Renormalization scale setting for D-meson mixing

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A naive application of the heavy quark expansion (HQE) yields theory estimates for the decay rate of neutral D mesons that are 4 orders of magnitude below the experimental determination. It is well known that this huge suppression results from severe Glashow-Iliopoulos-Maiani cancellations. We find that this mismatch can be solved by individually choosing the renormalization scale of the different internal quark contributions. For b and c hadron lifetimes, as well as for the decay rate difference of neutral B mesons, the effect of our scale setting procedure lies within the previously quoted theory uncertainties, while we get enlarged theory uncertainties for the semileptonic CP asymmetries in the B system.

I. INTRODUCTION

An improvement of our theoretical understanding of charm physics is crucial to make use of the huge amount of current and future experimental charm data obtained by LHCb [1], BESIII [2], and Belle II [3]. The recent discovery of direct CP violation in the charm system by the LHCb Collaboration [4] is an example of this necessity. Briefly after the announcement of a nonvanishing measurement of $\Delta A_{CP} = A_{CP}(D^0 \to K^+ K^-) - A_{CP}(D^0 \to \pi^+ \pi^-)$, both theory papers arguing for a beyond standard model [5,6] (partly based on the calculation of Ref. [7]) and a standard model (SM) [8–11] origin of this measurement appeared (a summary of references investigating a previous claim for evidence of CP violation can be found in Ref. [12]). Thus, a decisive conclusion about the potential size of the SM contribution to ΔA_{CP} is mandatory to fully exploit the significant experimental progress in this field. A long-standing puzzle in this regard is the theoretical description of mixing of neutral D mesons. Charm mixing is by now experimentally well established and the HFLAV [13] average of Refs. [4,14–46] (in the case of allowed CP violation) reads

$$x = \frac{\Delta M_D}{\Gamma_{D^0}} = (0.37 \pm 0.12)\%, \quad y = \frac{\Delta \Gamma_D}{2\Gamma_{D^0}} = 0.68^{+0.06}_{-0.07}\%, \quad (1)$$

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where ΔM_D is the mass difference of the two mass eigenstates of the neutral D^0 mesons and $\Delta \Gamma_D$ the corresponding decay rate difference. However, theory predictions for x and y cover a vast range of values—differing by several orders of magnitude, see e.g., the compilations of theory predictions in Refs. [47,48]. Future measurements will not only increase the precision of x and y, but also give stronger bounds or even a measurement of the CP violation in mixing [49] encoded e.g., in the phase ϕ_{12}, see Eq. (3), which is currently constrained to be within $[-2.5°, 1.8°]$ [13]. A reliable range of potential SM values is pivotal to benefit from the coming experimental improvements.

II. HQE

The heavy quark expansion (HQE) [50–56] (see Ref. [57] for a recent overview) describes the total decay rate of heavy hadrons and the decay rate difference of heavy neutral mesons as an expansion in inverse powers of the heavy quark mass. In the case of B_s-mixing and b hadron lifetimes, the HQE predicts values [57–63] which are in good agreement with the experimental ones [13].

	HFLAV 2019	HQE 2019
$\frac{\tau(B_s)}{\tau(B_d)}$	0.994(4)	1.0007(25)
$\frac{\tau(B^+)}{\tau(B_d)}$	1.076(4)	$1.082^{+0.022}_{-0.026}$
$\frac{\tau(\Lambda_b)}{\tau(B_d)}$	0.969(6)	0.935(54)
$\Delta \Gamma_{B_s}$	0.091(13) ps^{-1}	0.090(5) ps^{-1}

This impressive result when the expansion parameter is Λ/m_b (Λ denotes an hadronic scale of the order of Λ^{QCD}) suggests that one might still get reasonably well-behaving estimates moving to the charm system, where the expansion parameter increases by a factor of 3. For the lifetime ratio $\tau(D^+)/\tau(D^0)$, both NLO-QCD corrections to the

dimension-six contribution [64] and values for the non-perturbative matrix elements of four quark operators [58] are known—for all other charm hadrons, this is not yet the case; thus, corresponding theory estimates have to be taken with care—and one finds indeed a nice agreement within the huge theory uncertainties,

$$\frac{\tau(D^+)}{\tau(D^0)}\Big|^{\text{HFLAV 2019}} = 2.536(19), \quad \frac{\tau(D^+)}{\tau(D^0)}\Big|^{\text{HQE 2019}} = 2.7^{+0.7}_{-0.8}. \tag{2}$$

Hence, it is quite surprising that a naive application of the HQE fails completely for D-mixing.

III. CHARM MIXING

Diagonalizing the two-dimensional mixing matrix of the D^0 and the $\bar{D}^0$ meson—containing the off-diagonal matrix elements M_{12} and Γ_{12}—one gets

$$x_{12} = \frac{2|M_{12}|}{\Gamma_{D^0}}, \quad y_{12} = \frac{|\Gamma_{12}|}{\Gamma_{D^0}}, \quad \phi_{12} = \arg\left(\frac{M_{12}}{\Gamma_{12}}\right), \tag{3}$$

while x and y depend on both M_{12} and Γ_{12}. The calculation of M_{12} is beyond the scope of the present work; hence, we can only determine one contribution to the mixing phase ϕ_{12}, also for $\Delta\Gamma_D$ we will use the bound $\Delta\Gamma_D \leq 2|\Gamma_{12}|$ (see e.g., Refs. [65,66]). Within the HQE, Γ_{12} is expanded as

$$\Gamma_{12} = \left[\Gamma_3^{(0)} + \frac{\alpha_s}{4\pi}\Gamma_3^{(1)} + \ldots\right]\frac{\langle Q_6\rangle}{m_c^3} + \ldots, \tag{4}$$

where the ellipsis stands for terms of higher order. Equation (4) is diagrammatically represented in Fig. 1. The product of $\Delta C = 1$ operators in the effective
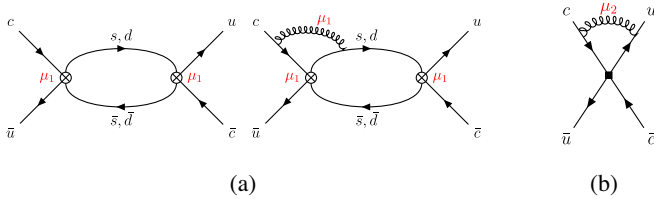

(a)   (b)

FIG. 1. (a) Diagrams describing mixing of neutral D mesons via intermediate $s\bar{s}$, $s\bar{d}$, $d\bar{s}$, and $d\bar{d}$ states in the full theory at LO-QCD (left) and NLO-QCD (right). The crossed circles denote the insertion of $\Delta C = 1$ operators of the effective Hamiltonian describing the charm-quark decay. The dependence on the renormalization scale μ_1 in the Wilson coefficients cancels against the μ_1 dependence of the QCD corrections. (b) Diagram describing mixing of neutral D mesons at NLO-QCD in the HQE. The full dot indicates the insertion of $\Delta C = 2$ operators. The dependence on the renormalization scale μ_2 cancels between the QCD corrections to the diagram and the matrix element of the corresponding operators.

Hamiltonian ("full" theory) is matched into a series of local $\Delta C = 2$ operators Q_n of increasing dimension $n \geq 6$, with the short distance coefficients denoted by $\Gamma_{n-3}^{(i)}$. The expressions for $\Gamma_3^{(i)}$ can be simply obtained from the corresponding ones for B-mixing given in Refs. [67–72] while the matrix elements of the dimension-six operators have been determined in e.g., Refs. [58,73]. Experiments yield a small value for the decay rate difference

$$\Delta\Gamma_D^{\text{Exp}} = 2y/\tau(D^0) = (0.032 \pm 0.003) \text{ ps}^{-1}, \tag{5}$$

which leads to the following bound:

$$\Delta\Gamma_D^{\text{Exp}} \geq 0.028 \text{ ps}^{-1} \tag{6}$$

at 1 standard deviation. Below we will investigate the quantities

$$\alpha = -\arg(\Gamma_{12}), \quad \Omega = \frac{2|\Gamma_{12}|^{\text{SM}}}{0.028 \text{ ps}^{-1}}, \tag{7}$$

where α contributes to CP violation in mixing and values of Ω smaller than 1 indicate a failure of our theoretical framework to describe D-mixing within the 1 sigma range. A naive application of the HQE leads to $\Omega = 3.4 \times 10^{-5}$ at leading order (LO)-QCD (6.2×10^{-5} at next-to-leading order (NLO)-QCD), i.e., the HQE prediction of the decay rate difference is more than 4 orders of magnitude smaller than the experimental determination. Correspondingly, the phase α is very large, i.e., $\alpha = 93°$ at LO-QCD ($\alpha = 99°$ at NLO-QCD). By default, in our numerical analysis, we use Particle Data Group [74] values for the quark ($\overline{\text{MS}}$) and meson masses as well as for the strong coupling, Cabibbo-Kobayashi-Maskawa (CKM) elements from Ref. [75], nonperturbative matrix elements from Ref. [58], and the D^0 decay constant from Ref. [76].

IV. GIM IN D-MIXING

In order to better understand the peculiarities of D-mixing, we decompose Γ_{12} according to the flavor of the internal quark pair. The three contributions are denoted Γ_{12}^{ss}, Γ_{12}^{dd}, and Γ_{12}^{sd},

$$\begin{aligned}\Gamma_{12} &= -(\lambda_s^2\Gamma_{12}^{ss} + 2\lambda_s\lambda_d\Gamma_{12}^{sd} + \lambda_d^2\Gamma_{12}^{dd}) \\ &= -\lambda_s^2(\Gamma_{12}^{ss} - 2\Gamma_{12}^{sd} + \Gamma_{12}^{dd}) \\ &\quad + 2\lambda_s\lambda_b(\Gamma_{12}^{sd} - \Gamma_{12}^{dd}) - \lambda_b^2\Gamma_{12}^{dd}.\end{aligned} \tag{8}$$

$\lambda_q = V_{cq}V_{uq}^*$ is the CKM element and we have used the unitarity relation $\lambda_d + \lambda_s + \lambda_b = 0$ to eliminate λ_d. Equation (8) shows very pronounced hierarchies,

$$-\lambda_s^2 = -4.791 \times 10^{-2} + 3.094 \times 10^{-6}I, \tag{9}$$

$$+2\lambda_s\lambda_b = +2.751 \times 10^{-5} + 6.121 \times 10^{-5}I, \quad (10)$$

$$-\lambda_b^2 = +1.560 \times 10^{-8} - 1.757 \times 10^{-8}I. \quad (11)$$

The CKM factor in the first term of Eq. (8) has by far the largest real part, while the second term has actually the largest imaginary part—it should thus be important for the determination of the potential size of CP violation in mixing. Since the relative imaginary part of λ_b is much larger than that of λ_s, we suggest to keep all terms in Eq. (8). Furthermore, extreme Glashow-Iliopoulos-Maiani (GIM) cancellations [77] affect the coefficients of the CKM elements in Eq. (8). Expanding in the small mass parameter $z = m_s^2/m_c^2$, we find at LO-QCD (top line) and at NLO-QCD (lower line),

$$\Gamma_{12}^{ss} = \begin{cases} 1.62 - 2.34z - 5.07z^2 + \ldots, \\ 1.42 - 4.30z - 12.45z^2 + \ldots, \end{cases} \quad (12)$$

$$\Gamma_{12}^{sd} - \Gamma_{12}^{dd} = \begin{cases} -1.17z - 2.53z^2 + \ldots, \\ -2.15z - 6.26z^2 + \ldots, \end{cases} \quad (13)$$

$$\Gamma_{12}^{ss} - 2\Gamma_{12}^{sd} + \Gamma_{12}^{dd} = \begin{cases} -13.38z^3 + \ldots, \\ 0.07z^2 - 29.72z^3 + \ldots. \end{cases} \quad (14)$$

It was observed before [78,79] that QCD corrections lower the GIM suppression by one power of z. The peculiarity of Eq. (8) is that the CKM dominant factor λ_s^2 multiplies the extremely GIM suppressed term given in Eq. (14), the CKM suppressed factor $\lambda_s\lambda_b$ multiplies the GIM suppressed term given in Eq. (13), and the very CKM suppressed factor λ_b^2 multiplies Γ_{12}^{dd}, where no GIM suppression is present. Thus, the three contributions in Eq. (8) have actually a similar size,

$$\Gamma_{12} = (2.08 \times 10^{-7} - 1.34 \times 10^{-11}I) \text{ (first term)}$$
$$- (3.74 \times 10^{-7} + 8.31 \times 10^{-7}I) \text{ (second term)}$$
$$+ (2.22 \times 10^{-8} - 2.5 \times 10^{-8}I) \text{ (third term)}. \quad (15)$$

It is also clear that a sizeable phase in D-mixing can only arise, if the slightly GIM suppressed term is enhanced. Different solutions have been suggested in order to explain the mismatch between the HQE prediction and experimental determination.

(i) Higher orders in the HQE could be less affected by the GIM suppression [80–82]—first estimates of the dimension nine contribution [83] to D-mixing show indeed such an enhancement, but not on a scale to reproduce the experimental number. For a final conclusion about this possibility, a full determination of dimension nine and twelve would be necessary.

(ii) Quark hadron duality violations as large as the deviation between experiments and the naive HQE prediction—yielding $\Omega \approx 10^{-5}$—seem unlikely given

the many successful tests of the HQE as stated above. In fact, in Ref. [66], it was shown that violations as small as 20 percent could be sufficient to explain the experimental value of D-mixing.

(iii) The HQE is not applicable and we have to rely on different methods, like summing over the exclusive decays channels contributing to the decay rate difference; see e.g., Refs. [84–86].

V. ALTERNATIVE SCALE SETTING

In Γ_{12}, the two renormalization scales μ_1 and μ_2 arise; see Fig. 1. μ_1 is the scale at which the $\Delta C = 1$ Wilson coefficients of the effective Hamiltonian and the QCD corrections in the full theory are computed and up to higher order terms, this scale dependence has to cancel between the two sources, Fig. 1(a). Similarly, radiative corrections to the HQE diagrams introduce the dependence on the scale μ_2, Fig. 1(b), the latter cancels against the scale dependence of the matrix element of the corresponding $\Delta C = 2$ operators. We will not discuss the μ_2-dependence any further since this cancellation is very effective. For the μ_1-dependence, in the B_s system, the cancellation is numerically only weakly realized when moving from LO-QCD to NLO-QCD, see Refs. [87,88]. This indicates the importance of higher order corrections and first steps in that direction show indeed large NNLO-QCD effects [87,88]. In the D system a reduction of the μ_1-dependence, when moving from LO-QCD to NLO-QCD, is present in the individual contributions $\Gamma_{12}^{ss,sd,dd}$ but not in Γ_{12}, see Fig. 2, which seems to be again a consequence of the severe GIM cancellations. Making the scale dependence explicit, we can write

$$\Gamma_{12} = \sum_{q_1q_2=ss,sd,dd} \Gamma_3^{q_1q_2}(\mu_1^{q_1q_2}, \mu_2^{q_1q_2})\langle Q_6\rangle(\mu_2^{q_1q_2})\frac{1}{m_c^3} + \ldots. \quad (16)$$

In general, different internal quark pairs contribute to different decay channels of the $D^0(\bar{D}^0)$ meson e.g., $s\bar{s}$ to
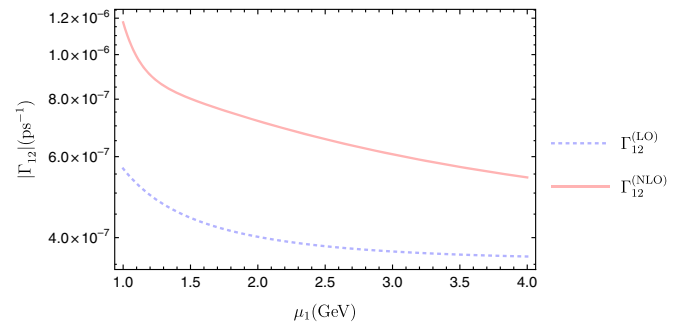

FIG. 2. Comparison of μ_1-dependence of $|\Gamma_{12}|$ at LO-QCD (dotted purple) and NLO-QCD (solid pink).

a K^+K^- final state and $s\bar{d}$ to a π^+K^- final state. For each of these different observables, the choice of the renormalisation scales is *a priori* arbitrary; nevertheless, one typically fixes $\mu_x^{ss} = \mu_x^{sd} = \mu_x^{dd} = \mu$ which is then chosen to be equal to the mass of the decaying heavy quark, i.e., $\mu = m_Q$ for Q quark decays, to minimize terms of the form $\alpha_s(\mu)\ln(\mu^2/m_Q^2)$. Uncertainties due to unknown higher order corrections are estimated varying μ between $m_Q/2$ and $2m_Q$—in the case of the charm quark, we fix the lower bound to 1 GeV in order to still ensure reliable perturbative results.

Here we propose two different ways to treat the renormalization scale $\mu_1^{q_1q_2}$, both will reduce the mismatch between the HQE prediction and the experimental determination of D-mixing, while leaving the other HQE predictions unchanged:

(i) μ_1^{ss}, μ_1^{sd}, and μ_1^{dd} are set to the common scale m_c but varied independently between 1 GeV and $2m_c$.

(ii) μ_1^{ss}, μ_1^{sd}, and μ_1^{dd} are set to different scales according to the size of the available phase space. In particular, we will evaluate Γ_3^{ss} at the scale $\mu_3^{ss} = \mu - 2\epsilon$, Γ_3^{sd} at the scale $\mu_1^{sd} = \mu - \epsilon$, and Γ_3^{dd} at the scale $\mu_1^{dd} = \mu$, where ϵ is related to the kinematics of the decays.

If ϵ is not too large, then both methods will yield results for the individual Γ_3^{ss}, Γ_3^{sd}, and Γ_3^{dd} which lie within the usually quoted theory uncertainties obtained following the prescription stated above, but they will clearly affect in a sizable way the severe GIM cancellations in Eqs. (13) and (14). The first method gives a considerably enhanced range of values for Ω,

$$\Omega \in [4.6 \times 10^{-5}, 1.3], \qquad (17)$$

which nicely covers also the experimental determination of the decay rate difference. Scanning independently over μ_1^{ss}, μ_1^{sd}, and μ_1^{dd} in 11 equidistant steps, we find that out of the 1331 points only 14 give a value of $\Omega < 0.001$, while 984 give a value of $\Omega > 0.1$. The very small HQE prediction seems thus to be an artifact of fixing the scales μ_1^{ss}, μ_1^{sd}, and μ_1^{dd} to be the same. The range of values shown in Eq. (17) is similar even if we use the pole scheme for the quark masses, lattice results instead of the HQET results, or a different $\Delta C = 2$ operator basis. In all these cases, the value $\Omega \geq 1$ can be obtained. For α, we get in general results ranging from $-\pi$ to π. A closer look, however, shows that for $\Omega > 0.5$ only values of $\alpha < 0.1°$ are allowed, while large values of α can only be obtained if the theory prediction for y is inconsistent with the experimental determination. The second method for the scale setting requires the introduction of a mass scale ϵ. A possible estimate for the size of this parameter could be the strange quark mass $\epsilon = m_s \approx 0.1$ GeV or the phase space difference of the corresponding exclusive decay channels: comparing the energy release of $D^0 \to K^+K^-$, $M_{D^0} - 2M_{K^+} = 0.88$ GeV, with that of $D^0 \to \pi^+\pi^-$,
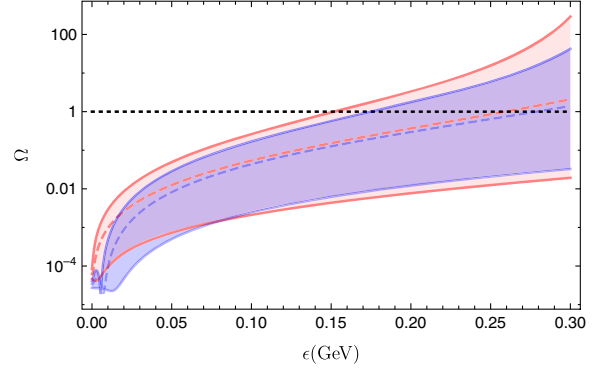

FIG. 3. Comparison of the ϵ dependence of Ω at LO-QCD (purple) and NLO-QCD (pink) for different values of μ: the dashed line corresponds to $\mu = m_c$ while the two solid lines to $\mu = 1$ GeV and $\mu = 2m_c$.

$M_{D^0} - 2M_{\pi^+} = 1.59$ GeV, we might expect that $\epsilon \approx 0.35$ GeV. Figure 3 shows how the HQE prediction of Ω would be affected in this scenario. Again, an enhancement up to the experimental value is possible for values of $\epsilon \approx 0.2$ GeV. Finally, we have to test the effect of our alternative scale setting procedure on all the other HQE predictions. For the lifetimes (i.e., $\tau(D^+)/\tau(D^0)$ as well as b hadron lifetimes) and the decay rate difference $\Delta\Gamma_s$, no GIM-like cancellations arise and we can only get a shift within the usually quoted theory range. But the semileptonic CP asymmetries are governed by the weakly GIM suppressed contribution in B_s-mixing. Within the SM, we get

$$\mathrm{Re}\left(\frac{\Gamma_{12}^q}{M_{12}^q}\right)^{\mathrm{SM}} = -\frac{\Delta\Gamma_q}{\Delta M_q} = \begin{cases} -(49.9 \pm 6.7) \times 10^{-4} & q = s \\ -(49.7 \pm 6.8) \times 10^{-4} & q = d \end{cases},$$

$$\mathrm{Im}\left(\frac{\Gamma_{12}^q}{M_{12}^q}\right)^{\mathrm{SM}} = a_{sl}^q = \begin{cases} (+2.2 \pm 0.2) \times 10^{-5} & q = s \\ (-5.0 \pm 0.4) \times 10^{-4} & q = d \end{cases}. \qquad (18)$$

Performing the ϵ analysis, we obtain

ϵ (GeV)	Γ_{12}^s/M_{12}^s	Γ_{12}^d/M_{12}^d
0.	**−0.00499+0.000022**I	**−0.00497−0.00050**I
0.2	**−0.00494+0.000023**I	**−0.00492−0.00053**I
0.5	**−0.00484** + 0.000026I	**−0.00482** − 0.00059I
1.0	**−0.00447** + 0.000037I	**−0.00448** − 0.00084I
1.5	−0.00287 + 0.000091I	−0.00309 − 0.0021I

We see that for values of ϵ up to 1 GeV the predictions for the real part lie with the usually quoted theory uncertainties (indicated in bold). The predictions for the semileptonic asymmetries can, however, be increased by almost 100% compared to the usually quoted values.

VI. CONCLUSIONS

Our main finding is that the range of the HQE uncertainty for y is much larger than previously thought, and it

covers the experimental value if we modify the usually adopted scale setting. For a full solution of the D-mixing puzzle, we nevertheless suggest a more precise estimate of higher order corrections in the HQE, as well as a completion of the NNLO-QCD corrections to the leading term.

The alternative scale setting procedure shows that a small contribution to CP violation in mixing stemming from the decay rate can be up to one per mille within in the SM, which agrees with estimates made in Refs. [89,90]. For a prediction of CP violation in mixing, the contribution coming from M_{12} has to be determined in addition. This might be done in future via the help of dispersion relations, see e.g., Refs. [85,90,91]. We would like to note that our suggested procedure still respects the GIM mechanism, because for vanishing internal strange quark mass, also the parameter ϵ will be zero.

Finally, this scale setting does not affect quantities like $\tau(D^+)/\tau(D^0)$, b hadron lifetimes, and $\Delta\Gamma_s$ outside the range of their quoted theoretical errors, but it affects the semileptonic CP asymmetries and we get enhanced SM ranges,

$$a_{sl}^d \in [-9.2; -4.6] \times 10^{-4}, \quad a_{sl}^s \in [2.0; 4.0] \times 10^{-5}. \quad (19)$$

ACKNOWLEDGMENTS

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