Analogue Hawking radiation from black hole solitons in quantum Josephson transmission lines

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The nonlinear LC circuit dual to the Toda lattice (voltage) soliton is studied using the reductive perturbation method. It is found that the current flowing through the circuit obeys the defocusing modified Korteweg–de Vries equation with shock-wave-type (current) solitons. The current soliton spatially modulates the nonlinear inductance, equivalently the spatial modulations of electromagnetic wave velocity in the circuit, resulting in the generation of analogue black holes. Hawking radiation from the black holes is also discussed based on the tunneling models.

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I. INTRODUCTION

A soliton is a solitary wave formed by balancing dispersion and nonlinearity of the nonlinear dispersive media and behaves as a particle against collisions between solitons [1–3]. This unique property is not only a fundamental concept for understanding nonlinear physical phenomena in various fields, but it also has been applied to information communication such as optical solitons [4]. In electrical circuits, the Toda solitons [5] have been studied by generating voltage solitons in the nonlinear LC circuit to explore the basic concept of soliton theory and have been utilized to signal transmissions. In the previous studies, the origin of the nonlinearity was due to nonlinear *capacitance* in the circuit. On the other hand, the system we consider is an electronic circuit dual to the nonlinear LC circuit mentioned above. The nonlinearity in the circuit is due to nonlinear *inductance* rather than capacitance. This type of circuit was studied in the 1990s. Although numerical calculations have revealed the existence of shock waves [6,7] for current in the circuit, the analytical solutions have not been successfully derived so far.

In this paper, we deal with two issues: new types of solitons and soliton-induced analogue black holes. First, we clarify the wave nature hidden in the circuit equation by using the reductive perturbation method [8]. As a result, it was found that modified Korteweg–de Vries (mKdV) equation with a *negative* nonlinear term is obtained with shock-wave-type current soliton solutions, which are consistent with numerical results obtained in the previous studies [6]. Second, we demonstrate that this current soliton behaves as black holes in the circuit [9,10]. In the LC circuit, the velocity c of the electromagnetic wave depends on both the inductance L and the capacitance C via $c = a/\sqrt{LC}$ with a being the unit length of the circuit. The current soliton produces spatially dependent inductance, resulting in the spatially varying velocity of electromagnetic waves. In our Josephson transmission line, it is clarified that analogue black holes can be created by spatially varying inductance induced by current solitons. In addition, the Hawking temperature is derived on the basis of tunneling mechanism and is evaluated to discuss the observability of Hawking radiation.

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II. JOSEPHSON TRANSMISSION LINES

Let us consider a coplanar Josephson transmission line indicated in Fig. 1, where all Josephson junctions are assumed to have identical critical current I_c and capacitance C.

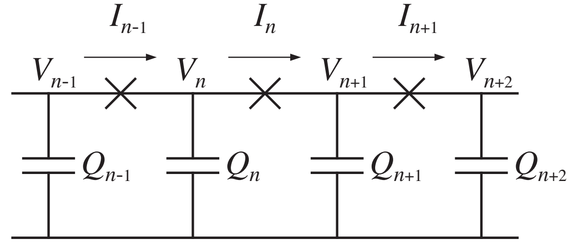

FIG. 1. Schematic diagram of a Josephson transmission line. The cross indicates a Josephson junction. I_n, V_n, and Q_n represent current, voltage, and electric charge of the nth junction, respectively.

From the Kirchhoff's law together with the Josephson relation, the circuit equations are given by

$$I_n(t) - I_{n-1}(t) = C\frac{dV_n(t)}{dt} \tag{1}$$

and

$$V_{n+1}(t) - V_n(t) = \frac{\hbar}{2e}\frac{d\theta_n(t)}{dt}, \tag{2}$$

where $\hbar$ and e are the Plank constant h divided by 2π and the elementary electric charge, respectively. I_n, V_n, and θ_n represent the current, voltage, and phase difference of the nth Josephson junction on the line, respectively. These equations lead to the discrete equation of motion for the phase difference as

$$\frac{\hbar}{2e}C\frac{d^2\theta_n(t)}{dt^2} = I_{n+1}(t) + I_{n-1}(t) - 2I_n(t). \tag{3}$$

In continuum approximation, i.e., $I_n(t) = I(x,t) \equiv I$ together with the Josephson relation,

$$\theta = \sin^{-1}\frac{I}{I_c} = \sum_{i=1}^{\infty}\frac{1}{(2i-1)!}\left(\frac{I}{I_c}\right)^{2i-1}. \tag{4}$$

Equation (3) is reduced to

$$\frac{\hbar}{2e}C\frac{\partial^2}{\partial t^2}\left\{\sum_{i=1}^{\infty}\frac{1}{(2i-1)!}\left(\frac{I}{I_c}\right)^{2i-1}\right\} = \sum_{i=1}^{\infty}\frac{2a^{2i}}{(2i)!}\frac{\partial^{2i}}{\partial x^{2i}}I. \tag{5}$$

By introducing the normalized variables $\bar{x} = x/a$ and $\bar{t} = t/\omega_J^{-1} = t\sqrt{I_c/(C\hbar/2e)}$ with ω_J being the Josephson plasma frequency, the equation reduces to

$$\frac{\partial^2}{\partial \bar{t}^2}\left\{\sum_{i=1}^{\infty}\frac{1}{(2i-1)!}\bar{I}^{2i-1}\right\} = \sum_{i=1}^{\infty}\frac{2}{(2i)!}\frac{\partial^{2i}}{\partial \bar{x}^{2i}}\bar{I}, \tag{6}$$

where $\bar{I} = I/I_c$.

III. MODIFIED KORTEWEG–DE VRIES EQUATION

A. Reductive perturbation method

Now let us find the wave nature hidden in the circuit equation by using the reductive perturbation method [8]. It derives a scale-invariant nonlinear evolution equation near a linear approximation by considering the balance between dispersion and nonlinearity in the comoving frame of reference. Changes in physical quantities appear to be gradual in coordinate systems that move together with phase velocities at long-wavelength limits. This requires stretched (slow) variables called the Gardner-Morikawa transformation [11,12] such as

$$\xi = \epsilon^{\frac{1}{2}}(\bar{x} - \bar{t}), \tag{7}$$

$$\tau = \epsilon^{\frac{3}{2}}\bar{t}, \tag{8}$$

where ϵ represents the small amplitude of the perturbation. The gradual temporal change in the higher order compared to $\epsilon^{1/2}$ represents the wave number dispersion because the phase velocity depending on the wave number deviates from the long-wavelength limit phase velocity. Inserting the transformation into the field equation (6), we have

$$\left(\epsilon\frac{\partial^2}{\partial\xi^2} - 2\epsilon^2\frac{\partial^2}{\partial\xi\partial\tau} + \epsilon^3\frac{\partial^2}{\partial\tau^2}\right)\left\{\sum_{i=1}^{\infty}\frac{1}{(2i-1)!}\bar{I}^{2i-1}\right\} = \left(\sum_{i=1}^{\infty}\frac{2}{(2i)!}\epsilon^i\frac{\partial^{2i}}{\partial\xi^{2i}}\right)\bar{I}. \tag{9}$$

Now let us assume that the normalized current $\bar{I}$ is expanded in a power series of a small parameter ϵ in order to include nonlinearity perturbatively as follows:

$$\bar{I} = \epsilon^{\frac{1}{2}}\bar{I}^{(1)} + \epsilon\bar{I}^{(2)} + \epsilon^{\frac{3}{2}}\bar{I}^{(3)} + \cdots. \tag{10}$$

Introducing the expansion Eq. (10) into Eq. (9) and setting the coefficients of similar powers equal to zero, we obtain a set of differential equations. For $\epsilon^{\frac{5}{2}}$,

$$2\frac{\partial^2\bar{I}^{(1)}}{\partial\xi\partial\tau} - \frac{1}{6}\frac{\partial^2(\bar{I}^{(1)})^3}{\partial\xi^2} + \frac{1}{12}\frac{\partial^4\bar{I}^{(1)}}{\partial\xi^4} = 0. \tag{11}$$

From the integration of Eq. (11) with respect to ξ, we get

$$2\frac{\partial\bar{I}^{(1)}}{\partial\tau} - \frac{1}{6}\frac{\partial(\bar{I}^{(1)})^3}{\partial\xi} + \frac{1}{12}\frac{\partial^3\bar{I}^{(1)}}{\partial\xi^3} = g(\tau), \tag{12}$$

where $g(\tau)$ is an arbitrary function of its argument and can be chosen to be zero. By rescaling our coordinates as $\bar{I}^{(1)} = 2u$, $\xi = \bar{\xi}/2$, and $\tau = 3\bar{\tau}$, this finally reduces to the modified Korteweg–de Vries equation with a negative nonlinear term [13],

$$\frac{\partial u}{\partial\bar{\tau}} - 6u^2\frac{\partial u}{\partial\bar{\xi}} + \frac{\partial^3 u}{\partial\bar{\xi}^3} = 0. \tag{13}$$

This equation is specifically called defocusing mKdV or mKdV$^-$, because the sign of the nonlinear term is negative. The difference in the sign of the nonlinear term in the KdV equation only changes its polarity, and no significant changes appear in the soliton solution. In contrast, the difference in the sign of the nonlinear term in the mKdV equation makes a drastic change. The mKdV$^-$ produces an entirely new solution set shown below.

B. Soliton solutions

Unlike the mKdV$^+$ equation, the mKdV$^-$ equation has kink (or shock-wave-type) traveling wave solutions of the form [13,14]

$$u(\bar{\xi}, \bar{\tau}) = \alpha \tanh(\alpha\bar{\xi} + 2\alpha^3\bar{\tau}), \tag{14}$$

where the parameter α characterizes solitonslike amplitude α and velocity $(-2\alpha^2)$. In terms of electric current in the system, the normalized current soliton in the $\bar{x} - \bar{t}$ coordinate is given as

$$\begin{aligned}\bar{I}(\bar{x},\bar{t}) &= 2\alpha\sqrt{\epsilon}\tanh\left[2\alpha\sqrt{\epsilon}\left\{\bar{x}-\left(1-\frac{(\alpha\sqrt{\epsilon})^2}{3}\right)\bar{t}\right\}\right] \\ &= 2\sqrt{3(1-\bar{v}_s)}\tanh[2\sqrt{3(1-\bar{v}_s)}\{\bar{x}-\bar{v}_s\bar{t}\}],\end{aligned} \tag{15}$$

where $\bar{v}_s$ stands for the normalized soliton velocity in the $\bar{x} - \bar{t}$ coordinate ($\bar{v}_s = 1 - \alpha'^2/3$) and $\alpha' = \alpha\sqrt{\epsilon}$. This is exactly in agreement with the numerical results in the previous studies [6,7]. Thus, we succeeded in demonstrating analytically the existence of shock-wave-type solutions in the superconducting circuits that have not been solved for many years. Note that the smaller shock-wave-type soliton moves more rapidly than the larger one, contrary to the solitons of both KdV and mKdV$^+$ equations. In addition, discrete treatment might be required for detailed analysis when the soliton speed is slow, since the soliton width d approaches the lattice spacing a.

IV. BLACK HOLE SOLITONS

Here let us discuss analogue black holes induced by current soliton in this system based on the seminar work by Unruh and Schützhold [15] rather than Gegenberg and Kunstatter using a duality between black holes in Jackiw-Teitelboim dilaton gravity and solitons in sine-Gordon field theory [16,17]. The basic idea of artificially creating a black hole in a laboratory system is as follows. Take a carp climbing a waterfall as an example. The flow velocity is gentle in the upstream, whereas it is high in the downstream. In this way, the flow velocity changes spatially. In other words, the velocity is different between upstream and downstream, and there are places where carps cannot climb. That corresponds to the horizon of the event. Therefore, if such a system in which the flow velocity changes spatially can be produced, a pseudo black hole can be created.

In our system, the current through the junction modifies the junction inductance according to the relation

$$L = \frac{\hbar}{2eI_c\cos(\arcsin\bar{I}(x,t))}, \tag{16}$$

resulting in the spatial modification of the velocity of the electromagnetic wave in the transmission line via $c = a/\sqrt{LC}$. Since the current soliton depends on the position, the velocity of the electromagnetic wave is modulated in space as

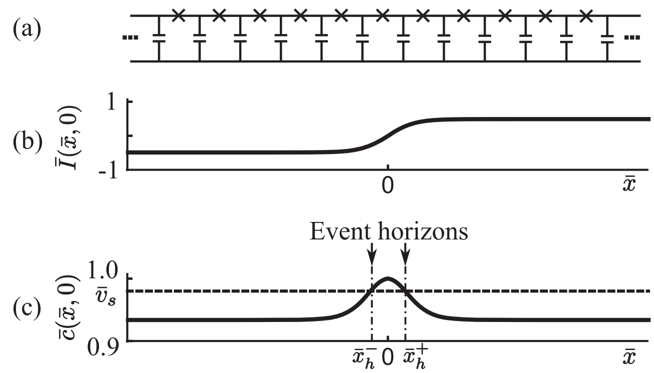

FIG. 2. (a) Josephson transmission line, (b) shock-wave-type normalized current soliton at $\bar{t} = 0$ with the velocity $\bar{v}_s = 0.98$, and (c) normalized velocity of the electromagnetic wave at $\bar{t} = 0$. The horizontal dotted line represents the normalized soliton velocity $\bar{v}_s$. $\bar{x}_h^-$ and $\bar{x}_h^+$ indicate the black hole and white hole horizon positions, respectively.

$$c(\bar{x},\bar{t}) = c_0\sqrt{\cos[\arcsin\{2\alpha'\tanh\{2\alpha'(\bar{x}-\bar{v}_s\bar{t})\}\}]}, \tag{17}$$

with $c_0 = a/\sqrt{L_1C}$ and $L_1 = \hbar/2eI_c$. The soliton parameter α' is restricted to $|\alpha'| \le 1/2$ because the electromagnetic wave velocity must be in real numbers. Equation (17) is rewritten by

$$\bar{c}(\bar{x},\bar{t}) = [1 - 12(1-\bar{v}_s)\tanh^2\{2\sqrt{3(1-\bar{v}_s)}(\bar{x}-\bar{v}_s\bar{t})\}]^{\frac{1}{4}}, \tag{18}$$

where $\bar{c}(\bar{x},\bar{t})$ is the normalized velocity of the electromagnetic wave, i.e., $\bar{c}(\bar{x},\bar{t}) = c(\bar{x},\bar{t})/c_0$, in the Josephson transmission lines.

Figure 2 shows the correlation diagram of a current soliton (b) and the velocity of the electromagnetic wave (c) in the Josephson transmission line (a). Two event horizons are formed due to the spatially varying velocities of the electromagnetic wave, leading to a pair of black hole and white hole as presented in nonlinear optical fibers [18]. The event horizon of analogue black hole appears at the place where the velocity of electromagnetic wave $\bar{c}(\bar{x},\bar{t})$ is equal to the velocity of the reference wave, i.e., the normalized current soliton $\bar{v}_s = v_s/c_0$. In short, the velocity of the current soliton determines the position of the event horizon. In our system, a pair of black holes occurs within $11/12 \le \bar{v}_s < 1$ as follows. The lower bound is given by the condition $|\alpha'| = |\sqrt{3(1-\bar{v}_s)}| \le 1/2$ demonstrated above. This is also equivalent to the condition that the soliton width is larger than the lattice spacing. In other words, the continuum approximation is justified for soliton velocities greater than this. On the other hand, the upper bound is determined by the condition that the velocity of the soliton is smaller than the velocity of the electromagnetic wave c_0.

V. HAWKING RADIATION

A black hole is an area of space where a gravitational field is so strong that no matter or radiation (including light) can escape. However, Hawking proposed that *virtual* particle pairs that arise quantum mechanically from a vacuum near the event horizon may result in one particle's escape as Hawking radiation in the vicinity of the black hole while the partner particle with negative energy falls into it before the pair annihilation can happen [19,20].

Hawking temperature is the key to assess the observability of Hawking radiation. In the previous papers [15], it was evaluated from the surface gravity of the effective horizon that depends on the rate of spatial change of the light velocity in the laboratory frame across the horizon. Here we employ the tunneling mechanism known as radial null geodesic method [21,22]: a particle with positive energy of virtual particle pairs tunnels through the event horizon toward the outside of the black hole. The tunneling probability Γ of the classically forbidden trajectory from inside ($x_{\rm in}$) to outside ($x_{\rm out}$) of the horizon, per unit time per unit volume, in the semiclassical approximation is given by the following formula:

$$\Gamma = \exp\left(-\frac{2\,\mathrm{Im}\,S}{\hbar}\right), \tag{19}$$

where S is classical action of the trajectory and is given as

$$\begin{aligned} S &= \int (p\dot{x} - H(x,p))dt \\ &= \int_{x_{\rm in}}^{x_{\rm out}} \int_0^p dp' dx - \int H(x,p)dt \\ &= \int_{x_{\rm in}}^{x_{\rm out}} \int_0^{E(t)} \frac{dH}{dx/dt} dx - \int H(x,p)dt, \end{aligned} \tag{20}$$

where Hamilton's equation of motion $\dot{x} = dH/dp$ is used. To evaluate the imaginary part of the action, we need to express dx/dt concretely.

In the Painléve-Gullstrand coordinates [9,10], the metric of this system is

$$ds^2 = -(v_s^2 - c^2)dt^2 + 2c\,dx\,dt + dx^2. \tag{21}$$

The radial null geodesic ($ds^2 = 0$) leads to

$$\left(\frac{dx}{dt}\right)^2 + 2c\frac{dx}{dt} - (v_s^2 - c^2) = 0. \tag{22}$$

The solution is then given as

$$\frac{dx}{dt} = -c \pm v_s, \tag{23}$$

where the positive (negative) sign in the second term on the right side represents the external (internal) mode. For the external mode, Eq. (20) is reduced to

$$S = \int_{x_{\rm in}}^{x_{\rm out}} \int_0^{E(t)} \frac{dH'}{v_s - c} dx - \int H(x,p)dt. \tag{24}$$

Hereafter, the second term of this equation is ignored since it is real. Taylor expansion of $c(x,t)$ near the position of the event horizon x_h is written as

$$\begin{aligned} c(x,t) &= c(x_h,t) + \frac{\partial c}{\partial x}\bigg|_{x=x_h}(x - x_h) + O((x-x_h)^2) \\ &\simeq v_s + \frac{\partial c}{\partial x}\bigg|_{x=x_h}(x - x_h). \end{aligned} \tag{25}$$

According to the Hilbert formula,

$$\frac{f(x)}{x - x_0 \mp i\varepsilon} = \mathcal{P}\frac{f(x)}{x - x_0} \pm i\pi\delta(x - x_0), \tag{26}$$

the action is described as

$$\begin{aligned} S &\simeq \int_{x_{\rm in}}^{x_{\rm out}} \int_0^H \frac{dH'}{-\frac{\partial c}{\partial x}|_{x=x_h}(x - x_h)} dx \\ &= -\mathcal{P}\int_{x_{\rm in}}^{x_{\rm out}} \frac{E(t)}{\frac{\partial c}{\partial x}|_{x=x_h}(x - x_h)} dx - i\pi\left(-\frac{E(t)}{\frac{\partial c}{\partial x}|_{x=x_h}}\right). \end{aligned} \tag{27}$$

Since the principal value is real, the imaginary part of the action is given as

$$\mathrm{Im}\,S = \frac{\pi E(t)}{\frac{\partial c}{\partial x}|_{x=x_h}}. \tag{28}$$

The tunneling rate is then expressed as

$$\Gamma \simeq \exp\left(-\frac{2\,\mathrm{Im}\,S}{\hbar}\right) = \exp\left(-\frac{2\pi E(t)}{\hbar\frac{\partial c}{\partial x}|_{x=x_h}}\right). \tag{29}$$

The Hawking temperature T_H is defined by assuming that the tunneling rate is related to the Boltzmann factor $\exp(-E(t)/k_B T_H)$ with k_B being the Boltzmann constant. That reads as

$$T_H = \frac{\hbar}{2\pi k_B}\left|\frac{\partial c}{\partial x}\right|_{x=x_h}. \tag{30}$$

This is exactly identical with the Unruh's expression [15] despite being derived in a different way. Higher-order quantum corrections will appear in the theory beyond

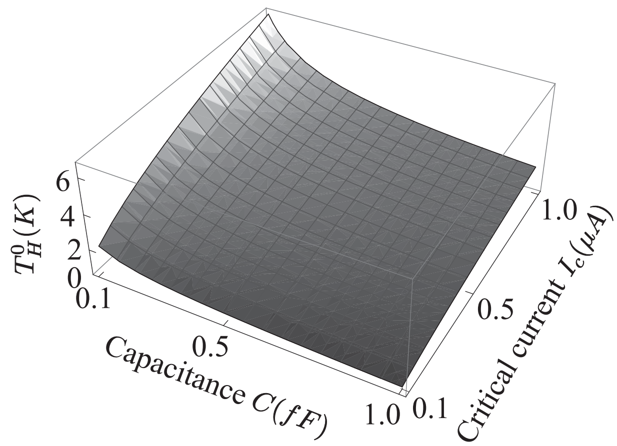

FIG. 3. Diagram of junction parameter (the capacitance C and critical current I_c) dependence on the bare Hawking temperature T_H^0 defined in Eq. (32).

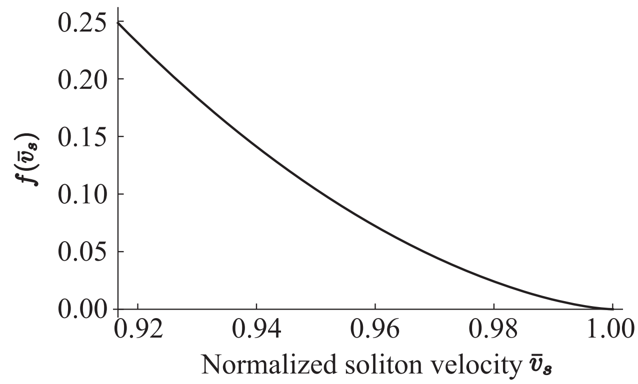

FIG. 4. Diagram of soliton velocity $\bar{v}_s$ dependence on Hawking temperature $f(\bar{v}_s)$ given in Eq. (33).

the semiclassical approximation. This will be discussed elsewhere. The Hawking temperature in our system is derived as

$$T_H = \frac{\hbar}{2\pi k_0}\left(c_0 \frac{d\bar{x}}{dx}\right)\left|\frac{\partial \bar{c}}{\partial \bar{x}}\right|_{\bar{x}=\bar{x}_h} = T_H^0 f(\bar{v}_s), \tag{31}$$

where the *bare* Hawking temperature T_H^0 is the dominant term of the Hawking temperature expression and determines the order of the temperatures, while $f(\bar{v}_s)$ represents the dynamical contribution to the Hawking temperature of solitons as defined below,

$$T_H^0 = \frac{\hbar}{2\pi k_B}\left(\frac{1}{\sqrt{L_1 C}}\right), \tag{32}$$

$$f(\bar{v}_s) = \left|\frac{\partial \bar{c}}{\partial \bar{x}}\right|_{\bar{x}=\bar{x}_h} = \left|\frac{\sqrt{1-\bar{v}_s^4}(12\bar{v}_s - \bar{v}_s^4 - 11)}{2\bar{v}_s^3}\right|. \tag{33}$$

Here we use the position of the event horizon arising under the condition $\bar{v}_s^2(\bar{x}) = \bar{c}^2(\bar{x})$ as follows:

$$\bar{x}_h^{\pm} = \pm \frac{1}{2\sqrt{3(1-\bar{v}_s)}} \tanh^{-1}\left(\sqrt{\frac{(1+\bar{v}_s)(1+\bar{v}_s^2)}{12}}\right). \tag{34}$$

The Hawking temperature is essentially determined by the characteristic time scale $(1/\sqrt{L_1 C})$ as shown in Eq. (32). This implies that the Hawking temperature can be controlled by designing the circuit. In fact, the bare Hawking temperature T_H^0 changes with the junction parameters C and I_c as shown in Fig. 3. Note that the bare Hawking temperature is in the order of the Kelvin temperature range that is accessible experimentally in existing technologies. This result shows that analogue Hawking radiation is observable in our system, as in the previous papers.

In addition, there is a completely new contribution to the Hawking temperature in our expression that has not been found in the previous theories. This is the dynamical contribution of the soliton represented by $f(\bar{v}_s)$ in Fig. 4. This stems from the change in the shape of soliton depending on the soliton velocity. This indicates that the Hawking temperature can be controlled simply by changing the soliton velocity without changing the circuit at all. Therefore, it can be confirmed that the experimentally detected radiation is indeed due to Hawking radiation through this change. This is the advantageous feature that has never existed before. Considering this effect, the Hawking temperature eventually reaches tens milli-Kelvin order for $v_s = 0.98c_0$ and is sufficiently observable.

Finally, we comment on the radiation power of Hawking radiation. The radiation power for black holes in $1+1$-dimensional asymptotically flat spacetime [23,24] is given by

$$\frac{dE}{dt} = \frac{k_B T}{2\pi}\int_0^\infty d\omega \frac{\frac{\hbar\omega}{k_B T}}{e^{\frac{\hbar\omega}{k_B T}} - 1} = \frac{\pi}{12\hbar}(k_B T_H)^2. \tag{35}$$

It is interesting to note that the radiation power depends only on Hawking temperature. Using the results obtained above, the radiation power can be estimated to be in the region of 10^{-17} W to 10^{-15} W for $v_s = 0.98c_0$. Therefore, Hawking radiation can be detected experimentally enough from the viewpoint of radiation power.

VI. SUMMARY

We have studied the current that flows through the Josephson transmission line dual to the LC circuit with nonlinear capacitance generating the Toda soliton (voltage soliton). Starting from the circuit equation, we revealed that the current obeys the mKdV$^-$ equation with a shock-wave-type soliton solution (current soliton) by using the reductive perturbation method. This proved the existence of such a solution known numerically in the Josephson transmission lines that had been unsolved analytically for a long time.

In addition, we found that the current soliton creates an analogue black hole and a white hole pair by the modification of the electromagnetic wave velocity through the soliton-induced spatially varying inductance in the circuit. This pair provides a new platform for exploring Hawking radiation. Then, we derived the Hawking temperature based on the tunneling mechanism for Hawking radiation. As a result, the Hawking temperature was found to be experimentally accessible. The resulting formula also showed that the experimentally observed temperature was indeed due to Hawking radiation by confirming the dependence of soliton velocity on the temperature without changing circuit parameters.

Black hole solitons might enable us to explore further fascinating features and unveil phenomena of black holes that have not been accessible in real black holes so far, such as dynamical event horizon and birth of micro black holes as quantum nucleation. These will be discussed in the forthcoming papers. Therefore, this study will open a new path to the *circuit* quantum gravity.

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