Pairs of surface wave packets with zero-sum energy in the Hawking radiation analog

Anirban Guha[*]
School of Science and Engineering, University of Dundee, Dundee DD1 4HN, United Kingdom

Eyal Heifetz
Department of Geophysics, Porter school of the Environment and Earth Sciences, Tel Aviv University, Tel Aviv 69978, Israel

Akanksha Gupta
Department of Mechanical Engineering, Indian Institute of Technology, Kanpur, U.P. 208016, India

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Here we propose a minimal analog gravity setup and suggest how to select two surface gravity wave packets in order to mimic some key aspects of Hawking radiation from the horizon of nonrotating black holes. Our proposed setup, unlike the scattering problem conventionally studied, constitutes of a constant mean flow over a flat bathymetry, in which the two wave packets possess the same amount of wave action but equal and opposite (sign) amount of energy, thereby mimicking virtual particles created out of near horizon vacuum fluctuations. Attention is given to the physical mechanism relating to the signs of the wave action and energy norm with the wave's intrinsic and total phase speeds. We construct narrow wave packets of equal wave action, the one with positive energy and group speed propagates against the mean flow and escapes from the black hole as Hawking radiation, while the other with negative energy and group speed is drifted by the mean flow and falls into it. Hawking's prediction of low frequency mode amplification is satisfied in our minimal model by construction. We find that the centroid wave numbers and surface elevation amplitudes of the wave packets are related by simple analytical expressions.

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I. INTRODUCTION

Direct probing of Hawking radiation in gravitational black holes (BHs) seems to be unlikely in the near future. Hence, laboratory studies of the phenomena in analogous physical systems, obeying similar equations of motion as the fields around BHs, provide tools to examine and demonstrate different features of Hawking radiation. In the pursuit of finding laboratory analogs of BH radiation (c.f. Barceló [1] for an updated review), Schutzhold and Unruh [2] theoretically demonstrated how surface gravity waves, in the presence of a countercurrent flow in a shallow basin, can be used to simulate phenomena around BHs in the laboratory. Rousseaux *et al.* [3] reported the first successful analog gravity experiment mimicking white hole (WH) horizons by surface gravity waves. Weinfurtner *et al.* [4] used localized obstacle to block the upstream propagation of a long wave, converting it into a pair of short waves with opposite-signed energy, one with positive and the other with negative energy. This experiment successfully demonstrated the thermal nature of the stimulated Hawking process at an analog WH horizon. Hawking radiation in analog wave-current systems have been further established experimentally and numerically in recent years, see Refs. [5–7]. Specifically, Euvé *et al.* [5] established analog quantum Hawking radiation using correlation of the randomly fluctuating free surface downstream of the obstacle.

The objective in this paper is more modest. It aims to propose a minimal water wave analog of pairs of virtual particles with equal and opposite energy, created out of near horizon vacuum fluctuations, where the particle with the positive energy escapes to infinity, and the one with negative energy falls into the BH, leading to BH evaporation [8,9]. As this phenomena by itself is not necessarily related to wave scattering, it is enough to assume here a flow system with a constant mean countercurrent over a flat bathymetry (i.e., constant water depth, see Fig. 1).

II. PSEUDOENERGY AND PSEUDOMOMENTUM

Consider for simplicity a rectangular quasi-2D domain (x, z) of the size $(0, L) \times (-H, \eta')$, filled with water (assumed here to be inviscid and incompressible), where L is the horizontal length, H is the mean fluid depth, and

[*]anirbanguha.ubc@gmail.com

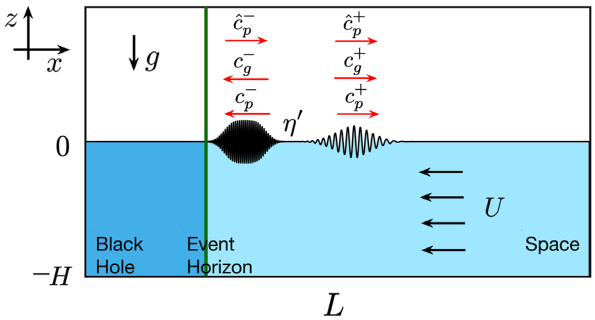

FIG. 1. Schematic diagram of the black hole analog setup. For details about the various symbols, see text.

$\eta'(x,t)$ denotes the free surface elevation about the mean depth (e.g., Fig. 1). For this setup the continuity and Euler's momentum equations read:

$$\nabla \cdot \mathbf{u} = 0; \qquad \frac{D\mathbf{u}}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}. \tag{1a, b}$$

Here $\nabla \equiv (\partial/\partial x, \partial/\partial z)$ is the 2D gradient operator, $\mathbf{u} = (u, w)$ denotes velocity, p denotes pressure, ρ is the density of water (assumed constant), and $\mathbf{g} = -g\hat{\mathbf{z}}$ is the gravity vector pointing downwards.

Assuming periodic boundary conditions at $x = 0$ and L, it is straightforward to show that both the domain-integrated momentum in the x direction (P) and the total fluid energy (E):

$$P = \rho \int_{x=0}^{L}\int_{z=-H}^{\eta'} u dx dz, \tag{2a}$$

$$E = \frac{\rho}{2}\int_{x=0}^{L}\left[\left(\int_{z=-H}^{\eta'} |\mathbf{u}|^2 dz\right) + g(\eta'^2 - H^2)\right]dx, \tag{2b}$$

are conserved [10]. The two terms in the rhs of Eq. (2b) are, respectively, the fluid kinetic and potential energy. Consider a steady mean current in the negative x direction: $\mathbf{u} = (-\bar{U}, 0)$ with $\bar{U} > 0$, and a constant mean height H satisfying hydrostatic balance. This flow is a solution of Eq. (1a,b) and posses the domain integrated momentum and energy

$$\bar{P} = -\rho L H \bar{U}, \qquad \bar{E} = \frac{\rho L H}{2}(\bar{U}^2 - gH). \tag{3a, b}$$

Now suppose that on top of this steady base state we add a perturbation that is composed of surface gravity waves of the form $\eta'(x,t) = a e^{i(kx-\omega t)} + \text{c.c.}$, where a and k, respectively, denote amplitude and wave number (defined positive here), $\omega = kc_p$ denotes frequency, c_p is the phase speed, and c.c. denotes complex conjugate. Then

$$\omega = \hat{\omega} - k\bar{U} = k(\hat{c}_p - \bar{U}) = kc_p, \tag{4}$$

where the intrinsic surface gravity wave frequency and phase speeds (denoted by hat) are given by the familiar dispersion relation:

$$\hat{\omega} = k\hat{c}_p = \pm\sqrt{gk \tanh kH}. \tag{5}$$

Denoting the wave fields by prime so that $\mathbf{u} = (-\bar{U} + u', w')$, we obtain

$$P = \bar{P} + \delta P, \qquad \delta P = \rho \int_{x=0}^{L}\int_{z=0}^{\eta'} u' dx dz, \tag{6a}$$

$$E = \bar{E} + \delta E, \qquad \delta E = E' - \bar{U}\delta P,$$
$$E' = \frac{\rho}{2}\int_{x=0}^{L}\left(\int_{z=-H}^{\eta'} |\mathbf{u}'|^2 dz + g\eta'^2\right)dx. \tag{6b}$$

The quantities δP and δE are, respectively, known by (the somewhat confusing terms) pseudomomentum and pseudoenergy. As is evident from Eqs. (6a) and (6b), they are simply the momentum and energy contribution of the waves to the system. Since $\bar{P}$ and $\bar{E}$ are constant, δP and δE are also conserved (in the Appendix we explicitly show that δE in the shallow water limit is equivalent to the energy density integral in Schützhold and Unruh [2] [Eqs. (67) and (68)]). Note that E'—the positive definite wave eddy energy—is only one of the contributions by the surface waves to the total change in the energy (as will be clarified further in the next section). Hence, neither the pseudomomentum nor the pseudoenergy are sign definite; negative pseudoenergy implies that the addition of linear waves to the base flow reduces the energy of the system below its mean value $\bar{E}$, whereas positive pseudoenergy increases the energy above its mean value.

III. PAIRS OF ZERO-SUM PSEUDOENERGY WAVE PACKETS

The essential idea in this analogy is that confined surface gravity wave packets represent virtual particles. Therefore we aim to choose superposition pairs of wave packets with equal and opposite values of pseudoenergy δE in a way that the sign of their group velocity (in the frame of rest) will be equal to the sign of their pseudoenergy. When this is achieved, the wave packet with the positive pseudoenergy manages to overcome the leftward countercurrent $-\bar{U}$ and escapes rightward (from the BH horizon into the outer space), whereas the negative pseudoenergy wave packet is drifted leftward with the base flow (into the BH). Consequently, the energy in the left region (inside the BH) is reduced on average and become $\bar{E} - |\delta E|$. Eventually when the leftward wave packet dissipates, it is expected to reduce the mean energy of BH, so that the new mean energy $\bar{E}_{\text{new}} \approx \bar{E} - |\delta E|$.

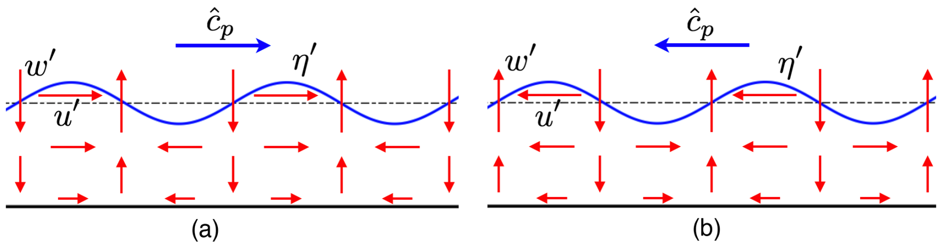

FIG. 2. Schematic description of the fact that (a) rightward propagating surface waves have a positive pseudomomentum, while (b) leftward propagating surface waves have a negative pseudomomentum.

Next we wish to suggest how to choose excited pairs of oppositely signed pseudoenergy wave packets based on their physical properties. We first note that for surface waves it can be shown, after some algebra, that the wave eddy energy satisfies

$$E' = \frac{1}{2}\rho g L a^2 = \hat{c}_p \delta P, \tag{7}$$

implying that $\hat{c}_p$ and δP are of the same sign. This sign agreement can be understood from Fig. 2. The mechanism of surface wave propagation is such that the horizontal convergence (divergence) results in upward (downward) motion that translates the vertical height anomaly η'. Hence for rightward or positive propagation, $\hat{c}_p > 0$ [Fig. 2(a)], and u' is in phase with η'. Therefore the vertical integration of positive u' from the bottom to the wave crests exceeds the vertical integration of negative u' from the bottom to the wave troughs and consequently δP is positive, in agreement with Eq. (6a). By the same argument it follows that δP is negative when $\hat{c}_p$ is negative [Fig. 2(b)]. Equations (4), (6b), and (7) then imply the following relations:

$$\delta E = (\hat{c}_p - \bar{U})\delta P = c_p \delta P = \left(1 - \frac{\bar{U}}{\hat{c}_p}\right) E'. \tag{8}$$

Consider then two waves with different wave numbers k^+ and k^- (both defined positive), where both waves have a positive $\hat{c}_p$ (and hence a positive δP). Thus both waves are "trying" to propagate to the right (in the positive x direction) against the mean current $-\bar{U}$, see Fig. 1. If we assume a situation such that

$$\hat{c}_p^- < \bar{U} < \hat{c}_p^+,$$

then Eq. (8) implies that $\delta E^+ > 0$ while $\delta E^- < 0$. In other words, the wave that manages to counterpropagate against the current with a positive phase speed in the rest frame ($c_p^+ > 0$) carries a positive pseudoenergy, whereas the wave whose intrinsic phase speed is not large enough to match the opposed current ($c_p^- < 0$) carries a negative pseudoenergy and consequently propagates to the left in the rest frame (despite that the pseudomomentum of both waves being positive), as shown in Fig. 1. This statement can be written in terms of frequency and wave action. Defining the wave action as $\delta A \equiv \delta P/k$, we obtain from Eq. (8) that $\delta E = \omega \delta A$. Consider δA as an analog for $\hbar$, then for positive δA the sign of the pseudoenergy is determined by the sign of its frequency ω. This suggests that we can set a perturbation of *zero* pseudoenergy composed of two waves ($\delta E = \delta E^+ + \delta E^- = 0$) with the same positive value of wave action $\delta A^+ = \delta A^- > 0$. These in combination yield

$$\Omega^+ = -\Omega^- > 0 \Rightarrow \hat{\Omega}^+ + \hat{\Omega}^- = \alpha^+ + \alpha^-, \tag{9a}$$

$$\left(\frac{a^-}{a^+}\right)^2 = \frac{\hat{\Omega}^-}{\hat{\Omega}^+} = \sqrt{\frac{\alpha^- \tanh \alpha^-}{\alpha^+ \tanh \alpha^+}}. \tag{9b}$$

Here we have used the following nondimensionalizations: $\alpha^{+(-)} \equiv k^{+(-)}H$, $\hat{\Omega}^{+(-)} \equiv \hat{\omega}^{+(-)}H/\bar{U}$ and $\Omega^{+(-)} \equiv \omega^{+(-)}H/\bar{U}$. Additionally Eq. (4) has also been used, from which we obtain $\Omega^{+(-)} = \hat{\Omega}^{+(-)} - \alpha^{+(-)}$, where $\hat{\Omega}^{+(-)} = Fr^{-1}\sqrt{\alpha^{+(-)} \tanh \alpha^{+(-)}}$, in which the Froude number $Fr \equiv \bar{U}/\sqrt{gH}$. According to Eq. (9a), the waves have equal and opposite frequencies. Hence in the rest frame, the "+" wave will propagate to the right against the mean current whereas the "−" wave will be drifted to the left, following the scenario depicted in Fig. 1. Furthermore, Eq. (9b) provides a direct relation of the amplitude ratio of the "+" and "−" waves. An interesting point to notice from Eq. (9b) is that the condition of zero pseudoenergy superposition does *not* imply that the free surface should be initially flat.

While the pseudomomentum of a monochromatic sinusoidal wave is perfectly well defined, its position is obviously not. Therefore, in order to generate an initial zero pseudoenergy perturbation whose position and momentum are both reasonably well defined, we should construct pairs of narrow wave packets rather than pairs of monochromatic waves. Hence, the positive (negative) pseudoenergy wave packet should propagate with a positive (negative) group speed c_g (or in nondimensional terms, $C_g^{+(-)} \equiv c_g^{+(-)}/\bar{U}$), satisfying:

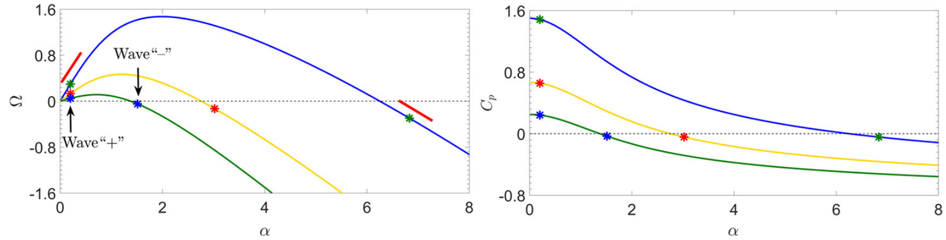

FIG. 3. Dispersion curves: (a) Ω versus α, and (b) C_p versus α. The blue, yellow and green curves, respectively, denote $Fr = 0.4$, 0.6 and 0.8. The short red lines in (a) denotes the slope of the blue curve, which equals to the group speed. The "*"s of same color denote a pair wave; the one above the zero line has $\delta A > 0$ and $\delta E > 0$, while that below the zero line has $\delta A > 0$ and $\delta E < 0$.

$$C_g^{+(-)} \equiv \frac{\partial \Omega^{+(-)}}{\partial \alpha^{+(-)}} = -1 + \frac{1}{2Fr}\sqrt{\frac{1}{\alpha^{+(-)}}\tanh \alpha^{+(-)}}\left[1 + \frac{2\alpha^{+(-)}}{\sinh 2\alpha^{+(-)}}\right]. \quad (10)$$

Furthermore, the centroid group and phase speeds of each wave packet should posses the same sign. This is because the sign of c_p (or in nondimensional terms, $C_p^{+(-)} \equiv c_p^{+(-)}/\bar{U}$) determines the sign of δE whereas the sign of c_g determines the wave packet's direction of propagation.

Consider the positive branch of Ω and address only subcritical flows, i.e., $Fr < 1$, in order to enable wave's counterpropagation. The variations of Ω and C_p with α for different Fr values are respectively plotted in Figs. 3(a) and 3(b). Two wave packets with equal wave action, and equal and opposite pseudoenergy, consist of a "pair wave" (denoted by the same colored "*"s), and therefore satisfies Eqs. (9a) and (9b). The "+" ("−") wave packet's frequency, phase and group speeds are all positive (negative), and hence escapes into space (falls into the BH), in analogy with Hawking radiation. Notice that for subcritical flows, this condition fails in the shallow-water limit (since the pseudoenergy is always positive); see the Appendix.

Figure 4 shows a pair of wave packets (both having positive wave action but equal and opposite pseudoenergy) in a countercurrent flow over a flat bathymetry. This configuration is numerically simulated using an in-house high-order spectral code, detailed in Raj and Guha [11]. As already mentioned, a zero-sum pseudoenergy does not necessarily imply that the superposition of the wave packet pair would render the free surface flat, as clearly shown in Fig. 4(a), which is the configuration at $t = 0$. The background flow is subcritical with $Fr = 0.7$. The "+" wave packet (centroid wave number $\alpha^+ = 0.8$) emits as Hawking radiation while the "−" wave packet (centroid wave number $\alpha^- = 2.47$) falls inside the BH; the wave pair has the same magnitude of centroid frequency as per Eq. (9a). Here the definition of the event horizon is arbitrary; however it must be located to the left of the superposed wave packets at $t = 0$. The fact that $\alpha^- > \alpha^+$ is evident from the dispersion curve in Fig. 3(a). A consequence of $\alpha^- > \alpha^+$ is that $a^- > a^+$ as per Eq. (9b), which is also clear from Fig. 4(b).

IV. PARALLELS WITH THE RATIO OF BOGOLIUBOV COEFFICIENTS AND LOW-FREQUENCY MODE AMPLIFICATION

The study of classical and quantum fields around BHs reveals that a pair wave created with a temporal frequency Ω satisfies [2,8]:

$$\left(\frac{\beta^-}{\beta^+}\right)^2 = \exp\left(-\frac{\Omega}{T}\right), \quad (11)$$

where $\beta^{+(-)}$ are referred to as the positive (negative) norm amplitudes (also known as the Bogoliubov coefficients),

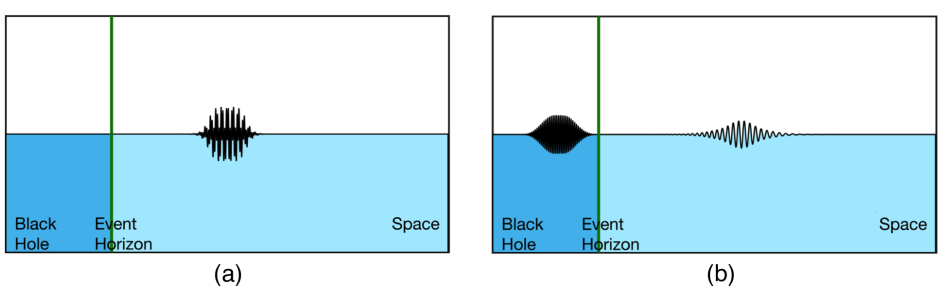

FIG. 4. Simulation of zero-sum pseudoenergy wave packet pair for $Fr = 0.7$. (a) Configuration at $t = 0$, and (b) configuration at a later time when the "+" wave packet escapes the BH while the "−" wave packet falls inside it.

and T denotes an effective temperature proportional to the surface gravity of a BH. According the Hawking's prediction $(\beta^-)^2 = [\exp(\Omega/T) - 1]^{-1}$, which implies divergence as $\Omega \to 0$ since for this limit, $(\beta^-)^2 \approx T/\Omega$.

In analog gravity experiments with surface waves in a countercurrent flow over a localized obstacle, parallels between Eq. (11) and the scattering coefficients were first established in Weinfurtner *et al.* [4], and then in subsequent studies, e.g., see Refs. [5,6]. The scattering coefficients in the analog-gravity experiments correspond to the wave action of the "+" and "−" waves [4]. We emphasize that here we have *not* solved a scattering problem, therefore its relevancy to Eq. (11) is somewhat limited. Yet, it is interesting to see that in the current analysis $\delta A^+ = \delta A^-$, hence the $\Omega \to 0$ limit of Eq. (11) is always satisfied. Furthermore, noting that

$$\delta A^{+(-)} = \frac{\rho g L}{2} \frac{\{a^{+(-)}\}^2}{\omega^{+(-)} + k^{+(-)}\bar{U}}, \tag{12}$$

we readily find that $\delta A^+ \to \infty$ when $\hat{\omega}^+ \to 0$, leading to both $k^+ \to 0$ and $\omega^+ \to 0$ [c.f. Fig. 3(a)]. Hence by construction $\delta A^- \to \infty$, however the denominator in Eq. (12) for this case does not vanish, rather $a^- \to \infty$. This fact can also be clearly observed from Eq. (9b). In summary, the aspect of low-frequency mode amplification in Hawking's prediction is satisfied by this minimal model.

V. DISCUSSION

The aim of this paper is to characterize the properties of zero-sum energy pair wave packets in the hydrodynamic analogy of Hawking radiation. First we wished to clarify the somewhat non-intuitive physical meaning of positive and negative energy norms (pseudoenergy), how those are related to the wave propagation mechanism, and how the general energy norm converges to the one suggested by Schützhold and Unruh [2] in the shallow water limit.

Next we considered a simple setup consisting of a constant subcritical countercurrent flow over a flat bathymetry; this setup was enough to demonstrate the analog phenomena where positive (negative) energy wave packets escape from (drifted into) the black hole. The combined requirements of a wave packet pair with equal (and positive in our case) wave action, and equal and opposite signed pseudoenergy, determine their centroid wave numbers as well as their surface elevation amplitude.

While forming such pairs of wave packets in the laboratory might not be a simple task, it is straight forward to numerically simulate stochastic generation of such zero-sum energy pairs, mimicking near-horizon vacuum fluctuations. The nonlinear effects of wave dissipation and wave-mean flow interaction, which feedback into the countercurrent and shift the horizon position, are under ongoing numerical investigation and will be published in a follow-up paper.

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APPENDIX: PSEUDOENERGY OF SHALLOW WATER GRAVITY WAVE

Writing the pseudoenergy explicitly, using Eqs. (6a) and (6b) we obtain

$$\delta E = \frac{\rho}{2} \int_{x=0}^{L} \left[\int_{z=-H}^{\eta'} (|\mathbf{u}'|^2 - 2\bar{U}u') dz + g\eta'^2 \right] dx. \tag{A1}$$

In the shallow water limit, $|\mathbf{u}'|^2 \Rightarrow u'^2$, and u' is not a function of z. Consequently the pseudo-energy expression for shallow water gravity waves for this setup becomes

$$\delta E_{\mathrm{SW}} = \frac{\rho}{2} \int_{x=0}^{L} (Hu'^2 + g\eta'^2 - 2\bar{U}u'\eta') dx. \tag{A2}$$

Let us define the perturbation velocity potential ϕ' to satisfy $\mathbf{u}' = \nabla\phi'$, then for the shallow water the linearized, time-dependent Bernoulli's potential equation (or equivalently, the linearized momentum in the x direction) implies

$$\left(\frac{\partial}{\partial t} - \bar{U}\frac{\partial}{\partial x}\right)\phi' = -g\eta'. \tag{A3}$$

This relation allows writing the integrand of Eq. (A2) solely in terms of ϕ'

$$\delta E_{\mathrm{SW}} = \frac{\rho}{2g} \int_{x=0}^{L} \left[gH\left(\frac{\partial \phi'}{\partial x}\right)^2 + \left(\frac{\partial \phi'}{\partial t}\right)^2 - \left(\bar{U}\frac{\partial \phi'}{\partial x}\right)^2 \right] dx, \tag{A4}$$

which is equivalent to the energy norm defined in Eqs. (67) and (68) in Schützhold and Unruh [2]. Furthermore, for the shallow water surface gravity wave, the amplitudes of the vertical displacement a, and the velocity potential amplitude $|\phi|$, are related by [12]

$$a = \frac{\alpha|\phi|}{\sqrt{gH}}.$$

Using Eq. (8) and $\hat{c}_p = \pm\sqrt{gH}$, we can express the pseudoenergy in terms of $|\phi|$ as

$$\delta E_{\mathrm{SW}} = \frac{\rho L}{2H} \alpha^2 |\phi|^2 (1 \mp Fr). \tag{A5}$$

Hence pseudoenergy for shallow-water waves is *always* positive for subcritical flows ($Fr < 1$). Therefore pairs of opposite pseudoenergy wave packets in subcritical flows require nonshallow water dynamics.

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