{c^2}}. \tag{4.13}$$

Notice that, for $x < 0$, being $|v| > c$, we obtain an exponentially increasing solution (called growing mode in [8]), and a decaying solution. We first take into account the latter solutions, and associate to them the so-called transport equation

$$\theta'^2(6\theta'' y_0 + 4\theta' y_0' + \theta'^2 y_1) + \left(1-\frac{v^2}{c^2}\right)(\theta'' y_0 + 2\theta' y_0' + \theta'^2 y_1) + 2\frac{v}{c}\left(i-\frac{v'}{c}\right)\theta' y_0 = 0, \tag{4.14}$$

and the next-to-leading-order equation

$$\theta'^2(6\theta'' y_1 + 4\theta' y_1' + \theta'^2 y_2) + \left(1-\frac{v^2}{c^2}\right)(\theta'' y_1 + 2\theta' y_1' + \theta'^2 y_2) + 2\frac{v}{c}\left(i-\frac{v'}{c}\right)\theta' y_1 + 5\theta'^2 y_0'' + \left(12\theta'\theta'' + 2\frac{v}{c}\left(i-\frac{v'}{c}\right)\right)y_0' + \left(3\theta''^2 + 4\theta'\theta'' + 1 + i\frac{v'}{c}\right)y_0 = 0. \tag{4.15}$$

Going back to the original coordinates, we find the solutions

$$\varphi_{\pm}(x) = C\left(\frac{1}{1-\frac{v^2(x)}{c^2}}\right)^{3/4} \exp\left(\pm\frac{i}{\epsilon}\frac{\omega}{c}\int^x ds\sqrt{1-\frac{v^2(s)}{c^2}}\right)\exp\left(i\frac{\omega}{c}\int^x ds\frac{v(s)}{c}\frac{1}{1-\frac{v^2(s)}{c^2}}\right)$$
$$\times\left(1+\epsilon C_1 \pm \frac{\epsilon}{2}\frac{c}{\omega}\int^x ds\frac{1}{i(1-\frac{v^2(s)}{c^2})^{\frac{3}{2}}}\left[\frac{1}{1-\frac{v^2(s)}{c^2}}\psi_1(s)+\psi_2(s)\right]+O(\epsilon^2)\right), \tag{4.16}$$

where

$$\psi_1(s) = (i2\omega + 3v'(s))\frac{v^2(s)}{c^4}\left(\frac{15}{4}v'(s)+\frac{3}{4}i\omega\right) + \frac{v^2(s)v'^2(s)}{c^4}, \tag{4.17}$$

$$\psi_2(s) = \frac{\omega^2}{c^2} - 4i\frac{\omega v'(s)}{c^2} - \frac{7}{2}\frac{v'^2(s)+v(s)v''(s)}{c^2}. \tag{4.18}$$

Omitting the terms of order $\epsilon$, the solutions correspond to the high wave number $k_{\pm}$ solutions appearing in [2,8]. $C$ is a normalization constant which, as in [2], we can put equal to one. $C_1$ is a second integration constant that also can be considered of order one. The $O(\epsilon)$ terms allow us to determine the conditions under which our approximations remain good. Since the integral diverges when $x \to 0$, this approximation fails at the TP. Still, it is assumed to hold in the linear region. When $x \to \infty$, the integral part of the order $\epsilon$ terms goes like

$$\sim \mp i\frac{1}{2}\frac{1}{(1-\frac{v_r^2}{c^2})^{\frac{3}{2}}}\left(1-\frac{3}{2}\frac{v_r^2}{c^2}\frac{1}{1-\frac{v_r^2}{c^2}}\right)\frac{\omega^2}{c^2}\frac{x}{k_0}, \tag{4.19}$$

where $-c < v_r < 0$ is the value assumed by $v$ far from the TP. We observe that the validity of the approximation requires

$$x \ll k_0\frac{c^2}{\omega^2}\left(1-\frac{v_r^2}{c^2}\right)^{\frac{5}{2}} = \frac{1}{\epsilon}\frac{c}{\omega}\left(1-\frac{v_r^2}{c^2}\right)^{\frac{5}{2}}. \tag{4.20}$$

Since, as in [2], we are interested in very low frequencies, $\omega \sim 0$; this is not a strong restriction at all, at least if the asymptotic velocity is not too close to $-c$.

It is also interesting to write the leading terms of $\varphi_{\pm}(x)$ in the linear region:

$$\varphi_{\pm}(x) \simeq \left(\frac{2\kappa}{c}x\right)^{-3/4} x^{-\frac{i\omega}{2\kappa}} \exp\left(\pm\frac{i}{\epsilon}\frac{2}{3}\sqrt{\frac{2\kappa}{c}}x^{\frac{3}{2}}\right). \tag{4.21}$$

Two further solutions occur when $\theta' = 0$ can be obtained from the reduced equation. The corresponding momenta are indicated, for a better comparison with [2], as $k_{\pm s}$ (in literature one finds also the following correspondence: $k_{+s} \mapsto k_u$, $k_{-s} \mapsto k_v$). In order to maintain the same order of approximation in our WKB expansion, one would need exact solutions, in order to avoid the introduction of a further expansion parameter. Nevertheless, we can appeal to the general features of the equation itself. Indeed, we obtain near the regular singular point $x = 0$ (our TP) the following series expansions:

$$\varphi_{-s}(x) = 1 + \sum_{n=1}^{\infty} c_n x^n, \tag{4.22}$$

$$\varphi_{+s}(x) = x^{i\frac{\omega}{\kappa}}\left(1+\sum_{n=1}^{\infty} d_n x^n\right). \tag{4.23}$$

By comparing, as in [2,8], the behavior of the above four solutions in the linear region where (4.8) holds, with the solutions near the TP (to be discussed in the following subsection), one finds both thermality and the gray-body factor.

It is useful to provide approximate solutions of the reduced equation even for large $x$ (in the external region with respect to the black hole). It is easy to show that for large $x$ in the above sense we have $v(x) \sim$ const, and then $v' = 0$. As a consequence, e.g., under the conditions of theorem 1.9.1 of [31], we get as $x \to \infty$

$$\varphi_{-s}(x) \sim \exp\left(-i\omega\frac{1}{c-v_r}x\right), \tag{4.24}$$

$$\varphi_{+s}(x) \sim \exp\left(i\omega\frac{1}{c+v_r}x\right), \tag{4.25}$$

and this completes our asymptotic basis of solutions together with $\varphi_-(x)$ and $\varphi_+(x)$. As useful interpolating formulas (WKB-like, but they cannot be rigorously obtained by using the $\epsilon$ expansion as in the above framework) we could also use

$$\varphi^{\rm int}_{-s}(x) \sim \exp\left(-i\omega\int^x dy\frac{1}{c-v(y)}\right), \tag{4.26}$$

$$\varphi^{\rm int}_{+s}(x) \sim \exp\left(i\omega\int^x dy\frac{1}{c+v(y)}\right), \tag{4.27}$$

which still display the correct behavior both in the linear region and in the asymptotic one.

For $x < 0$, the reduced equation provides us two further solutions

$$\varphi_d(x) = 1 + \sum_{n=1}^{\infty} e_n x^n, \tag{4.28}$$

$$\varphi_l(x) = x^{\frac{i\omega}{\kappa}}\left(1 + \sum_{n=1}^{\infty} f_n x^n\right), \tag{4.29}$$

with the asymptotic behavior

$$\varphi_d(x) \sim \exp\left(-i\omega \frac{1}{c - v_l} x\right), \tag{4.30}$$

$$\varphi_l(x) \sim \exp\left(i\omega \frac{1}{c + v_l} x\right), \tag{4.31}$$

with $\lim_{x\to-\infty} v(x) =: v_l < -c < 0$. These solutions correspond to left-moving modes in the superluminal region, and they are the only propagating modes in that region. We notice that the mode $\varphi_l(x)$ is a negative-norm mode. We remark that the modes we obtain are the same as in [2,8], albeit obtained through a different approach to the WKB approximation. This confirms also at this level the performance of our general framework.

### C. Approximation near the turning point

We first point out that, for the present case, we have[2]

$$z = \left(\frac{2\kappa}{c}\right)^{1/3} \epsilon^{-2/3} x, \tag{4.32}$$

and we choose to construct directly the relevant physical states by exploiting the method of the steepest descents [32–34]. The analysis proceeds as in the original paper by Corley [2], with the relevant difference that a different parameter of the asymptotic expansion is proposed (e.g., in [2] the nonlinearity scale $k_0$, which plays a fundamental role in our analysis, is put equal to one); furthermore, the near horizon equation allows one to take into account the $-s$ mode, albeit in the form of a constant solution, which still matches the WKB behavior in the matching region. A different but rigorous tool for evaluating the branch cut contribution (see below) is exploited. In previous literature, starting from [2], the so-called boundary condition for the subluminal case required a decaying mode beyond the horizon ($x < 0$), described by a path in the complex plane that can be deformed into the ones in the external region. The mathematical root of this condition will be discussed below.

From a physical point of view, this condition fixes the relative amplitudes of the involved modes near the turning point. Strictly speaking, it is not a boundary condition in itself, but it indicates how the modes involved in the process at hand actually participate to the process itself. Figure 2 amounts to the diagram introduced by Corley [2], in which the homotopic deformation of the decaying mode for $x < 0$ gives the modes with momenta $k_+$, $k_-$, $k_{+s}$ appearing in the external region ($x > 0$). They represent the high momentum incoming modes $k_\pm$, one of which having negative norm ($k_-$), and the outgoing Hawking mode $k_{+s}$. Since these modes must implement such a diagram near the TP, they participate with the same relative amplitude to the scattering process. The fourth mode, i.e., the short momentum regular mode, $k_{-s}$, corresponds to a solution in the Laplace space (or, equivalently, in the Fourier space) that is of different nature. Therefore, it cannot be included in the diagram as a mode resulting from the homotopic deformation of the decaying mode. With respect to the analysis carried out in [8,14], which is instead based on a Fourier transform analysis, the diagrams involved in the Hawking effect are the same as in Fig. 2. In order to match the WKB solutions, we are interested in an asymptotic expansion for large $z$ (notice that this can be obtained also by leaving $x$ suitably small in order to allow the linear approximation to hold true).

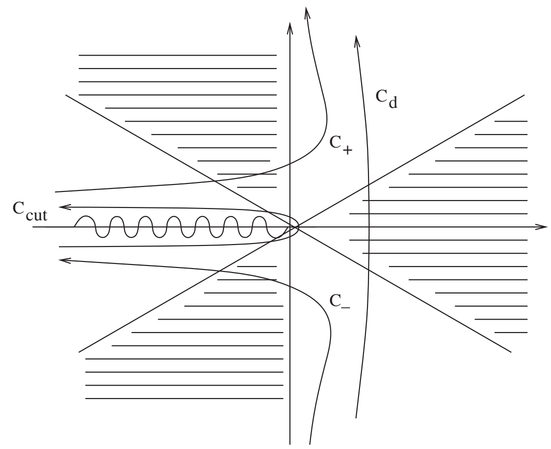


FIG. 2. Paths used in the subluminal case in Corley's work [2]. $C_\pm$ correspond to the dispersive modes, $C_{\rm cut}$ to the Hawking mode, and $C_d$ to the decaying mode. The last mode is the one in the inner region $x < 0$. As remarked by Corley [2], $C_d$ can be deformed in the paths $C_+$, $C_-$, $C_{\rm cut}$.

The $k_\pm$ contribution can be evaluated by means of the saddle point approximation, as well as the aforementioned decaying mode. We need the following formal expression:

$$w_j(z) = \frac{1}{2\pi i}\int_{C_j} dt t^{\lambda-2} \exp\left(zt + \frac{1}{3}t^3\right), \tag{4.33}$$

which, by putting $t = \sqrt{|z|}u$, can be rewritten as

[2]We use (3.13) working with the coordinate $x$ in place of $\xi$.

$$w_j(z) = \frac{1}{2\pi i}|z|^{\frac{\lambda-1}{2}} I_j(z), \quad (4.34)$$

where

$$I_j(z) = \int_{\bar{C}_j} du g(u) \exp(|z|^{3/2} h_\pm(u)), \quad (4.35)$$

and

$$g(u) \coloneqq u^{\lambda-2}, \quad (4.36)$$

$$h_\pm(u) \coloneqq \pm u + \frac{u^3}{3}; \quad (4.37)$$

here $\pm = \mathrm{sign}(x)$ and $C_j$ are the paths defined in [2]. Compare also with [8]. We have for $x < 0$ the decaying mode passing through the saddle point $u = 1$

$$w_{\mathrm{decaying}}(z) \simeq \frac{1}{2\sqrt{\pi}} |z|^{-\frac{i\omega}{2\kappa}-\frac{3}{4}} e^{-\frac{2}{3}|z|^{3/2}}. \quad (4.38)$$

The other saddle point $u = -1$ corresponds to the growing mode (which diverges at infinity), whose coefficient in the scattering matrix is zero (cf. e.g., [8,21]).[3]

For $x > 0$ we have the modes $k_\pm$ in correspondence with the steepest descents passing through the saddle points $u_\pm = \pm i$, and we get

$$w_+(z) \simeq \frac{1}{2\sqrt{\pi}} e^{-\frac{3}{4}\pi i} e^{\frac{\pi\omega}{2\kappa}} |z|^{-\frac{i\omega}{2\kappa}-\frac{3}{4}} e^{i\frac{2}{3}|z|^{3/2}}, \quad (4.39)$$

$$w_-(z) \simeq \frac{1}{2\sqrt{\pi}} e^{\frac{1}{4}\pi i} e^{-\frac{\pi\omega}{2\kappa}} |z|^{-\frac{i\omega}{2\kappa}-\frac{3}{4}} e^{-i\frac{2}{3}|z|^{3/2}}. \quad (4.40)$$

It is nice to notice that, thanks to relation (4.32), the amplitude of the decaying mode and of the $k_\pm$ modes above are proportional to $\sqrt{\epsilon}$ and then vanish as $\epsilon \to 0$, as expected. We can also provide a bound on the error occurring in neglecting higher order contributions to the saddle point approximation. Following e.g., [32] we find

$$x^{3/2} \gg \frac{1}{k_0}\sqrt{\frac{c}{2\kappa}\frac{1}{8}\left(\frac{1681}{36} + 4\frac{\omega^4}{\kappa^4} + \frac{110}{3}\frac{\omega^2}{\kappa^2}\right)^{1/2}}. \quad (4.41)$$

As to the ratio $\omega/\kappa$, it is known that the Hawking effect is mostly peaked for $\omega \simeq \kappa$. As is well known, there is also a maximal value of $\omega$ beyond which no Hawking effect occurs. See the following subsection for more details.

As to the cut contribution, it represents the Hawking mode, as is well known. It is remarkable that the branch cut lies along the steepest descent. Indeed, we have that for the subluminal case the imaginary part of $u + u^3/3$ is $b(1 + a^2 - b^2/3)$, where $u = a + ib$. As a consequence, $b = 0$ is a steepest descent line. This allows us to compute the cut contribution along the lines suggested in [34], chapter 4, section 4. 8, finding thus

$$w_{\mathrm{cut}}(z) \simeq -\frac{1}{i\pi}|z|^{i\frac{\omega}{k}} \Gamma\left(-i\frac{\omega}{\kappa}\right) \sinh\left(\frac{\pi\omega}{\kappa}\right), \quad (4.42)$$

which coincides (apart for the factor $2\pi i$ we introduced) with the approximation given in [2], but on more rigorous grounds. This result is compatible with the analogous one obtained in [8], with the difference that the Fourier transform formalism is adopted and a dominated convergence must be used therein.

For $x < 0$, it is easy to realize that the constant solution still appears. And one may also simply consider the contribution (4.42) by choosing a suitable analytical continuation for $x < 0$. It turns out that, by choosing the branch where $-1 = e^{-i\pi}$, the further solution one obtains,

$$w_{\mathrm{cut}-l}(z) \colon \simeq -\frac{1}{i\pi} e^{\pi\frac{\omega}{\kappa}} z^{i\frac{\omega}{k}} \Gamma\left(-i\frac{\omega}{\kappa}\right) \sinh\left(\frac{\pi\omega}{\kappa}\right), \quad (4.43)$$

is such that it corresponds to the Hawking partner, living on a different branch (cf. also [2]); furthermore, one is enabled to obtain the so-called mode which straddles the horizon [36]. See the following subsection.

### D. Matching: Complete solutions

A careful comparison with the WKB expansion displayed in the previous section provides us the connection formulas (cf. the so-called central connections in [25]). It has to be remarked that, as a consequence of the Corley's black hole boundary condition, in the external region near the turning point we have

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) + \phi_3(x,t) + h\phi_4(x,t), \quad (4.44)$$

with $\phi_1 \mapsto w_+$, $\phi_2 \mapsto w_-$, $\phi_3 \mapsto w_{\mathrm{cut}}$, $\phi_4 \mapsto 1$ and where $h$ remains undetermined by adopting the diagram of Fig. 2. Therefore, the modes corresponding to $w_\pm, w_{\mathrm{cut}}$ enter with the same amplitude in the scattering matrix. Instead, for the fourth constant mode, room is left for a different amplitude, as indicated by the factor $h$ in front of it. Eventually, $h$ might even be set equal to zero; see also the discussion below Eq. (4.46). From a mathematical point of view, the solutions $w_\pm, w_{\mathrm{cut}}, w_{\mathrm{decay}}$, for $z = 0$, where they are regular, as a consequence of Cauchy's theorem, satisfy

$$w_+(0) + w_-(0) + w_{\mathrm{cut}}(0) = w_{\mathrm{decay}}(0). \quad (4.45)$$

For what concerns strictly the problem of fixing the relative amplitudes of the respective modes, this amounts to the

[3]The growing mode could still give some contributions in other context, see e.g., [35]. We thank the anonymous referee for pointing this out.

above boundary condition stated by considering modes on different sides of the real turning point. Condition (4.45) works as well as the original condition by Corley.

A complete description of the matching is described in Appendix B. By comparing with the WKB solutions again in the matching region we find

$$\begin{aligned}\phi(x,t) &= e^{-\frac{3}{4}\pi i}\frac{e^{\frac{\pi\omega}{2\kappa}}}{2\sqrt{\pi}}\left(\frac{2\kappa}{c}\right)^{-\frac{i\omega}{6\kappa}+\frac{1}{2}}\epsilon^{\frac{i\omega}{3\kappa}+\frac{1}{2}}\varphi_{+}(x,t)\\ &+ e^{\frac{1}{4}\pi i}\frac{e^{-\frac{\pi\omega}{2\kappa}}}{2\sqrt{\pi}}\left(\frac{2\kappa}{c}\right)^{-\frac{i\omega}{6\kappa}+\frac{1}{2}}\epsilon^{\frac{i\omega}{3\kappa}+\frac{1}{2}}\varphi_{-}(x,t)\\ &- \frac{\sinh(\frac{\pi\omega}{\kappa})}{\pi i}\Gamma\left(-\frac{i\omega}{\kappa}\right)\left(\frac{2\kappa}{c}\right)^{\frac{i\omega}{3\kappa}}\epsilon^{-\frac{2i\omega}{3\kappa}}\varphi_{+s}(x,t)\\ &+ h\varphi_{-s}(x,t). \end{aligned} \tag{4.46}$$

$h$ is still undetermined. The fact that the fourth mode $\varphi_{-s}(x,t)$ is not involved in the Corley's black hole boundary condition, suggests the following interpretation: it does not participate in the process of Hawking particle production very near the TP, but it still might participate at a subsequent stage when scattering on the geometry depletes the flux of Hawking particles by "barrier reflection." This is what consistently appears to hold true for the model at hand, as also a direct calculation of the emitted flux confirms.

In literature, there exist two models where the fourth mode appears in the Corley diagram, see [16,18], where the fourth mode solution near the horizon has the same functional dependence of the other three solutions, and $h=1$ occurs. Homotopic deformation from the decaying mode involves also the fourth mode, and its direct contribution to the gray-body factor appears [16]. On the grounds of the comparison with these models, being the fourth mode not present in the Corley's diagram, the absence of the mode leaves $h$ undetermined; we can also infer that there is no contribution to $h$ for what strictly concerns the pair-creation process at least at the leading order. We shall discuss the problem further in the following.

As to the modes $d$, $l$ in the black hole region, their matching is analogous to the one described above. The mode $d$, as discussed above, may be considered, together with its counterpart $-s$ on the external side of the horizon, a single mode representing the particle entering the hole, and, as such, it passes without any relevant effect. Of course, it can also participate in the whole scattering process for Hawking particles as the backward mode originated from scattering on the geometry of Hawking particles. The other mode $l$ can be again straightforwardly matched with its WKB part, and together with the $+s$ mode one may define the so-called straddle mode:

$$\phi_{\text{straddle}}(x,t) \coloneqq \phi_{+s}(x,t)\theta(x) + \phi_{l}(x,t)\theta(-x), \tag{4.47}$$

where $\theta(x)$ is the Heaviside function. This mode, starting from the matching regions on both sides of the turning point, is composed by the Hawking mode on the external side, and of the Hawking partner on the black hole side. It contains a Planckian distribution of Hawking modes in the external region [36]. With respect to the standard case, there is of course a near-horizon regular part of the mode which is missing in the standard black hole case. See also the discussion in [14].

It may be noticed that, due to the transformation defined in (3.5), each solution in the near horizon approximation should be multiplied by an overall factor $\epsilon^{4/3}$. We can reabsorb this factor in the normalization. We shall adopt this convention henceforth in all the models we take into consideration.

### E. Thermality

As usual, for thermality one may verify that

$$\frac{|J_x^{-}|}{|J_x^{+}|} = e^{-\beta\omega}, \tag{4.48}$$

where

$$\beta \coloneqq \frac{2\pi}{\kappa} \tag{4.49}$$

is the inverse Hawking temperature. We stress that, in this sense, thermality is unaffected by the still undetermined value of $h$. The current conservation provides

$$|J_x^{+s}| = |J_x^{+}| - |J_x^{-}| + |J_x^{-s}|, \tag{4.50}$$

which amounts to the usual relation between the Bogoliubov coefficients involved in the process. If we separate each contribution by $|J_x^{+s}|$ we obtain the square modulus of the amplitudes in (B16).

We note that there is also the contribution of the regular mode $-s$, which is missing in the near-horizon diagram 2. The subtle point is that *a priori*, the flux at infinity of the Hawking mode $+s$ can be depleted because of scattering (reflection) on a potential barrier emerging as an effect of the geometry. It has nothing to do with the horizon itself, as in the well-known astrophysical case: in four dimensions, e.g., a scalar particle on the Schwarzschild background is affected by the presence of a centrifugal barrier in the external region of the black hole (apart for $l=0$ modes), which can reflect back to the horizon the Hawking quanta. Of course, in the 2D Schwarzschild case this phenomenon is absent (no centrifugal contribution). This discussion is in agreement with the one carried out in [8,14], where only scattering effects are present in the depletion of the Hawking flux.

### F. Gray-body factor

In order to get also the gray-body factor one must evaluate the ratio

$$R \coloneqq \frac{|J_x^{-s}|}{|J_x^{+s}|}, \tag{4.51}$$

which indicates the fraction of particles reflected back, and then obtain the gray-body factor as

$$\Gamma = 1 - R = 1 - \frac{|J_x^{-s}|}{|J_x^{+s}|}. \tag{4.52}$$

In line of principle, one might deduce the gray-body factor from the direct calculation of

$$|\beta_\omega|^2 \coloneqq \frac{|J_x^{-}|}{|J_x^{+s}|} = |\bar{C}_-|^2, \tag{4.53}$$

which represents the number of created particles, as known [for the second equality cf. (B17)]. In the case of the present model, one would obtain a perfectly Planckian spectrum with $\Gamma = 1$, which implies $h = 0$. Still, even if this route is viable, there is the risk of a poor approximation (as in the standard Hawking effect calculations). We notice that fluxes in (4.52) are both calculated at $x = \infty$, which is the only asymptotic region available to both the modes at hand. Our strategy in the present framework for the calculation of the gray-body factor consists in taking account of the scattering contribution to the geometry simply by studying the reduced equation for the $\pm s$ modes, reducing it in the form of a Schrödinger equation. This might be obtained by means of a suitable variable transformation on the geometry associated with the reduced equation, which is the geometry of the analog black hole, allowing one to switch to Schwarzschild-like coordinates where the metric is diagonal and only a second order term in spatial derivatives appears. Indeed, the reduced equation, which is valid in the WKB approximation, couples the short wave number modes $\pm s$ to each other. Given a $+s$ mode entering from the part of the linear region, where the WKB approximation is valid, we are enabled to calculate

$$R \eqqcolon R_{\text{reduced}} \coloneqq \left(\frac{|J_x^{-s}|}{|J_x^{+s}|}\right)_{\text{reduced}}, \tag{4.54}$$

with the fluxes computed asymptotically, using e.g.,[4] tortoiselike coordinate $\rho$ [see (4.59) below], and with $|J_x^{-s}|$ measured at $\rho = -\infty$ (i.e., near the horizon, but still in a region where the WKB works well). That value would give a mechanism of interplay between the two short wave number modes, which should be taken properly into account. Compare again the discussion in [8,14]. Notice that, in general, the reduced equation has a quite involved form, and it is not easy to solve exactly, except for particular cases. As in the astrophysical case, it allows also further approximations with respect to the weak dispersion scheme we propose herein. Indeed, even a limit of low frequency can be adopted, as in the astrophysical case, without making it difficult to ascertain if thermality is present, as thermality is anyway granted by the calculations above. There is also a further possible interpretation, indeed one may also choose to measure the flux of particles entering the horizon by measuring the flux of modes $d$ at $x = -\infty$, as the flux of entering particles generated by the backscattering and measured by the static observer must coincide with the one of modes $d$ arriving at $x = -\infty$, so that $R = |J_x^d|/|J_x^{+s}|$.

[4]Notice that this is not mandatory; cf. e.g., [37] for the BEC case.

To be more explicit, (4.6) is of course equivalent to the Klein-Gordon equation

$$\Box\phi(x,t) = 0, \tag{4.55}$$

on the curved background metric

$$ds^2 = c^2 dt^2 - (v(x)dt - dx)^2, \tag{4.56}$$

when static solutions $\phi(x,t) = e^{-i\omega t}\varphi(x)$ are considered. A standard coordinate transformation

$$dt = d\tau - \frac{g_{01(x)}}{g_{00}(x)} dx \tag{4.57}$$

carries the metric in the diagonal Schwarzschild-like form

$$ds^2 = \left(1 - \frac{v(x)^2}{c^2}\right) c^2 d\tau^2 - \frac{1}{1 - \frac{v(x)^2}{c^2}} dx^2, \tag{4.58}$$

so that, by choosing the tortoiselike coordinate

$$\rho \coloneqq \int \frac{dx}{1 - \frac{v(x)^2}{c^2}}, \tag{4.59}$$

one obtains the following Schrödinger-like equation

$$\frac{1}{1 - \frac{v(x(\rho))^2}{c^2}} \left(\frac{d^2\varphi(\rho)}{d\rho^2} + \omega^2 \varphi(\rho)\right) = 0, \tag{4.60}$$

which amounts to a free equation in the external region. Therefore, there is no barrier, i.e., no reflection, and the gray-body factor is trivially

$$\Gamma = 1. \tag{4.61}$$

As a consequence, $h = 0$ and then, in this framework the model at hand is purely thermal, at least at the leading order in $\epsilon$.

As is well known from former studies on the dispersive models, there exists a maximal frequency $\omega_{\max}$ such that, for $\omega > \omega_{\max}$, only two modes participate in the scattering process and the Hawking effect is no more present [22]. It is also known that $\omega_{\max}$ is proportional to the dispersive scale $k_0$ both in the subluminal and in the superluminal cases [38,39], and then it goes to infinity in the limit as $k_0 \to \infty$ (i.e., as $\epsilon \to 0$). One has to expect that the spectrum is truncated at $\omega_{\max}$ for nonzero values of $\epsilon$.

## V. THE DIELECTRIC CASE

This case is more tricky, since one has to deal with a system of differential equations instead of a single equation. Indeed, in the so-called $\phi$–$\psi$ model [40], one has

$$\mathcal{L}_{\varphi\psi} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2\chi\omega_0^2}[(v^\alpha\partial_\alpha\psi)^2 - \omega_0^2\psi^2] + \frac{g}{c}(v^\alpha\partial_\alpha\psi)\phi, \tag{5.1}$$

where $\phi, \psi$ play the role of electromagnetic field and polarization field respectively, $\chi$ plays the role of the dielectric susceptibility, $v^\mu$ is the usual four-velocity vector of the dielectric, $\omega_0$ is the proper frequency of the medium, and $g$ is the coupling constant between the fields. We get the system

$$\Box\phi - \frac{g}{c}(v^\mu\partial_\mu\psi) = 0, \tag{5.2}$$

$$\left(\frac{1}{\chi\omega_0^2}(v^\mu\partial_\mu)^2 + \frac{1}{\chi}\right)\psi + \frac{g}{c}(v^\mu\partial_\mu\phi) = 0. \tag{5.3}$$

For simplicity, we put $g = 1$ in what follows (as this parameter, introduced in [15], is no more necessary herein). We proceed as in [18], by considering that the spatial dependence appears in $\chi$ and in $\omega_0$ in such a way that $\chi\omega_0^2 = \text{const}$. See also [40], chapter 10. In this case, we identify

$$\epsilon^2 \coloneqq \frac{1}{\chi\omega_0^2} \tag{5.4}$$

as the small parameter occurring in the problem.

### A. A separated equation for $\psi$

We apply the operator $\Box$ on the left of equation (5.3) (cf. [16]), and by taking into account the stationary case, where $\phi = \varphi(x)e^{i\omega t}$, $\psi = f(x)e^{i\omega t}$ are in the kernel of the operator $[\Box, v^\mu\partial_\mu]$, one obtains the following fourth order ordinary differential equation:

$$-\epsilon^2\partial_x^4 f - 2i\epsilon^2\frac{\omega}{v}\partial_x^3 f + \frac{1}{\chi\gamma^2 v^2}\left(-\left(1-\chi\gamma^2\frac{v^2}{c^2}\right) + \epsilon^2\chi\omega^2\right)\partial_x^2 f + 2\left(i\frac{\omega}{v}\frac{1}{c^2}(1-\epsilon^2\omega^2) - \frac{1}{\gamma^2 v^2}\left(\partial_x\frac{1}{\chi}\right)\right)\partial_x f + \left(\epsilon^2\frac{\omega^4}{v^2c^2} - \frac{1}{\gamma^2 v^2}\left(\partial_x^2\frac{1}{\chi}\right) - \frac{\omega^2}{\chi\gamma^2 v^2 c^2} - \frac{\omega^2}{c^2 v^2}\right)f = 0. \tag{5.5}$$

In order to obtain a form reproducing the original master equation (2.1), we need a further step: we define $f(x) = h(x)\zeta(x)$, with

$$h(x) = A\exp\left(-i\frac{\omega}{2v}x\right), \tag{5.6}$$

where $A$ is a constant. $h(x)$ is chosen such that the third order term vanishes, and the procedure is analogous to the Liouville transformation which eliminates the first order term in a second order linear ordinary differential equation. This leads to the following quartic equation, which is just of the type "Orr–Sommerfeld" in the sense described in the previous sections,

$$\begin{aligned}
&-\epsilon^2\partial_x^4\zeta + \left[-\frac{1}{\chi\gamma^2 v^2}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right) + \epsilon^2\frac{1}{\gamma^2 v^2}\left(1-\frac{3}{2}\gamma^2\right)\omega^2\right]\partial_x^2\zeta \\
&+\left(i\frac{\omega}{v}\frac{1}{\chi\gamma^2 v^2}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right) - 2\frac{1}{\gamma^2 v^2}\left(\partial_x\frac{1}{\chi}\right) - i\epsilon^2\frac{\omega^3}{vc^2}\right)\partial_x\zeta \\
&+\left[\frac{1}{\gamma^2 v^2}\left(i\frac{\omega}{v}\left(\partial_x\frac{1}{\chi}\right) - \left(\partial_x^2\frac{1}{\chi}\right)\right) + \frac{1}{\gamma^2 v^2}\left(\frac{1}{4\chi}\frac{\omega^2}{v^2}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right) - \frac{\omega^2}{\chi c^2}\right)\right. \\
&\left.+\epsilon^2\left(\frac{\omega^4}{v^4}\left(-\frac{1}{16}+\frac{1v^2}{4c^2}\right)\right)\right]\zeta = 0.
\end{aligned} \tag{5.7}$$

The TPs occur for

$$1-\chi(x)\gamma^2\frac{v^2}{c^2}=0, \tag{5.8}$$

and we consider only the black hole solution.

The reduced equation is

$$\frac{1}{\chi\gamma^2 v^2}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right)\partial_x^2\zeta - \frac{1}{\gamma^2 v^2}\left(i\frac{\omega}{v}\frac{1}{\chi}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right)-2\left(\partial_x\frac{1}{\chi}\right)\right)\partial_x\zeta+[\cdots]\zeta=0, \tag{5.9}$$

where the limit $\epsilon\to 0$ is taken and $[\cdots]$ is a contribution readable just from (5.7), and which does not participate to the indicial equation, whose roots are

$$\alpha_1=0,\quad \alpha_2=-1-i\frac{\omega c}{\gamma^2 v^2 n'}. \tag{5.10}$$

$n$ is the refractive index, which is defined such that

$$n^2=1+\chi; \tag{5.11}$$

then we find

$$\lambda=2+i\frac{\omega c}{\gamma^2 v^2 n'}. \tag{5.12}$$

Thanks to such knowledge, one is able to find out the behavior of $\psi$ in all regions of interest, and in particular in the matching region.

### 1. WKB approximation

As in Sec. IV B, we put

$$\zeta(x)=\exp\left(\frac{\theta(x)}{\epsilon}\right)\sum_{i=0}^{\infty}\epsilon^i y_i(x), \tag{5.13}$$

and obtain

$$\theta'^4+\frac{1}{\chi\gamma^2 v^2}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right)\theta'^2=0, \tag{5.14}$$

whose solutions are $\theta'=0$ (multiplicity two), and for $x>0$

$$\theta'_\pm=\pm i\frac{1}{\sqrt{\chi}\gamma v}\sqrt{1-\chi\gamma^2\frac{v^2}{c^2}}. \tag{5.15}$$

The latter solutions are associated with the transport equation

$$y_0'+\frac{1}{(1-\chi\gamma^2\frac{v^2}{c^2})}\left[-\frac{1}{4}\frac{\chi'}{\chi}+i\frac{\omega}{2v}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right)\right]y_0=0, \tag{5.16}$$

and the next-to-leading-order equation

$$\begin{aligned} y_1'+\frac{1}{1-a}\left[-\frac{1}{4}\frac{\chi'}{\chi}+i\frac{\omega}{2v}(1+a)\right]y_1 &= \mp i\frac{\sqrt{\chi}v\gamma}{(1-a)^{\frac{3}{2}}}\left[\frac{9}{4}\frac{\chi''}{\chi}-\frac{1}{1-a}\frac{37}{16}\frac{\chi'^2}{\chi^2}+i\frac{\omega}{v}\frac{\chi'}{\chi}\frac{4+7a}{1-a}\right.\\ &\left.\quad+\frac{\omega^2}{v^2}\frac{a}{1-a}+\frac{\omega^2}{c^2}-\frac{\omega^2}{c^2}\left(\frac{3}{2}\gamma^2-1\right)\frac{\chi}{a}(1-a)\right]y_0, \end{aligned} \tag{5.17}$$

where we have defined

$$a(x)\coloneqq\chi(x)\gamma^2\frac{v^2}{c^2}. \tag{5.18}$$

Solutions are of the form

$$y_0(x)=B\chi(x)^{1/4}(1-a(s))^{-1/4}e^{-i\frac{\omega}{2v}\int^x ds\frac{1+a(s)}{1-a(s)}}, \tag{5.19}$$

$$\begin{aligned} y_1(x)=y_0(x)\Bigg\{D\mp i\int^x ds\frac{\sqrt{\chi}v\gamma}{(1-a(s))^{\frac{3}{2}}}\Bigg[&\frac{9}{4}\frac{\chi''(s)}{\chi(s)} -\frac{1}{1-a(s)}\frac{37}{16}\frac{\chi'^2(s)}{\chi^2(s)}+i\frac{\omega}{v}\frac{\chi'(s)}{\chi(s)}\frac{4+7a(s)}{1-a(s)}\\ &+\frac{\omega^2}{v^2}\frac{a(s)}{1-a(s)}+\frac{\omega^2}{c^2}\\ &-\frac{\omega^2}{c^2}\left(\frac{3}{2}\gamma^2-1\right)\frac{\chi(s)}{a(s)}(1-a(s))\Bigg]\Bigg\}, \end{aligned} \tag{5.20}$$

where $B$ and $D$ are constants. Then we obtain the high momentum modes

$$\zeta_\pm(x)=e^{\pm\frac{i}{\epsilon}\frac{1}{\gamma v}\int^x ds\frac{1}{\sqrt{\chi(s)}}\sqrt{1-a(s)}}(y_0(x)+\epsilon y_1(x)+O(\epsilon^2)). \tag{5.21}$$

If we require $\epsilon y_1<y_0$ in the region where $\chi$ is essentially constant, we get the restriction

$$x<\frac{\omega_0 v}{\omega^2}(1-a_{as})^{\frac{5}{2}}, \tag{5.22}$$

where $a_{as}$ is the asymptotic value of $a(x)$. Since typically $1-a_{as}\sim 10^{-2}$, by $v/\omega=\lambda/(2\pi)$ we can also write

$$x<\frac{\omega_0}{\omega}\frac{\lambda}{2\pi}10^{-5}. \tag{5.23}$$

This implies that the approximation is valid for frequencies such that $\omega\ll\omega_0$.

Near the TP one obtains

$$|f_\pm(x)|\propto x^{-1/4}, \tag{5.24}$$

as found in [18] for the electromagnetic case and in the $\phi$–$\psi$ model, see [40], chapter 10. For $x < 0$, the solutions with $\theta' \neq 0$ are exponentially decaying (decaying mode) and growing (growing mode) respectively.

Two further solutions occur from the reduced equation, when $\theta' = 0$. We find near the regular singular point $x = 0$ (our TP) the following series expansions for $x > 0$:

$$\zeta_{-s}(x) = 1 + \sum_{n=1}^{\infty} c_n x^n, \tag{5.25}$$

$$\zeta_{+s}(x) = x^{-1-i\frac{\omega c}{\gamma^2 v^2 n'}} \left(1 + \sum_{n=1}^{\infty} d_n x^n\right). \tag{5.26}$$

Still, we get the same behavior near the TP as calculated in [18] for the electromagnetic case and in [40], chapter 10, for the simpler $\phi$–$\psi$ model

$$|f_{-s}(x)| \propto \text{const}, \tag{5.27}$$

$$|f_{+s}(x)| \propto x^{-1}. \tag{5.28}$$

As in the Corley model discussed in the previous section, we can also obtain for $x < 0$ two further modes $d$, $l$ which asymptotically propagate towards $x = -\infty$. We do not provide details, as they are straightforward.

### 2. Approximation near the turning point

Solutions near the TP have the following behavior in the matching region, and we recall that $\lambda = 2 + i\frac{\omega c}{\gamma^2 v^2 n'} \coloneqq 2 - i\frac{\omega c}{\kappa}$, where

$$\kappa \coloneqq \gamma^2 v^2 |n'| \tag{5.29}$$

amounts to the surface gravity of the dielectric black hole (see e.g., [41]). Because

$$z = \left(\frac{2\kappa}{vc^3}\right)^{1/3} \epsilon^{-2/3} x, \tag{5.30}$$

we can exploit the solutions we found in the previous section, as formally we have the same equation and then the same solutions [with different explicit values of $p'_{30}(0)$ and of $\epsilon$]. As a consequence, we obtain for $x < 0$ the decaying mode in an analogous way as for (4.38), and it provides us the black hole boundary condition for the present model (which is subluminal, too). For $x > 0$ we have the modes $k_\pm$ in correspondence of the steepest descents passing through the saddle points $u_\pm = \pm i$, i.e.,

$$w_+(z) \simeq \frac{1}{2\sqrt{\pi}} e^{-\frac{3}{4}\pi i} e^{\frac{\pi\omega c}{2\kappa}} |z|^{-\frac{i\omega c}{2\kappa} - \frac{3}{4}} e^{i\frac{2}{3}|z|^{3/2}}, \tag{5.31}$$

$$w_-(z) \simeq \frac{1}{2\sqrt{\pi}} e^{\frac{1}{4}\pi i} e^{-\frac{\pi\omega c}{2\kappa}} |z|^{-\frac{i\omega c}{2\kappa} - \frac{3}{4}} e^{-i\frac{2}{3}|z|^{3/2}}. \tag{5.32}$$

As to the cut contribution, we find

$$|w_{\rm cut}(z)| \simeq \left| \frac{1}{i\pi} \Gamma\left(1 - i\frac{\omega c}{\kappa}\right) \sinh\left(\frac{\pi\omega c}{\kappa}\right) \right|. \tag{5.33}$$

It is easy to show that a matching is possible in the linear region, and thermality can be easily verified. Still, as the polarization field is substantially an "ancillary field" in the model, the really propagating field being the electromagnetic one, we prefer to calculate the matching and thermality of the spectrum by following a different route.

## B. A separated equation for $\phi$

One might get an equation for $\phi$ as in [16], with the drawback of a tricky complication for dealing the limit as $\omega \to 0$. Hence we prefer to proceed in a different way, and obtain a fourth order equation for $\phi$ from the original system of differential equations (5.2) and (5.3).

Our trick is again to separate the variables in the comoving frame, with $\phi = \varphi(x)e^{i\omega t}$, $\psi = f(x)e^{i\omega t}$. A quartic equation is obtained as follows: we apply the operator $(i\omega + v\partial_x)$ to both the members of (5.2)

$$(i\omega + v\partial_x)\left(-\frac{\omega^2}{c^2} - \partial_x^2\right)\varphi = \frac{1}{c}\gamma(i\omega + v\partial_x)^2 f; \tag{5.34}$$

from (5.3) one can isolate the term $f/\chi$ on the left side, and by finding $(i\omega + v\partial_x)^2 f$ from (5.34) one obtains

$$f = -\frac{1}{c}\chi\gamma(i\omega + v\partial_x)\varphi - \chi\epsilon^2\gamma c(i\omega + v\partial_x)\left(-\frac{\omega^2}{c^2} - \partial_x^2\right)\varphi. \tag{5.35}$$

Then one can exploit Eq. (5.2) on the separated variables

$$\left(-\frac{\omega^2}{c^2} - \partial_x^2\right)\varphi = \frac{1}{c}\gamma(i\omega + v\partial_x)f, \tag{5.36}$$

together with the above expression for $f$, and get the fourth order equation

$$
\begin{aligned}
&-\epsilon^2\gamma^2 v^2\chi\partial_x^4\varphi-\epsilon^2\gamma^2(2i\omega\chi v+v^2(\partial_x\chi))\partial_x^3\varphi-\left(1-\chi\gamma^2\frac{v^2}{c^2}+\epsilon^2\left(i\frac{\omega}{v}\gamma^2v^2(\partial_x\chi)-\chi\omega^2\right)\right)\partial_x^2\varphi\\
&+\left[\frac{1}{c^2}\chi\gamma^2v^2\left(2i\frac{\omega}{v}+\frac{1}{\chi}(\partial_x\chi)\right)-\epsilon^2\gamma^2\left(2i\frac{\omega^3}{c^2}\chi v+v^2(\partial_x\chi)\frac{\omega^2}{c^2}\right)\right]\partial_x\varphi\\
&+\left[-\frac{\omega^2}{c^2}-\frac{1}{c^2}\chi\gamma^2\omega^2+iv\frac{1}{c^2}\gamma^2\omega(\partial_x\chi)+\epsilon^2\gamma^2\chi\frac{\omega^4}{c^2}-\epsilon^2 i\gamma^2 v\frac{\omega^3}{v^2}(\partial_x\chi)\right]\varphi=0. \qquad (5.37)
\end{aligned}
$$

In order to eliminate the third order term, we put $\varphi = h(x)\eta(x)$, and in this case the function $h(x)$ must satisfy the differential equation

$$4h' + \left(2i\frac{\omega}{v}+\frac{1}{\chi}(\partial_x\chi)\right)h = 0, \qquad (5.38)$$

whose solution is

$$h = A\chi^{-1/4}e^{-i\frac{\omega}{2v}x}. \qquad (5.39)$$

Then one obtains the fourth order differential equation for $\eta$ in the desired form:

$$
\begin{aligned}
&-\epsilon^2\gamma^2v^2\chi\partial_x^4\eta-\left(1-\chi\gamma^2\frac{v^2}{c^2}+O(\epsilon^2)\right)\partial_x^2\eta\\
&+\left[i\frac{\omega}{v}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right)+\frac{1}{2}\gamma^2\frac{v^2}{c^2}(\partial_x\chi)+\frac{1}{2\chi}(\partial_x\chi)+O(\epsilon^2)\right]\eta'\\
&+(\cdots+O(\epsilon^2))\eta=0, \qquad (5.40)
\end{aligned}
$$

where we have not written explicitly the $O(\epsilon^2)$ terms and the last contribution because they are not useful herein. In particular, the last contribution does not affect the indicial equation for the reduced equation

$$
\begin{aligned}
&-\left(1-\chi\gamma^2\frac{v^2}{c^2}\right)\partial_x^2\eta\\
&+\left[i\frac{\omega}{v}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right)+\frac{1}{2}\gamma^2\frac{v^2}{c^2}(\partial_x\chi)+\frac{1}{2\chi}(\partial_x\chi)\right]\eta'\\
&+(\cdots)\eta=0. \qquad (5.41)
\end{aligned}
$$

We find

$$\alpha_1 = 0, \qquad \alpha_2 = -i\frac{\omega c}{\gamma^2 v^2 n'}, \qquad (5.42)$$

from which

$$\lambda = 1 + i\frac{\omega c}{\gamma^2 v^2 n'}. \qquad (5.43)$$

### *1. WKB approximation*

In this case we put

$$\eta(x) = \exp\left(\frac{\theta(x)}{\epsilon}\right)\sum_{i=0}^{\infty}\epsilon^i y_i(x), \qquad (5.44)$$

and obtain again

$$\theta'^4 + \frac{1}{\chi\gamma^2v^2}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right)\theta'^2 = 0. \qquad (5.45)$$

By now, we consider just the case $x > 0$, as the case $x < 0$ is analogous to the one of the Corley model. Coming back to (5.45), its solutions are $\theta' = 0$ (multiplicity two), and

$$\theta'_{\pm} = \pm i\frac{1}{\sqrt{\chi}\gamma v}\sqrt{1-\chi\gamma^2\frac{v^2}{c^2}}. \qquad (5.46)$$

The latter solutions are associated with the transport equation

$$
\begin{aligned}
&y_0' + \frac{1}{(1-\chi\gamma^2\frac{v^2}{c^2})}\left[-\frac{3}{4}\frac{\chi'}{\chi}+i\frac{\omega}{2v}\left(1+\chi\gamma^2\frac{v^2}{c^2}\right)\right.\\
&\left.-\frac{1}{4}\left(1-\chi\gamma^2\frac{v^2}{c^2}\right)\frac{\chi'}{\chi}\right]y_0=0. \qquad (5.47)
\end{aligned}
$$

Solutions are of the form

$$y_0(x) = A\chi(1-a)^{-\frac{3}{4}}e^{-i\frac{\omega}{2v}\int^x ds\frac{1+a(s)}{1-a(s)}}, \qquad (5.48)$$

and the high momentum modes are

$$\eta_{\pm}(x) = e^{\pm\frac{i}{\epsilon}\frac{1}{\gamma v}\int^x ds\frac{1}{\sqrt{\chi(s)}}\sqrt{1-a(s)}}y_0(x). \qquad (5.49)$$

Near the TP one obtains

$$|\eta_{\pm}(x)| \propto x^{-3/4}. \qquad (5.50)$$

Two further solutions occur when $\theta' = 0$ are obtained from the reduced equation. We find near the regular singular point $x = 0$ (our TP) the series expansions

$$\eta_{-s}(x) = 1 + \sum_{n=1}^{\infty} c_n x^n, \tag{5.51}$$

$$\eta_{+s}(x) = x^{-i\frac{\omega c}{\gamma^2 v^2 n'}} \left(1 + \sum_{n=1}^{\infty} d_n x^n\right). \tag{5.52}$$

Near the TP we get

$$|\eta_{-s}(x)| \propto \text{const}, \tag{5.53}$$

$$|\eta_{+s}(x)| \propto \text{const}, \tag{5.54}$$

and all the aforementioned asymptotics have the same behavior as calculated in [18] for the electromagnetic case and for the simpler $\phi$–$\psi$ model in [40], chapter 10.

#### 2. Approximation near the turning point

We recall that $\lambda = 1 + i\frac{\omega c}{\gamma^2 v^2 n'} \coloneqq 1 - i\frac{\omega c}{\kappa}$; because $z = (\frac{2\kappa}{vc^3})^{1/3}\epsilon^{-2/3}x$, we find in the external region $x > 0$

$$w_+(z) \simeq \frac{1}{2\sqrt{\pi}} e^{-\frac{3}{4}\pi i} e^{\frac{\pi\omega c}{2\kappa}} |z|^{-\frac{i\omega c}{2\kappa}-\frac{3}{4}} e^{i\frac{2}{3}|z|^{3/2}}, \tag{5.55}$$

$$w_-(z) \simeq \frac{1}{2\sqrt{\pi}} e^{\frac{1}{4}\pi i} e^{-\frac{\pi\omega c}{2\kappa}} |z|^{-\frac{i\omega c}{2\kappa}-\frac{3}{4}} e^{-i\frac{2}{3}|z|^{3/2}}. \tag{5.56}$$

As to the cut contribution, we find

$$w_{\text{cut}}(z) \simeq -\frac{1}{i\pi}\Gamma\left(-i\frac{\omega c}{\kappa}\right)\sinh\left(\frac{\pi\omega c}{\kappa}\right)|z|^{i\frac{\omega c}{\kappa}}. \tag{5.57}$$

Also in this case, we obtain for $x < 0$ the decaying mode in an analogous way as for (4.38). It is worthwhile noting that, due to the universal form of Eq. (3.12) governing the near-horizon approximation, the approximate expressions for the aforementioned modes near horizon are of the same type as for the simpler Corley model [2,8,14], as the latter is a subcase of the general framework we are discussing.

Near the turning point we obtain from the black hole boundary condition and in the external region

$$\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) + \phi_3(x,t) + h\phi_4(x,t), \tag{5.58}$$

where $\phi_1 \mapsto w_+$, $\phi_2 \mapsto w_-$, $\phi_3 \mapsto w_{\text{cut}}$, and $\phi_4 \mapsto 1$. As far as the factor $h$ is concerned, analogous considerations as in the case of the previous section hold true. By comparing with the WKB solutions again in the matching region, we find

$$\begin{aligned}\phi(x,t) = {}& \frac{1}{2\sqrt{\pi}} e^{\frac{\omega c}{2\kappa}\pi} e^{-i\frac{3}{4}\pi} \frac{v^2\gamma^2}{c^2}\sqrt{\frac{2\kappa}{v}}\epsilon^{i\frac{\omega c}{3\kappa}+\frac{1}{2}}\left(\frac{2\kappa}{vc^3}\right)^{-i\frac{\omega c}{6\kappa}}\varphi_+(x,t) \\ & + \frac{1}{2\sqrt{\pi}} e^{-\frac{\omega c}{2\kappa}\pi} e^{i\frac{1}{4}\pi} \frac{v^2\gamma^2}{c^2}\sqrt{\frac{2\kappa}{v}}\epsilon^{i\frac{\omega c}{3\kappa}+\frac{1}{2}}\left(\frac{2\kappa}{vc^3}\right)^{-i\frac{\omega c}{6\kappa}}\varphi_-(x,t) \\ & - \frac{\sinh(\frac{\omega c}{\kappa})}{\pi i}\Gamma\left(-i\frac{\omega c}{\kappa}\right)\left(\frac{2\kappa}{vc^3}\right)^{i\frac{\omega c}{\kappa}}\epsilon^{-i\frac{2\omega c}{3\kappa}}\varphi_{+s}(x,t) \\ & + h\varphi_{-s}(x,t).\end{aligned} \tag{5.59}$$

A trivial matching involves also the fourth mode $\varphi_{-s}(x,t)$, which is regular everywhere.

### C. Thermality

We can identify the aforementioned solutions as corresponding to the backward state $B \mapsto \phi_{-s}(x,t)$, the positive high-momentum state $P \mapsto \phi_+(x,t)$, the negative norm high-momentum state $N \mapsto \phi_-(x,t)$ and the Hawking state $H \mapsto \phi_{+s}(x,t)$, respectively. We obtain

$$\frac{|N|^2}{|P|^2} \coloneqq \frac{|J_x^-|}{|J_x^+|} = e^{-\frac{2\pi c}{\kappa}\omega}, \tag{5.60}$$

which corresponds to the standard signal of the thermal character of the black hole horizon. The current density has the following structure [15]:

$$\begin{aligned}J^\mu \coloneqq \frac{i}{2}\Big[&\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi + \frac{1}{\chi\omega_0^2}v^\mu\psi^* v^\alpha\partial_\alpha\psi \\ & - \frac{1}{\chi\omega_0^2}v^\mu\psi v^\alpha\partial_\alpha\psi^* + \frac{1}{c}v^\mu(\psi^*\phi - \psi\phi^*)\Big].\end{aligned} \tag{5.61}$$

One considers the fields in the asymptotic (homogeneous) region in the comoving frame, where they are normalized as in [42]. Furthermore, the term quadratic in $\psi$ in the present expansion at the leading order is suppressed, as is $O(\epsilon^2)$. One obtains

$$\begin{aligned}|J_x| = {}& \left|\left(-k_x - \frac{1}{c^2}\chi\gamma v\frac{k_\alpha v^\alpha}{1-\frac{(k_\alpha v^\alpha)^2}{\omega_0^2}} + O(\epsilon^2)\right)\right| \\ & \times\left|\left(-k_x - \frac{1}{c^2}\chi\gamma v(k_\alpha v^\alpha)\right)\varphi^*\varphi\right|.\end{aligned} \tag{5.62}$$

In particular, in the asymptotic region $x \to \infty$ we have

$$k_x^{+s} = \frac{\omega}{v}\frac{n - \frac{v}{c}}{\frac{c}{v} - n}, \tag{5.63}$$

$$k_x^{-s} = -\frac{\omega}{v}\frac{n + \frac{v}{c}}{\frac{c}{v} + n}. \tag{5.64}$$

### D. The gray-body factor

Of course, one may study the problem of determining $h$ directly by considering the reduced equation and its solutions. This might be a nontrivial route, as the equation is quite involved. Alternatively, in order to calculate the gray-body factor at least in an approximate way, we could first identify the metric associated with the model at hand. From Eqs. (5.2), (5.3), in the approximation where the term $\propto \epsilon^2$ is neglected and in the eikonal approximation, we get the metric also deduced in [41]

$$ds^2 = c^2\gamma^2 \frac{1}{n^2}\left(1+\frac{nv}{c}\right)\left(1-\frac{nv}{c}\right)dt^2 + 2\gamma^2\frac{v}{n^2}(1-n^2)dtdx - \gamma^2\left(1+\frac{v}{nc}\right)\left(1-\frac{v}{nc}\right)dx^2, \tag{5.65}$$

where we are in the comoving frame of the pulse generating a propagating dielectric perturbation and the refractive index depends on $x$: $n = n(x)$. Differently from the Corley model, the metric is not exact but approximated, and holds only in the eikonal approximation. The above metric is conformally related to the one deduced in [16]. There exists a coordinate transformation carrying the metric into a static form; even if they are singular, we carry out the relative transformation because it allows a direct computation of the gray-body coefficient. The following coordinate change

$$dt = d\tau - \alpha(x)dx, \tag{5.66}$$

where

$$\alpha(x) = \frac{g_{01}(x)}{g_{00}(x)} \tag{5.67}$$

carries the metric to the static form [41]

$$ds^2 = \frac{c^2}{n^2(x)}g_{\tau\tau}(x)d\tau^2 - \frac{1}{g_{\tau\tau}(x)}dx^2, \tag{5.68}$$

where

$$g_{\tau\tau}(x) \coloneqq \gamma^2\left(1+n(x)\frac{v}{c}\right)\left(1-n(x)\frac{v}{c}\right). \tag{5.69}$$

We do not delve into the explicit calculation, as is the same displayed in [41], which confirms in the present two-dimensional model that $\Gamma = 1$, and then, in this approximation, $h = 0$ once more, and that there is a divergence as $\omega \to 0$ in the number of created particles, as numerically tested in [43] and then also found in [16] in a different approximation scheme (see also [13]). This approximation might be too crude, and $\Gamma < 1$ could also be allowed by a better approximation. Still, again, the leading contribution to $h$ as arising from the pair creation process is vanishing.

Also in this case, a maximal frequency $\omega_{\max}$ exists [43] (see also [15]) beyond which no Hawking effect is expected, and then a truncation of the spectrum for $\omega > \omega_{\max}$ is to be taken into account. One may wonder which differences occur with respect to the calculation in [16]. Therein, the fourth backward mode participates to the Corley's diagram near the TP, as it appears as a further cut integral in the Fourier space. It is remarkable that this diagram was calculated in the approximation where the square of the resonance frequency is a linear function in $x$, which is of course different from the case at hand. But this is not the only source of differences, as it is the approximation we perform herein in itself which is able to leave just a cut integral (in the Laplace dual space), with the other short wave number mode (the backward one) absent from the diagram. Analogous considerations can be made in a comparison with the calculations developed for the Hopfield model discussed in [18], where the gray-body factor was not available.

## VI. CONCLUSIONS

We have explored a further way to approach analytical calculations for the Hawking effect in analog gravity. A fourth order equation, which is of the Orr–Sommerfeld type, has been shown to play the role of master equation in analog gravity, with reference to the analogous Hawking effect. The approximation adopted is the one of weak dispersive effects, where the suitable coupling of the fourth order term is associated with the parameter $\epsilon$ entering the equation. This kind of approximation is not new in literature, see e.g., [8,14], but it is applied in the framework provided by Nishimoto's analysis [25] for equations of Orr-Sommerfeld. This allows us to achieve a suitable approximation near the turning point (horizon), and we are enabled to provide a complete study of thermality for both the subluminal fluid model of [2,22] and for the dielectric one. Indeed, we can provide a scheme for the calculation of an analytic expression of the gray-body factor, which is in agreement with the analysis carried out in [8,14], as far as the Corley's model is concerned, but is more general and allows one to encompass important physical models which cannot be included by the Corley's model itself: dielectrics, BEC, and water waves with varying speed of sound velocity $c(x)$. Indeed, the same calculational scheme can be adopted successfully also in the case of BEC and of surface waves in the companion paper [23].

Then a more complete study of the Hawking emission in condensed matter systems is achieved when dispersion is weak, which provides the most direct correspondence with the standard Hawking effect, with an enhanced role of the reduced equation (i.e., the equation one obtains in absence of dispersion).

It is remarkable that the geometrical setting of the analogous Hawking effect in this scheme arises in the WKB approximation which holds near but not too near the horizon. The model of course leaves open the possibility to explore more sophisticated situations where dispersive effects are strong, which would provide regimes for Hawking-like radiation which are more far from the standard case.

The perspective is open also for a more sophisticated analog black hole spectroscopy, allowing a more precise comparison between experimental measurements and theoretical computations.

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## APPENDIX A: A FURTHER JUSTIFICATION OF THE NEAR HORIZON APPROXIMATION

We provide a further justification of the near horizon approximation, which allows us also to show that the Orr-Sommerfeld form of the equation is not mandatory, in the sense that one can allow also for third order terms in the derivative, with the only restriction that they are at least of the same order of the fourth order one in the suitable coupling and that they do not vanish at the TP.

We start by a slight generalization of (2.1)

$$\delta^2\frac{d^4\Phi}{dx^4}\pm\left[\delta^2 p_4(x,\delta)\frac{d^3\Phi}{dx^3}+p_3(x,\delta)\frac{d^2\Phi}{dx^2}+p_2(x,\delta)\frac{d\Phi}{dx}+p_1(x,\delta)\Phi\right]=0, \tag{A1}$$

where we have changed the power of the expansion parameter with respect to [44,45], in order to allow a direct comparison with the framework discussed in the paper. The new term in the third order derivative has been added. We first introduce for simplicity of notation

$$f(x)\coloneqq\frac{p_{30}(x)}{p'_{30}(0)}, \tag{A2}$$

where we have shifted the turning point [where $p_{30}(x)=0$] at $x=0$.

Then we define a Langer-like variable, adapting the definition assumed in [44,45]:

$$\eta(x)\coloneqq\left[\frac{3}{2}\int_0^x dy\sqrt{f(y)}\right]^{2/3}. \tag{A3}$$

For definiteness, we consider the subluminal case (the superluminal one is obtained in a straightforward way). We shall indicate with $\Phi^{(i)}$, $i=1, 2, 3, 4$ the derivatives with respect to the new variable, and by $\Phi'$, $\Phi''$, $\Phi'''$, $\Phi''''$ the derivatives with respect to $x$. We notice that

$$\eta'(x)=\sqrt{\frac{f(x)}{\eta(x)}}, \tag{A4}$$

which is regular as $x\to 0$. Furthermore, due to (A2), also $\eta'\to 1$ as $x\to 0$ holds true.

As to (A1), considering only the leading order terms, we obtain

$$\begin{aligned}&\delta^2\Phi^{(4)}+\delta^2\left(6\frac{\eta''}{(\eta')^2}+p_{40}\frac{1}{\eta'}+O(\delta)\right)\Phi^{(3)}\\&+(p'_{30}(0)\eta+O(\delta))\Phi^{(2)}\\&+\left(p_{20}\frac{1}{(\eta')^3}+p_{30}\frac{\eta''}{(\eta')^4}+O(\delta)\right)\Phi^{(1)}\\&+\left(p_{10}\frac{1}{(\eta')^4}+O(\delta)\right)\Phi=0.\end{aligned} \tag{A5}$$

We now define

$$\epsilon_R\coloneqq\frac{\delta}{(p'_{30}(0))^{1/2}}, \tag{A6}$$

in order to mimic the behavior occurring in [44,45]. An equation holding in the near horizon approximation is obtained by means of the following definition (with some abuse of notation)

$$\Phi(\zeta,\epsilon_R)\coloneqq\Phi\left(\frac{\eta}{(\epsilon_R)^{2/3}},\epsilon_R\right), \tag{A7}$$

and also the new variable

$$\zeta\coloneqq\frac{\eta}{(\epsilon_R)^{2/3}}=(p'_{30}(0))^{1/3}\delta^{-2/3}\eta. \tag{A8}$$

Furthermore, one has to take into account that

$$p_{30}=(\eta')^2p'_{30}(0)\eta. \tag{A9}$$

Then one finds the following equation:

$$\begin{aligned}&(p'_{30}(0))^{4/3}\Phi^{(4)}+\delta^{2/3}\left(6\frac{\eta''}{(\eta')^2}+p_{40}\frac{1}{\eta'}+O(\delta)\right)(p'_{30}(0))\Phi^{(3)}\\&+((p'_{30}(0))^{4/3}\zeta+O(\delta^{1/3}))\Phi^{(2)}\\&+\left[(p_{20}\left(p'_{30}(0))^{1/3}\Phi^{(4)}\frac{1}{(\eta')^3}\right)+O(\delta^{2/3})\right]\Phi^{(1)}\\&+O(\delta^{2/3})\Phi=0.\end{aligned} \tag{A10}$$

At the leading order and assuming that $p_2$ (and then also $p_{20}$) is analytic in a neighborhood of the TP we obtain

$$\Phi^{(4)} + \zeta\Phi^{(2)} + \frac{p_{20}(0)}{p'_{30}(0)}\Phi^{(1)} = 0, \tag{A11}$$

which by taking into account that

$$\lambda := \frac{p_{20}(0)}{p'_{30}(0)}, \tag{A12}$$

coincides with the equation obtained by means of the method borrowed from [25]. The relation between $\epsilon$ in the previous sections and $\epsilon_R$ is simply

$$\epsilon = \epsilon_R\sqrt{p'_{30}(0)} = \delta, \tag{A13}$$

and also $\zeta = z$ holds.

## APPENDIX B: MATCHING CONDITIONS

Let us now further discuss the matching conditions underlying the scattering process at hand. We take into consideration states living on the right side of the turning point, directly involved in the Hawking effect. We have to match in a single solution the WKB part and the near horizon part of the modes introduced above, in such a way to obtain basis functions which are defined in the whole domain. For the WKB part, we have to consider the basis

$$\{\varphi_+^{\mathrm{WKB}}(x), \varphi_-^{\mathrm{WKB}}(x), \varphi_{+s}^{\mathrm{WKB}}(x), \varphi_{-s}^{\mathrm{WKB}}(x)\}, \tag{B1}$$

whereas for the near horizon (NH) region we get the further basis

$$\{\varphi_+^{\mathrm{NH}}(x), \varphi_-^{\mathrm{NH}}(x), \varphi_{+s}^{\mathrm{NH}}(x), \varphi_{-s}^{\mathrm{NH}}(x)\}. \tag{B2}$$

Let us denote $\varphi_i^{\mathrm{WKB}}(x)$ and $\varphi_i^{\mathrm{NH}}(x)$ the parts to be joined for the $i$ mode, with $i = \pm, \pm s$. The general WKB solution has the form

$$\varphi^{\mathrm{WKB}}(x) = \sum_i C_i\varphi_i^{\mathrm{WKB}}(x), \tag{B3}$$

where $C_i$ are constant (i.e., independent from $x$), and the general NH solution is

$$\varphi^{\mathrm{NH}}(x) = \sum_i D_i\varphi_i^{\mathrm{NH}}(x), \tag{B4}$$

where also $D_i$ are constant. In the matching region, where the two approximations coexist, we have

$$\varphi_i^{\mathrm{WKB}}(x) \sim a_i h_i(x), \tag{B5}$$

and also

$$\varphi_i^{\mathrm{NH}}(x) \sim b_i h_i(x), \tag{B6}$$

with the same functional dependence $h_i(x)$. Then, matching in the linear region requires

$$C_i = \frac{b_i}{a_i}D_i. \tag{B7}$$

Notice that, compared to the standard matching of the WKB solutions with Airy functions for the Schrödinger equation in quantum mechanics, in place of fixing the constant for the near turning point solutions as functions of the ones in the WKB-allowed regions, in agreement with Corley's ideas, we proceed in the complementary direction, as an indication that part of the amplitudes arises from what happens at the turning point.

Moreover, for $x \to \infty$ the propagating modes participating to the Hawking process behave as plane waves:

$$\varphi_i^{\mathrm{WKB}}(x) \sim \bar{a}_i e^{ik_i(\omega)x}, \tag{B8}$$

so that

$$\varphi^{\mathrm{WKB}}(x) = \sum_i C_i\bar{a}_i e^{ik_i(\omega)x}, \tag{B9}$$

and we may define

$$c_i := \frac{b_i}{a_i}D_i\bar{a}_i, \tag{B10}$$

in order to compare with the amplitudes defined in [2]. In order to get scattering amplitudes, let us write

$$C_i = \bar{C}_i N_i, \tag{B11}$$

where $N_i$ are the normalizations of the modes in the asymptotic region, which are consistent with the quantization of the field in the $\omega$ representation [39]. In particular, we have

$$N_i = \frac{1}{\sqrt{4\pi|v_g(k_i(\omega))(\omega - vk_i(\omega))|}}, \tag{B12}$$

where $v_g(k_i(\omega))$ is the group velocity of the $i$th mode. $\bar{C}_j$ represent the actual amplitudes:

$$\bar{C}_i = \frac{b_i}{N_i a_i}D_i\bar{a}_i. \tag{B13}$$

Because of the black hole boundary condition, we have

$$D_+ = D_- = D_{+s} := D, \tag{B14}$$

whereas the fourth mode has a different amplitude, that we put equal to

$$D_{-s} =: hD. \tag{B15}$$

Therefore, by comparison with the asymptotic behavior of the field, one obtains

$$\bar{C}_{+} = \frac{b_{+}\bar{a}_{+}}{N_{+}a_{+}}\frac{N_{+s}a_{+s}}{b_{+s}\bar{a}_{+s}}, \tag{B16}$$

$$\bar{C}_{-} = \frac{b_{-}\bar{a}_{-}}{N_{-}a_{-}}\frac{N_{+s}a_{+s}}{b_{+s}\bar{a}_{+s}}, \tag{B17}$$

$$\bar{C}_{+s} = 1, \tag{B18}$$

$$\bar{C}_{-s} = h\frac{b_{-s}\bar{a}_{-s}}{N_{-s}a_{-s}}\frac{N_{+s}a_{+s}}{b_{+s}\bar{a}_{+s}}. \tag{B19}$$

See also the comment below Eq. (4.50). As regards the complete solution, we have a basis

$$\{\varphi_{+}(x), \varphi_{-}(x), \varphi_{+s}(x), \varphi_{-s}(x)\}, \tag{B20}$$

which reduces to the aforementioned bases in the different regions: of course

$$\varphi_i(x) \sim \varphi_i^{\mathrm{WKB}}(x) \tag{B21}$$

asymptotically, and also

$$\varphi_i(x) \sim \varphi_i^{\mathrm{NH}}(x) \tag{B22}$$

near the turning point. In the matching region it holds

$$\varphi_i(x) \sim \bar{C}_i N_i a_i h_i(x) = D_i b_i h_i(x). \tag{B23}$$

For the process at hand, the general solution

$$\varphi(x) = \sum_i A_i \varphi_i(x) \tag{B24}$$

must be such that, asymptotically, one gets again

$$A_i = \bar{C}_i = \frac{b_i}{N_i a_i} D_i \bar{a}_i. \tag{B25}$$

It is worthwhile mentioning that, in more rigorous mathematical terms, we have been discussing the topic of central connections in terms of the language adopted in [25]. Therein, one considers a fundamental matrix $\Phi^{\mathrm{NH}}$ of solutions near the TP, a fundamental matrix $\Phi^{\mathrm{WKB}}$ of solutions in the WKB region, and then matches through a matrix $\Lambda$ according to

$$\Phi^{\mathrm{WKB}} = \Phi^{\mathrm{NH}}\Lambda, \tag{B26}$$

where $\Lambda$ is asymptotically diagonal [25]. It is easily verified that this condition is equivalent to the one we discussed above.


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