Novel dual relation and constant in Hawking-Page phase transitions

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Universal relations and constants have important applications in understanding a physical theory. In this article, we explore this issue for Hawking-Page phase transitions in Schwarzschild anti–de Sitter black holes. We find a novel exact dual relation between the minimum temperature of the $(d + 1)$-dimensional black hole and the Hawking-Page phase transition temperature in d dimensions, reminiscent of the holographic principle. Furthermore, we find that the normalized Ruppeiner scalar curvature is a universal constant at the Hawking-Page transition point. Since the Ruppeiner curvature can be treated as an indicator of the intensity of the interactions amongst black hole microstructures, we conjecture that this universal constant denotes an interaction threshold, beyond which a virtual black hole becomes a real one. This new dual relation and universal constant are fundamental in understanding Hawking-Page phase transitions, and might have new important applications in the black hole physics in the near future.

I. INTRODUCTION

Phase transitions are indicative of the competition between various internal interactions within a system and play a crucial role in contributing to our understanding of its macroscopic and microscopic properties. This becomes particularly pertinent when universal relations and constants emerge from such investigations, the most celebrated of which are critical exponents in mean field theory that are believed to depend only on general features of a system but not on its particular details.

In black hole physics the most well-known phase transition is that observed by Hawking and Page [1], which is between a thermal radiation phase and a stable large black hole phase in an anti–de Sitter (AdS) spacetime. This Hawking-Page (HP) phase transition was explained as a confinement/deconfinement phase transition in gauge theory by Witten [2] in the AdS/CFT correspondence, which is a holographic duality between a $(d + 1)$-dimensional quantum gravity and a d-dimensional quantum field theory [3–5]. It can also be understood as a solid/liquid phase transition in the context of black hole chemistry [6], where the cosmological constant Λ can regard as the system's pressure [7]:

$$P = -\frac{\Lambda}{8\pi G_d} = \frac{(d-1)(d-2)}{16\pi l^2}, \qquad (1)$$

where l is the AdS radius. We take the d-dimensional gravitational constant $G_d = 1$ for simplicity. If the black hole is charged, a complete analogy holds between the phases of a van der Waals fluid and the black hole [8].

We demonstrate here some universal relations and constants associated with the HP phase transition. We find for the first time a duality relation between the minimum temperature of a Schwarzschild-AdS black hole in $(d + 1)$ dimensions and the HP phase transition temperature in d dimensions. This provides us with a new interpretation of the metastable large black hole phase, that we expect is indicative of some new duality relation. We also find that the Ruppeiner scalar curvature at the HP transition point is a universal constant, independent of any parameters other than the spacetime dimension. Since the Ruppeiner scalar curvature is indicative of the microscopic interactions of the constituents of a system, we conjecture that there are universal properties underlying black hole microstructures that are key to the formation of black holes. We expect that our perspective on this phase transition will provide us with a foundation for understanding the special properties of other black holes in AdS spacetime.

We depict the HP phase transition and its properties in Fig. 1. It can be seen that there are two branches corresponding to small and large black hole phases. The small black hole has negative heat capacity, and thus it is

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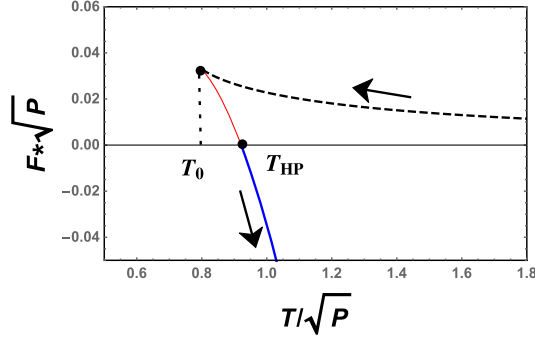weishw@lzu.edu.cn
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FIG. 1. Free energy of the four-dimensional Schwarzschild-AdS black hole. The arrows indicate increasing black hole horizon radius, and T_{HP} and T_0 denote the HP phase transition temperature and minimum temperature. The black dashed curve is the unstable small black hole branch, whereas the metastable and stable large black hole branches are described by the red thin solid and blue solid curves, respectively.

thermodynamically unstable, whereas the large black hole has positive specific heat and is stable. There exists a minimum temperature $T_0 = \sqrt{2P/\pi}$ below which no black hole can exist. In this parameter range the spacetime has only one phase, the thermal radiation phase, characterized by vanishing free energy. As the temperature of the system increases from zero, it becomes possible to form a large black hole for $T > T_0$, whose free energy is larger than that of the thermal radiation. As such, this black hole is metastable. Further increasing the temperature leads to a situation in which both the radiation and black hole have vanishing free energy, which is where the HP transition takes place, at $T_{\mathrm{HP}} = \sqrt{8P/3\pi}$. Above this temperature it is thermodynamically favorable for the radiation to collapse into a large black hole, which is the most stable phase.

The metastable large black hole branch (red thin curve in Fig. 1) that exists for $T_0 < T < T_{\mathrm{HP}}$ is often neglected. However for continuous decreasing temperature it is possible for a large black hole to pass through the HP point and for the metastable phase to emerge just like a supercooled liquid phase of water below its freezing point. We will show that this branch has a new and interesting interpretation.

II. NOVEL DUALITY IN THE HP TRANSITION

In d dimensions, the Hawking temperature of a Schwarzschild-AdS black hole is

$$T = \frac{4Pr_h}{d-2} + \frac{d-3}{4\pi r_h}, \tag{2}$$

with r_h the radius of the black hole horizon. The pressure P is defined in Eq. (1). The temperature is obviously dependent on the dimension of the spacetime. The free energy has the following form [9,10]:

$$F = \frac{\omega_{d-2} r_h^{d-3}((d-3)d - 16\pi P r_h^2 + 2)}{16\pi(d-2)(d-1)}, \tag{3}$$

where $\omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$ is the volume of the unit $(d-2)$-sphere. Combining (2) and (3), we can obtain the black hole minimum temperature and HP phase transition temperature

$$T_0 = \sqrt{\frac{4(d-3)}{(d-2)\pi}} \times \sqrt{P}, \tag{4}$$

$$T_{\mathrm{HP}} = \sqrt{\frac{4(d-2)}{(d-1)\pi}} \times \sqrt{P}. \tag{5}$$

These two temperatures share similar dependence on the pressure, $T \sim \sqrt{P}$ with different d-dependent coefficients. We illustrate the complete phase diagram in Fig. 2 for $d = 4$; the same qualitative picture holds for any $d > 4$. Indeed it is easy to see that

$$T_0(d) = \sqrt{\frac{2(d-3)}{d-2}} T_0(4), \tag{6}$$

$$T_{\mathrm{HP}}(d) = \sqrt{\frac{3(d-2)}{2(d-1)}} T_{\mathrm{HP}}(4) \tag{7}$$

allowing these temperatures to be computed in any dimension from knowledge of the $d = 4$ case; note that T_{HP}/T_0 is a decreasing function of d for $d > 3$. Three system phases—thermal radiation, a stable large black hole, and a metastable large black hole—are present. The minimum temperature T_0 and the HP transition temperature T_{HP} are respectively indicated by the blue solid and red dashed curves. These two curves extend to infinity [6], consistent with solid/supercooled-liquid/liquid phase behavior at any temperature.
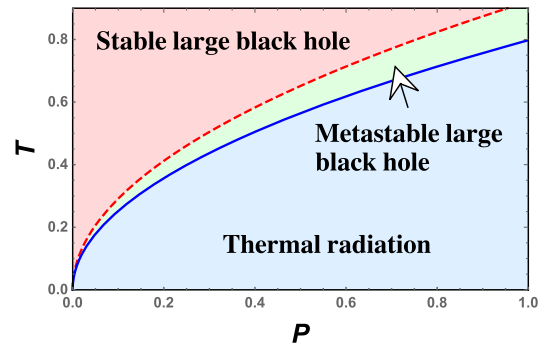

FIG. 2. Phase diagram for a four-dimensional Schwarzschild-AdS black hole. The blue solid and red dashed curves respectively correspond to the black hole minimum temperature and the HP phase transition temperature. Both curves extend to infinity.

Simple inspection of (4) and (5) indicates an underlying interesting relation

$$T_{\mathrm{HP}}(d) = T_0(d+1) \qquad (8)$$

stating that the d-dimensional HP phase transition temperature exactly equals to the black hole minimum temperature in one larger dimension. This relation is also independent of the pressure P and it is reminiscent of the AdS/CFT correspondence. If T_0 is the temperature of a physical quantity in the bulk, then T_{HP} can be treated as the temperature of the dual physical quantity on the boundary. We conjecture that this universal relation will have important holographic applications and that some holographic properties should originate from it. On thing worth emphasizing is that different from the AdS/CFT correspondence, both sides of (8) are in gravitational theories. Furthermore, since T_0 is the minimum temperature of a black hole, we can treat this as the ground state of the AdS black hole [11], with the black hole at T_{HP} as one of its excited states. Thus Eq. (8) can also be interpreted as the duality between the ground state and an excited state of a physical system in two successive dimensions.

We sketch this dual relation in Fig. 3. The metastable large black hole branch is the horizontal line and from left to right the horizon radius increases from r_0 to r_{HP}. Correspondingly the temperature increases from $T_0(d+1)$ to $T_{\mathrm{HP}}(d+1)$ (see above the horizontal line). According to the dual relation (8), we have $T_0(d+1) = T_{\mathrm{HP}}(d)$. By writing

$$\epsilon = \frac{r_h - r_0}{r_{\mathrm{HP}} - r_0} \in (0,1), \qquad (9)$$

we see that as ϵ varies from 0 to 1 we observe a new dual relation

$$T_{\mathrm{HP}}(d) \rightarrow T_{\mathrm{HP}}(d+1), \qquad (10)$$

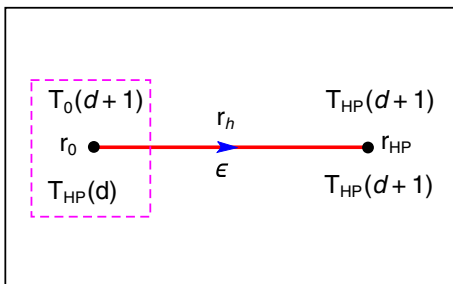see below the horizontal line. A similar process can also be applied to the minimum temperature of the black hole.

FIG. 3. Schematic depiction of the dual relation. The red horizontal line denotes the metastable black hole branch, with r_0 and r_{HP} the horizon radii of the black holes at the minimum temperature and HP phase transition point respectively. The dotted box on the left is realized by the new dual relation (8).

Put succinctly, we can say that the metastable black hole branch indeed has a physical meaning insofar as it realizes the new dual relation $T_{\mathrm{HP}}(d) \rightarrow T_{\mathrm{HP}}(d+1)$ parametrized by ϵ.

III. UNIVERSAL CONSTANT FOR BLACK HOLE MICROSTRUCTURE.

Understanding black hole microstructure, while a challenge in the absence of a quantum theory of gravity, has a wide range of applications, including for example the formations of black holes from gravitational collapse. Making use of Ruppeiner geometry [12], constructed from thermodynamic fluctuation theory, negative or positive Ruppeiner scalar curvature respectively indicates (empirically) attractive or repulsive interaction among a system's microstructures. This approach was taken for the charged AdS black hole [13,14], which was shown to exhibit repulsive interactions that dominate in a certain parameter range, a phenomenon that has no counterpart in a van der Waals fluid [14,15], for which all microstructure interactions are attractive.

The Schwarzschild-AdS black hole is the simplest such black hole. As such we expect its microstructures to be fundamental, having characteristic properties that should be shared amongst all AdS black holes. The line element of its thermodynamic geometry reads

$$dl^2 = -\frac{1}{T}\left(\frac{\partial^2 F}{\partial T^2}\right)_V dT^2 + \frac{1}{T}\left(\frac{\partial^2 F}{\partial V^2}\right)_T dV^2, \qquad (11)$$

where the thermodynamic volume is $V = \omega_{d-2} r_h^{d-1}/(d-1)$. Note that in the original Ruppeiner geometry, the black hole mass and volume, both extensive quantities, are chosen as the fluctuation coordinates, and then transformed to temperature and volume. Using (11) it is straightforward to construct the scalar curvature, where we normalize it as in our previous study [14,15], while it is different from that of Ref. [16].

For the d-dimensional Schwarzschild-AdS black hole, the corresponding normalized scalar curvature is

$$R = -\frac{d-3}{2} \frac{4\pi T \hat{V}^{\frac{1}{d-1}} - d + 3}{\left(2\pi T \hat{V}^{\frac{1}{d-1}} - d + 3\right)^2}, \qquad (12)$$

with $\hat{V} = (d-1)V/\omega_{d-2} = r_h^{d-1}$. We see that R has both vanishing and divergent behaviors. Since the small black hole branch is unstable (as shown above) we will not consider that branch. From (2), we find that the black hole temperature is functions of the pressure P and horizon radius r_h, so we can say that the normalized scalar curvature is dependent on the pressure P.

In Fig. 4 we plot R as a function of V for $d = 4$; in other dimensions the behavior is similar. The red dashed and blue solid curves correspond to metastable and stable large black
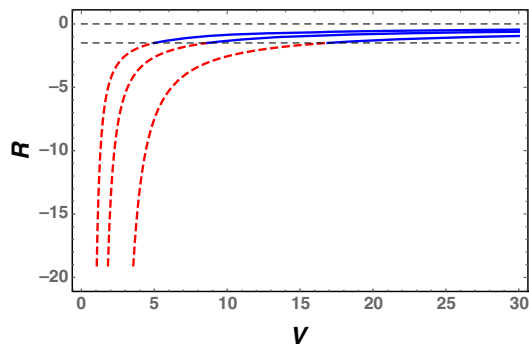
FIG. 4. The scalar curvature R as a function of the volume V for $d = 4$ and $T = 0.2$, 0.25, and 0.3 from bottom to top. The red dashed and blue solid curves are for the metastable and stable large black holes, respectively. Since the small black hole is unstable, we have excluded them in this picture.

holes respectively. For both of these large black hole branches the scalar curvature is negative, and the attractive interaction always dominates for both. This result is not hard to understand: in this spacetime there exists a competition between thermal radiation and black hole phases. Thermal radiation has effectively repulsive microstructure interactions and so in order to form a black hole these must become attractive. A detailed study also shows that at the black hole minimum temperature, the normalized scalar curvature diverges. If we conjecture that the absolute value of R is an indicator of the intensity of the interaction, this black hole case is more like an ideal rigid body.

Note that if we exclude the metastable large black hole branch, the divergent behavior is also removed. The curves then begin at the HP phase transition points, and upon inserting the temperature (5) into the scalar curvature (12), we obtain the amazing result

$$R_{\mathrm{HP}} = -\frac{(d-1)(d-3)}{2}, \quad (13)$$

namely that the Ruppeiner curvature at the HP transition point is a universal constant, independent of the temperature, pressure, or the horizon radius, while only dependent on the dimension. We expect this constant may indicate the fundamental nature of the HP phase transition, and we now consider one possible interpretation of this result.

In physics, there exist many thresholds, beyond or below which we have rather different physical behavior. We contend that the HP transition is one such threshold. In forming or destroying a massive black hole in the AdS space, the HP phase transition is the formation of a black hole from pure thermal radiation. At this point the repulsive microstructure interactions of the radiation must become attractive microstructure interactions for the black hole. The lack of dependence on thermodynamic parameters of the Ruppeiner curvature at this point strongly suggests a

universal property in the formation of black hole degrees of freedom.

IV. SUMMARY

We have shown that a d-dimensional Schwarzschild-AdS black hole exhibits a universal behavior at the HP transition point. For any given dimension, the system is characterized by two special temperatures: the HP phase transition temperature and the black hole minimum temperature. Although each is pressure dependent, we find an exact universal relation (8) between them: the minimum temperature T_0 in $(d+1)$ dimensions equals to the HP phase transition temperature T_{HP} in d dimensions. Similar to the AdS/CFT correspondence, the temperatures T_0 and T_{HP} can be regarded as the physical quantities residing in the bulk and boundary respectively. The duality between two HP phase transition temperatures in successive dimensions is realized by the metastable black hole branch, which has habitually been ignored. We expect that this dual relation discloses deeper fundamental aspects of the HP phase transition.

Furthermore, when applying methods of Ruppeiner thermodynamic geometry to Schwarzschild-AdS black holes, we find attractive interactions dominate amongst the microstructures for both the metastable and stable large black hole branches. Moreover, the normalized scalar curvature at the HP phase transition point is a universal dimension-dependent negative constant. We suggest that this constant can be understood as a pressure independent critical point where a virtual black hole turns to a real one from the thermal radiation. Therefore, we presented a corresponding microscopic threshold in contrast to the macroscopic phase transition. This may be very helpful for understanding the black hole formation at the microscopic level, especially for supermassive black holes formed from the huge interstellar clouds of radiation as expected.

In summary, a novel dual relation between the black hole minimum temperature and HP phase transition temperature in two successive dimensions was discovered. Another universal constant indicating a threshold of the formation of a black hole in the microscopic level was also found. It would be interesting to apply the study of the HP phase transition to other black hole backgrounds. Of further interest is the study of the holographic aspects of the HP phase transition and universal underlying black hole microstructures with different thresholds.

Our results for the dual relation can be extended to the charged case, where the lowest temperature T_0 and the HP phase transition temperature are

$$T_0 = \frac{2\sqrt{(d-3)P(-2(d-3)\Phi^2 + d - 2)}}{\sqrt{\pi}(d-2)}, \quad (14)$$

$$T_{\mathrm{HP}} = \frac{2\sqrt{P(-2(d-3)\Phi^2 + d - 2)}}{\sqrt{\pi}\sqrt{d-1}}, \qquad (15)$$

in the grand canonical ensemble, where the electric potential Φ is fixed. This implies

$$\frac{T_0}{T_{\mathrm{HP}}} = \frac{\sqrt{(d-3)(d-1)}}{d-2}, \qquad (16)$$

which is independent of Φ, and

$$T_0\left(d+1, \frac{\sqrt{(d-1)(d-3)}}{d-2}\Phi\right) = T_{\mathrm{HP}}(d, \Phi) \qquad (17)$$

is the generalization of (8).

ACKNOWLEDGMENTS

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