Quantum fields in the static de Sitter universe

E. T. Akhmedov[1,2] K. V. Bazarov,[1,2] D. V. Diakonov,[1,2] and U. Moschella[3,4]

[1]*Institutskii per. 9, Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia*
[2]*B. Cheremushkinskaya, 25, Institute for Theoretical and Experimental Physics, 117218 Moscow, Russia*
[3]*Università degli Studi dell'Insubria—Dipartimento DiSAT, Via Valleggio, 11—22100 Como, Italy*
[4]*INFN, Sez di Milano, Via Celoria 16, 20146 Milano, Italy*

We construct explicit mode expansions of various tree-level propagators in the Rindler–de Sitter universe, also known as the static (or compact) patch of the de Sitter spacetime. We construct in particular the Wightman functions for thermal states having a generic temperature T. We give a fresh simple proof that the only thermal Wightman propagator that respects the de Sitter isometry is the restriction to the Rindler–de Sitter wedge of the propagator for the Bunch-Davies state. It is the thermal state with $T = (2\pi)^{-1}$ in the units of de Sitter curvature. We show that propagators with $T \neq (2\pi)^{-1}$ are only time translation invariant and have extra singularities on the boundary of the static patch. We also construct the expansions for the so-called alpha-vacua in the static patch and discuss the flat limit.

I. INTRODUCTION

Notwithstanding the existence of a vast literature on de Sitter quantum fields there is still no consensus as regard to their infrared behavior which is very much different from the behavior of Minkowski and anti–de Sitter fields. The root of the problem is the absence of a globally defined timelike Killing vector field on the de Sitter manifold: the components of the metric in general depend on the time variable t of the chosen coordinate system.

The nonstationarity of the de Sitter metric indicates that to compute loop quantum corrections one should make use of the Schwinger-Keldysh rather than the Feynman diagrammatic technique; in the first step, initial data have to be imposed on a Cauchy surface at some time t_0 to define the correlation functions. No matter what state is chosen, secular growing infrared contributions appear in the loops (see e.g., [1] for a review); these effects are global and sensitive to the initial conditions [1–4]. Summarizing, when quantizing fields in curved space-times, the field dynamics may and in general does depend on the choice of coordinates through the choice of the initial data and this is crucial, in particular, for understanding the properties of de Sitter quantum physics.

Cosmologists usually make use either of the spatially flat Poincaré coordinates [5–7] or of the global spherical coordinates [6–8] for the de Sitter manifold. To build correlation functions, the common initial choice is the Bunch-Davies (also called Euclidean) de Sitter invariant state [9–14]. What makes it special is that this is the only state among the de Sitter invariant ones—the so-called alpha vacua [15–17]—to be maximally analytic [18,19].

There is however a particular chart—the "static patch" discovered by de Sitter in 1917 [20]—which admits a timelike Killing vector field (see Fig. 1). The field of course is not globally timelike: it becomes lightlike on the boundary of the static patch (the horizons) and spacelike beyond it. A celebrated result by Gibbons and Hawking[1] [14,18,19,22,23] says that the restriction to the static patch of the Bunch-Davies state is a thermal equilibrium state at the temperature $1/2\pi R$ with respect to the relevant time coordinate, where R is the de Sitter radius.

As one can see, it is a very specific statement that (of course) is true only when all the above conditions are fulfilled. Nonetheless, it is not uncommon to hear or read the (incorrect) general statement that the de Sitter space has a temperature, which is sometimes taken as the starting point of vague speculations about cosmology and quantum gravity.

Apart from the Gibbons-Hawking result, not very much is known about quantum fields in the Rindler–de Sitter wedge (the static patch). The relevance of this model for cosmology and black hole physics makes this lack of knowledge even more surprising.

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[1]This result was actually predated by another important result by Figari *et al.* [21] who studied interacting quantum fields in the wedge in two dimensions by applying constructive methods on the Euclidean sphere.
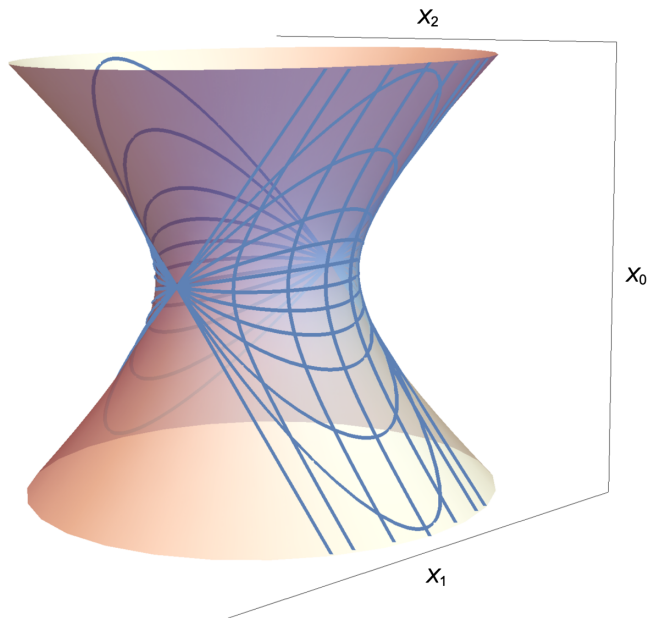
FIG. 1. The static patch and its opposite represented by embedding the two-dimensional de Sitter manifold in a three-dimensional Minkowski spacetime. The right (left) static patch is the intersection of the de Sitter manifold with the right (left) wedge of the ambient spacetime. The integral curves of the Killing vector field are the vertical hyperbolae obtained by intersecting the manifold with the planes $X^1 = \text{const.}$

In this paper, we partially fill this gap by constructing all the time translation invariant states at tree level. In particular, we produce new (to the best of our knowledge) formulas giving explicit mode expansions of the correlation functions where the coordinates of the static patch are separated. This is an obviously necessary preliminary step to apply perturbation theory and calculate loops as described above.

Our class of equilibrium states contains in particular all the thermal states, and we give a new direct proof that the complete de Sitter invariance is recovered only at $T = 1/2\pi R$. All the other states, including the zero temperature vacuum (pure) state, are not de Sitter invariant and have unusual singularities at the horizon, giving retrospectively some support to Einstein's suspicions about the equator of the patch [24].

We restrict our attention to the two-dimensional case to avoid unnecessary complications in the equations. Most of our results can be straightforwardly extended to other dimensions. Loop corrections to tree-level propagators will be investigated in a companion paper.

II. GEOMETRY

The coordinate system which is of interest for us here was introduced as early as 1917 by Willem de Sitter in the course of the famous debate on the relativity of inertia [24].

By visualizing the two-dimensional de Sitter space as the one-sheeted hyperboloid embedded in a three-dimensional ambient Minkowski space

$$dS_2 = \{X \in \mathbf{R}^3, \quad X^\alpha X_\alpha = X_0^2 - X_1^2 - X_2^2 = -R^2\} \quad (2.1)$$

(capital X^α denote the coordinates of a given Lorentzian frame of the ambient spacetime), the de Sitter static coordinates are

$$X\left(\frac{t}{R}, \frac{x}{R}\right) = \begin{cases} X^0 = R \sinh\frac{t}{R} \operatorname{sech}\frac{x}{R} \\ X^1 = R \tanh\frac{x}{R} = u \\ X^2 = R \cosh\frac{t}{R} \operatorname{sech}\frac{x}{R} \end{cases},$$

$$t \in (-\infty, \infty), x \in (-\infty, \infty). \quad (2.2)$$

In the following, we will set $R = 1$ and $\tanh x = u$.

From a group theoretical viewpoint, the new time coordinate t parametrizes the one-parameter subgroup of the de Sitter group stabilizing the equator. The action of that subgroup to points of any spherical spatial section containing the equator gives coordinates to two opposite static patches. In two dimensions, the spatial sections are ellipses; the equator degenerates in the two points where all the ellipses meet (see Fig. 1). A static patch is in fact the intersection of the de Sitter manifold with a Rindler wedge in one dimension more and this is why we also call it as Rindler–de Sitter wedge. The above coordinates thus cover only the region $\{|X^1| \langle 1\} \cap \{X^2 \rangle |X^0|\}$ of the *real* de Sitter manifold, the right wedge in Fig. 1, the shaded region in Fig. 2.

A Rindler–de Sitter wedge is itself a globally hyperbolic space-time, but a Cauchy surface for the wedge is incomplete with respect to the whole de Sitter manifold, being only "one half" of a bona fide Cauchy surface[2]; see Figs. 1 and 2. On the other hand, quantization in the static coordinates has an advantage in comparison with other coordinate systems: the Hamiltonian operator is time independent. The metric

$$ds^2 = \frac{dt^2 - dx^2}{\cosh x^2} \quad (2.3)$$

is time independent and conformal to the flat metric. The static patch is bordered by a bifurcate Killing horizon

$$x \to \pm\infty, \qquad t = \pm x,$$

where the metric degenerates. The corresponding Killing vector is not timelike when extended outside the static patch. The de Sitter invariant scalar product is given by

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[2]In saying this, we *suppose* that the geodesical completion of the wedge is the de Sitter manifold. Would we suppose that the geodesical completion be, say, its double covering, the result would change completely. In particular, there would be no thermal state at all [25].
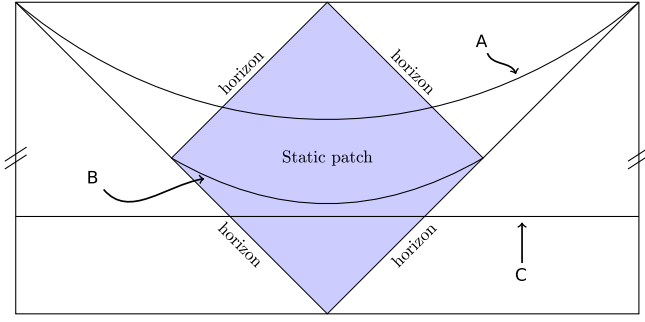
FIG. 2. Penrose diagram of the de Sitter manifold with Cauchy surfaces of different patches. A denotes a Cauchy surface for the expanding Poincaré patch, B for the static patch, and C for the global de Sitter space. The static patch is bordered by a bifurcate Killing horizon.

$$\zeta = \zeta_{12} = X_1^\alpha X_{2\alpha} = -\frac{\cosh(t_1 - t_2) + \sinh x_1 \sinh x_2}{\cosh x_1 \cosh x_2}.$$

(2.4)

The geodesic distances L and ζ are related as follows: $\zeta = -\cosh(L)$ for timelike geodesics, $\zeta = \cos(L)$ for spacelike ones, $\zeta = -1$ for lightlike separations or coincident points.

III. CANONICAL QUANTIZATION

In this section, we outline the canonical quantization of the Klein-Gordon field in the static chart coordinates,

$$\left(\partial_t^2 - \partial_x^2 + \frac{m^2}{\cosh^2 x}\right)\phi(t, x) = 0,$$

(3.1)

$$[\phi(t, x_1), \phi(t, x_2)] = 0,$$
$$[\phi(t, x_1), \dot\phi(t, x_2)] = i\delta(x_1 - x_2).$$

(3.2)

The static chart is in itself a globally hyperbolic manifold, though geodesically incomplete. We may apply standard methods of canonical quantization and look for a complete set of modes by separating the variables. Of course, the so-constructed set of modes will be incomplete when considered with respect to the whole de Sitter manifold [26,27].

Let us consider factorized modes which have *positive frequencies* with respect to the time coordinate t,

$$\varphi(t, x) = e^{-i\omega t}\psi_\omega(u), \qquad u = \tanh x.$$

(3.3)

$\psi_\omega(u)$ are eigenfunctions of the continuous spectrum of the well-known quantum mechanical scattering problem,

$$\left[-\partial_x^2 + \frac{m^2}{\cosh^2 x}\right]\psi_\omega(u) = \omega^2\psi_\omega(u), \qquad u = \tanh x,$$

$$m^2 = \frac{1}{4} + \nu^2.$$

(3.4)

For any given $\omega \geq 0$, the Ferrers functions $\mathsf{P}^{i\omega}_{-\frac{1}{2}+i\nu}(\pm u)$— also known as Legendre functions on the cut [28]—are two independent solutions of the above equation. The double degeneracy of the energy level ω points toward the introduction of two pairs of creation and annihilation operators for each level,

$$[a_{\omega_1}, a^\dagger_{\omega_2}] = \delta(\omega_1 - \omega_2), \qquad [b_{\omega_1}, b^\dagger_{\omega_2}] = \delta(\omega_1 - \omega_2),$$
$$[a_{\omega_1}, b_{\omega_2}] = [a_{\omega_1}, b^\dagger_{\omega_2}] = 0.$$

(3.5)

The mode expansion of the field operator $\phi(t, x)$ can then be written as follows:

$$\phi(t, x) = \int_0^\infty \frac{d\omega}{2\pi}[e^{-i\omega t}(\psi_\omega(u)a_\omega + \psi_\omega(-u)b_\omega)$$
$$+ e^{i\omega t}(\psi_\omega^*(u)a_\omega^\dagger + \psi_\omega^*(-u)b_\omega^\dagger)],$$

(3.6)

where

$$\psi_\omega(u) = \sqrt{\sinh(\pi\omega)}\,\Gamma\left(\frac{1}{2} + i\nu - i\omega\right)$$
$$\times \Gamma\left(\frac{1}{2} - i\nu - i\omega\right)\mathsf{P}^{i\omega}_{-\frac{1}{2}+i\nu}(u).$$

(3.7)

The normalization was chosen according with the completeness relation (A1) shown in the Appendix. At large positive x, the wave

$$\psi_\omega(\tanh x) \sim e^{i\omega x} \qquad x \to \infty$$

(3.8)

is purely right moving (at large negative $x \to -\infty$, the wave $\psi_\omega(-\tanh x) \sim e^{-i\omega x}$ is purely left moving).

By normal ordering with respect to the vacuum of the a_ω and b_ω operators, we get the free Hamiltonian in the standard form

$$:H := \int_{-\infty}^{+\infty} dx \sqrt{g} :T_0^0 := \int_0^{+\infty} d\omega\,\omega(a_\omega^\dagger a_\omega + b_\omega^\dagger b_\omega).$$

(3.9)

Note that the range of integration over ω starts from zero (rather than m as for a massive field in flat space). This is because the "mass" term in the action

$$S_m = \int d^2x\sqrt{g}\,m^2\,\varphi^2(t, x)$$

vanishes near the horizon [recall that $\sqrt{g} = (\cosh x)^{-2}$].

A. Thermal two-point functions

The quantum mechanical average over a thermal state of inverse temperature β is given by

$$\langle\mathcal{O}\rangle_\beta = \frac{\mathrm{Tr}\,\rho\mathcal{O}}{\mathrm{Tr}\,\rho}, \qquad \rho \equiv e^{-\beta H}.$$

(3.10)

Although the previous expression is ill-defined in quantum field theory, it still allows to compute the thermal two-point function at inverse temperature β by assuming the Bose-Einstein distribution of the energy levels,

$$\langle a_\omega^\dagger a_{\omega'} \rangle_\beta = \langle b_\omega^\dagger b_{\omega'} \rangle_\beta = (e^{\beta\omega} - 1)^{-1} \delta(\omega - \omega'). \tag{3.11}$$

Equations (3.6) and (3.9) give the following expression:

$$W_\beta(t_1 - t_2, x_1, x_2) = \langle \phi(t_1, x_1)\phi(t_2, x_2) \rangle_\beta = \int_0^\infty \frac{d\omega}{4\pi^2} \left[\frac{e^{-i\omega(t_1-t_2)}}{1 - e^{-\beta\omega}} (\psi_\omega(u_1)\psi_\omega^*(u_2)) \right.$$

$$+ \psi_\omega(-u_1)\psi_\omega^*(-u_2)) + \frac{e^{i\omega(t_1-t_2)}}{e^{\beta\omega} - 1} \left. (\psi_\omega^*(u_1)\psi_\omega(u_2) + \psi_\omega^*(-u_1)\psi_\omega(-u_2)) \right]$$

$$= \int_{-\infty}^\infty e^{-i\omega(t_1-t_2)} \frac{1 - e^{-2\pi\omega}}{1 - e^{-\beta\omega}} \tilde{P}_\nu(\omega, u_1, u_2) d\omega, \tag{3.12}$$

where

$$\tilde{P}_\nu(\omega, u_1, u_2) = \frac{e^{\pi\omega}(\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(u_1)\mathsf{P}_{-\frac{1}{2}-i\nu}^{-i\omega}(u_2) + \mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(-u_1)\mathsf{P}_{-\frac{1}{2}-i\nu}^{-i\omega}(-u_2))}{8\cosh\pi(\nu - \omega)\cosh\pi(\nu + \omega)}. \tag{3.13}$$

The states defined by the above two-point functions are mixed. The only pure state is obtained in the limit $\beta \to \infty$.

In Sec. V, we will prove that for $\beta = 2\pi$, the above two-point function is de Sitter invariant and coincides with the restriction to the static patch of the Bunch-Davies two-point function,

$$W_{2\pi}(t_1 - t_2, x_1, x_2) = W_{\mathrm{BD}}(\zeta) = \frac{1}{4\cosh\pi\nu} P_{-\frac{1}{2}+i\nu}(\zeta), \tag{3.14}$$

where ζ is the de Sitter invariant variable defined in (2.4). On the other hand, for arbitrary β, the two-point function $W_\beta(t_1 - t_2, x_1, x_2)$ and its permuted function do not respect the de Sitter isometry because their periodicity thermal property in imaginary time $t \to t + i\beta$ is incompatible with the geometry of the global de Sitter manifold, the only exception being $\beta = 2\pi$.

IV. MODE EXPANSION OF THE HOLOMORPHIC PLANEPLANE WAVES

Let us now move to the complex two-dimensional de Sitter spacetime,

$$dS_2^c = \{Z \in \mathbf{C}^3, \qquad Z_0^2 - Z_1^2 - Z_2^2 = -1\}. \tag{4.1}$$

We may use the same coordinate chart as in Eq. (2.2),

$$Z(t, x) = \begin{cases} Z^0 = \sinh t \ \mathrm{sech} x \\ Z^1 = \tanh x \\ Z^2 = \cosh t \ \mathrm{sech} x \end{cases}, \tag{4.2}$$

but now t and x are complex. In particular,

(1) For $0 < \mathrm{Im} t < \pi$ and $x \in \mathbf{R}$, the point $Z(t, x)$ belongs to the forward tube

$$\mathcal{T}_+ = \{Z = X + iY \in dS_2^c, \quad Y^2 > 0, \quad Y^0 > 0\}. \tag{4.3}$$

(ii) For $-\pi < \mathrm{Im} t < 0$ and $x \in \mathbf{R}$, the point $Z(t, x)$ belongs to the backward tube

$$\mathcal{T}_- = \{Z = X + iY \in dS_2^c, \quad Y^2 > 0, \quad Y^0 < 0\}. \tag{4.4}$$

There exists a remarkable set of solutions of the de Sitter Klein-Gordon equation which may be interpreted as de Sitter plane waves [18,19,29]. Their definition makes no appeal to any particular coordinate system and may be given just in terms of the ambient space-time coordinates: given a forward pointing lightlike real vector ξ in the ambient space-time[3] and a complex number $\lambda \in C$, let us construct the homogeneous function

$$Z \in dS_2^c: \quad Z \mapsto (\xi \cdot Z)^\lambda. \tag{4.5}$$

For any given ξ and λ, the above functions are holomorphic in the tuboids $\mathcal{T}_\pm$ [18,19] and satisfy the massive (complex) de Sitter Klein-Gordon equation,

$$(\Box - \lambda(\lambda - 1))(\xi \cdot Z)^\lambda = 0 \tag{4.6}$$

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[3]ξ is a real vector belonging to the forward light cone $C^+ = \{\xi \in \mathbf{R}^3, \quad (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 = 0, \xi^0 > 0\}.$

(we may write $\lambda = -\frac{1}{2} + i\nu$; in the following, we will take for simplicity $\nu \in \mathbf{R}$). The boundary values

$$(\xi \cdot X)^{\lambda}_{\pm} = \lim_{Z \in \mathcal{T}_{\pm}, Z \to X} (\xi \cdot Z)^{\lambda} \qquad (4.7)$$

are homogeneous distributions of degree λ in the ambient spacetime and their restrictions to the real manifold dS_2 are solutions of the real de Sitter Klein-Gordon equation. All these objects are entire functions of λ.

Let us now expand the above plane wave into modes of the static patch. The first thing to be done is to choose a basis manifold of the forward light cone; the convenient choice is the hyperbolic basis $\Gamma = \Gamma_l \cup \Gamma_r$ "parallel" to the coordinate system (2.2) of the static chart,

$$\xi_l(w) = \begin{cases} \xi^0 = \cosh w \\ \xi^1 = -1 \\ \xi^2 = \sinh w \end{cases} \qquad \xi_r(w) = \begin{cases} \xi^0 = \cosh w \\ \xi^1 = +1 \\ \xi^2 = \sinh w \end{cases}. \qquad (4.8)$$

With all the above specifications, we get

$$\xi_l \cdot Z = \tanh x + \operatorname{sech} x \sinh(t - w), \qquad \xi_r \cdot Z = -\tanh x + \operatorname{sech} x \sinh(t + w). \qquad (4.9)$$

Let us take $Z(t + i\epsilon, x)$ with t real. Since $Z(t + i\epsilon, x) \in \mathcal{T}_+$, the wave $(\xi_l \cdot Z)^{\lambda}$ is a regular function of t decreasing at infinity; its Fourier transform is given by

$$\int_{-\infty}^{\infty} e^{-i\omega t}(\xi_l(w) \cdot Z(t + i\epsilon, x))^{-\frac{1}{2}+i\nu} dt = \frac{2e^{-i\omega w}\Gamma(\frac{1}{2} - i\nu + i\omega)}{\Gamma(\frac{1}{2} - i\nu)} e^{\frac{1}{2}\pi\omega} (e^{-\pi\omega} Q^{-i\omega}_{-\frac{1}{2}-i\nu}(u + i\epsilon)). \qquad (4.10)$$

Here Q is the associated Legendre function of the second kind[4] [28] defined on the complex plane cut on the real axis from $-\infty$ to 1. Inversion gives

$$(\xi_l(w) \cdot Z(t + i\epsilon, x))^{-\frac{1}{2}+i\nu} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\omega(t-w)} \frac{\Gamma(\frac{1}{2} - i\nu + i\omega)}{\Gamma(\frac{1}{2} - i\nu)} e^{\frac{1}{2}\pi\omega} (e^{-\pi\omega} Q^{-i\omega}_{-\frac{1}{2}-i\nu}(u + i\epsilon)) d\omega. \qquad (4.12)$$

For $Z \in \mathcal{T}^-$, an analogous computation gives

$$(\xi_l(w) \cdot Z(t - i\epsilon, x))^{-\frac{1}{2}-i\nu} = \frac{1}{\pi} \int e^{-i\omega(t-w)} \frac{\Gamma(\frac{1}{2} + i\nu - i\omega)}{\Gamma(\frac{1}{2} + i\nu)} e^{\frac{1}{2}\pi\omega} (e^{\pi\omega} Q^{i\omega}_{-\frac{1}{2}+i\nu}(u - i\epsilon)) d\omega. \qquad (4.13)$$

Similarly,

$$(\xi_r(w) \cdot Z(t \pm i\epsilon, x))^{-\frac{1}{2}\pm i\nu} = \pm \frac{e^{-\nu\pi}}{i\pi} \int e^{\pm i\omega(t+w)} \frac{\Gamma(\frac{1}{2} \mp i\nu \pm i\omega)}{\Gamma(\frac{1}{2} \mp i\nu)} e^{\frac{1}{2}\pi\omega} (e^{\mp\pi\omega} Q^{\mp i\omega}_{-\frac{1}{2}\mp i\nu}(u \mp i\epsilon)) d\omega. \qquad (4.14)$$

V. MODE EXPANSIONS OF THE MAXIMALLY ANALYTIC TWO-POINT FUNCTION

The maximally analytic (Bunch-Davies) two-point function admits the following *global* manifestly de Sitter invariant integral representation, valid for $Z_1 \in \mathcal{T}^-$ and $Z_2 \in \mathcal{T}^+$ [18,19]:

$$W_{\text{BD}}(Z_1, Z_2) = \frac{e^{\pi\nu}}{8\pi \cosh \pi\nu} \int_{\Sigma} (\xi \cdot Z_1)^{-\frac{1}{2}-i\nu} (\xi \cdot Z_2)^{-\frac{1}{2}+i\nu} d\sigma(\xi). \qquad (5.1)$$

Here Σ is any basis manifold of the forward light cone C_+ and $d\sigma$ the corresponding induced measure [18]. In the symbol W_{BD} referring to the Bunch-Davies Wightman function, we left the mass parameter is $m = \sqrt{\frac{1}{4} + \nu^2}$ implicit.

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[4]Note that the above Legendre functions are related by complex conjugation as follows:

$$\left(Q^{-i\omega}_{-\frac{1}{2}-i\nu}(u + i\epsilon)\right)^* = e^{2\pi\omega} Q^{i\omega}_{-\frac{1}{2}+i\nu}(u - i\epsilon). \qquad (4.11)$$

By using the static coordinates (2.2), the basis $\Gamma = \Gamma_l \cup \Gamma_r$ for the light cone (with $d\sigma_{\Gamma_i} = dw$) and by inserting Eqs. (4.12)–(4.14) in Eq. (5.1), we get that the boundary value on the reals in the static chart of the above global holomorphic two-point function can be represented as follows:

$$W_{\mathrm{BD}}(X_1, X_2) = \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} \tilde{W}_{\mathrm{BD}}(\omega, u_1, u_2) d\omega, \tag{5.2}$$

$$\tilde{W}_{\mathrm{BD}}(\omega, u_1, u_2) = \frac{e^{\pi\omega}}{4\pi^2 \cosh \pi(\nu - \omega)} [e^{\pi\nu} Q_{-\frac{1}{2}+i\nu}^{i\omega}(u_1 - i\epsilon) Q_{-\frac{1}{2}-i\nu}^{-i\omega}(u_2 + i\epsilon) + e^{-\pi\nu} Q_{-\frac{1}{2}+i\nu}^{i\omega}(u_1 + i\epsilon) Q_{-\frac{1}{2}-i\nu}^{-i\omega}(u_2 - i\epsilon)]. \tag{5.3}$$

By using the identity [28]

$$Q_{-\frac{1}{2}+i\nu}^{i\omega}(u \pm i0) = \frac{\pi}{2\cosh(\pi(\nu + \omega))} e^{-\pi\omega \mp \frac{\pi\omega}{2}} (\mp \, ie^{\pm\pi(\nu+\omega)} \mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(u) + \mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(-u)), \tag{5.4}$$

a straightforward calculation shows that

$$\tilde{W}_{\mathrm{BD}}(\omega, u_1, u_2) = \tilde{P}_{\nu}(\omega, u_1, u_2). \tag{5.5}$$

When $\beta = 2\pi$, Eqs. (3.12) and (5.3) do coincide proving the claimed identification.

The permuted two-point function is in turn represented as follows:

$$W_{\mathrm{BD}}(X_2, X_1) = \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} \tilde{P}_{\nu}(-\omega, u_2, u_1) d\omega = \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} e^{-2\pi\omega} \tilde{P}_{\nu}(\omega, u_1, u_2) d\omega. \tag{5.6}$$

In the above chain of identities, we changed the integration variable $\omega \to -\omega$ and—in the second step—used the symmetry of the two-point function $\nu \to -\nu$. As a by-product—by the Riemann-Lebesgue theorem—we get also the following crucial identity (which may also be checked directly):

$$\tilde{P}_{\nu}(-\omega, u_2, u_1) = e^{-2\pi\omega} \tilde{P}_{\nu}(\omega, u_1, u_2). \tag{5.7}$$

Equation (5.6) encodes the Kubo-Martin-Schwinger property of the restriction of the maximal analytic two-point function (5.3) to the static patch: a geodetic observer in the static patch "perceives" a thermal bath of particles at inverse temperature $2\pi R$.

VI. MORE ABOUT THE VACUUM OF THE STATIC GEODETIC OBSERVER

By using Eqs. (5.2) and (5.6), we obtain the following new integral representation of the covariant commutator in the static chart:

$$C_{\nu}(X_1, X_2) = W_{\mathrm{BD}}(X_1, X_2) - W_{\mathrm{BD}}(X_2, X_1) = \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} \tilde{C}_{\nu}(\omega, u_1, u_2) d\omega, \tag{6.1}$$

where

$$\tilde{C}_{\nu}(\omega, u_1, u_2) = (1 - e^{-2\pi\omega}) \tilde{P}_{\nu}(\omega, u_1, u_2) = -\tilde{C}_{\nu}(-\omega, u_2, x_1). \tag{6.2}$$

Let us take the zero temperature limit $\beta \to \infty$ in Eq. (3.12); only positive energies survive,

$$W_{\infty}(X_1, X_2) = \int_0^{\infty} e^{-i\omega(t_1-t_2)} (1 - e^{-2\pi\omega}) \tilde{P}_{\nu}(\omega, u_1, u_2) d\omega$$

$$= \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} \theta(\omega) \tilde{C}_{\nu}(\omega, u_1, u_2) d\omega. \tag{6.3}$$

$\theta(\omega)$ is Heaviside's step function. The above equation points toward the following natural family of Rindler–de Sitter positive frequency modes ($\omega \geq 0$):

$$\varphi_{\omega,1}(t,u) = e^{\frac{1}{2}\pi\nu}e^{\pi\omega}\sqrt{\frac{\sinh \pi\omega}{2\pi^3}}\Gamma\left(\frac{1}{2}-i\nu+i\omega\right)e^{-i\omega t}Q^{i\omega}_{-\frac{1}{2}+i\nu}(u-i\epsilon),$$

$$\varphi_{\omega,2}(t,u) = e^{-\frac{1}{2}\pi\nu}e^{\pi\omega}\sqrt{\frac{\sinh \pi\omega}{2\pi^3}}\Gamma\left(\frac{1}{2}-i\nu+i\omega\right)e^{-i\omega t}Q^{i\omega}_{-\frac{1}{2}+i\nu}(u+i\epsilon),$$

$$\varphi^*_{\omega,1}(t,u) = e^{\frac{1}{2}\pi\nu}e^{-\pi\omega}\sqrt{\frac{\sinh \pi\omega}{2\pi^3}}\Gamma\left(\frac{1}{2}+i\nu-i\omega\right)e^{i\omega t}Q^{-i\omega}_{-\frac{1}{2}-i\nu}(u+i\epsilon),$$

$$\varphi^*_{\omega,2}(t,u) = e^{-\frac{1}{2}\pi\nu}e^{-\pi\omega}\sqrt{\frac{\sinh \pi\omega}{2\pi^3}}\Gamma\left(\frac{1}{2}+i\nu-i\omega\right)e^{i\omega t}Q^{-i\omega}_{-\frac{1}{2}-i\nu}(u-i\epsilon), \tag{6.4}$$

equivalent to the one used in Sec. III. Using the above modes, we may represent the field operator in the static patch in the usual way,

$$\phi(t,x) = \int_0^\infty (\varphi_{\omega,1}(t,u)a_1(\omega) + \varphi_{\omega,2}(t,u)a_2(\omega) + \varphi^*_{\omega,1}(t,u)a_1^\dagger(\omega) + \varphi^*_{\omega,2}(t,u)a_2^\dagger(\omega))d\omega. \tag{6.5}$$

The state $W_\infty(X_1,X_2)$ is characterized by the conditions $a_1(\omega)\Psi_0 = a_2(\omega)\Psi_0 = 0$; it is a *pure state*,

$$W_\infty(X_1,X_2) = \sum_i \int_0^\infty \varphi_{\omega,i}(t_1,u_1)\varphi^*_{\omega,i}(t_2,u_2)d\omega. \tag{6.6}$$

Positive definiteness is also clear from (6.6). The state defined by W_∞ may be interpreted as the vacuum state for the geodesic observer in the Rindler–de Sitter wedge and is the close analogous of the fulling vacuum of Rindler quantum field theory [30,31].

Finally, the covariant commutator (6.1), which is of course independent from the chosen state, can be written as follows:

$$C_\nu(X_1,X_2) = W_\infty(X_1,X_2) - W_\infty(X_2,X_1)$$

$$= \sum_{i=1}^2 \int_0^\infty [\varphi_{\omega,i}(t_1,u_1)\varphi^*_{\omega,i}(t_2,u_2) - \varphi_{\omega,i}(t_2,u_2)\varphi^*_{\omega,i}(t_1,u_1)]d\omega. \tag{6.7}$$

VII. OTHER TIME-TRANSLATION INVARIANT STATES

More generally, we may introduce the two-point functions

$$W_F(X_1,X_2) = \int_{-\infty}^\infty e^{-i\omega(t_1-t_2)}F(\omega)\tilde{C}_\nu(\omega,u_1,u_2)d\omega, \tag{7.1}$$

$$W_F(X_2,X_1) = \int_{-\infty}^\infty e^{-i\omega(t_1-t_2)}F(-\omega)\tilde{C}_\nu(-\omega,u_2,u_1)d\omega, \tag{7.2}$$

where $F(\omega)$ is a real function or a distribution such that the product $F(\omega)\tilde{C}(\omega,u_1,u_2)$ is well defined. Equation (6.2) implies that it must be

$$F(\omega) + F(-\omega) = 1. \tag{7.3}$$

In particular,

(1) The vacuum (6.3) of the static geodetic observer corresponds to

$$F(\omega) = \theta(\omega). \tag{7.4}$$

(2) The Bunch-Davies maximally analytic state (5.3) corresponds to

$$F(\omega) = \frac{1}{1 - e^{-2\pi\omega}}. \tag{7.5}$$

(3) An antisymmetric function $\beta(-\omega) = -\beta(\omega)$ defines a time invariant state

$$F(\omega) = \frac{1}{1 - e^{-\beta(\omega)}}. \tag{7.6}$$

(4) The thermal equilibrium state (3.12) at inverse temperature β corresponds to $\beta(\omega) = \beta\omega$.

All the above two-point functions have the following general structure:

$$W(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \cosh^2(\gamma(\omega))\varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^*(t_2, u_2)d\omega + \sum_{i=1,2} \int_0^\infty \sinh^2(\gamma(\omega))\varphi_{\omega,i}^*(t_1, u_1)\varphi_{\omega,i}(t_2, u_2)d\omega, \quad (7.7)$$

with in particular $\cosh\gamma(\omega) = (1 - e^{-\beta\omega})^{-\frac{1}{2}}$ for the thermal state,

$$W_\beta(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \frac{\varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^*(t_2, u_2)}{1 - e^{-\beta\omega}} d\omega + \sum_{i=1,2} \int_0^\infty \frac{\varphi_{\omega,i}^*(t_1, u_1)\varphi_{\omega,i}(t_2, u_2)}{e^{\beta\omega} - 1} d\omega. \quad (7.8)$$

The latter formula in turn allows to write $W_\beta(t, x_1, x_2)$ as a Matsubara sum over imaginary frequencies as follows:

$$W_\beta(X_1, X_2) = \sum_{n=0}^\infty W_\infty(t_1 - in\beta, x_1, t_2, x_2) + \sum_{n=1}^\infty W_\infty(t_2, x_2, t_1 + in\beta, x_1). \quad (7.9)$$

The above representations clearly show that all such states (but the vacuum $\gamma = 0$) are mixed states. In particular, the maximally analytic Bunch-Davies two-point function is written as

$$W_{\mathrm{BD}}(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \frac{\varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^*(t_2, u_2)}{1 - e^{-2\pi\omega}} d\omega + \sum_{i=1,2} \int_0^\infty \frac{\varphi_{\omega,i}^*(t_1, u_1)\varphi_{\omega,i}(t_2, u_2)}{e^{2\pi\omega} - 1} d\omega. \quad (7.10)$$

These are simple examples of what has been called a generalized Bogoliubov transformation [26,27], a construction that directly provides mixed states by suitably extending the canonical quantization formalism.

VIII. ALPHA STATES IN THE STATIC CHART

The set of states (7) does not contain every time translation invariant state. There is still the freedom to add to the two-point function a symmetric part that, as such, does not contribute to the commutator. The so-called α-vacua [4,12,16,17,32] belong to this second class of states. Let us briefly sketch their construction in the static patch coordinates.

The two-point Wightman functions of the α-vacua may be written in terms of the Bunch-Davies two-point function as follows [17,32]:

$$W^{(\alpha)}(X_1, X_2)$$
$$= \cosh^2\alpha W_{\mathrm{BD}}(X_1, X_2) + \sinh^2\alpha W_{\mathrm{BD}}(X_2, X_1)$$
$$+ \frac{1}{2}\sinh 2\alpha[W_{\mathrm{BD}}(X_1, -X_2) + W_{\mathrm{BD}}(-X_1, X_2)]. \quad (8.1)$$

We are left with the task of expanding $W_{\mathrm{BD}}(X_1, -X_2)$ in the modes (6.4). To do it, let us introduce the parity automorphism of the static patch,

$$X(t, x) \rightarrow \tilde{X}(t, x) = X(t, -x). \quad (8.2)$$

The curve $s \rightarrow \tilde{X}(t + is, x)$ for $0 < s < \pi$ is entirely contained in $\mathcal{T}_+$ and ends at

$$\tilde{X}(t + i\pi, x) = -X(t, x) \quad (8.3)$$

in the left Rindler–de Sitter wedge (see Fig. 1). Similarly, the curve $s \rightarrow \tilde{X}(t + is, x)$ for $0 > s > -\pi$ is entirely contained in $\mathcal{T}_-$ and ends again at $\tilde{X}(t - i\pi, x) = -X(t, x)$ but from the opposite tube.

Given any two points X_1 and X_2 in the right Rindler–de Sitter wedge, we may use again the maximally analytic global two-point function (5.1) and get

$$W_{\mathrm{BD}}(X_1, -X_2) = W_{\mathrm{BD}}(X_1, \tilde{X}_2(t_2 + i\pi, x_2)) = W_{\mathrm{BD}}(-X_1, X_2) = W_{\mathrm{BD}}(\tilde{X}_1(t_1 - i\pi, x_1), X_2)$$

$$= -\frac{i}{4\pi^2} \int_{-\infty}^\infty \frac{e^{-i\omega(t_1 - t_2)}}{\cosh\pi(\nu - \omega)} Q_{-\frac{1}{2}+i\nu}^{i\omega}(u_1 - i\epsilon)Q_{-\frac{1}{2}-i\nu}^{-i\omega}(u_2 - i\epsilon)d\omega$$

$$+ \frac{i}{4\pi^2} \int_{-\infty}^\infty \frac{e^{-i\omega(t_1 - t_2)}}{\cosh\pi(\nu - \omega)} Q_{-\frac{1}{2}+i\nu}^{i\omega}(u_1 + i\epsilon)Q_{-\frac{1}{2}-i\nu}^{-i\omega}(u_2 + i\epsilon)d\omega \quad (8.4)$$

$$= -i \int_0^\infty \frac{\varphi_{\omega,1}(t_1, x_1)\varphi_{\omega,2}^*(t_2, x_2) - \varphi_{\omega,2}(t_1, x_1)\varphi_{\omega,1}^*(t_2, x_2)}{2 \sinh \pi\omega} d\omega$$

$$+ i \int_0^\infty \frac{\varphi_{\omega,1}^*(t_1, u_1)\varphi_{\omega,2}(t_2, u_2) - \varphi_{\omega,2}^*(t_1, u_1)\varphi_{\omega,1}(t_2, u_2)}{2 \sinh \pi\omega} d\omega. \tag{8.5}$$

In the second step, we used Eq. (7.10) and the following relations:

$$\varphi_{\omega,1}(t \pm i\pi, -u) = ie^{\pm\pi\omega}\varphi_{\omega,2}(t, u), \qquad \varphi_{\omega,2}(t \pm i\pi, -u) = -ie^{\pm\pi\omega}\varphi_{\omega,1}(t, u),$$

$$\varphi_{\omega,1}^*(t \pm i\pi, -u) = -ie^{\mp\pi\omega}\varphi_{\omega,2}^*(t, u), \qquad \varphi_{\omega,2}^*(t \pm i\pi, -u) = ie^{\mp\pi\omega}\varphi_{\omega,1}^*(t, u). \tag{8.6}$$

Integration is (8.5), the over positive energies only. Putting everything together, we get

$$W^{(\alpha)}(X_1, X_2) = \sum_i \int_0^\infty \cosh^2(\gamma(\omega))\varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^*(t_2, u_2)d\omega$$

$$+ \sum_i \int_0^\infty \sinh^2(\gamma(\omega))\varphi_{\omega,i}^*(t_1, u_1)\varphi_{\omega,i}(t_2, u_2)d\omega$$

$$- i \sinh 2\alpha \int_0^\infty \frac{\varphi_{\omega,1}(t_1, x_1)\varphi_{\omega,2}^*(t_2, x_2) - \varphi_{\omega,2}(t_1, x_1)\varphi_{\omega,1}^*(t_2, x_2)}{2 \sinh \pi\omega} d\omega$$

$$+ i \sinh 2\alpha \int_0^\infty \frac{\varphi_{\omega,1}^*(t_1, u_1)\varphi_{\omega,2}(t_2, u_2) - \varphi_{\omega,2}^*(t_1, u_1)\varphi_{\omega,1}(t_2, u_2)}{2 \sinh \pi\omega} d\omega, \tag{8.7}$$

where

$$\cosh(\gamma(\omega)) = \sqrt{\frac{e^{\pi\omega}\cosh^2\alpha + e^{-\pi\omega}\sinh^2\alpha}{2\sinh(\pi\omega)}}. \tag{8.8}$$

As expected, the α-vacua are translation invariant with respect to the time variable of the Rindler—de Sitter wedge; here the generalized Bogoliubov transformation of the positive energy modes is more general than the one exhibited in Eq. (7.7). The extra terms that do not contribute to the commutator are altogether symmetric in the exchange of X_1 and X_2.

IX. MORE ABOUT THERMAL PROPAGATORS

In this section, we examine some properties of the thermal correlation functions and discuss various limiting behaviors. This study is to better characterize them and also to lay the ground for the study of the IR loop contributions which will be the matter of a companion paper.

A. Wightman propagators for large timelike separation

Let us consider the limit $t = t_1 - t_2 \to \infty$, $x_1 = x_2 = 0$ (the general case $x_1 \neq x_2$ being essentially the same). The integrand in Eq. (3.12) has poles at

$$\omega = \pm\nu - \frac{i}{2} + in, \qquad n \in \mathbb{Z}, \qquad \omega = \frac{2\pi ik}{\beta},$$

$$k \in \mathbb{Z}, \qquad k \neq 0. \tag{9.1}$$

In Eq. (3.12), there is no pole at $\omega = 0$; still, $\omega = 0$ has a role to play in calculating the spacelike asymptotics.

In the limit $t \to \infty$, the leading contributions come from the poles which are closer to the real axis,

$$W_\beta(t, x_1 = x_2 = 0)$$

$$\approx \begin{cases} e^{-\frac{t}{2}}(C_+e^{i\nu t} + C_-e^{-i\nu t}) & \text{for } \beta < 4\pi \\ C_\beta e^{-t\frac{2\pi}{\beta}} & \text{for } \beta > 4\pi \end{cases}, \tag{9.2}$$

where

$$C_+ = C_-^* = \frac{1 - e^{-2\pi(\nu+\frac{i}{2})}}{1 - e^{-\beta(\nu+\frac{i}{2})}} \frac{e^{-\pi\nu}\Gamma(i\nu)\Gamma(\frac{1}{2} - i\nu)}{2\pi\sqrt{\pi}}, \tag{9.3}$$

$$C_\beta = \frac{\sin(\frac{2\pi^2}{\beta})}{4\pi^2\beta} \left| \Gamma\left(\frac{1}{4} - \frac{i\nu}{2} - \frac{\pi}{\beta}\right)\Gamma\left(\frac{1}{4} - \frac{i\nu}{2} + \frac{\pi}{\beta}\right) \right|^2. \tag{9.4}$$

The asymptotic behavior of the propagator changes at $\beta = 4\pi$. In the limit $\beta \to \infty$, the constant C_β tends to zero and the Wightman function asymptotics are given again in the upper line in (9.2).

The first set of poles in (9.1) is also actually related to the transmission and reflection coefficients of the quantum

mechanical scattering problem (3.4). For instance, a straightforward computation based on Eqs. (9.6) and (9.7) gives the transmission coefficient

$$T = \frac{\sinh^2(\pi\omega)}{\cosh[\pi(\nu - \omega)]\cosh[\pi(\nu + \omega)]}. \qquad (9.5)$$

The second set of poles in (9.1) depends also on the inverse temperature β. As β increases, the poles move toward the real axis. When $\beta > 4\pi$ poles of the second set dominate, the large t behavior of the propagator changes accordingly.

B. Large spacelike separation

We now evaluate the asymptotic behaviour of the correlators when one of the spacelike coordinates goes to infinity in two distinct ways.

By use of the asymptotic behaviour of the Ferrers function at $x \to \infty$, we get

$$\mathsf{P}^{i\omega}_{-\frac{1}{2}+i\nu}(\tanh x) \underset{x\to\infty}{\approx} \frac{e^{i\omega x}}{\Gamma(1 - i\omega)}, \qquad (9.6)$$

$$\mathsf{P}^{i\omega}_{-\frac{1}{2}+i\nu}(-\tanh x) \underset{x\to\infty}{\approx} \left[\frac{\Gamma(-i\omega)e^{-i\omega x}}{\Gamma(\frac{1}{2} + i\nu - i\omega)\Gamma(\frac{1}{2} - i\nu - i\omega)} + \frac{\cosh(\nu\pi)\Gamma(i\omega)e^{i\omega x}}{\pi}\right]. \qquad (9.7)$$

The singularities at $\omega = 0$ in the latter equation cancel each other, but, in the limit under consideration, the two terms contribute separately. By substituting the above expressions into (3.12) and making the shift $\omega \to \omega + i\epsilon$, we see that the dominant contribution comes from the lower half plane. We get that in the limit $x_1 \to \infty$ the Wightman propagator still depends on the temperature,

$$\lim_{x_2\to\infty} W_\beta(t_1 - t_2, x_1, x_2) = \frac{2\pi}{\beta}\frac{1}{4\cosh\nu\pi}P_{-\frac{1}{2}+i\nu}(-\tanh x_1) = \frac{2\pi}{\beta}W_{\mathrm{BD}}(-\tanh x_1). \qquad (9.8)$$

Alternatively, we may consider the formal Taylor expansion of Eq. (3.12),

$$W_\beta(t, x_1, x_2) = \frac{2\pi}{\beta}W_{\mathrm{BD}}(\zeta) + \left(\pi - \frac{2\pi^2}{\beta}\right)i\frac{\partial}{\partial t}W_{\mathrm{BD}}(\zeta)$$
$$- \left(\frac{\pi\beta}{6} + \frac{4\pi^3}{3\beta} - \pi^2\right)\frac{\partial^2}{\partial t^2}W_{\mathrm{BD}}(\zeta) - \left(-\frac{\pi^2\beta}{6} - \frac{2\pi^4}{3\beta} + \frac{2\pi^3}{3}\right)i\frac{\partial^3}{\partial t^3}W_{\mathrm{BD}}(\zeta) + \dots. \qquad (9.9)$$

For $\beta = 2\pi$, all the de Sitter breaking terms (i.e., every term but the first) at the rhs cancel, as expected. Also, when $t = t_1 - t_2$ is held constant and either x_1 or x_2 tends to plus or minus infinity, only the first terms at the rhs survive, with $\zeta = -\tanh x$.

C. Lightlike separation

For lightlike separations, the propagators should behave as in Minkowski space. In the Bunch-Davies invariant case, this comes immediately from Eq. (3.14),

$$W_{\mathrm{BD}}(\zeta \approx -1) \approx -\frac{1}{4\pi}\log(1 + \zeta) \approx -\frac{1}{4\pi}\log[t^2 - (x_1 - x_2)^2]. \qquad (9.10)$$

For arbitrary β at lightlike separation, large values of ω's dominate in the integral (3.12). For large ω, we may approximate $\mathsf{P}^{i\omega}_{-\frac{1}{2}+i\nu}(\tanh x_1) \approx e^{i\omega x_1}/\Gamma(1 - i\omega)$ and get the leading term

$$W_\beta(t, x_1, x_2) \approx \int_1^\infty \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{2\omega}\left(e^{-i\omega(x_1 - x_2)} + e^{i\omega(x_1 - x_2)}\right)$$
$$\approx -\frac{1}{4\pi}\log[t^2 - (x_2 - x_1)^2].$$

The cutoff in this integral is order of R—the radius of the de Sitter universe, which we set equal to one. The approximation works for $|\omega|$ much larger than m and R. The dependence on the temperature is lost in this high energy limit: only the Hadamard term survives.

D. Anomalous singularities at the horizon

When the temperature is an integer multiple of the Hawking-Gibbons temperature, i.e., when $\beta = 2\pi/N$, we may use Eq. (3.12) to derive another representation of the two-point function as a finite sum of Legendre functions [as opposed to the infinite Matsubara-type series (7.9)]; this is obtained by translating the Bunch-Davies maximal analytic two-point function in the imaginary time variable within the analyticity strip $(-2\pi < \mathrm{Im}\, t < 0)$ (see also [4,33]),

$$W_{\frac{2\pi}{N}}(t_1 - t_2, x_1, x_2)$$

$$= \int_{-\infty}^{+\infty} e^{-i\omega(t_1-t_2)} \frac{1 - e^{-2\pi\omega}}{1 - e^{-\frac{2\pi\omega}{N}}} \tilde{P}_{\omega,\nu}(u_1, u_2) d\omega$$

$$= \frac{1}{4\cosh \pi\nu} P_{-\frac{1}{2}+i\nu}(\zeta(t_1 - t_2 - i\epsilon, x_1, x_2))$$

$$+ \frac{1}{4\cosh \pi\nu} \sum_{n=1}^{N-1} P_{-\frac{1}{2}+i\nu}\left(\zeta\left(t_1 - t_2 - i\frac{2\pi n}{N}, x_1, x_2\right)\right).$$

$$(9.11)$$

The first term on the rhs is exactly the Bunch-Davies de Sitter invariant Wightman function; this is singular at $\zeta = -1$. The extra terms become singular when the two points approach either the left or the right horizon,

$$X_1 = X(\lambda + c_1, \lambda), \qquad X_2 = X(\lambda + c_2, \lambda + \Delta\lambda). \quad (9.12)$$

In the limit $\lambda \to \pm\infty$, the above events belong to the horizons. Then,

$$\zeta\left(c_1 - c_2 - i\frac{2\pi n}{N}, \lambda, \lambda + \Delta\lambda\right)$$

$$= -\frac{\cosh(c_1 - c_2 - i\frac{2\pi n}{N}) + \sinh\lambda \sinh(\lambda + \Delta\lambda)}{\cosh\lambda \cosh(\lambda + \Delta\lambda)} \to -1.$$

For generic β, the limit $\lambda \to \infty$ may be obtained by performing manipulations similar to those which led to (9.8),

$$W_\beta(\lambda \to \infty) \approx -\frac{1}{2} \int_{-\infty}^{+\infty} d\omega$$

$$\times \frac{1}{(e^{\beta(\omega+i0)} - 1) \sinh \pi(\omega + i0)} e^{-2i\omega\lambda}.$$

Due to presence of the double pole at $\omega = -i0$, the answer is as follows:

$$W_\beta(\lambda \to \infty) \approx \frac{2\pi}{\beta} \frac{\lambda}{\pi} \approx \frac{2\pi}{\beta} W_{BD}(\lambda \to \infty).$$

Note that taking in Eq. (9.11) the horizon limit also gives $W_{\frac{2\pi}{N}}(\lambda \to \infty) \approx N W_{BD}(\lambda \to \infty)$.

A remarkable fact is the following: for lightlike separations inside the static patch, the dominant contribution to the propagator comes from large ω's; on the contrary, at the horizon, small ω's provide the leading contribution. This is because the horizon is the boundary of the patch; the main contribution comes from the infrared rather than ultraviolet frequencies.

E. Flat space limit

Here we consider the flat space limit, i.e., we let the de Sitter radius go to infinity $(R \to \infty)$. Let us start by discussing the flat limit of the modes (4.5) and of the Bunch-Davies two-point function, following the treatment given in [18]. To this aim, it is better to use another orbital basis of the forward light cone C^+,

$$\xi_+(k) = \begin{cases} \xi^0 = \sqrt{k^2 + m^2}/m \\ \xi^1 = k/m \\ \xi^2 = -1 \end{cases}$$

$$\xi_-(k) = \begin{cases} \xi^0 = \sqrt{k^2 + m^2}/m \\ \xi^1 = -k/m \\ \xi^2 = +1 \end{cases}, \quad (9.13)$$

$$\lim_{R\to\infty} \left(\frac{\xi_+(k) \cdot X(\frac{t-i\epsilon}{R}, \frac{x}{R})}{R}\right)^{-\frac{1}{2}-imR} = e^{-it\sqrt{k^2+m^2}+ikx}, \quad (9.14)$$

$$\lim_{R\to\infty} \left(\frac{\xi_-(k) \cdot X(\frac{t-i\epsilon}{R}, \frac{x}{R})}{R}\right)^{-\frac{1}{2}-imR} = 0, \quad (9.15)$$

and so on (recall that $\nu = \sqrt{m^2 R^2 - \frac{1}{4}}$).

It follows that when $R \to \infty$ [18],

$$W_{BD}\left(X\left(\frac{t_1 - i\epsilon}{R}, \frac{x_1}{R}\right), X\left(\frac{t_2 + i\epsilon}{R}, \frac{x_2}{R}\right)\right) \to \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-i\sqrt{k^2+m^2}(t_1-t_2-i\epsilon)+ik(x_1-x_2)} \frac{dk}{\sqrt{k^2+m^2}}, \quad (9.16)$$

which is the standard Fourier representation of the (positive energy) Wightman function in Minkowski space.

To find the flat limit of the Wightman function W_β for arbitrary β in the same way, we may start by rewriting the integral representation (3.12) in the following way:

$$W_\beta(t, x_1, x_2) = \int_0^\infty d\omega \left[e^{-i\omega t} \tilde{P}_\nu^R(\omega, x_1, x_2) \frac{1 - e^{-2\pi R\omega}}{1 - e^{-\beta\omega}} + e^{i\omega t} \tilde{P}_\nu^R(\omega, x_1, x_2) \frac{1 - e^{-2\pi R\omega}}{e^{\beta\omega} - 1}\right], \quad (9.17)$$

the superscript R indicates explicitly the restored dependence of (3.12) on the radius R. For $\beta = 2\pi R$, the limit $R \to \infty$ (formally) gives

$$\lim_{R\to\infty} W_{\mathrm{BD}}(t, x_1, x_2) = \int_0^\infty d\omega e^{-i\omega t} \tilde{P}_\nu^\infty(\omega, x_1, x_2). \tag{9.18}$$

Taking into account Eq. (9.16), it follows that

$$\lim_{R\to\infty} W_\beta(t, x_1, x_2) = \int_0^\infty d\omega \left[e^{-i\omega t} P_\nu^\infty(\omega, x_1, x_2) \frac{1}{1 - e^{-\beta\omega}} + e^{i\omega t} P_\nu^\infty(\omega, x_1, x_2) \frac{1}{e^{\beta\omega} - 1} \right]$$

$$= \int_{-\infty}^\infty \frac{dk}{4\pi\sqrt{k^2 + m^2}} \left[\frac{e^{-i\sqrt{k^2+m^2}t + ik(x_1-x_2)}}{1 - e^{-\beta\sqrt{k^2+m^2}}} + \frac{e^{i\sqrt{k^2+m^2}t - ik(x_1-x_2)}}{e^{\beta\sqrt{k^2+m^2}} - 1} \right], \tag{9.19}$$

which is precisely the flat space thermal propagator with temperature $1/\beta$. In the Bunch-Davies, the temperature scales together with R and this maintains invariance at every stage, while in generic case, β does not scale with R. On the other hand, scaling $\beta = \beta' R$ with constant β' provides in the vacuum positive energy Wightman function. This is true also for $\beta = \infty$.

X. CONCLUSIONS AND OUTLOOK

Cauchy surfaces in the Rindler–de Sitter wedge are not Cauchy's for the geodesically complete global de Sitter universe. Giving initial data on such surfaces completely determines the classical dynamics of fields in the Rindler–de Sitter wedge only. By applying the formalism of canonical quantization and Bogoliubov transformations, we may construct all the pure Fock states representing quantum Klein-Gordon fields in the wedge. Generalized Bogoliubov transformations [26,27], however, allow for the construction of a much wider set of states which are, generally speaking, mixed. In this paper, we have explicitly constructed all the above states by separating the variables in the static chart (2.2); the construction was exhibited for the two-dimensional de Sitter space not to burden the presentation with unnecessary complications.

In particular, we gave integral representations of all the Kubo-Martin-Schwinger states including the Bunch-Davies state at temperature $T = 1/2\pi R$. All of them are directly seen to be mixed states, the only pure state in that family being obtained in the zero temperature limit. We also provided explicit formulas for the alpha states which include also nondiagonal terms.

The thermal propagators have unusual pathological singularities on the horizons (vaguely remembering Einstein's suspicions about the presence of matter on the horizons [24]). We mention also that, while these propagators obey the fluctuation-dissipation theorem, the de Sitter invariant Bunch-Davies state, restricted to the wedge, does not possess at least one of the properties of Minkowskian thermal states [34], because de Sitter invariance forbids the Debye screening. So there is room for further study.

The important question for cosmology is: what about the initial state of our Universe? The difference between the static patch, the Poincaré patch, and the global de Sitter universe [1,2] will appear in the infrared loops which are sensitive to the initial (and to the boundary conditions).

In flat space-time (at least in a box), an initial arbitrary state (within a reasonable class) will thermalize sooner or latter. The temperature of the final state depends on the initial conditions and may be arbitrary. What about thermalization in de Sitter space? Is there thermalization to a state with an arbitrary temperature? How does the answer to these questions depends on the choice of patch (type of initial Cauchy surface)?

To answer these questions, one has to resum secularly growing loop corrections. It is the Boltzmann's equation which allows to do that in Minkowski space [1]. What is the analog of the flat space Boltzmann's equation in the static de Sitter space?

We will address some of the above questions in a forthcoming companion paper.

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APPENDIX: COMPLETENESS RELATION OF ASSOCIATED LEGENDRE FUNCTIONS ON THE CUT

Here we provide an explicit (formal) calculation of the canonical commutation relations (3.2) which, by introducing $\cos\theta = \tanh x = u$, we rewrite as follows:

$$\sin\theta_1 \sin\theta_2 \delta(\cos\theta_1 - \cos\theta_2) = \int_{-\infty}^{\infty} \frac{\omega d\omega}{4\pi\sinh(\pi\nu)} \Gamma\left(\frac{1}{2} + i\nu - i\omega\right) \Gamma\left(\frac{1}{2} - i\nu + i\omega\right)$$

$$\times \left[\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(\cos\theta_1) \left(\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(\cos\theta_2) \right)^* + \mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(-\cos\theta_1) \left(\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(-\cos\theta_2) \right)^* \right]. \tag{A1}$$

Using the holomorphic plane waves introduced in Sec. IV, we get the following integral representation for $\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(\cos\theta)$ [see Eq. (4.10) and the following ones]:

$$\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(\cos\theta) = \frac{i\Gamma(\frac{1}{2} + i\nu)}{2\pi\Gamma(\frac{1}{2} + i\nu - i\omega)} \int_{-\infty}^{\infty} dt e^{-i\omega t} \Delta f(t,\theta), \tag{A2}$$

where we set

$$f_\pm(t,\theta) = (\xi_l(0) \cdot Z(t \pm i\epsilon, \theta))^{-\frac{1}{2}-i\nu} = [\cos\theta + \sin\theta \sinh(t \pm i\epsilon)]^{-\frac{1}{2}-i\nu}, \tag{A3}$$

$$\Delta f(t,\theta) = (f_+(t,\theta) - f_-(t,\theta)). \tag{A4}$$

$\mathsf{P}_{-\frac{1}{2}+i\nu}^{i\omega}(\cos\theta)$ is therefore the Fourier transform of the discontinuity of the holomorphic plane waves on the real de Sitter manifold. Let us insert (A2) in Eq. (A1); let us consider for instance the first term on the rhs of Eq. (A1). By performing the integration over ω, we get

$$(A.1) = -\frac{i}{16\pi\sinh^2\pi\nu} \int_{-\infty}^{\infty} dt[(\partial_t \Delta f(t,\theta_1))\Delta f(t,\theta_1)^* - \Delta f(t,\theta_1)\partial_t \Delta f(t,\theta_2)^*] +$$

$$- \frac{i}{16\pi\sinh^2\pi\nu} \int_{-\infty}^{\infty} dt[(\partial_t \Delta f(t,\pi-\theta_1))\Delta f(t,\pi-\theta_1)^* - \Delta f(t,\pi-\theta_1)\partial_t \Delta f(t,\pi-\theta_2)^*]$$

$$= -\frac{i}{16\pi\sinh^2\pi\nu} \sum_{k=-\pm} \int_{-\infty}^{\infty} dt[(\partial_t f_k(t,\theta_1))f_k(t,\theta_1)^* - f_k(t,\theta_1)\partial_t f_k(t,\theta_2)^*] +$$

$$- \frac{i}{16\pi\sinh^2\pi\nu} \sum_{k=-\pm} \int_{-\infty}^{\infty} dt[(\partial_t f_k(t,\pi-\theta_1))f_k(t,\pi-\theta_1)^* - f_k(t,\pi-\theta_1)\partial_t f_k(t,\theta_2)^*]. \tag{A5}$$

In the second step, we used the analyticity properties of the plane waves; this simplification is valid in the two-dimensional spacetime and in any even dimensional spacetime as well. By introducing the Mellin representation of the plane wave,

$$f_\pm(t,\theta) = \frac{e^{\mp\frac{i\pi}{2}(\frac{1}{2}+i\nu)}}{\Gamma(\frac{1}{2}+i\nu)} \int_0^{\infty} du\, u^{-\frac{1}{2}+i\nu} e^{\pm iu(\cos\theta + \sin\theta \sinh(t \pm i\epsilon))}, \quad 0 < \theta < \pi, \tag{A6}$$

a few easy integrations show the validity of Eq. (A1) and the completeness of the modes.

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