General gravitational Lagrangian with deformed covariance

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We derive the gravitational Lagrangian to all orders of curvature when the canonical constraint algebra is deformed by a phase space function as predicted by some studies into loop quantum cosmology. The deformation function seems to be required to satisfy a nonlinear conservation equation usually found in fluid mechanics and can form discontinuities quite generally. These results arise from attempting to consistently incorporate general spatial inhomogeneities in effective models of loop quantum cosmology rather than directly investigating the nature of signature change in such models. We work within the restriction of not allowing additional degrees of freedom.

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I. INTRODUCTION

Many effective models of quantum gravity work from the hypothesis that the symmetries of general relativity should be deformed by quantum effects. The mechanism by which it is implemented at the effective level is diverse and often only considers individual particles. This includes deformed special relativity [1] and rainbow gravity [2], which struggle to go beyond describing particles coupled to a metric dependent on the particle's energy. Such models can suffer from a breakdown of causality [3] or find it difficult to describe multiparticle states [4]. The model we consider in this paper, deformed general relativity, should not suffer from these problems by construction since it is energy density and curvature that the deformation depends on. However, it includes the possibility of metric signature change, which has a different implication for causality [5].

A specific kind of deformation consistently appears in some investigations of loop quantum cosmology, when loop quantization effects are introduced into minisuperspace models without causing anomalies [5–11]. The constraint algebra,[1] which ensures spacetime covariance is maintained when we have made a spacetime decomposition [12], is deformed by a phase space function $\beta(q, p)$. For a more in-depth review, please see Ref. [13].

In this paper we seek to derive the most general effective gravitational action that satisfies the deformed constraint algebra without introducing additional degrees of freedom and only includes the Ricci scalar for spatial curvature. We derive the restrictions on the Lagrangian in Sec. II. In Sec. III we use them to find the allowed forms of the deformation and the general Lagrangian. Curiously, we find the deformation function must satisfy the inviscid form of the Burgers' equation in curvature space. This may be related to the curved phase space hypothesis [14], which is known to be linked with similar models of deformed relativity. These calculations generalize those presented in Ref. [15] where the fourth order gravitational Lagrangian was perturbatively derived from the constraint algebra.[2] This is a companion paper to Ref. [16], wherein we calculate the general scalar-tensor Hamiltonian with deformed general covariance.

We foliate spacetime into a stack of time-labeled spatial hypersurfaces as usual for canonical relativity. We are using the same definitions as Refs. [16,17], but for full details, see Refs. [13,18]. Each spatial hypersurface has a metric q_{ab}, and the spatial slicing is characterized by the lapse N and the shift N^a. These act as Lagrange multipliers in the classical action, and they produce constraints (i.e., they vanish in the dynamical regime) definable from the total Hamiltonian,

$$C := \frac{\delta H}{\delta N}, \qquad D_a := \frac{\delta H}{\delta N^a}, \tag{1.1}$$

which are, respectively, known as the Hamiltonian constraint and the diffeomorphism constraint. The classical Poisson bracket structure of these constraints forms a Lie algebroid [19],

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[1]Also known as the hypersurface deformation algebra.

[2]An updated version of the calculation of the fourth order perturbative gravitational Lagrangian can be found in [13].

$$\{D_a[N^a], D_b[M^b]\} = D_a[\mathcal{L}_M N^a], \tag{1.2a}$$

$$\{C[N], D[M^a]\} = C[\mathcal{L}_M N], \tag{1.2b}$$

$$\{C[N], C[M]\} = D_a[q^{ab}(N\partial_b M - \partial_b N M)]. \tag{1.2c}$$

Since there are no anomalous terms (those not constrained to vanish), N and N^a are gauge functions, and therefore the observable dynamics are unaffected by the spatial slicing.

Classical general relativity with a spacetime decomposition can be formulated equivalently using different variables. Geometrodynamics usually uses the spatial metric q_{ab} and the extrinsic curvature $K_{ab} = \frac{1}{2}\mathcal{L}_n q_{ab}$. Connection dynamics uses the Ashtekar-Barbero connection A_a^I and densitized triads E_I^a [20,21]. The connection has an ambiguity in its definition given by the Barbero-Immirzi parameter γ, which parametrizes the contribution of the extrinsic curvature relative to the triad's spin connection, but the exact value of γ should not affect the dynamics [22]. The other prominent alternative is loop dynamics, which uses integrated versions of the A_a^I and E_I^a. If the integration regions are taken to be infinitesimal, then the one can easily relate loop dynamics and connection dynamics [23, p. 21].

Upon quantization, the different variables no longer describe equivalent dynamics, particularly with a dependence on γ [24,25]. Significantly, quantizing loop variables (loop quantum gravity) explicitly discretize the geometry, and so the integration regions cannot be taken to be infinitesimal [23, p. 105]. In this work, we use the metric variables of geometrodynamics because the comparison to classical models of gravity should be clearer, and there is no ambiguity arising from γ.

We are considering only the spatial metric field q_{ab} and its normal derivative $v_{ab} = \mathcal{L}_n q_{ab}$. Time derivatives above first-order, mixed-type derivatives such as $\nabla_c v_{ab}$ and tensor contractions of derivatives above second order are associated with additional degrees of freedom [26], and for simplicity we do not consider such terms in this paper. The only index-free quantities we can form up to second order in derivatives from the spatial metric are the determinant $q = \det q_{ab}$ and the Ricci curvature scalar R. The normal derivative can be split into its trace and traceless components, $v_{ab} = v_{ab}^{\mathsf{T}} + \frac{1}{3} v q_{ab}$, so it can form scalars from the trace v and a variety of contractions of the traceless tensor v_{ab}^{T}. However, to second order we only need to consider $w := v_{ab}^{\mathsf{T}} v_{\mathsf{T}}^{ab}$. Therefore, we consider the Lagrangian given by $L = L(q, v, w, R)$. Throughout this investigation we often need to take the logarithmic derivative with respect to the metric determinant, so we will use the simplifying definition $\partial_q := \frac{\partial}{\partial \log q}$.

Some studies into loop quantum cosmology predict that the classical constraint algebra should be deformed in a specific way when loop-quantization effects are included without introducing anomalies [5–11]. This retains gauge invariance, and therefore arbitrariness of the lapse and shift. Were the constraint algebra to contain anomalous terms, then the theory would not be gauge invariant and solving the constraints would determine specific values for N and N^a. This privileging of a particular frame of reference demonstrates the breaking of general covariance.

In the referenced studies, the Poisson bracket of two Hamiltonian constraints (1.2c) is deformed by a phase space function β,

$$\{C[N], C[M]\} = D_a[\beta q^{ab}(N\partial_b M - \partial_b N M)]. \tag{1.3}$$

This has not been demonstrated to be generally true, but it is a feature that has appeared in several independent models. Since the constraint algebra is without anomaly, the model technically remains generally covariant. As interpreted in Ref. [12], Eq. (1.2c) specifies the form of C such that the spatial hypersurfaces can be embedded in spacetime geometry. So it may be that the deformed form of this as given by (1.3) implies that the embeddability is in some way no longer valid.

This deformation appears to be necessary only when $\gamma \in \mathbb{R}$. By comparison, when $\gamma = \pm i$, the deformation does not appear in similar calculations [27]. However, the latter case does not seem to resolve curvature singularities, and obtaining the correct classical limit is nontrivial [25] and therefore does not seem as desirable. The fact that our calculations consider $\beta \neq 1$ and assume the correct classical limit means that, though we use metric variables, there should be relevance to the models of loop quantum cosmology with $\gamma \in \mathbb{R}$.

From the constraint algebra, we can derive equations which restrict the relationship between the deformation β and either the Hamiltonian constraint C or the Lagrangian L. The diffeomorphism constraint D_a is not affected when the deformation is a scalar[3] and so is completely determined [16]. With D_a as an input, and with assumptions about field content, we can find how β relates to C and thereby to L by manipulating (1.3).

Firstly, we must find the unsmeared form of the deformed algebra. For general canonical variables (q_I, p_I),

$$0 = \{C[N], C[M]\} - D_a[\beta q^{ab}(N\partial_b M - \partial_b N M)] \tag{1.4a}$$

$$= \int d^3z \left\{ \sum_I \frac{\delta C[N]}{\delta q_I(z)} \frac{\delta C[M]}{\delta p_I(z)} - (D^a \beta N \partial_a M)_z \right\} - (N \leftrightarrow M). \tag{1.4b}$$

For when we wish to derive the action instead of the constraint, we can transform the equation by first noting that

[3] That is, when β has a density weight of zero [18].

$$\frac{\delta C[N]}{\delta q_I} = -\frac{\delta L[N]}{\delta q_I}, \qquad N v_I = \frac{\delta C[N]}{\delta p_I}, \tag{1.5}$$

where $v_I \coloneqq \mathcal{L}_n q_I$ and the Lagrangian is defined such that $S = \int \mathrm{d}t\mathrm{d}^3xNL = \int \mathrm{d}tL[N]$. We substitute these into (1.4b), then take functional derivatives to remove N and M, and are left with a distribution equation,

$$0 = \sum_I \frac{\delta L(x)}{\delta q_I(y)} v_I(y) + (\beta D^a \partial_a)_x \delta(x,y) - (x \leftrightarrow y). \tag{1.6}$$

To find a useful form for this, we need to use a specific form for the diffeomorphism constraint. Because it depends on momenta, we must replace them using

$$p_I \coloneqq \frac{\delta S}{\delta \dot{q}_I} = \frac{1}{N}\frac{\delta L[N]}{\delta v_I}, \tag{1.7}$$

and, as before, if we note that we will only consider actions without mixed derivatives, this simplifies to

$$p_I = \frac{\partial L}{\partial v_I}. \tag{1.8}$$

Therefore, substituting the diffeomorphism constraint, and the momenta (1.8) into (1.6), we find the distribution equation which can be used for restricting the form of the deformed action.

It would be helpful to have a general understanding of how a deformation function that depends on curvature affects the form of the action before we calculate the specific solutions. In particular, can a curvature-deformed Lagrangian be a finite-order polynomial of curvature? Consider the distribution equation (1.6) with only a scalar field,

$$0 = \frac{\delta L(x)}{\delta \psi(y)} v_\psi(y) + \left(\beta \frac{\partial L}{\partial v_\psi} \partial^a \psi \partial_a\right)_x \delta(x,y) - (x \leftrightarrow y), \tag{1.9}$$

where we have used the diffeomorphism constraint for a scalar field $D_a = p_\psi \partial_a \psi$ [16], where $p_\psi = \frac{\partial L}{\partial v_\psi}$. Let us consider a simplified model to match the derivative orders for the deformation and the derivative orders for the Lagrangian in a way analogous to dimensional analysis. First order time derivatives are given by v_ψ, and two orders of spatial derivatives are given by Δ. We can collect terms in the distribution equation of the same order of time derivatives as they are linearly independent. Schematically, the distribution equation is given by

$$0 = \frac{\partial L}{\partial \Delta} v_\psi + \frac{\partial L}{\partial v_\psi} \beta, \tag{1.10}$$

and expanding the Lagrangian and deformation in powers of v_ψ,

$$L = \sum_{m=0}^{n_L} L^{(m)} v_\psi^m, \qquad \beta = \sum_{m=0}^{n_\beta} \beta^{(m)} v_\psi^m, \tag{1.11}$$

the coefficient of v_ψ^n is then given by

$$0 = \frac{\partial L^{(n-1)}}{\partial \Delta} + \sum_{m=0}^{n_\beta} (n-m+1) L^{(n-m+1)} \beta^{(m)}. \tag{1.12}$$

We can relabel and rearrange to find a schematic solution for the highest order of L appearing here,

$$L^{(n)} = \frac{-1}{n\beta^{(0)}} \left\{ \frac{\partial L^{(n-2)}}{\partial \Delta} + \sum_{m=1}^{n_\beta} (n-m) \beta^{(m)} L^{(n-m)} \right\}. \tag{1.13}$$

We can see that if $n_\beta > 0$, then this equation is recursive and $n_L \to \infty$ because there is no natural cutoff, suggesting that a deformed L is required to be nonpolynomial. If we wish to truncate the action at some order, then it must be treated as a perturbative approximation. We considered a perturbative fourth order Lagrangian in Ref. [15]; footnote 2, and the nonperturbative gravitational Lagrangian is considered in this paper.

II. SOLVING THE DISTRIBUTION EQUATION

The general deformed Lagrangian must satisfy the distribution equation from (1.6), which when we are only considering metric variables is given by

$$0 = \frac{\delta L(x)}{\delta q_{ab}(y)} v_{ab}(y) + (\beta D^a \partial_a)_x \delta(x,y) - (x \leftrightarrow y). \tag{2.1}$$

As shown in Ref. [16] the diffeomorphism constraint for a metric is uniquely given by

$$D^a = -2\nabla_b p^{ab} = -2(\delta^a_{(b}\partial_{c)} + \Gamma^a_{bc}) \frac{\partial L}{\partial v_{bc}}. \tag{2.2}$$

First, we integrate (2.1) by parts to move spatial derivatives from L onto the delta functions. We discard the surface term and find

$$0 = \frac{\delta L(x)}{\delta q_{ab}(y)} v_{ab}(y) - 2\left(\beta \frac{\partial L}{\partial v_{bc}} \Gamma^a_{bc} \partial_a\right)_x \delta(x,y) + 2\left(\frac{\partial L}{\partial v_{ab}} \partial_b\right)_x [(\beta \partial_a)_x \delta(x,y)] - (x \leftrightarrow y); \tag{2.3}$$

from this we take the functional derivative with respect to $v_{ab}(z)$ (after relabeling the other indices),

$$0 = \frac{\delta L(x)}{\delta q_{ab}(y)}\delta(y,z) + \left\{\frac{\delta\partial L(x)}{\delta q_{cd}(y)\partial v_{ab}(x)}v_{cd}(y) + 2\left(\frac{\partial\beta_{,d}}{\partial v_{ab,e}}\frac{\partial L}{\partial v_{cd}}\partial_c\right)_x\delta(x,y)\partial_{d(x)} + 2\left[\frac{\partial}{\partial v_{ab}}\left(\partial_d\beta\frac{\partial L}{\partial v_{cd}} - \beta\frac{\partial L}{\partial v_{de}}\Gamma^c_{de}\right)\partial_c + \frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{cd}}\right)\partial_{cd}\right]_x\delta(x,y)\right\}\delta(x,z) - (x\leftrightarrow y). \tag{2.4}$$

Following a procedure similar to that used in [28], we move the derivative from $\delta(x,z)$ and exchange some terms using the $(x\leftrightarrow y)$ symmetry to find it in the form

$$0 = A^{ab}(x,y)\delta(y,z) - A^{ab}(y,x)\delta(x,z), \tag{2.5}$$

where

$$A^{ab}(x,y) = \frac{\delta L(x)}{\delta q_{ab}(y)} - v_{cd}(x)\frac{\delta\partial L(y)}{\delta q_{cd}(x)\partial v_{ab}(y)} + 2\left\{\frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{de}}\Gamma^c_{de} - \partial_d\beta\frac{\partial L}{\partial v_{cd}}\right)\partial_c - \frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{cd}}\right)\partial_{cd} + \partial_e\left(\frac{\partial\beta_{,d}}{\partial v_{ab,e}}\frac{\partial L}{\partial v_{cd}}\right)\partial_d\right\}_y\delta(y,x). \tag{2.6}$$

Integrating over y, we find that part of the equation can be combined into a tensor dependent only on x,

$$0 = A^{ab}(x,z) - \delta(z,x)\int \mathrm{d}^3y A^{ab}(y,x) = A^{ab}(x,z) - \delta(z,x)A^{ab}(x), \quad \text{where } A^{ab}(x) = \int \mathrm{d}^3y A^{ab}(y,x). \tag{2.7}$$

Substituting in the definition of $A^{ab}(x,z)$ and then relabeling,

$$0 = \frac{\delta L(x)}{\delta q_{ab}(y)} - v_{cd}(x)\frac{\delta\partial L(y)}{\delta q_{cd}(x)\partial v_{ab}(y)} + 2\left\{\frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{de}}\Gamma^c_{de} - \partial_d\beta\frac{\partial L}{\partial v_{cd}}\right)\partial_c - \frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{cd}}\right)\partial_{cd} + \partial_e\left(\frac{\partial\beta_{,d}}{\partial v_{ab,e}}\frac{\partial L}{\partial v_{cd}}\right)\partial_c\right\}_y\delta(y,x) - A^{ab}(x)\delta(x,y). \tag{2.8}$$

To find this in terms of one independent variable, we multiply by the test tensor $\theta_{ab}(y)$, and integrate by parts over y. Then collecting derivatives of θ_{ab},

$$0 = \theta_{ab}(\cdots)^{ab} + \partial_c\theta_{ab}\left\{\frac{\partial L}{\partial q_{ab,c}} + v_{de}\frac{\partial^2 L}{\partial q_{de,c}\partial v_{ab}} - 2v_{ef}\partial_d\left(\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}}\right) + 2\frac{\partial}{\partial v_{ab}}\left(\partial_d\beta\frac{\partial L}{\partial v_{cd}} - \beta\frac{\partial L}{\partial v_{de}}\Gamma^c_{de}\right) - 4\partial_d\left[\frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{cd}}\right)\right] + 2\partial_e\left(\frac{\partial\beta_{,d}}{\partial v_{ab,c}}\frac{\partial L}{\partial v_{de}}\right)\right\} + \partial_{cd}\theta_{ab}\left\{\frac{\partial L}{\partial q_{ab,cd}} - v_{ef}\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}} - 2\frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{cd}}\right) + 2\frac{\partial\beta_{,e}}{\partial v_{ab,(c}}\frac{\partial L}{\partial v_{d)e}}\right\}, \tag{2.9}$$

where we have discarded the terms containing θ_{ab} without derivatives, because they do not provide any restrictions on the form of the Lagrangian. This is simplified by noting that ∂_c and $\frac{\partial}{\partial v_{ab}}$ commute, and that $\frac{\partial\beta_{,e}}{\partial v_{ab,c}} = \delta^c_e\frac{\partial\beta}{\partial v_{ab}}$. Therefore, the solution is given by

$$0 = \theta_{ab}(\cdots)^{ab} + \partial_c\theta_{ab}\left\{\frac{\partial L}{\partial q_{ab,c}} + v_{de}\frac{\partial^2 L}{\partial q_{de,c}\partial v_{ab}} - 2v_{ef}\partial_d\left(\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}}\right) - 2\Gamma^c_{de}\frac{\partial}{\partial v_{ab}}\left(\beta\frac{\partial L}{\partial v_{de}}\right) - 2\partial_d\beta\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}} - 4\beta\partial_d\left(\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}}\right) - 2\frac{\partial\beta}{\partial v_{ab}}\partial_d\left(\frac{\partial L}{\partial v_{cd}}\right)\right\} + \partial_{cd}\theta_{ab}\left\{\frac{\partial L}{\partial q_{ab,cd}} - v_{ef}\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}} - 2\beta\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}}\right\}. \tag{2.10}$$

To find the derivatives with respect to spatial derivatives of the metric, we must use equations from Ref. [16] for decomposing the Ricci curvature scalar.

A. Finding the conditions on the Lagrangian

Unlike similar previous studies such as Refs. [15,29], to reach this point we have not used an ansatz for the Lagrangian. Going forward, we make assumptions about what scalar variables the Lagrangian depends on, but we do not make a finite-order expansion in the hope of finding as general a solution as possible.

Substituting the variables into (2.10), the resulting equation contains a series of unique tensor combinations. The test tensor θ_{ab} is completely arbitrary so the coefficient of each unique tensor contraction with it must independently vanish. For example, suppose that we find the equation $0 = B^{ab}\theta_{ab}$. Decomposing B^{ab} in terms of the only available tensors, q_{ab} and v^{T}_{ab}, we find

$$0 = q^{ab}\theta_{ab}B_0 + v_{\mathsf{T}}^{ab}\theta_{ab}B_1 + v_{\mathsf{T}}^{ac}v_{\mathsf{T}}^{bd}q_{cd}\theta_{ab}B_2 + v_{\mathsf{T}}^{ac}v_{\mathsf{T}}^{bd}v^{\mathsf{T}}_{cd}\theta_{ab}B_3 + \cdots. \tag{2.11}$$

For this to be satisfied for general metrics, the coefficient of each unique term must vanish independently, so we can conclude in this example that $B_I = 0$ for all I.

First, we focus on the terms depending on the second order derivative $\partial_{cd}\theta_{ab}$. We evaluate each individual term in Appendix. Substituting (A3) into (2.10), we find the following independent conditions:

$$q^{ab}\partial^2\theta_{ab}: 0 = \frac{\partial L}{\partial R} - \frac{2v}{3}\frac{\partial^2 L}{\partial R\partial v} + 2\beta\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right), \tag{2.12a}$$

$$q^{ac}q^{bd}\partial_{cd}\theta_{ab}: 0 = \frac{\partial L}{\partial R} - 4\beta\frac{\partial L}{\partial w}, \tag{2.12b}$$

$$q^{ab}v_{\mathsf{T}}^{cd}\partial_{cd}\theta_{ab}: 0 = \frac{\partial^2 L}{\partial R\partial v} + 4\beta\frac{\partial^2 L}{\partial w\partial v}, \tag{2.12c}$$

$$v_{\mathsf{T}}^{ab}\partial^2\theta_{ab}: 0 = \frac{v}{3}\frac{\partial^2 L}{\partial R\partial w} - \beta\frac{\partial^2 L}{\partial v\partial w}, \tag{2.12d}$$

$$v_{\mathsf{T}}^{ab}v_{\mathsf{T}}^{cd}\partial_{cd}\theta_{ab}: 0 = \frac{\partial^2 L}{\partial R\partial w} + 4\beta\frac{\partial^2 L}{\partial w^2}. \tag{2.12e}$$

Before we analyze these equations, we will find the conditions from the first order derivative part of (2.10). There are many different complicated configurations of tensors, so we define $X_a \coloneqq q^{bc}\partial_a q_{bc}$ and $Y_a \coloneqq q^{bc}\partial_c q_{ab}$ for convenience. We evaluate the individual terms in Appendix. When we substitute the results (A4) into (2.10), we once again find a series of unique tensor combinations with their own coefficient that vanishes independently. Most of these conditions are the same as those found in (2.12) so we will not bother duplicating them again here. However, we do find the following new conditions:

$$X^a\partial^b\theta_{ab}: 0 = \frac{\partial L}{\partial R} - 4(\partial_q\beta + 2\beta\partial_q)\frac{\partial L}{\partial w}, \tag{2.13a}$$

$$q^{ab}X^c\partial_c\theta_{ab}: 0 = \frac{-1}{2}\frac{\partial L}{\partial R} + \frac{v}{3}(4\partial_q - 1)\frac{\partial^2 L}{\partial v\partial R} + \frac{\partial\beta}{\partial v}(1 - 2\partial_q)\frac{\partial L}{\partial v} + (\beta - 2\partial_q\beta - 4\beta\partial_q)\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right), \tag{2.13b}$$

$$v_{\mathsf{T}}^{ab}X^c\partial_c\theta_{ab}: 0 = \frac{v}{3}(4\partial_q - 1)\frac{\partial^2 L}{\partial w\partial R} + \frac{\partial\beta}{\partial w}(1 - 2\partial_q)\frac{\partial L}{\partial v} + (\beta - 2\partial_q\beta - 4\beta\partial_q)\frac{\partial^2 L}{\partial v\partial w}, \tag{2.13c}$$

$$q^{ab}v_{\mathsf{T}}^{cd}X_d\partial_c\theta_{ab}: 0 = (1 - 2\partial_q)\frac{\partial^2 L}{\partial v\partial R} - 4(\partial_q\beta + 2\beta\partial_q)\frac{\partial^2 L}{\partial v\partial w} - 4\frac{\partial\beta}{\partial v}\partial_q\frac{\partial L}{\partial w}, \tag{2.13d}$$

$$v_{\mathsf{T}}^{ab}v_{\mathsf{T}}^{cd}X_d\partial_c\theta_{ab}: 0 = (1 - 2\partial_q)\frac{\partial^2 L}{\partial w\partial R} - 4(\partial_q\beta + 2\beta\partial_q)\frac{\partial^2 L}{\partial w^2} - 4\frac{\partial\beta}{\partial w}\partial_q\frac{\partial L}{\partial w}, \tag{2.13e}$$

$$q^{ab}v_{\mathsf{T}}^{cd}Y_d\partial_c\theta_{ab}: 0 = 2\beta\frac{\partial^2 L}{\partial v\partial w} + \frac{\partial\beta}{\partial v}\frac{\partial L}{\partial w}, \tag{2.13f}$$

$$v_{\mathsf{T}}^{ab}v_{\mathsf{T}}^{cd}Y_d\partial^c\theta_{ab}: 0 = 2\beta\frac{\partial^2 L}{\partial w^2} + \frac{\partial\beta}{\partial w}\frac{\partial L}{\partial w}, \tag{2.13g}$$

$$\partial^a F\partial^b\theta_{ab}: 0 = \left(\frac{\partial\beta}{\partial F} + 2\beta\frac{\partial}{\partial F}\right)\frac{\partial L}{\partial w}, \tag{2.13h}$$

$$q^{ab}\partial^c F\partial_c\theta_{ab}: 0 = \frac{2v}{3}\frac{\partial^3 L}{\partial F\partial v\partial R} - \frac{\partial\beta}{\partial v}\frac{\partial^2 L}{\partial F\partial v} - \left(\frac{\partial\beta}{\partial F} + 2\beta\frac{\partial}{\partial F}\right)\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right), \tag{2.13i}$$

$$v_{\mathsf{T}}^{ab}\partial^c F\partial_c\theta_{ab}: 0 = \frac{2v}{3}\frac{\partial^3 L}{\partial F\partial w\partial R} - \frac{\partial\beta}{\partial w}\frac{\partial^2 L}{\partial F\partial v} - \left(\frac{\partial\beta}{\partial F} + 2\beta\frac{\partial}{\partial F}\right)\frac{\partial^2 L}{\partial v\partial w}, \tag{2.13j}$$

$$q^{ab}v_{\mathsf{T}}^{cd}\partial_d F\partial_c\theta_{ab}: 0 = \frac{1}{2}\frac{\partial^3 L}{\partial F\partial v\partial R} + \frac{\partial\beta}{\partial v}\frac{\partial^2 L}{\partial F\partial w} + \left(\frac{\partial\beta}{\partial F} + 2\beta\frac{\partial}{\partial F}\right)\frac{\partial^2 L}{\partial v\partial w}, \tag{2.13k}$$

$$v_{\mathsf{T}}^{ab}v_{\mathsf{T}}^{cd}\partial_d F\partial_c\theta_{ab}: 0 = \frac{1}{2}\frac{\partial^3 L}{\partial F\partial w\partial R} + \frac{\partial\beta}{\partial w}\frac{\partial^2 L}{\partial F\partial w} + \left(\frac{\partial\beta}{\partial F} + 2\beta\frac{\partial}{\partial F}\right)\frac{\partial^2 L}{\partial w^2}, \tag{2.13l}$$

where $F \in \{v, w, R\}$.

We have now acquired all conditions on the form of the Lagrangian for our choice of variables. The next step is to try and consolidate them.

III. DERIVING THE LAGRANGIAN

As shown in (1.13), the deformed Lagrangian must be calculated either perturbatively, as has been done in Refs. [13,15], or completely generally. Before we attempt the general calculation, we note the results found for the perturbative case which were derived in Ref. [13] (though an incomplete form of the calculation was first done in Ref. [15]). For a deformation function that depends quadratically on derivatives and an action which depends quartically on derivatives, we found that a deformed covariance was perturbatively maintained for the solutions,

$$\beta = \beta_\varnothing + \beta_{(R)}\left(R + \frac{\mathcal{K}}{\beta_\varnothing}\right) + \mathcal{O}(\partial q^3), \tag{3.1a}$$

$$L = L_\varnothing + \xi v\sqrt{q} + \frac{\omega}{2}\sqrt{q|\beta_\varnothing|}\left\{R - \frac{\mathcal{K}}{\beta_\varnothing} - \frac{\beta_{(R)}}{4\beta_\varnothing}\left(R + \frac{\mathcal{K}}{\beta_\varnothing}\right)^2\right\} + \mathcal{O}(\partial q^5), \tag{3.1b}$$

where $\mathcal{K}$ is what we call the standard extrinsic curvature contraction,

$$\mathcal{K} = K^2 - K_{ab}K^{ab} = \frac{1}{4}(v^2 - v_{ab}v^{ab}) = \frac{v^2}{6} - \frac{w}{4}. \tag{3.2}$$

We now turn to the calculation of the general deformed Lagrangian. Take Eqs. (2.12) and (2.13), which solve the distribution equation for the gravitational Lagrangian when we expand it in terms of the variables $\{q, v, w, R\}$, and see what can be deduced about the effective action when it is treated nonperturbatively. Starting with the condition for $\partial^a F\partial^b\theta_{ab}$ where $F \in \{v, w, R\}$, Eq. (2.13h), this can be rewritten as

$$0 = \beta\left(\frac{\partial L}{\partial w}\right)^2\frac{\partial}{\partial F}\log\left\{\beta\left(\frac{\partial L}{\partial w}\right)^2\right\}, \tag{3.3}$$

which implies that

$$\beta\left(\frac{\partial L}{\partial w}\right)^2 = \lambda_1(q), \tag{3.4}$$

and so we can solve up to a sign, $\sigma_L = \pm 1$,

$$\frac{\partial L}{\partial w} = \sigma_L\sqrt{\left|\frac{\lambda_1}{\beta}\right|}. \tag{3.5}$$

Then, from $q^{ac}q^{bd}\partial_{cd}\theta_{ab}$, Eq. (2.12b), we find

$$\frac{\partial L}{\partial R} = 4\beta\frac{\partial L}{\partial w} = 4\sigma_L\sigma_\beta\sqrt{|\lambda_1\beta|}, \tag{3.6}$$

where $\sigma_\beta := \mathrm{sgn}(\beta(q, v, w, R))$. If we then match the second derivative of the Lagrangian, $\frac{\partial^2 L}{\partial w\partial R}$, using both equations, we find a nonlinear partial differential equation for the deformation function,

$$0 = \frac{\partial\beta}{\partial R} + 4\beta\frac{\partial\beta}{\partial w}, \tag{3.7}$$

which is the same form as Burgers' equation for a fluid with vanishing viscosity [30]. However, before we attempt to interpret this, we will find further restrictions on the Lagrangian and deformation.

We now seek to find how the trace of the metric's normal derivative, v, appears. Take the condition for $v_{\mathsf{T}}^{ab}\partial^2\theta_{ab}$, Eq. (2.12d),

$$0 = \frac{v}{3}\frac{\partial^2 L}{\partial R\partial w} - \beta\frac{\partial^2 L}{\partial v\partial w} = \frac{\sigma_L}{2}\sqrt{\left|\frac{\lambda_1}{\beta}\right|}\left(\frac{4v}{3}\frac{\partial\beta}{\partial w} + \frac{\partial\beta}{\partial v}\right), \tag{3.8}$$

which we can solve to find that $\beta = \beta(q, \bar{w}, R)$, where $\bar{w} = w - 2v^2/3$. So in the deformation, the trace v must always be paired with the traceless tensor squared w like this. We can see that this is related to the standard extrinsic curvature contraction by $\bar{w} = -4\mathcal{K}$. To find how the trace appears in the Lagrangian, we look at the condition from $q^{ab}\partial^2\theta_{ab}$, Eq. (2.12a),

$$0 = \frac{\partial L}{\partial R} - \frac{2v}{3}\frac{\partial^2 L}{\partial v \partial R} + 2\beta\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right). \qquad (3.9)$$

Substituting in our solutions so far, we can solve for the second derivative with respect to the trace,

$$\frac{\partial^2 L}{\partial v^2} = \frac{-4\sigma_L}{3}\sqrt{\left|\frac{\lambda_1}{\beta}\right|}\left(1 - \frac{v}{2}\frac{\partial \beta}{\partial v}\right). \qquad (3.10)$$

We integrate over v to find the first derivative,

$$\frac{\partial L}{\partial v} = \frac{-4v\sigma_L}{3}\sqrt{\left|\frac{\lambda_1}{\beta}\right|} + \xi_1(q,w,R) = \frac{-4v}{3}\frac{\partial L}{\partial w} + \xi_1(q,w,R), \qquad (3.11)$$

where ξ_1 is a function arising as an integration constant. To make sure that the solutions (3.5), (3.6), and (3.11) match for the second derivatives $\frac{\partial^2 L}{\partial v \partial R}$ and $\frac{\partial^2 L}{\partial v \partial w}$, we find that $\xi_1 = \xi_1(q)$. Therefore, from this we can see that the Lagrangian should have the time derivatives only appear in the combined form $\bar{w}$ apart from a single linear term $L \supset v\xi_1(q)$.

Now we just have to determine what restrictions there are on how the metric determinant appears in the Lagrangian. First, we have the condition from $X^a\partial^b\theta_{ab}$, Eq. (2.13a),

$$\begin{aligned} 0 &= \frac{\partial L}{\partial R} - 4(\partial_q \beta + 2\beta\partial_q)\frac{\partial L}{\partial w} \\ &= 4\sigma_L\sigma_\beta\sqrt{|\lambda_1\beta|}\left(1 - \frac{\partial_q\lambda_1}{\lambda_1}\right), \quad \therefore \lambda_1(q) = \lambda_2 q, \end{aligned} \qquad (3.12)$$

and second, from $v_{\mathsf{T}}^{ab}X^c\partial_c\theta_{ab}$, Eq. (2.13c),

$$\begin{aligned} 0 &= \frac{v}{3}(4\partial_q - 1)\frac{\partial^2 L}{\partial w \partial R} + \frac{\partial \beta}{\partial w}(1 - 2\partial_q)\frac{\partial L}{\partial v} \\ &\quad + (\beta - 2\partial_q\beta - 4\beta\partial_q)\frac{\partial^2 L}{\partial v \partial w} \\ &= \frac{\partial \beta}{\partial w}(\xi_1 - 2\partial_q\xi_1), \quad \therefore \xi_1(q) = \xi_2\sqrt{q}. \end{aligned} \qquad (3.13)$$

Both these results show that our Lagrangian will indeed have the correct density weight when $\beta \to 1$, that is, $L \propto \sqrt{q}$.

All the remaining conditions from the distribution equation that have not been explicitly referenced are already solved by what we have found so far, so to make progress we must now attempt to consolidate our equations to find an explicit form for the Lagrangian. If we integrate (3.5), we find

$$L = \sigma_L\sqrt{|q\lambda_2|}\int_0^{\bar{w}}\frac{\mathrm{d}\bar{w}'}{\sqrt{|\beta(q,\bar{w}',R)|}} + f_1(q,v,R), \qquad (3.14)$$

and then if we match the derivative of this with respect to v with (3.11), we find the v dependence of the function which appeared as an integration constant,

$$f_1(q,v,R) = v\xi_2\sqrt{q} + f_2(q,R). \qquad (3.15)$$

If we then match the derivative of (3.14) with respect to R with (3.6), we see that

$$\begin{aligned} \frac{\partial L}{\partial R} &= 4\sigma_L\sigma_\beta\sqrt{|q\lambda_2\beta|} \\ &= \frac{\partial f_2}{\partial R} - \frac{\sigma_L}{2}\sqrt{|q\lambda_2|}\int_0^{\bar{w}}\frac{\sigma_\beta \mathrm{d}\bar{w}'}{|\beta(q,\bar{w}',R)|^{3/2}}\frac{\partial}{\partial R}\beta(q,\bar{w}',R), \end{aligned} \qquad (3.16)$$

and using (3.7) to change the derivative of β,

$$\begin{aligned} &4\sigma_L\sigma_\beta\sqrt{|q\lambda_2\beta|} \\ &\quad = \frac{\partial f_2}{\partial R} + 2\sigma_L\sqrt{|q\lambda_2|}\int_0^{\bar{w}}\frac{\mathrm{d}\bar{w}'}{\sqrt{|\beta(q,\bar{w}',R)|}}\frac{\partial}{\partial \bar{w}'}\beta(q,\bar{w}',R), \end{aligned} \qquad (3.17)$$

from which we see we can change the integration variable,

$$4\sigma_L\sigma_\beta\sqrt{|q\lambda_2\beta|} = \frac{\partial f_2}{\partial R} + 2\sigma_L\sqrt{|q\lambda_2|}\int_{\beta(q,0,R)}^{\beta(q,\bar{w},R)}\frac{\mathrm{d}\beta'}{\sqrt{|\beta'|}}. \qquad (3.18)$$

The upper integration limit cancels with the left-hand side of the equality, and therefore

$$\frac{\partial f_2}{\partial R} = 4\sigma_L\sigma_0\sqrt{|q\lambda_2\beta(q,0,R)|}, \qquad (3.19)$$

where $\sigma_0 \coloneqq \mathrm{sgn}(\beta(q,0,R))$. Then integrating this over R,

$$f_2(q,R) = 4\sigma_L\sqrt{|q\lambda_2|}\int_0^R \sigma_0\sqrt{|\beta(q,0,R')|}\mathrm{d}R' + f_3(q), \qquad (3.20)$$

which means that finally we have our solution for the Lagrangian to all orders of curvature,

$$L = \sigma_L \sqrt{|q\lambda_2|}\left(\int_0^{\bar{w}} \frac{\mathrm{d}\bar{w}'}{\sqrt{|\beta(q,\bar{w}',R)|}} + 4\int_0^R \sigma_0 \sqrt{|\beta(q,0,R')|}\mathrm{d}R'\right) + v\xi_2\sqrt{q} + f_3(q). \tag{3.21}$$

Now, we test this with a zeroth order deformation so we can match terms with our previous results. Using $\beta = \beta_\varnothing(q)$,

$$L = \sigma_L \sqrt{|q\lambda_2|}\left(\frac{\bar{w}}{\sqrt{|\beta_\varnothing|}} + 4R\mathrm{sgn}(\beta_\varnothing)\sqrt{|\beta_\varnothing|}\right) + v\xi_2\sqrt{q} + f_3(q), \tag{3.22}$$

comparing this to Refs. [13,17] and using $\bar{w} = -4\mathcal{K}$ leads to

$$\sigma_L = \sigma_\beta, \qquad \sqrt{|\lambda_2|} = \frac{\omega}{8}, \qquad f_3 = -\sqrt{q}U(q), \tag{3.23}$$

and therefore, the full solution is given by

$$L = \frac{\omega\sigma_\beta\sqrt{q}}{2}\left(\int_0^R \sigma_0\sqrt{|\beta(q,0,R')|}\mathrm{d}R' - \int_0^{\mathcal{K}} \frac{\mathrm{d}\mathcal{K}'}{\sqrt{|\beta(q,\mathcal{K}',R)|}}\right) + \sqrt{q}(v\xi - U(q)), \tag{3.24}$$

where we have relabeled $\xi_2 \to \xi$, and the deformation function must satisfy the nonlinear partial differential equation,

$$\frac{\partial\beta}{\partial R} = \beta\frac{\partial\beta}{\partial\mathcal{K}}. \tag{3.25}$$

By performing a Legendre transform, we can see that the Hamiltonian constraint associated with this Lagrangian (3.24) is given by

$$C = \frac{\omega\sigma_\beta\sqrt{q}}{2}\left\{\int_0^{\mathcal{K}} \frac{\mathrm{d}\mathcal{K}'}{\sqrt{|\beta(q,\mathcal{K}',R)|}} - \frac{2\mathcal{K}}{\sqrt{|\beta(q,\mathcal{K},R)|}} - \int_0^R \sigma_0\sqrt{|\beta(q,0,R')|}\mathrm{d}R'\right\} + \sqrt{q}U. \tag{3.26}$$

A. Solving for the deformation function

The nonlinear partial differential equation for the deformation function (3.25) is an unexpected result and invites a comparison to a very different area of physics. We can compare it to Burgers' equation for nonlinear diffusion [30]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \eta\frac{\partial^2 u}{\partial x^2}, \tag{3.27}$$

(where u is a density function) and see that our equation is very similar to the "inviscid" limit of vanishing viscosity $\eta \to 0$. This equation is not trivial to solve because it can develop discontinuities where the equation breaks down, termed "shock waves." Returning to our own equation (3.25), we analyze its characteristics. It implies that there are trajectories parametrized by s given by

$$\frac{\mathrm{d}q}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}R}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}s} = -\beta(q,\mathcal{K},R), \tag{3.28}$$

along which β is constant. These trajectories have gradients given by

$$\frac{\mathrm{d}R}{\mathrm{d}\mathcal{K}} = \frac{-1}{\beta(q,\mathcal{K},R)}, \tag{3.29}$$

and because β is constant along the trajectories, they are a straight line in the $(\mathcal{K}, R)$ plane. We must have an "initial" condition in order to solve the equation, and because R is here the analogue of $-t$ in (3.27) we define the initial function when $R = 0$, given by $\beta(q,\mathcal{K},0) =: \alpha(q,\mathcal{K})$. Since there are trajectories along which β is constant, we can use α to solve for $R(\mathcal{K})$ along those curves, given an initial value $\mathcal{K}_0$,

$$R = \frac{\mathcal{K}_0 - \mathcal{K}}{\alpha(\mathcal{K}_0)}. \tag{3.30}$$

We reorganize to get $\mathcal{K}_0 = \mathcal{K} + R\alpha(\mathcal{K}_0)$, and then substitute into β. This leads to the implicit relation,

$$\beta(q,\mathcal{K},R) = \alpha(q,\mathcal{K} + R\beta(q,\mathcal{K},R)). \tag{3.31}$$

We invoke the implicit function theorem to calculate the derivatives of β,

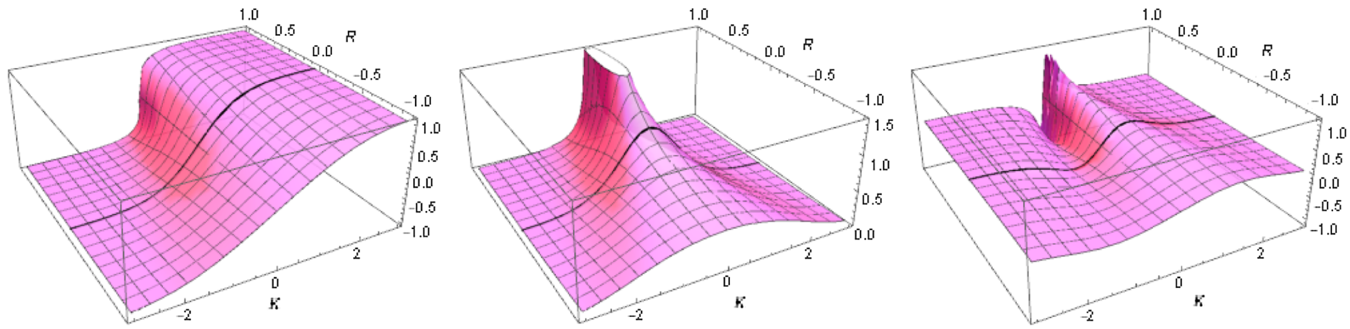
$$\frac{\partial\beta}{\partial\mathcal{K}} = \frac{\alpha'}{1 - R\alpha'}, \qquad \frac{\partial\beta}{\partial R} = \frac{-\beta\alpha'}{1 - R\alpha'}, \tag{3.32}$$

which show that a discontinuity develops when $R\alpha' \to 1$. This is the point where the characteristic trajectories along which β is constant converge to form a caustic. Beyond this point, β seems to become a multivalued function.

An analytic solution to β exists only when α is linear in $\mathcal{K}$,

$$\alpha = \alpha_\varnothing(q) + \alpha_2(q)\mathcal{K}, \qquad \beta = \frac{\alpha_\varnothing(q) + \alpha_2(q)\mathcal{K}}{1 - \alpha_2(q)R}, \tag{3.33}$$

which matches the equations for linear $\beta(\mathcal{K}, 0)$ and the corresponding $\beta(0, R)$ found in Ref. [16]. When $|\alpha_2 R| \ll 1$, we can expand β into a series,



(a) β (b) $\partial\beta/\partial\mathcal{K}$ (c) $\partial\beta/\partial R$

FIG. 1. Numerically solved deformation function for initial function $\alpha = \tanh(\omega\mathcal{K})$. The numerical evolution breaks for $\omega R > 1$ because a discontinuity has developed. The initial function is indicated by the black line. The plots are in $\omega = 1$ units.

$$\beta \simeq \alpha_\varnothing + \alpha_2(\mathcal{K} + \alpha_\varnothing R)\sum_{n=0}^{\infty} R^n \alpha_2^n, \tag{3.34}$$

and by comparing this to the perturbative deformation found in Ref. [13] and written in Eq. (3.1) we can see the correspondence $\alpha_\varnothing = \beta_\varnothing$ and $\alpha_2 = \beta_{(R)}/\beta_\varnothing$. For nonlinear initial functions, the deformation must be found numerically. As a test, in Fig. 1, we numerically solve for β when $\alpha = \tanh(\omega\mathcal{K})$. We see that, as R increases, the positive gradient in $\mathcal{K}$ intensifies to form a discontinuity and softens as R decreases.

We have also numerically solved for the deformation when the initial function is given by $\alpha = \cos(\omega\mathcal{K})$, shown in Fig. 2. This function is somewhat motivated by loop quantum cosmology models with holonomy corrections [5,10,11]. As with the tanh numerical solution in Fig. 1, we see the positive gradient intensify and the negative gradient soften as R becomes more positive. We could not evolve the equations past the formation of the shock wave so from this we cannot determine for certain whether a periodicity emerges in R, but we can compare the cross sections for β in Fig. 2(d). This cross section appears to match that found in Ref. [16] when we attempted to find the correspondence between $\beta(\mathcal{K}, 0)$ and $\beta(\mathcal{R})$ for nonlinear functions. In that, $\mathcal{R}$ is a function of the canonical metric momentum and R. It would seem that $\beta(0, R)$ should be a nonvanishing function even when $\beta(\mathcal{K}, 0)$ does vanish for some values of $\mathcal{K}$.

When the inviscid Burgers' equation is being simulated in the context of fluid dynamics, a choice must be made on how to model the shock wave [30]. The direct continuation of the equation means that the density function u becomes multivalued, and the physical interpretation of it as a density breaks down. The alternative is to then propagate the shock wave as a singular object, which requires a modification to the equations.

Considering our case of the deformation function, allowing a shock wave to propagate does not seem to make sense. It might require being able to interpret β as a density function and the space of $(\mathcal{K}, R)$ to be interpreted as a medium. Whether the shock wave remains singular or becomes multivalued, the most probable interpretation is that it represents a disconnection between different branches of curvature configurations. That is, for a universe to transition from one side of the discontinuity to the other may require taking an indirect path through the phase space. It may be that the behavior in $(\mathcal{K}, R)$ is an embodiment of the curved momentum space hypothesis [14].

B. Linear deformation

If we take the analytic solution for the deformation function when its initial condition is linear (3.33), we can substitute it into the general form for the gravitational Lagrangian (3.24). If we assume we are in a region where $1 - \alpha_2 R > 0$, we get the solution

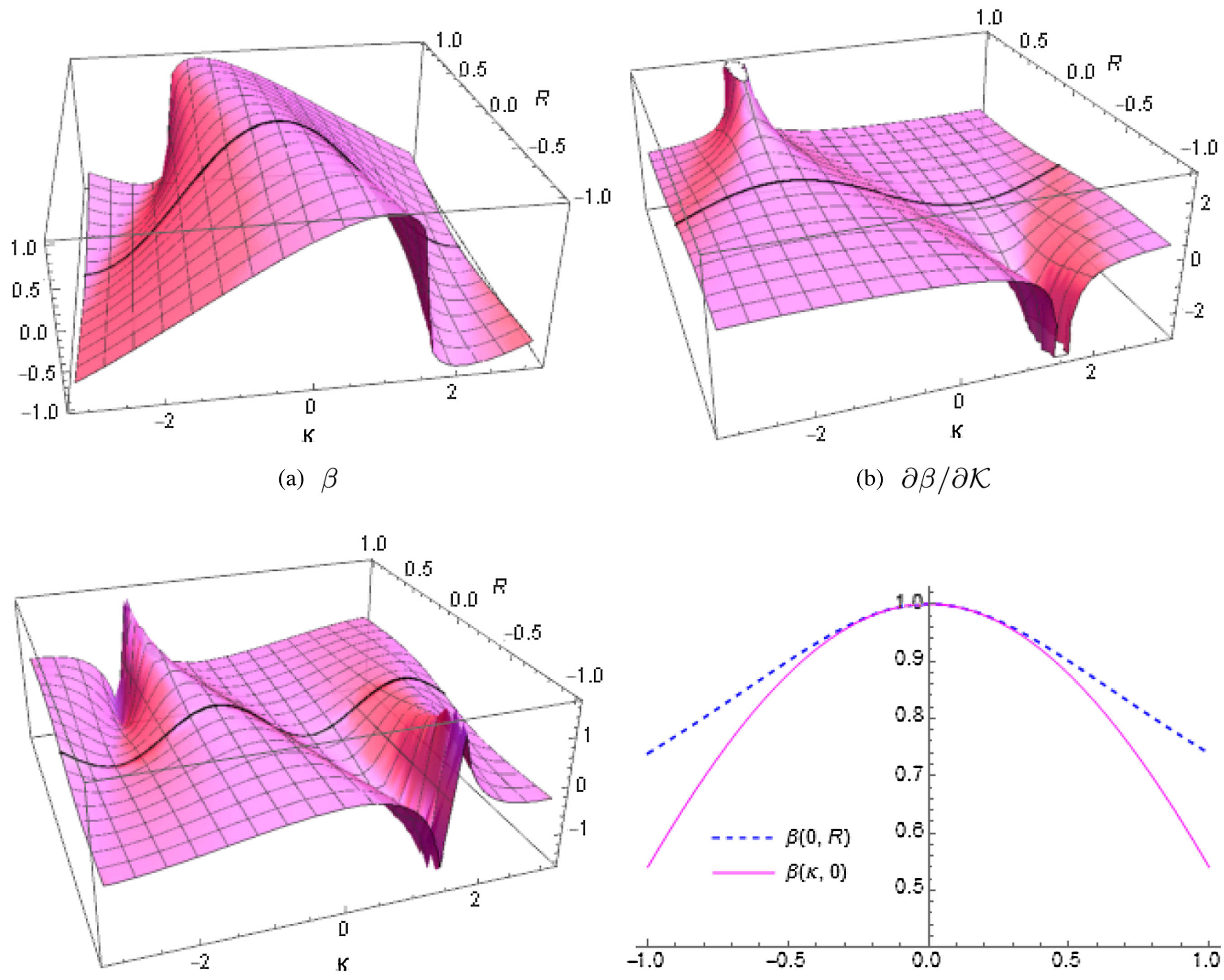
$$L = \frac{\omega\sqrt{q}}{\alpha_2}\left\{\mathrm{sgn}\left(1 + \frac{\alpha_2\mathcal{K}}{\alpha_\varnothing}\right)\sqrt{|\alpha_\varnothing|} - \sqrt{|\alpha_\varnothing + \alpha_2\mathcal{K}|}\sqrt{1 - \alpha_2 R}\right\} + \sqrt{q}(v\xi - U), \tag{3.35}$$

and expanding in series for small derivatives of the metric when we are in a region where $|\alpha_\varnothing| \gg |\alpha_2\mathcal{K}|$,

$$L = \frac{\omega}{2}\sqrt{q|\alpha_\varnothing|}\left(R - \frac{\mathcal{K}}{\alpha_\varnothing} - \frac{\alpha_2}{4}\left(R + \frac{\mathcal{K}}{\alpha_\varnothing}\right)^2 + \mathcal{O}(\partial q^5)\right) + \sqrt{q}(v\xi - U), \tag{3.36}$$

which matches exactly the previously calculated fourth order perturbative Lagrangian (3.1) when $\alpha_\varnothing = \beta_\varnothing$ and $\alpha_2 = \beta_{(R)}/\beta_\varnothing$.

The Hamiltonian constraint associated with the nonperturbative effective action can be found by using (3.26). Substituting in the Lagrangian for a linear deformation (3.35), we can solve for $\mathcal{K}$ when the constraint vanishes (as long as we specify that it must be finite in the limit $\alpha_2 \to 0$),



(a) β (b) $\partial\beta/\partial\mathcal{K}$

(c) $\partial\beta/\partial R$ (d) β cross sections

FIG. 2. Numerically solved deformation function with an initial function $\alpha = \cos(\omega\mathcal{K})$ and periodic boundary conditions. The numerical evolution breaks for $|\omega R| > 1$ because discontinuities have developed. The initial function is indicated by the black line. The plots are in $\omega = 1$ units.

$$\mathcal{K} = \left\{\frac{2}{\omega}\mathrm{sgn}(\alpha_\varnothing)\sqrt{|\alpha_\varnothing|}U\left(1 - \frac{\alpha_2 U}{2\omega\sqrt{|\alpha_\varnothing|}}\right) - \alpha_\varnothing R\right\} \times \left(1 - \frac{\alpha_2 U}{\omega\sqrt{|\alpha_\varnothing|}}\right)^{-2}. \tag{3.37}$$

If we restrict to the Friedmann-Lemaître-Robertson-Walker metric, where $q_{ab} = a^2\Sigma_{ab}$, $R = 6ka^{-2}$, $\mathcal{K} = 6\mathcal{H}^2$, and $U = \rho(a)$, as described in Ref. [16], we find the modified Friedmann equation,

$$\mathcal{H}^2 = \left\{\frac{\mathrm{sgn}(\alpha_\varnothing)\sqrt{|\alpha_\varnothing|}}{3\omega}\rho\left(1 - \frac{\alpha_2\rho}{2\omega\sqrt{|\alpha_\varnothing|}}\right) - \frac{\alpha_\varnothing k}{a^2}\right\} \times \left(1 - \frac{\alpha_2\rho}{\omega\sqrt{|\alpha_\varnothing|}}\right)^{-2}. \tag{3.38}$$

There is a correction term similar to that found for the fourth order perturbative Lagrangian [15] which suggests there could be a bounce when $\rho \to 2\omega\sqrt{|\alpha_\varnothing|}/\alpha_2$. However, there is also an additional factor that causes $\mathcal{H}$ to diverge when $\rho \to \omega\sqrt{|\alpha_\varnothing|}/\alpha_2$, which is before that potential bounce.

This is directly comparable to the modified Friedmann equation found for the deformation function $\beta(\mathcal{R}) = \beta_\varnothing(1 + \beta_2\mathcal{R})^{-1}$, investigated in Ref. [16], with $\alpha_\varnothing = \beta_\varnothing$ and $\alpha_2 = \omega\beta_2/2$. As is found here, those results suggested a sudden singularity where $\mathcal{H}$ diverges when a and ρ remain finite. Note that this is for the deformation function with a linear dependence on $\mathcal{K}$ which, unlike the cosine deformation, is not motivated by loop quantum cosmology. It does, however, demonstrate the difference that higher order corrections can have on dynamics.

IV. CONCLUSIONS

We derived the deformed effective gravitational action to all orders of scalar curvature from the deformed constraint algebra. The way the deformation function is differently affected by extrinsic and intrinsic curvature (i.e., by time and space derivatives) was found to be similar to a differential equation that usually appears in fluid mechanics. Discontinuities in the deformation function seem to be inevitable, but the interpretation of what they mean is not clear. The discontinuities might be avoided if there were natural restrictions on the sign of the deformation's coefficients or the curvature. This effect may be linked to the curved phase space hypothesis.

We sought to provide insight into the problem of incorporating spatial inhomogeneities into models of loop quantum cosmology with a deformed constraint algebra. From our results, we can see that it is indeed possible to determine the dependence of the deformation on spatial derivatives from its dependence on time derivatives. However, the lack of analytical solutions, and numerical solutions which tend toward discontinuities, means that determining general behavior is difficult. Unfortunately this study does not seem to provide much new information on the nature of signature change or the ultralocal regime in loop quantum cosmology.

There are important caveats to these conclusions. The use of metric variables rather than connection or loop variables might hinder comparison to the motivating studies. Considering the deformed algebra implicitly restricts us to real values of γ. Last, the order of tensor combinations and derivatives considered were limited, even though higher orders are likely to appear from quantum corrections.

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APPENDIX: EXTRA CALCULATIONS

For convenience, we use the abbreviating definitions,

$$Q_{abcd} \coloneqq q_{a(c}q_{d)b}, \qquad X_a \coloneqq q^{bc}\partial_a q_{bc}, \qquad Y_a \coloneqq q^{bc}\partial_c q_{ba} = \partial^b q_{ab}, \qquad Z_a \coloneqq v_{\mathsf{T}}^{bc}\partial_a q_{bc}, \qquad W_a \coloneqq v_{\mathsf{T}}^{bc}\partial_c q_{ba}. \tag{A1}$$

Evaluating each term in the $\partial_{cd}\theta_{ab}$ bracket of (2.10), by substituting in the variables

$$q \coloneqq \det q_{ab}, \qquad v \coloneqq q^{ab}v_{ab}, \qquad w \coloneqq v^{\mathsf{T}}_{ab}v_{\mathsf{T}}^{ab} = v_{ab}v^{ab} - \frac{1}{3}v^2, \qquad R \coloneqq q^{bc}R^a{}_{bac} \tag{A2}$$

and using the equations derived for decomposing R in Ref. [16],

$$\frac{\partial L}{\partial q_{ab,cd}} = (Q^{abcd} - q^{ab}q^{cd})\frac{\partial L}{\partial R}, \tag{A3a}$$

$$v_{ef}\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}} = \left(v_{\mathsf{T}}^{cd} - \frac{2}{3}vq^{cd}\right)\left(q^{ab}\frac{\partial^2 L}{\partial v\partial R} + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial w\partial R}\right), \tag{A3b}$$

$$\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}} = q^{ab}q^{cd}\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right) + 2Q^{abcd}\frac{\partial L}{\partial w} + 2(q^{ab}v_{\mathsf{T}}^{cd} + v_{\mathsf{T}}^{ab}q^{cd})\frac{\partial^2 L}{\partial v\partial w} + 4v_{\mathsf{T}}^{ab}v_{\mathsf{T}}^{cd}\frac{\partial^2 L}{\partial w^2}. \tag{A3c}$$

Evaluating each term in the $\partial_c\theta_{ab}$ bracket of (2.10),

$$\frac{\partial L}{\partial q_{ab,c}} = \frac{\partial L}{\partial R}\left(\frac{3}{2}Q^{abde}\partial^c q_{de} - q^{c(d}q^{e)(a}\partial^{b)}q_{de} + q^{ab}Y^c - \frac{1}{2}q^{ab}X^c - 2q^{c(a}Y^{b)} + q^{c(a}X^{b)}\right), \tag{A4a}$$

$$v_{ef}\frac{\partial^2 L}{\partial q_{ef,c}\partial v_{ab}} = \left(\frac{3}{2}Z^c - W^c - 2v_{\mathsf{T}}^{cd}Y_d + v_{\mathsf{T}}^{cd}X_d + \frac{v}{3}X^c\right)\left(q^{ab}\frac{\partial^2 L}{\partial v\partial R} + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial w\partial R}\right), \tag{A4b}$$

$$
\begin{aligned}
v_{ef}\partial_d\left(\frac{\partial^2 L}{\partial q_{ef,cd}\partial v_{ab}}\right) = {}& \left(Z^c - W^c + \frac{v}{3}X^c + \frac{v}{3}Y^c - v_{\mathsf{T}}^{cd}Y_d\right)\left(q^{ab}\frac{\partial^2 L}{\partial v\partial R} + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial w\partial R}\right)\\
&+ \left(v_{\mathsf{T}}^{cd} - \frac{2v}{3}q^{cd}\right)\left\{(q^{ab}\partial_d - Q^{abef}\partial_d q_{ef})\frac{\partial^2 L}{\partial v\partial R}\right.\\
&\left.+ 2(v_{\mathsf{T}}^{ab}\partial_d + Q^{abef}\partial_d v_{ef}^{\mathsf{T}} - 2v_{\mathsf{T}}^{e(a}q^{b)f}\partial_d q_{ef})\frac{\partial^2 L}{\partial w\partial R}\right\},
\end{aligned} \tag{A4c}
$$

$$
\begin{aligned}
\Gamma^c_{de}\frac{\partial^2 L}{\partial v_{ab}\partial v_{de}} = {}& (2q^{cd}q^{e(a}\partial^{b)}q_{de} - Q^{abde}\partial^c q_{de})\frac{\partial L}{\partial w} + (2W^c - Z^c)\left(q^{ab}\frac{\partial^2 L}{\partial v\partial w} + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial w^2}\right)\\
&+ \left(Y^c - \frac{1}{2}X^c\right)\left\{q^{ab}\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right) + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial v\partial w}\right\},
\end{aligned} \tag{A4d}
$$

$$
\Gamma^c_{de}\frac{\partial\beta}{\partial v_{ab}}\frac{\partial L}{\partial v_{cd}} = \left(q^{ab}\frac{\partial\beta}{\partial v} + 2v_{\mathsf{T}}^{ab}\frac{\partial\beta}{\partial w}\right)\left\{\left(Y^c - \frac{1}{2}X^c\right)\frac{\partial L}{\partial v} + (2W^c - Z^c)\frac{\partial L}{\partial w}\right\}, \tag{A4e}
$$

$$
\partial_d\beta\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}} = \partial^c\beta\left\{q^{ab}\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right) + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial v\partial w}\right\} + 2q^{c(a}\partial^{b)}\beta\frac{\partial L}{\partial w} + 2v_{\mathsf{T}}^{cd}\partial_d\beta\left(q^{ab}\frac{\partial^2 L}{\partial v\partial w} + 2v_{\mathsf{T}}^{ab}\frac{\partial^2 L}{\partial w^2}\right), \tag{A4f}
$$

$$
\begin{aligned}
\partial_d\left(\frac{\partial^2 L}{\partial v_{ab}\partial v_{cd}}\right) = {}& (q^{ab}\partial^c - q^{ab}Y^c - Q^{abef}\partial^c q_{ef})\left(\frac{\partial^2 L}{\partial v^2} - \frac{2}{3}\frac{\partial L}{\partial w}\right) + 2(q^{c(a}\partial^{b)} - q^{c(a}Y^{b)} - q^{c(e}q^{f)(a}\partial^{b)}q_{ef})\frac{\partial L}{\partial w}\\
&+ 2\left\{q^{ab}(v_{\mathsf{T}}^{cd}\partial_d - v_{\mathsf{T}}^{cd}Y_d - W^c + q^{cd}\partial^e v_{de}^{\mathsf{T}}) + v_{\mathsf{T}}^{ab}\partial^c - v_{\mathsf{T}}^{ab}Y^c + Q^{abef}(\partial^c v_{ef}^{\mathsf{T}} - v_{\mathsf{T}}^{cd}\partial_d q_{ef})\right.\\
&\left.- 2v_{\mathsf{T}}^{e(a}q^{b)f}\partial^c q_{ef}\right\}\frac{\partial^2 L}{\partial v\partial w} + 4\left\{v_{\mathsf{T}}^{ab}(v_{\mathsf{T}}^{cd}\partial_d - W^c - v_{\mathsf{T}}^{cd}Y_d + q^{cd}\partial^e v_{de}^{\mathsf{T}})\right.\\
&\left.+ Q^{abef}v_{\mathsf{T}}^{cd}\partial_d v_{ef}^{\mathsf{T}} - 2v_{\mathsf{T}}^{e(a}q^{b)f}v_{\mathsf{T}}^{cd}\partial_d q_{ef}\right\}\frac{\partial^2 L}{\partial w^2},
\end{aligned} \tag{A4g}
$$

$$
\frac{\partial\beta}{\partial v_{ab}}\partial_d\left(\frac{\partial L}{\partial v_{cd}}\right) = \left(q^{ab}\frac{\partial\beta}{\partial v} + 2v_{\mathsf{T}}^{ab}\frac{\partial\beta}{\partial w}\right)\left\{(\partial^c - Y^c)\frac{\partial L}{\partial v} + 2(v_{\mathsf{T}}^{cd}\partial_d + q^{cd}\partial^e v_{de}^{\mathsf{T}} - v_{\mathsf{T}}^{cd}Y_d - W^c)\frac{\partial L}{\partial w}\right\}. \tag{A4h}
$$

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