Equivalence between Horndeski and beyond Horndeski theories and imperfect fluids

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In this paper we approach to Horndeski and beyond Horndeski theories from the effective fluid perspective. We make explicit the formal equivalence between these theories and general relativity with an effective imperfect fluid. It is shown that, for the viable Horndeski theories, in the general case (arbitrary geometry) the nonvanishing contribution from the higher-order derivative terms to the imperfect fluidlike behavior affects only the heat flux vector but not the anisotropic stresses. The only contribution to the anisotropic stress tensor is due to the nonminimal coupling of the scalar field to the curvature as it is in standard scalar-tensor theories. For the viable beyond Horndeski theories the higher-order derivatives contribute both to the heat flux and to the anisotropic stresses. The effective fluid description is applied to several particular cases of interest. It is corroborated that, in Friedmann-Robertson-Walker background space, due to the underlying symmetries, the effective stress-energy tensor of viable Horndeski and beyond Horndeski theories is formally equivalent to that of a perfect fluid. This result might not be true for other less symmetric backgrounds such as the anisotropic Bianchi I space.

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I. INTRODUCTION

Scalar fields have played a very important role in the study of gravitational theories beyond Einstein's general relativity (GR). Among these we may mention the Brans-Dicke (BD) theory [1–4], the scalar-tensor theories (STTs) [5–16], the $f(R)$ theory [17–34], extended theories of gravity (ETGs) [35–42], Horndeski [43–55], and beyond Horndeski theories [55–66]. For purpose of comparison with well-understood GR results it is customary to write the field equations of the above mentioned theories in the form of Einstein's GR equations (here we use the units system where $8\pi G_N = 1$, with G_N as Newton's constant):

$$G_N^{\rm eff} G_{\mu\nu} + \mathcal{F}_{\mu\nu}(R, R_{\sigma\tau}R^{\sigma\tau}, R_{\sigma\tau\lambda\kappa}R^{\sigma\tau\lambda\kappa}, \nabla^2 R, \ldots, \nabla^{2l}R, \phi, \nabla_\mu\phi, \nabla^2\phi, \ldots, \nabla^{2m}\phi) = T_{\mu\nu}^{\rm mat},$$

where the additional scalar field related and curvature terms are appropriately grouped and organized in the form of an effective stress-energy tensor (SET):

$$G_{\mu\nu} = \frac{1}{G_N^{\rm eff}}(T_{\mu\nu}^{\rm mat} + T_{\mu\nu}^{\rm eff}),$$
$$T_{\mu\nu}^{\rm eff} = -\mathcal{F}_{\mu\nu}(R, R_{\sigma\tau}R^{\sigma\tau}, R_{\sigma\tau\lambda\kappa}R^{\sigma\tau\lambda\kappa}, \nabla^2 R, \ldots, \nabla^{2l}R, \phi, \nabla_\mu\phi, \nabla^2\phi, \ldots, \nabla^{2m}\phi). \quad (1)$$

In the above equations the generic tensor $\mathcal{F}_{\mu\nu}$ contains the contributions coming from higher-order curvature invariants and/or higher-order derivatives of the curvature scalar R and/or from the scalar field ϕ and its higher-order derivatives, while $T_{\mu\nu}^{\rm mat}$ accounts for the SET of the matter degrees of freedom: photons, baryons, dark matter, etc. Besides, $R_{\mu\nu}$ is the Ricci tensor, $R^\lambda_{\mu\kappa\nu}$ is the Riemann-Christoffel curvature tensor, $\nabla^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and l, m are nonvanishing integers. The effective gravitational coupling $G_N^{\rm eff}$ in the above equations can be, in principle, a function of the curvature invariants and their higher-order derivatives and of the scalar field and its higher-order derivatives,

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as well. The question is how to interpret the effective SET $T^{\rm eff}_{\mu\nu}$ and whether it could be formally regarded as the SET of a fluid? It happens that the usefulness of Eq. (1) relies, precisely, on the existing formal equivalence existing between $T^{\rm eff}_{\mu\nu}$ and the SET of perfect and imperfect fluids. The equivalence has been established for scalar-tensor theories [67–75] as well as for other modifications of gravity containing higher order derivatives such as the k essence [76–78] and its further generalization known as kinetic gravity braiding [79,80], the $f(R)$ theory [81], the $f(R,G)$ theories (G is the Gauss-Bonnet term) [82] and the ETGs [83]. The effective fluid picture has been proved to be useful also within the context of the so-called quantum modification of general relativity [84].

We want to point out that any symmetric rank 2 tensor, such as the stress-energy tensor $T_{\mu\nu}$, can be decomposed relative to a fixed timelike vector u^μ, into two scalars, a transverse vector and a transverse, trace-free tensor, i.e., into the general form of an imperfect fluid:

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}+2q_{(\mu}u_{\nu)}+\pi_{\mu\nu},$$

where ρ is the energy density and p is the pressure, q_μ is the heat flux vector and $\pi_{\mu\nu}$ is the anisotropic pressure tensor. Hence, the possibility to decompose the mentioned contribution is a consequence of making a specific choice of the timelike vector u^μ.[1] Yet establishing the precise form in which the equivalence with perfect/imperfect fluids is realized is not a trivial task since, besides of the mathematical complexity, the fact that one deals with an effective stress-energy tensor is to be taken with care. The non-negativity of the effective energy density, for instance, entails nontrivial mathematical conditions on the derivatives of the fields, etc. In addition, finding the explicit form of the equivalence between the fields and their derivatives and the effective fluid variables, ρ, p, q_μ, and $\pi_{\mu\nu}$, is of interest for the applications since the fluid variables have a more immediate and clear physical meaning. As a matter of fact, despite that the effective fluid equivalence is purely mathematical and not physical, the effective fluid picture is very useful when one compares different cosmological models. One may compare functions describing the effective fluid such as, for instance, the equation of state $p_{\rm eff}/\rho_{\rm eff}$, the anisotropic stresses $\pi^{\rm eff}_{\mu\nu}$, the sound speed c_s^2, etc. The fluid variables not only have a clear physical meaning but also greatly simplify the analysis of the system [79].

It is a well-known fact that when we deal with GR with a minimally coupled self-interacting scalar field φ, obeying the Einstein's equations of motion (here we omit other matter sources):

$$G_{\mu\nu}=T^{(\varphi)}_{\mu\nu}=\nabla_\mu\varphi\nabla_\nu\varphi-\frac{1}{2}g_{\mu\nu}(\nabla\varphi)^2-Vg_{\mu\nu},\qquad(2)$$

where $(\nabla\varphi)^2\equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and $V=V(\phi)$ is the self-interacting potential, the scalar field's SET $T^{(\varphi)}_{\mu\nu}$ can be written in the equivalent form of a relativistic perfect fluid [67,68,71,72,75,85,86] after identifying a timelike four-velocity vector [68,71,72,85]:

$$u_\mu=\frac{\nabla_\mu\phi}{\sqrt{-(\nabla\phi)^2}},-(\nabla\phi)^2\geq 0.\qquad(3)$$

This timelike vector determines the $3+1$ splitting of the spacetime into a three-space seen by comoving observers and the time direction [85]. The metric is written accordingly:

$$g_{\mu\nu}=h_{\mu\nu}-u_\mu u_\nu=h_{\mu\nu}+\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^2},\qquad(4)$$

where $h_{\mu\nu}$ is the metric of the three space (h^μ_ν is the projector onto the three space orthogonal to the time direction of comoving observers):

$$h^\mu_\lambda h^\lambda_\nu=h^\mu_\nu,\qquad g_{\mu\lambda}h^\lambda_\nu=h_{\mu\lambda}h^\lambda_\nu=h_{\mu\nu},$$
$$h_{\mu\nu}u^\nu=0,h^\mu_\mu=3.\qquad(5)$$

Besides, for a given vector v^μ:

$$h^\lambda_\nu\nabla_\lambda v_\mu=\nabla_\nu v_\mu+u_\nu u^\lambda\nabla_\lambda v_\mu.$$

After the choice (3) as the four velocity of observers comoving with the scalar field fluid we can rewrite the SET of the scalar field in the form of a perfect fluid SET [67,68,71,72,75,86]:

$$T^{(\varphi)}_{\mu\nu}=-(\nabla\varphi)^2u_\mu u_\nu+\left[-\frac{1}{2}(\nabla\varphi)^2-V\right]g_{\mu\nu}$$
$$=(\rho_\varphi+p_\varphi)u_\mu u_\nu+p_\varphi g_{\mu\nu},\qquad(6)$$

where we identify:

$$\rho_\varphi+p_\varphi=-(\nabla\varphi)^2,p_\varphi=-\frac{1}{2}(\nabla\varphi)^2-V\Rightarrow\rho_\varphi$$
$$=-\frac{1}{2}(\nabla\varphi)^2+V.\qquad(7)$$

In this effective (perfect fluid) picture ρ_φ and p_φ represent the energy density and pressure of the fluid. But, in general, when nonminimal coupling of the scalar field with the curvature is considered, the SET of the scalar field is equivalent to the one of an imperfect fluid as shown in Refs. [69,70,74] (see the next section).

[1]This argument has been suggested to us by one referee.

In this paper we wonder what the above mentioned formal equivalence with an effective perfect/imperfect fluid would look like for other more general STTs. Given that both Horndeski [43–55] and beyond Horndeski [55–66] theories represent further generalization of scalar-tensor theories, here we shall look for the mentioned kind of equivalence within the framework of these generalized STTs. We are going to consider a viable Horndeski subclass that is described by the Lagrangian [55]:

$$\mathcal{L}_{\rm vhorn} = G_2(\phi, X) - G_3(\phi, X)\nabla^2\phi + G_4(\phi)R, \quad (8)$$

where $X \equiv -(\nabla\phi)^2/2$, as well as a viable subclass of beyond Horndeski theories depicted by the Lagrangian:

$$\mathcal{L}_{\rm vbhorn} = f(\phi, X)R + A(\phi, X)(\nabla X)^2, \quad (9)$$

where

$$(\nabla X)^2 \equiv \nabla X \cdot \nabla X = \nabla^\lambda\phi\nabla_\mu\nabla_\lambda\phi\nabla^\mu\nabla^\kappa\phi\nabla_\kappa\phi.$$

These subclasses are the only ones that survive the cosmological observational tests, in particular the one related with the nearly simultaneous detection of gravitational waves GW170817 and the γ-ray burst GRB 170817A [87,88] (see the related discussion in Subsection (4.2) of reference [55], specifically the Eqs. (118)–(121) for the Horndeski theories and (123)–(127) for beyond Horndeski theories).[2] Given that the Horndeski and beyond Horndeski theories admit higher-order derivatives of the scalar field as well as self-couplings, here we shall address the question of how these higher derivative contributions affect the effective fluid picture.

It has to be mentioned that for a particular subclass in the Horndeski theories known as kinetic gravity braiding [79] the equivalence with general relativity with an effective imperfect fluid has been established in Ref. [80]. This subclass is given by the following Lagrangian:

$$\mathcal{L}_{\rm kgb} = R/2 + K(\phi, X) + G(\phi, X)\nabla^2\phi. \quad (10)$$

Notice that the choice $G_2 = K$, $G_3 = -G$, $G_4 = 1/2$ in (8) leads to (10). The effective fluid approach to Horndeski theories within the cosmological setup has been investigated also in Ref. [90] by means of the cosmological perturbations approach. In the present paper we want to approach the issue from the point of view of relativistic dynamics and, in addition, we are going to go further to include the beyond Horndeski theories also. Our results will generalize those of previous works in [69,70,74,80,90].

We have organized the paper in the following way. In the next section we shall apply the procedure we shall use in the paper in order to make explicit the formal equivalence with general relativity with an imperfect fluid to the very well-known example of Brans-Dicke theory. In Sec. III, for completeness, the basic elements of Horndeski theories are exposed. The formal equivalence between viable Horndeski theories and GR with an imperfect fluid is settled in Sec. IV, while in Sec. V the mentioned formal equivalence is made explicit for the viable beyond Horndeski theories. The effective "imperfect fluid" picture is explored in the cosmological setting in Sec. VI. Several important aspects of the explored picture are discussed in Sec. VII where brief conclusions are also given. Finally, for completeness we added an Appendix where the motion equation for the scalar field (the generalized Klein-Gordon equation) is derived, both for the Horndeski and for the beyond Hordeski theories, by taking the divergence of the effective stress-energy tensor.

II. THE SCALAR FIELD AS AN IMPERFECT FLUID

As already mentioned, the equations of motion of the scalar-tensor theories, where the scalar field is nonminimally coupled to the curvature, can be written as those of GR with an effective imperfect fluid. Here we explain the basis of the formalism in the particular case of the BD theory.

The effective imperfect fluid picture for the BD theory has been developed in Refs. [70,74]. In this case the effective stress-energy tensor is given by:

$$T^{(\phi)}_{\mu\nu} = \frac{\omega}{\phi^2}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] - \frac{V}{2\phi}g_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi), \quad (11)$$

where ω is the BD coupling parameter and V is the self-interacting potential for the scalar field. The above effective SET can be written in the form of the stress-energy tensor of an imperfect fluid:

$$T^{(\rm if)}_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}, \quad (12)$$

where ρ, p are the energy density and pressure of the fluid, q_μ is the heat flux vector which is, by definition, transversal to the timelike four velocity $(q_\mu u^\nu = 0)$ and $\pi_{\mu\nu}$ is the anisotropic SET:

[2] It should be noted that the bounds on the speed of the gravitational waves obtained in the event GW170817 are valid in the frequency range spanning from 24 Hz to a few hundred Hz. For this reason, it has been pointed out in reference [89] that there is a set of Horndeski theories which can evade this bound because its prediction on the speed of gravitational waves depends on the frequency k and an energy cutoff $M \lesssim \Lambda_{\rm Horndeski} \sim (M_{\rm PL}H_0^2)^{1/3} \sim 260$ Hz, where $\Lambda_{\rm Horndeski}$ is the strong coupling scale associated with many Horndeski dark energy models, H_0 is the Hubble parameter today, and $M_{\rm PL}$ is the Planck mass. The bound is evaded for frequencies $k \ll M$ where the speed can be subluminal and luminality is recovered for $k \gtrsim M$.

$$\pi_{\mu\nu} = \Pi_{\mu\nu} - \frac{1}{3}\Pi h_{\mu\nu}, \qquad \pi = \pi^\mu{}_\mu = 0, \qquad \pi_{\mu\nu}u^\nu = 0, \tag{13}$$

with

$$\Pi_{\mu\nu} = T_{\lambda\kappa}h^\lambda{}_\mu h^\kappa{}_\nu = p h_{\mu\nu} + \pi_{\mu\nu}, \qquad \Pi = \Pi^\mu{}_\mu = 3p,$$
$$\Pi_{\mu\nu}u^\nu = 0. \tag{14}$$

By comparing Eqs. (11) and (12) one gets that

$$(\rho + p)u_\mu u_\nu = \frac{\omega}{\phi^2}\nabla_\mu\phi\nabla_\nu\phi \Rightarrow u_\mu u_\nu = -\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^2},$$

from where the timelike four-velocity vector of the fluid u^μ is defined as in (3). Following the approach explained in the introduction, the energy density and pressure of the fluid as well as the heat flux vector q_μ, are defined as it follows:

$$\rho = T_{\mu\nu}u^\mu u^\nu, \qquad p = \frac{1}{3}\Pi, \qquad q_\mu = -T_{\lambda\kappa}u^\lambda h^\kappa{}_\mu. \tag{15}$$

Other kinematic quantities of the fluid are the following:

$$\begin{aligned}
\dot{u}_\mu &= u^\nu\nabla_\nu u_\mu, \\
\theta &= \nabla_\mu u^\mu, \\
\sigma_{\mu\nu} &= \nabla_{(\mu}u_{\nu)} + \dot{u}_{(\mu}u_{\nu)} - \frac{1}{3}\theta h_{\mu\nu}, \\
\omega_{\mu\nu} &= \nabla_{[\mu}u_{\nu]} + \dot{u}_{[\mu}u_{\nu]},
\end{aligned} \tag{16}$$

where $\dot{u}_\mu$ is the acceleration of the fluid, θ is the expansion, $\sigma_{\mu\nu}$ is the shear tensor of the fluid, while $\omega_{\mu\nu}$ accounts for the vorticity tensor. Under the choice (3), since the four-velocity is the gradient of a scalar, the vorticity tensor $\omega_{\mu\nu}$ vanishes identically. This is true for any scalar-tensor theories and their higher-derivative modifications: Horndeski and beyond Horndeski theories.

When we take into account the definition of the stress-energy tensor of the BD scalar field (11), the tensor (14) is given by this [74]:

$$\begin{aligned}
\Pi^{(\phi)}_{\mu\nu} = &-\left[\frac{\omega}{2\phi^2}(\nabla\phi)^2 + \frac{V}{2\phi} + \frac{2}{3}\frac{\nabla^2\phi}{\phi} + \frac{\nabla^\kappa\phi\nabla^\lambda\phi\nabla_\lambda\nabla_\kappa\phi}{3\phi(\nabla\phi)^2}\right]h_{\mu\nu} + \frac{1}{\phi}\left[\nabla_\mu\nabla_\nu - \frac{1}{3}h_{\mu\nu}\nabla^2\right]\phi \\
&- \frac{\nabla^\lambda\phi}{\phi(\nabla\phi)^2}\left[\nabla_\lambda\nabla_\mu\phi\nabla_\nu\phi + \nabla_\lambda\nabla_\nu\phi\nabla_\mu\phi - \frac{1}{3}h_{\mu\nu}\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa\phi - \frac{\nabla_\mu\phi\nabla_\nu\phi\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa\phi}{(\nabla\phi)^2}\right].
\end{aligned} \tag{17}$$

If we compare this latter equation with (14), for the anisotropic SET we obtain:

$$\pi^{(\phi)}_{\mu\nu} = -\frac{\nabla^\lambda\phi}{\phi(\nabla\phi)^2}\left[\nabla_\lambda\nabla_\mu\phi\nabla_\nu\phi + \nabla_\lambda\nabla_\nu\phi\nabla_\mu\phi - \frac{1}{3}h_{\mu\nu}\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa\phi - \frac{\nabla_\mu\phi\nabla_\nu\phi\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa\phi}{(\nabla\phi)^2}\right] + \frac{1}{\phi}\left[\nabla_\mu\nabla_\nu - \frac{1}{3}h_{\mu\nu}\nabla^2\right]\phi, \tag{18}$$

while for the pressure of the fluid:

$$p_\phi = -\left[\frac{\omega}{2\phi^2}(\nabla\phi)^2 + \frac{V}{2\phi} + \frac{2}{3}\frac{\nabla^2\phi}{\phi} + \frac{\nabla^\kappa\phi\nabla^\lambda\phi\nabla_\lambda\nabla_\kappa\phi}{3\phi(\nabla\phi)^2}\right]. \tag{19}$$

For other relevant quantities appearing in (12) we get:

$$\rho_\phi = -\left[\frac{\omega}{2\phi^2}(\nabla\phi)^2 - \frac{V}{2\phi} - \frac{\nabla^2\phi}{\phi} + \frac{\nabla^\kappa\phi\nabla^\lambda\phi\nabla_\lambda\nabla_\kappa\phi}{\phi(\nabla\phi)^2}\right], \tag{20}$$

for the energy density of the BD field, while for the heat flux vector [74]:

$$q^{(\phi)}_\mu = -\frac{\nabla^\lambda\phi}{\phi\sqrt{-(\nabla\phi)^2}}\left[\nabla_\lambda\nabla_\mu\phi - \frac{\nabla_\mu\phi\nabla^\kappa\phi\nabla_\kappa\nabla_\lambda\phi}{(\nabla\phi)^2}\right] = -\frac{\sqrt{-(\nabla\phi)^2}}{\phi}\dot{u}_\mu, \tag{21}$$

where we have taken into account that for the choice (3) the acceleration of the fluid is given by:

$$\dot{u}_\mu = u^\lambda\nabla_\lambda u_\mu = -\frac{\nabla^\lambda\phi}{(\nabla\phi)^2}\left[\nabla_\lambda\nabla_\mu\phi - \frac{\nabla_\mu\phi\nabla^\kappa\phi\nabla_\kappa\nabla_\lambda\phi}{(\nabla\phi)^2}\right]. \tag{22}$$

Below we shall apply this formalism to the Horndeski and beyond Horndeski theories in order to extend the effective imperfect fluid picture to these generalizations of scalar-tensor theories.

III. HORNDESKI THEORIES

According to [48], further generalization of four-dimensional scalar-tensor theories having second-order motion equations is achieved if considering the linear combinations of the following Lagrangians:

$$\mathcal{L}_2 = K, \qquad \mathcal{L}_3 = -G_3(\nabla^2\phi),$$
$$\mathcal{L}_4 = G_4 R + G_{4,X}[(\nabla^2\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2],$$
$$\mathcal{L}_5 = G_5 G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}[(\nabla^2\phi)^3 - 3\nabla^2\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \tag{23}$$

where $K = K(\phi, X)$ and $G_i = G_i(\phi, X)$ ($i = 3, 4, 5$), are functions of the scalar field ϕ and of its kinetic energy density X, while $G_{i,\phi}$ and $G_{i,X}$ represent the derivatives of the functions G_i with respect to ϕ and X, respectively. In the Lagrangian $\mathcal{L}_5$ above, for compactness of writing, we have adopted the same definitions used in Ref. [55]:

$$(\nabla_\mu\nabla_\nu\phi)^2 \coloneqq \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi,$$
$$(\nabla_\mu\nabla_\nu\phi)^3 \coloneqq \nabla^\mu\nabla_\alpha\phi\nabla^\alpha\nabla_\beta\phi\nabla^\beta\nabla_\mu\phi. \tag{24}$$

Note that in (23) we have slightly modified the notation with respect to (8), since we have replaced $K \to G_2$. We have done this in order to meet the notation most frequently found in the bibliography.

The general action for the Horndeski theories can be written as follows:

$$S_{\rm Horn} = \int d^4x\sqrt{|g|}(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\rm mat}), \tag{25}$$

where the $\mathcal{L}_i$ ($i = 2, 3, 4, 5$) are given by (23) and $\mathcal{L}_m$ stands for the Lagrangian of the matter degrees of freedom. The motion equations that can be derived from the above action read:

$$G_{\mu\nu} = \frac{1}{2G_4}T^{\rm mat}_{\mu\nu} + \sum_i T^{(i)}_{\mu\nu}, \quad i = 2, 3, 4, \tag{26}$$

where

$$\frac{\delta(\sqrt{|g|}\mathcal{L}_{\rm mat})}{\sqrt{|g|}\delta g^{\mu\nu}} = -\frac{1}{2}T^{\rm mat}_{\mu\nu},$$

and we have considered the following definitions of the effective stress-energy tensors related with the $\mathcal{L}_i$s in (23):

$$T^{(2)}_{\mu\nu} = \frac{1}{2G_4}(K_{,X}\nabla_\mu\phi\nabla_\nu\phi + Kg_{\mu\nu}),$$
$$T^{(3)}_{\mu\nu} = \frac{1}{2G_4}\{-(2G_{3,\phi} + G_{3,X}\nabla^2\phi)\nabla_\mu\phi\nabla_\nu\phi - 2G_{3,X}\nabla_{(\mu}\phi\nabla_{\nu)}X + g_{\mu\nu}[G_{3,\phi}(\nabla\phi)^2 + G_{3,X}(\nabla\phi\cdot\nabla X)]\},$$
$$T^{(4)}_{\mu\nu} = \frac{G_{4,\phi}}{G_4}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) + \frac{G_{4,\phi\phi}}{G_4}[\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}(\nabla\phi)^2], \tag{27}$$

where $(\nabla\phi\cdot\nabla X) \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu X$, with $\nabla_\mu X \equiv -\nabla^\lambda\phi(\nabla_\mu\nabla_\lambda\phi)$. Notice that, in the definition of the SETs $T^{(i)}_{\mu\nu}$ above, we have already included the contribution coming from the effective gravitational coupling G_4. Besides, since the latter is a function of the scalar field only: $G_4 = G_4(\phi)$, the resulting theory is in the viable subclass of Horndeski theories (8) mentioned in the introduction. For the same reason we have not considered the contribution coming from the Lagrangian $\mathcal{L}_5$ in (25).

As an aside let us to mention that taking the divergence of (26) multiplied by $2G_4$, up to a vector field $\partial^\mu\phi$, leads to the modified Klein-Gordon equation for the galileon (see the Appendix):

$$G_{\mu\nu}\nabla^\mu G_4 = \sum_i \nabla^\mu[G_4 T^{(i)}_{\mu\nu}], \tag{28}$$

i.e., the same that can be obtained through variation of the action (25) with respect to the scalar field ϕ. This is true also for the beyond Horndeski theories. In the Appendix, the detailed computation of the divergence of the effective stress-energy tensor, both for the Horndeski and the beyond Horndeski theories, is performed.

IV. EQUIVALENCE BETWEEN VIABLE HORNDESKI THEORIES AND IMPERFECT FLUIDS

Here, as before, we consider the timelike four-velocity vector defined as in Eq. (3): $u^\mu = \nabla^\mu\phi/\sqrt{2X}$, with non-negative $X \geq 0$, in order to determine the $3 + 1$ splitting of the spacetime [recall that $X \equiv -(\nabla\phi)^2/2$ is the kinetic energy of the scalar field]. Given the adopted definition of the timelike four velocity (3), we can rewrite the kinematic quantities in (16) in terms of our notation as it follows. For the expansion we have this:

$$\theta = \frac{1}{\sqrt{2X}}\left[\nabla^2\phi - \frac{(\nabla\phi\cdot\nabla X)}{2X}\right], \tag{29}$$

while, for the components of the shear and vorticity tensors, we have

$$\sigma_{\mu\nu} = \frac{1}{\sqrt{2X}}\left\{\nabla_\mu\nabla_\nu\phi - \frac{(\nabla\phi\cdot\nabla X)}{4X^2}\nabla_\mu\phi\nabla_\nu\phi - \frac{\nabla_{(\nu}\phi\nabla_{\mu)}X}{X} - \frac{\theta}{3}h_{\mu\nu}\right\},$$

$$\omega_{\mu\nu} = \frac{2}{(2X)^{3/2}}\nabla_{[\mu}\phi\nabla_{\nu]}X, \tag{30}$$

respectively. Above we have taken into account that

$$\nabla_\mu u_\nu = \frac{1}{\sqrt{2X}}\left(\nabla_\mu\nabla_\nu\phi - \frac{\nabla_\nu\phi\nabla_\mu X}{2X}\right). \tag{31}$$

so that the acceleration can be written as

$$\dot{u}_\mu = u^\lambda\nabla_\lambda u_\mu = -\frac{1}{2X}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right]. \tag{32}$$

In order to show the equivalence between the viable Horndeski theories (26), (27) and imperfect fluids we shall consider each of the effective SETs in (27) separately. Let us start with $T^{(2)}_{\mu\nu}$. This effective tensor corresponds to the so-called k-essence models. Although the equivalence between these models and a perfect fluid has been already demonstrated [76,78], here we write the basic equations in terms of our notation. Following the procedure exposed in Sec. II we obtain the following results:

$$\Pi^{(2)}_{\mu\nu} = T^{(2)}_{\lambda\kappa}h^\lambda{}_\mu h^\kappa{}_\nu = \frac{K}{2G_4}h_{\mu\nu}, \qquad \pi^{(2)}_{\mu\nu} = 0,$$

$$q^{(2)}_\mu = -T^{(2)}_{\lambda\kappa}u^\lambda h^\kappa{}_\mu = 0, \qquad p_{(2)} = \frac{1}{3}\Pi^{(2)} = \frac{K}{2G_4},$$

$$\rho_{(2)} = T^{(2)}_{\lambda\kappa}u^\lambda u^\kappa = \frac{1}{2G_4}(2XK_{,X} - K). \tag{33}$$

In what regards to the piece $T^{(3)}_{\mu\nu}$, the calculations are a bit more complicated. Let us start by computing the tensor:

$$\Pi^{(3)}_{\mu\nu} = T^{(3)}_{\lambda\kappa}h^\lambda{}_\mu h^\kappa{}_\nu = -\frac{1}{G_4}\left[G_{3,\phi}X - \frac{G_{3,X}}{2}(\nabla\phi\cdot\nabla X)\right]h_{\mu\nu}, \tag{34}$$

so that the effective pressure $p_{(3)}$ is given by the following:

$$p_{(3)} = \frac{1}{3}\Pi^{(3)} = -\frac{1}{G_4}\left[G_{3,\phi}X - \frac{G_{3,X}}{2}(\nabla\phi\cdot\nabla X)\right], \tag{35}$$

meanwhile the calculation of effective energy density $\rho_{(3)}$ gives this:

$$\rho_{(3)} = T^{(3)}_{\lambda\kappa}u^\lambda u^\kappa = -\frac{1}{G_4}\left[G_{3,\phi}X - \frac{G_{3,X}}{2}(\nabla\phi\cdot\nabla X) + G_{3,X}X(\nabla^2\phi)\right]. \tag{36}$$

It can be shown that $\pi^{(3)}_{\mu\nu} = 0$, so that the effective fluid does not have anisotropic stresses. However, there is a nonvanishing heat flux given by

$$q^{(3)}_\mu = -\frac{G_{3,X}}{2\sqrt{2X}G_4}[2X\nabla_\mu X + \nabla_\mu\phi(\nabla\phi\cdot\nabla X)]. \tag{37}$$

The effective SET tensor $T^{(4)}_{\mu\nu}$ in Eq. (27) can be written in the alternative way:

$$T^{(4)}_{\mu\nu} = \frac{G_{4,\phi}}{G_4}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) + \frac{G_{4,\phi\phi}}{G_4}2Xh_{\mu\nu}. \tag{38}$$

Hence

$$\Pi^{(4)}_{\mu\nu} = T^{(4)}_{\lambda\kappa}h^\lambda{}_\mu h^\kappa{}_\nu = \left(\frac{G_{4,\phi\phi}}{G_4}2X - \frac{G_{4,\phi}}{G_4}\nabla^2\phi\right)h_{\mu\nu} + \frac{G_{4,\phi}}{G_4}\left[\nabla_\mu\nabla_\nu\phi - \frac{1}{X}\nabla_{(\mu}\phi\nabla_{\nu)}X - \frac{(\nabla\phi\cdot\nabla X)}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right]. \tag{39}$$

Since the resulting effective pressure $p_{(4)} = \Pi_{(4)}/3$, we get the following:

$$p_{(4)} = -\frac{G_{4,\phi}}{G_4}\left[\frac{(\nabla\phi\cdot\nabla X)}{6X} + \frac{2}{3}\nabla^2\phi\right] + \frac{G_{4,\phi\phi}}{G_4}2X. \tag{40}$$

This entails that, since $\pi^{(4)}_{\mu\nu} = \Pi^{(4)}_{\mu\nu} - p_{(4)}h_{\mu\nu}$, for the tensor of anisotropic stresses we obtain the following expression:

$$\pi^{(4)}_{\mu\nu} = \frac{G_{4,\phi}}{G_4}\left\{\left[\frac{(\nabla\phi\cdot\nabla X)}{6X} - \frac{1}{3}\nabla^2\phi\right]h_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \frac{1}{X}\nabla_{(\mu}\phi\nabla_{\nu)}X - \frac{(\nabla\phi\cdot\nabla X)}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right\}. \tag{41}$$

The effective energy density is given by

$$\rho_{(4)} = T^{(4)}_{\lambda\kappa} u^\lambda u^\kappa = -\frac{G_{4,\phi}}{G_4}\left[\frac{(\nabla\phi\cdot\nabla X)}{2X} - \nabla^2\phi\right], \quad (42)$$

while the heat flux vector is given by

$$q^{(4)}_\mu = -T^{(4)}_{\lambda\kappa} u^\lambda h^\kappa{}_\mu = \frac{G_{4,\phi}}{\sqrt{2X}G_4}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right]. \quad (43)$$

In consequence the viable Horndeski theory (26) is equivalent to GR with the effective stress-energy tensor of an imperfect fluid:

$$T^{\rm eff}_{\mu\nu} = (\rho_{\rm eff} + p_{\rm eff})u_\mu u_\nu + p_{\rm eff} g_{\mu\nu} + 2q^{\rm eff}_{(\mu}u_{\nu)} + \pi^{\rm eff}_{\mu\nu}, \quad (44)$$

where

$$\rho_{\rm eff} = \sum_i \rho_{(i)}, \qquad p_{\rm eff} = \sum_i p_{(i)},$$
$$q^{\rm eff}_\mu = \sum_i q^{(i)}_\mu, \qquad \pi^{\rm eff}_{\mu\nu} = \sum_i \pi^{(i)}_{\mu\nu}, \quad i = 2,3,4. \quad (45)$$

We have, in particular, that for the effective heat flux vector [for the effective pressure and energy density see Eqs. (76) and (77) in Sec. VII]:

$$q^{\rm eff}_\mu = \frac{G_{4,\phi} - XG_{3,X}}{\sqrt{2X}G_4}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right], \quad (46)$$

while the effective anisotropic stress coincides with $\pi^{(4)}_{\mu\nu}$ in (41):

$$\pi^{\rm eff}_{\mu\nu} = \frac{G_{4,\phi}}{G_4}\left\{\left[\frac{(\nabla\phi\cdot\nabla X)}{6X} - \frac{1}{3}\nabla^2\phi\right]h_{\mu\nu} + \nabla_\mu\nabla_\nu\phi - \frac{1}{X}\nabla_{(\mu}\phi\nabla_{\nu)}X - \frac{(\nabla\phi\cdot\nabla X)}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right\}. \quad (47)$$

Nonvanishing of any of the quantities $q^{\rm eff}_\mu$ and $\pi^{\rm eff}_{\mu\nu}$ is what distinguishes an imperfect effective fluid from a perfect one. This means that, in the present case, while the heat flux gets contributions from both the nonminimal coupling $G_4 = G_4(\phi)$ and the higher-derivative coupling $G_3 = G_3(\phi, X)$, the anisotropic stresses are the consequence of the non-minimal coupling only, as it is in standard scalar-tensor theories. In particular, for constant $G_4 = 1/2$ (minimal coupling), the anisotropic stresses vanish. In summary, for the viable Horndeski theories, the higher-order derivative contributions affect only the heat flux.

A. Particular cases

Let us check several particular cases in the Horndeski class of theories [52,54]:

(1) General relativity with a minimally coupled scalar field: this is given by the following choice of the relevant functions in (23): $G_4 = 1/2$, $G_3 = G_5 = 0$,

$$S = \int d^4x\sqrt{|g|}\left[\frac{1}{2}R + K(\phi,X) + \mathcal{L}_m\right].$$

This choice comprises quintessence, $K(\phi,X) = X - V$, and k essence, for instance, $K(\phi,X) = f(\phi)g(X)$, where f and g are arbitrary functions of their arguments. The most important kinematic quantities for this case are given in Eq. (33):

$$\Pi^{\rm eff}_{\mu\nu} = Kh_{\mu\nu}, \qquad \pi^{\rm eff}_{\mu\nu} = 0, \qquad q^{\rm eff}_\mu = 0,$$
$$p_{\rm eff} = K, \qquad \rho_{\rm eff} = 2XK_{,X} - K.$$

(2) Brans-Dicke theory: the following choice corresponds to the BD theory [2] (here ω is the BD coupling parameter): $K(\phi,X) = 2\omega X/\phi - V(\phi)$, $G_3 = G_5 = 0$, $G_4 = \phi$,

$$S = \int d^4x\sqrt{|g|}[\phi R + 2\omega X/\phi - V]. \quad (48)$$

Although the kinematic quantities for this case have been already computed in Sec. II, here we rewrite them in terms of the present notation. In this case, the quantities (45) in the effective stress energy tensor (44) read as follows:

$$\rho_{\rm eff} = \frac{\omega X}{\phi^2} + \frac{V}{2\phi} + \frac{\nabla^2\phi}{\phi} - \frac{(\nabla\phi\cdot\nabla X)}{2\phi X}, \qquad p_{\rm eff} = \frac{\omega X}{\phi^2} - \frac{V}{2\phi} - \frac{2\nabla^2\phi}{3\phi} - \frac{(\nabla\phi\cdot\nabla X)}{6\phi X},$$
$$\pi^{\rm eff}_{\mu\nu} = \left[\frac{(\nabla\phi\cdot\nabla X)}{6\phi X} - \frac{\nabla^2\phi}{3\phi}\right]h_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi} - \frac{\nabla_{(\mu}\phi\nabla_{\nu)}X}{\phi X} - \frac{(\nabla\phi\cdot\nabla X)}{4\phi X^2}\nabla_\mu\phi\nabla_\nu\phi,$$
$$q^{\rm eff}_\mu = \frac{1}{\phi\sqrt{2X}}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right]. \quad (49)$$

Notice that the above expressions coincide with the corresponding ones in (18)–(21).

(3) Nonminimal coupling theory: this is described by the functions: $K = \omega(\phi)X - V(\phi)$, $G_4 = (1 - \xi\phi^2)/2$, $G_3 = G_5 = 0$,

$$S = \int dx^4 \sqrt{|g|}\left[\frac{1 - \xi\phi^2}{2} R + \omega(\phi)X - V(\phi)\right].$$

The main kinematic quantities in (44) are as follows:

$$\begin{aligned}
\rho_{\rm eff} &= \frac{1}{1-\xi\phi^2}\left\{\omega X + V - 2\xi\phi\nabla^2\phi + 2\xi\phi\frac{(\nabla\phi\cdot\nabla X)}{2X}\right\},\\
p_{\rm eff} &= \frac{1}{1-\xi\phi^2}\left\{\omega X - V + \frac{4\xi\phi}{3}\nabla^2\phi + \frac{2}{3}\xi\phi\frac{(\nabla\phi\cdot\nabla X)}{2X} - 4\xi X\right\},\\
\pi^{\rm eff}_{\mu\nu} &= -\frac{2\xi\phi}{1-\xi\phi^2}\left\{\left[\frac{(\nabla\phi\cdot\nabla X)}{6X} - \frac{1}{3}\nabla^2\phi\right]h_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi} - \frac{\nabla_{(\mu}\phi\nabla_{\nu)}X}{\phi X} - \frac{(\nabla\phi\cdot\nabla X)}{4\phi X^2}\nabla_\mu\phi\nabla_\nu\phi\right\},\\
q^{\rm eff}_\mu &= -\frac{2\xi\phi}{(1-\xi\phi^2)\sqrt{2X}}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right].
\end{aligned} \tag{50}$$

(4) Cubic galileon: for this particular case in the functions in (23) one sets: $K = 2\omega X/\phi - 2\Lambda\phi$, $G_3 = -2f(\phi)X$, $G_4 = \phi$, $G_5 = 0$, and the resulting Jordan frame action reads as follows [47]:

$$S = \int d^4x\sqrt{|g|}[\phi R + 2\omega X/\phi - 2\Lambda\phi - 2Xf(\phi)\nabla^2\phi].$$

We obtain the following expressions for the fundamental kinematic quantities in (44), (45):

$$\begin{aligned}
\rho_{\rm eff} &= \frac{\omega X}{\phi^2} + \Lambda + 2\frac{f_{,\phi}}{\phi}X^2 + [1 + 2f(\phi)X]\left[\frac{\nabla^2\phi}{\phi} - \frac{(\nabla\phi\cdot\nabla X)}{2\phi X}\right],\\
p_{\rm eff} &= \frac{\omega X}{\phi^2} - \Lambda + 2\frac{f_{,\phi}}{\phi}X^2 - \frac{2\nabla^2\phi}{3\phi} - \left[\frac{1}{3} + 2f(\phi)X\right]\frac{(\nabla\phi\cdot\nabla X)}{2\phi X},\\
\pi^{\rm eff}_{\mu\nu} &= \left[\frac{(\nabla\phi\cdot\nabla X)}{6\phi X} - \frac{\nabla^2\phi}{3\phi}\right]h_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu\phi}{\phi} - \frac{\nabla_{(\mu}\phi\nabla_{\nu)}X}{\phi X} - \frac{(\nabla\phi\cdot\nabla X)}{4\phi X^2}\nabla_\mu\phi\nabla_\nu\phi,\\
q^{\rm eff}_\mu &= \frac{1 + 2f(\phi)X}{\phi\sqrt{2X}}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right].
\end{aligned} \tag{51}$$

The above examples belong in the viable Horndeski theories (8), where by viable we mean that the speed of propagation of the tensor perturbations coincides with the speed of light: $c^2_{\rm gw} = 1$. However, there are a few interesting cases that do not belong in (8) but that can evade the bound on the speed of gravitational waves (see footnote 2). One interesting example is given by the choice: $K = X - V$, $G_3 = 0$, $G_4 = 1/2$, $G_5 = -\alpha\phi/2$. The corresponding theory is known as kinetic coupling to the Einstein's tensor and is given by the following action:

$$S = \frac{1}{2}\int d^4x\sqrt{|g|}[R + 2(X - V) + \alpha G_{\mu\nu}\partial^\mu\phi\partial^\nu\phi]. \tag{52}$$

As it is discussed in [89] in detail, this theory—formulated as a effective field theory—has a region of its parameter space where the bound on the speed of gravitational waves is evaded.

V. VIABLE BEYOND HORNDESKI THEORIES AS IMPERFECT FLUIDS

Here we consider the viable beyond Horndeski theories [60–63,66,91] (also known as degenerate higher-order scalar-tensor theories) with Lagrangian (9). The following effective Einstein's equation can be derived from the latter Lagrangian: $G_{\mu\nu} = T^{\rm vbhorn}_{\mu\nu}$, where the effective SET for the viable beyond Horndeski fluid is given by the following expression:

$$
\begin{aligned}
T_{\mu\nu}^{\rm vbhorn} = {} & \frac{1}{2f}[f_{,X}R - A_{,X}(\nabla X)^2 - 2A_{,\phi}(\nabla\phi\cdot\nabla X) - 2A\nabla^2 X]\nabla_\mu\phi\nabla_\nu\phi \\
& + \frac{f_{,\phi\phi}}{f}[\nabla_\mu\phi\nabla_\nu\phi + 2Xg_{\mu\nu}] + \frac{f_{,\phi}}{f}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) \\
& + 2\frac{f_{,\phi X}}{f}[\nabla_{(\mu}\phi\nabla_{\nu)}X - g_{\mu\nu}(\nabla\phi\cdot\nabla X)] + \frac{(f_{,XX}-A)}{f}\nabla_\mu X\nabla_\nu X \\
& - \frac{(2f_{,XX}-A)}{2f}g_{\mu\nu}(\nabla X)^2 + \frac{f_{,X}}{f}(\nabla_\mu\nabla_\nu X - g_{\mu\nu}\nabla^2 X),
\end{aligned}
\tag{53}
$$

where

$$
A = \frac{3f_{,X}^2}{2f}; \qquad A_{,X} = 3\left(\frac{f_{,X}}{f}\right)f_{,XX} - \frac{3}{2}\left(\frac{f_{,X}}{f}\right)^2 f_{,X}.
$$

The above stress-energy tensor (53) can be written in the form of an effective SET for an imperfect fluid (44) with effective energy density:

$$
\begin{aligned}
\rho_{\rm eff} = {} & \frac{f_{,X}}{f}XR + \frac{f_{,\phi}}{f}\nabla^2\phi + \frac{2f_{,XX} - A - 2XA_{,X}}{2f}(\nabla X)^2 + \frac{f_{,X} - 2XA}{f}\nabla^2 X \\
& - \frac{f_{,\phi} + 4X^2A_{,\phi}}{2Xf}(\nabla\phi\cdot\nabla X) + \frac{f_{,XX}-A}{2Xf}(\nabla\phi\cdot\nabla X)^2 + \frac{f_{,X}}{2Xf}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu X,
\end{aligned}
\tag{54}
$$

effective pressure:

$$
\begin{aligned}
p_{\rm eff} = {} & \frac{f_{,\phi\phi}}{f}2X - \frac{2f_{,\phi}}{3f}\nabla^2\phi + \frac{1}{6f}(A - 4f_{,XX})(\nabla X)^2 - \frac{2f_{,X}}{3f}\nabla^2 X - \frac{1}{6Xf}(f_{,\phi} + 12Xf_{,\phi X})(\nabla\phi\cdot\nabla X) \\
& + \frac{1}{6Xf}(f_{,XX} - A)(\nabla\phi\cdot\nabla X)^2 + \frac{f_{,X}}{6Xf}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu X,
\end{aligned}
\tag{55}
$$

effective heat flux vector:

$$
\begin{aligned}
q_\mu^{\rm eff} = {} & \frac{1}{\sqrt{2X}f}\{[f_{,\phi} + 2Xf_{,\phi X} - (f_{,XX} - A)(\nabla\phi\cdot\nabla X)]\nabla_\mu - f_{,X}\nabla^\lambda\phi\nabla_\lambda\nabla_\mu\}X \\
& + \frac{1}{(2X)^{3/2}f}\{(f_{,\phi} + 2Xf_{,\phi X})(\nabla\phi\cdot\nabla X) - (f_{,XX} - A)(\nabla\phi\cdot\nabla X)^2 - f_{,X}\nabla^\lambda\phi\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa X\}\nabla_\mu\phi,
\end{aligned}
\tag{56}
$$

and effective anisotropic stress tensor:

$$
\pi_{\mu\nu}^{\rm eff} = \Pi_{\mu\nu}^{\rm eff} - p_{\rm eff}h_{\mu\nu},
\tag{57}
$$

where $p_{\rm eff}$ is given by (55) and

$$
\begin{aligned}
\Pi_{\mu\nu}^{\rm eff} = {} & \left[\frac{2f_{,\phi\phi}}{f}X - \frac{f_{,\phi}}{f}\nabla^2\phi - \frac{2f_{,\phi X}}{f}(\nabla\phi\cdot\nabla X) - \frac{2f_{,XX} - A}{2f}(\nabla X)^2 - \frac{f_{,X}}{f}\nabla^2 X\right]h_{\mu\nu} \\
& + \frac{f_{,\phi}}{f}\left[\nabla_\mu\nabla_\nu\phi - \frac{\nabla_{(\mu}\phi\nabla_{\nu)}X}{X} - \frac{(\nabla\phi\cdot\nabla X)}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right] + \frac{f_{,XX} - A}{f}\left[\nabla_\mu X\nabla_\nu X + \frac{(\nabla\phi\cdot\nabla X)}{X}\nabla_{(\mu}\phi\nabla_{\nu)}X\right. \\
& \left. + \frac{(\nabla\phi\cdot\nabla X)^2}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right] + \frac{f_{,X}}{f}\left[\nabla_\mu\nabla_\nu X + \frac{\nabla^\lambda\phi\nabla_{(\mu}\phi\nabla_{\nu)}\nabla_\lambda X}{X} + \frac{\nabla^\lambda\phi\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa X}{4X^2}\nabla_\mu\phi\nabla_\nu\phi\right].
\end{aligned}
\tag{58}
$$

It is evident from the above equations that, unlike as it was for the viable Horndeski theories, in the present case, the higher-order derivatives contribute both to the heat flux and to the anisotropic stresses.

VI. HORNDESKI AND BEYOND HORNDESKI COSMOLOGICAL PERFECT FLUIDS

In Secs. IV and V we have shown that, in the general case, both the viable Horndeski and beyond Horndeski theories admit an imperfect fluid representation. However, this is a correct statement only if we consider a scalar field with nonvanishing spatial gradient. This means that the four-velocity $u_\mu = \nabla_\mu\phi/\sqrt{-(\nabla\phi)^2}$ cannot be that of free-falling observers. In a Friedmann-Robertson-Walker (FRW) cosmological framework where the background metric is $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ (i, $j = 1, 2, 3$ and, as usual, t is the cosmic time, $a = a(t)$ is the scale factor, and $H \equiv \dot{a}/a$ is the Hubble parameter), on the contrary, the timelike four-velocity $u_\mu = \nabla_\mu\phi/\dot{\phi} = \delta^0_\mu$, is that of a free-falling observer (the overdot accounts for derivative in respect to the cosmic time). Hence, the acceleration of the comoving observers vanishes since along geodesics, necessarily: $\dot{u}_\mu = 0$. This is also true for the Horndeski and beyond Horndeski theories since the four-velocity vector is the same: $u_\mu = \delta^0_\mu$.

In general, the symmetry of FRW spacetime implies that the heat flux vector q_μ, which is transversal to the four-velocity u_μ ($u^\mu q_\mu = 0$), must always vanish since, otherwise, it would yield a preferred direction and thus break the isotropy.[3] This means, in turn, that the effective energy-momentum tensor always has the perfect fluid form with only the effective pressure p and the effective energy density ρ nonvanishing. Accordingly, in a FRW background the heat-flux vectors for the BD theory (21):

$$q^{\rm BD}_\mu = \frac{1}{\phi\sqrt{2X}}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right] = -\frac{1}{\phi}\sqrt{2X}\dot{u}_\mu, \tag{59}$$

as well as for the viable Horndeski theories:

$$q^{\rm vhorn}_\mu = \frac{G_{4,\phi} - XG_{3,X}}{\sqrt{2X}G_4}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right] = -\frac{G_{4,\phi} - XG_{3,X}}{G_4}\sqrt{2X}\dot{u}_\mu, \tag{60}$$

both vanish: $q^{\rm BD}_\mu = q^{\rm vhorn}_\mu = 0$, where we have taken into account the expression (32) for the acceleration:

$$\dot{u}_\mu = u^\lambda\nabla_\lambda u_\mu = -\frac{1}{2X}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right].$$

For the beyond Horndeski theories we have that the heat flux vector (56) can be written as follows:

$$q^{\rm vbhorn}_\mu = \frac{f_{,\phi} + 4Xf_{,\phi X} - (f_{,XX} - A)(\nabla\phi\cdot\nabla X)}{\sqrt{2X}f}\left[\nabla_\mu X + \frac{(\nabla\phi\cdot\nabla X)}{2X}\nabla_\mu\phi\right] - \frac{f_{,X}}{\sqrt{2X}f}\nabla^\lambda\phi\left[\nabla_\lambda\nabla_\mu X + \frac{\nabla^\kappa\phi\nabla_\lambda\nabla_\kappa X}{2X}\nabla_\mu\phi\right].$$

However, since $u^\mu = \nabla^\mu\phi/\sqrt{2X} = g^{0\mu}$, then the second term in the rhs of the above equation exactly vanishes:

$$-\frac{f_{,X}}{f}g^{0\lambda}[\nabla_\lambda\nabla_\mu X + g^{0\kappa}\delta^0_\mu\nabla_\lambda\nabla_\kappa X] = -\frac{f_{,X}}{f}[\nabla^0\dot{X} + \ddot{X}]\delta^0_\mu = -\frac{f_{,X}}{f}[-\ddot{X} + \ddot{X}]\delta^0_\mu = 0,$$

so that

$$q^{\rm vbhorn}_\mu = -\frac{f_{,\phi} + 4Xf_{,\phi X} - (f_{,XX} - A)(\nabla\phi\cdot\nabla X)}{f}\sqrt{2X}\dot{u}_\mu = 0, \tag{61}$$

as it was for the BD and the viable Horndeski theories.

Let us obtain the expressions of the remaining effective kinematic quantities for the viable Horndeski and beyond Horndeski theories in the FRW background. The timelike FRW four-velocity is given by $u_\mu = \nabla_\mu\phi/\dot{\phi} = \delta^0_\mu$, so that for the components of the three-metric $h_{\mu\nu}$ we obtain $h_{00} = 0$, $h_{ij} = a^2(t)\delta_{ij}$. We also have

$$X = \dot{\phi}^2/2, \qquad \nabla^2\phi = -(\ddot{\phi} + 3H\dot{\phi}), \qquad (\nabla\phi\cdot\nabla X) = -\dot{\phi}^2\ddot{\phi}, \qquad \nabla_\mu\nabla_\nu\phi = \ddot{\phi}\delta^0_\mu\delta^0_\nu - H\dot{\phi}h_{\mu\nu}, \tag{62}$$

[3]This argument has been suggested by one referee.

so that, by making the appropriate substitutions, we get the following expressions for the effective kinematic quantities of viable Horndeski theories (8) in FRW spacetimes (recall that $q_\mu^{\rm eff}=0$):

$$\rho_{\rm eff}=\frac{1}{2G_4}[(K_{,X}-G_{3,\phi}+3G_{3,X}H\dot\phi)\dot\phi^2-K-6G_{4,\phi}H\dot\phi], \tag{63}$$

$$p_{\rm eff}=\frac{1}{2G_4}[(2G_{4,\phi\phi}-G_{3,\phi}-G_{3,X}\ddot\phi)\dot\phi^2+K+2G_{4,\phi}(\ddot\phi+2H\dot\phi)], \tag{64}$$

$$\pi_{\mu\nu}^{\rm eff}=0. \tag{65}$$

In what regards to the viable beyond Horndeski effective cosmological fluid we have that, for the effective energy density,

$$\rho_{\rm eff}=\frac{3f_{,X}}{f}(\dot H+2H^2)\dot\phi^2-\frac{3f_{,\phi}}{f}H\dot\phi-\frac{3}{f}(f_{,X}-A\dot\phi^2)H\dot\phi\ddot\phi+\frac{1}{f}(2f_{,\phi X}+A_{,\phi}\dot\phi^2)\dot\phi^2\ddot\phi+\frac{1}{2f}(A+A_{,X}\dot\phi^2)\dot\phi^2\ddot\phi^2+\frac{A}{f}\dot\phi^3\dddot\phi, \tag{66}$$

while for the effective pressure:

$$p_{\rm eff}=\frac{2f_{,\phi}}{f}H\dot\phi+\frac{2f_{,X}}{f}H\dot\phi\ddot\phi+\frac{f_{,\phi\phi}}{f}\dot\phi^2+\frac{1}{f}(f_{,\phi}+2\dot\phi^2f_{,\phi X})\ddot\phi+\frac{1}{2f}(2f_{,XX}-A)\dot\phi^2\ddot\phi^2+\frac{f_{,X}}{f}[\ddot\phi^2+\dot\phi(\dddot\phi)], \tag{67}$$

where we have taken into account that for the FRW metric the curvature scalar $R=6(\dot H+2H^2)$, while

$$(\nabla X)^2=-\dot X^2=-(\dot\phi\ddot\phi)^2,\qquad \nabla_\mu\nabla_\nu X=\ddot X\delta_\mu^0\delta_\nu^0-H\dot Xh_{\mu\nu},$$
$$\nabla^2X=-\ddot X-3H\dot X=-\ddot\phi(\ddot\phi+3H\dot\phi)-\dot\phi(\dddot\phi).$$

For the flux vector, as already mentioned, we get $q_\mu^{\rm eff}=0$, while since

$$\Pi_{\mu\nu}^{\rm eff}=\left\{\frac{2f_{,\phi}}{f}H\dot\phi+\frac{2f_{,X}}{f}H\dot\phi\ddot\phi+\frac{f_{,\phi\phi}}{f}\dot\phi^2+\frac{1}{f}(f_{,\phi}+2\dot\phi^2f_{,\phi X})\ddot\phi+\frac{1}{2f}(2f_{,XX}-A)\dot\phi^2\ddot\phi^2+\frac{f_{,X}}{f}[\ddot\phi^2+\dot\phi(\dddot\phi)]\right\}\times h_{\mu\nu}=p_{\rm eff}h_{\mu\nu}, \tag{68}$$

the anisotropic stresses vanish as well: $\pi_{\mu\nu}^{\rm eff}=0$. This means that, in a FRW spacetime, the viable beyond Horndeski theories are formally equivalent to GR with an effective perfect fluid.

As we have discussed here, in the FRW cosmological setup, due to the symmetries, both the effective heat flux vector and the anisotropic stresses vanish, so that the resulting effective picture is that of a perfect fluid.[4] In a general cosmological setup the higher-derivative terms that arise in the Horndeski and beyond Horndeski theories do not contribute to the effective imperfect fluid behavior unless the spatial gradient of the scalar field cannot be ignored, as in situations of astrophysical interest. Otherwise these will contribute only to the effective energy density and the effective pressure of the perfect fluid, respectively.

VII. DISCUSSION AND CONCLUSION

In this paper we have approached to the viable Horndeski and beyond Horndeski theories from the perspective of the effective (imperfect) fluid approach. In this regard, as it happens in the framework of the STTs, the divergence of the effective stress-energy tensor leads to the equation of motion of the scalar field:

$$\nabla^\mu T_{\mu\nu}^{\rm eff}-2G_{4,\phi}\partial^\mu\phi G_{\mu\nu}=0,\quad T_{\mu\nu}^{\rm eff}=2G_4\sum_{i=2}^4T_{\mu\nu}^{(i)},$$
$$\nabla^\mu T_{\mu\nu}^{\rm eff}-(f_{,\phi}\partial^\mu\phi+f_{,X}\partial^\mu X)G_{\mu\nu}=0,\quad T_{\mu\nu}^{\rm eff}=f\sum_{i=1}^2T_{\mu\nu}^{(i)}, \tag{69}$$

where the first line above is for Horndeski theories with the contributions to the $T_{\mu\nu}^{(i)}$s given by (27), while the second line is for beyond Horndeski theories with the respective contributions $T_{\mu\nu}^{(i)}$ given by (A14) and (A15) in the Appendix. As a matter of fact, up to a vector field $\partial^\mu\phi$, (69) coincides with the motion equation that is derived from either of the following actions by varying with respect to the scalar field ϕ:

$$S_{\rm vhorn}=\int d^4x\sqrt{|g|}\mathcal{L}_{\rm vhorn},\quad S_{\rm vbhorn}=\int d^4x\sqrt{|g|}\mathcal{L}_{\rm vbhorn}, \tag{70}$$

where $\mathcal{L}_{\rm vhorn}$ is given by (8) while $\mathcal{L}_{\rm vbhorn}$ is given by (9). Our results represent further generalization of previously published works on the effective fluid equivalences [70,74,76–83].

[4] It should be expected that, if considering other less symmetric geometric backgrounds such as, for instance, the Bianchi I anisotropic space, the anisotropic stress tensor would has nonvanishing contribution to the effective imperfect fluid.

Although the effective perfect/imperfect fluid description is always possible, making an appropriate choice of the timelike vector u^μ is central to the explicit form of the equivalence. In [81], for instance, it is shown that an n-dimensional generalized Robertson-Walker (GRW) spacetime with divergence-free conformal curvature tensor exhibits a perfect fluid stress-energy tensor for any $f(R)$ gravity model. The demonstration relies on the notion of GRW spacetime and on the condition that a timelike unit vector u^μ $(u_\mu u^\mu = -1)$ exists such that $\nabla_\mu u_\nu = \varphi(g_{\mu\nu} + u_\mu u_\nu)$, where the scalar field φ is called as "perfect scalar" since it obeys [81–83]:

$$\nabla_\mu \varphi = -u_\mu (u^\lambda \nabla_\lambda \varphi). \tag{71}$$

In [74], on the other hand, it has been shown that, if introduce a timelike unit vector field,

$$u_\mu = \frac{\nabla_\mu R}{\sqrt{-(\nabla R)^2}}, \qquad (\nabla R)^2 \equiv \nabla_\mu R \nabla^\mu R < 0, \tag{72}$$

where R is the Ricci scalar, the $f(R)$ theory can be written in the form of general relativity with an effective imperfect fluid. These examples show that fulfillment of certain conditions is required in order to obtain the explicit form in which the effective fluid equivalence is realized and that, depending on these requirements, the same theory might admit different explicit forms of the equivalence. We want to mention that whenever the effective fluid is generated by a purely scalar degree of freedom, it is irrotational [74], so that not any kind of fluid with physical sense can be reproduced with scalars.

In this regard we point out that the effective fluid equivalence is purely mathematical and not physical because imperfect and perfect fluids obey thermodynamic laws that, in general, have no equivalent in the framework of the STTs. As an illustration, the particle number density cannot be defined for a real scalar field. Another illustration can be based on the energy density of the effective fluid. Let us rewrite the gravitational equations of Horndeski theory (26) in the following form:

$$T^{\rm mat}_{\mu\nu} = 2G_4 G_{\mu\nu} - T^{\rm eff}_{\mu\nu}, \qquad T^{\rm eff}_{\mu\nu} = 2G_4 \sum_i T^{(i)}_{\mu\nu}, \tag{73}$$

where the effective stress-energy tensor $T^{\rm eff}_{\mu\nu}$ is contributed by curvature quantities. For the energy density measured by observers with four-velocity u^μ one gets

$$\rho_{\rm mat} = u^\mu u^\nu T^{\rm mat}_{\mu\nu} = 2G_4 G_{\mu\nu} u^\mu u^\nu - \rho_{\rm eff}. \tag{74}$$

The requirement that for a standard matter fluid $\rho_{\rm mat} \geq 0$, translates into the following requirement: $2G_4 G_{\mu\nu} u^\mu u^\nu \geq \rho_{\rm eff}$. Apart from this the energy density of the effective fluid measured by the observers $\rho_{\rm eff}$ can be a negative quantity without violating any known physical laws. Hence, the energy density of the effective fluid might not have the usual physical sense assigned to it in fluid dynamics. Despite of this the effective fluid picture is very useful when one compares different cosmological models. One may compare functions describing the effective fluid such as, for instance, the equation of state, the anisotropic stresses, the sound speed, etc. The fluid variables have a more immediate and clear physical meaning and also simplify the analysis of the system [80]. Besides, the effective fluid description of Horndeski and beyond Horndeski theories represents an alternative opportunity to deal with cosmological perturbations within the higher-derivative generalizations of scalar-tensor theories among others because in this framework it is relatively easy to identify the contribution of each term in the stress energy tensor of the imperfect fluid to the scalar, vectorial, and tensorial cosmological perturbations. For example, a heat flux contributes to the vectorial cosmological perturbations while the anisotropic stresses contribute to both scalar and tensorial perturbations.

Although we have been able to settle the explicit form of the imperfect/perfect fluid equivalence specifically for the viable Horndeski and beyond Horndeski theories, the present results could be applied to other modifications of gravity such as the extended theories of gravity that are based in the Lagrangian $\mathcal{L} \propto F(R, \nabla^2 R, \nabla^4 R, \ldots, \nabla^{2k} R)$. The ETGs are equivalent to multi-STTs [37]. One example is the sixth-order gravity given by the choice: $F = R + \alpha R \nabla^2 R$, which is equivalent to Brans-Dicke theory with vanishing coupling parameter $\omega = 0$, with a BD scalar field ϕ and an additional canonical scalar field φ as matter source. Hence, in principle it could be put into the form of GR with a mixture of an effective imperfect and a perfect fluids. Another example is given by the $f(R)$ theory. Under the replacement $\phi \to f_{,R}$, $V(\phi) \to f_{,R} - f$, the $f(R)$ theory can be written in the equivalent form of BD theory with vanishing coupling parameter ($\omega = 0$). Hence, the above mentioned modifications of general relativity could be given the form of GR with an effective perfect/imperfect fluid. In particular, for the $f(R)$ theory, the kinematic quantities that appear in the effective SET (12) are those given by Eqs. (18)–(21) with the substitutions $\omega = 0$ and $\phi = f_{,R}$. If we take into account that $\nabla_\mu \phi = f_{,RR} \nabla_\mu R \to (\nabla\phi)^2 = f^2_{,RR} (\nabla R)^2$, etc., we get the same expressions of Ref. [74]. Notice, in particular, that if make these substitutions in (3), we obtain the definition of the four velocity in [74]: $u_\mu = \nabla_\mu R / \sqrt{-(\nabla R)^2}$. The effective fluid approach of $f(R)$ theories has been investigated also in Ref. [92] from the perspective of the cosmological perturbations.

An important aspect of the higher-derivative theories is related with the speed of propagation of scalar and tensor cosmological perturbations. For the Horndeski theories the speed of propagation of the gravitational waves is given by [55,93]:

$$c_{\rm gw}^2=\frac{G_4-X(\ddot\phi G_{5,X}+G_{5,\phi})}{G_4-2XG_{4,X}-X(\dot\phi HG_{5,X}-G_{5,\phi})}. \tag{75}$$

Hence, after the simultaneous detection of gravitational waves GW170817 and the γ-ray burst GRB 170817A [87,88], leading to inferring that the speed of propagation of the tensor perturbations coincides with the speed of light in vacuum $c_{\rm gw}^2=1$ (recall that in this paper we work in the units system where the speed of light in vacuum $c=1$), only the theories with $G_5=0$, $G_4=G_4(\phi)$ survive the observational checks. These are known as viable Horndeski theories [55]. In a similar fashion the only beyond Horndeski theories that survive the observational checks related to the GW170817 event are the viable beyond Horndeski theories given by the Lagrangian (9). Yet, the surviving higher-derivative generalizations of STTs have to be consistent with the limits $0\leq c_s^2\leq 1$ on the squared sound speed (also the squared speed of propagation of the scalar perturbations), $c_s^2=p_{,X}/\rho_{,X}$, in order to avoid gradient instability and to obey causality [54,94]. These additional bounds establish conditions on the derivatives.

For the viable Horndeski theories (8) we have, the effective pressure ($p^{\rm eff}=p_{(2)}+p_{(3)}+p_{(4)}$) is given by the following expression:

$$p^{\rm eff}=\frac{1}{G_4}\left[\frac{K}{2}+(2G_{4,\phi\phi}-G_{3,\phi})X+\left(G_{3,X}X-\frac{1}{3}G_{4,\phi}\right)\frac{(\nabla\phi\cdot\nabla X)}{2X}-\frac{2}{3}G_{4,\phi}\nabla^2\phi\right], \tag{76}$$

while for the effective energy density ($\rho^{\rm eff}=\rho_{(2)}+\rho_{(3)}+\rho_{(4)}$), we have

$$\rho^{\rm eff}=\frac{1}{G_4}\left[XK_{,X}-\frac{K}{2}-G_{3,\phi}X+(G_{3,X}X-G_{4,\phi})\frac{(\nabla\phi\cdot\nabla X)}{2X}+(G_{4,\phi}-G_{3,X}X)\nabla^2\phi\right]. \tag{77}$$

Hence, the speed of propagation of the scalar perturbations in the viable Horndeski theories reads

$$c_s^2=\frac{p^{\rm eff}_{,X}}{\rho^{\rm eff}_{,X}}=\frac{\frac{K_{,X}}{2}-G_{3,\phi X}X-G_{3,\phi}+(G_{3,XX}X^2+\frac{1}{3}G_{4,\phi})\frac{(\nabla\phi\cdot\nabla X)}{2X^2}+2G_{4,\phi\phi}}{\frac{K_{,X}}{2}-G_{3,\phi X}X-G_{3,\phi}+(G_{3,XX}X^2+G_{4,\phi})\frac{(\nabla\phi\cdot\nabla X)}{2X^2}+K_{,XX}X-(G_{3,X}+G_{3,XX}X)\nabla^2\phi}. \tag{78}$$

Causality ($c_s^2\leq 1$) leads to the following condition on the derivatives of the scalar field:

$$\nabla^2\phi\leq\frac{G_{4,\phi}(\nabla\phi\cdot\nabla X)+3X^2(XK_{,XX}-2G_{4,\phi\phi})}{3X^2(XG_{3,XX}+G_{3,X})}, \tag{79}$$

while the absence of gradient instability ($c_s^2\geq 0$) requires that

$$(\nabla\phi\cdot\nabla X)\geq\frac{[2XG_{3,\phi X}+2G_{3,\phi}-K_{,X}-4G_{4,\phi\phi}]X^2}{X^2G_{3,XX}+\frac{1}{3}G_{4,\phi}}, \tag{80}$$

and that (79) is satisfied.

Similar conditions on the derivatives can be found for the viable beyond Horndeski theories. This means that these "viable" theories as a matter of fact can be nonviable if the above conditions on the squared sound speed are not satisfied. In other words: the bounds (79) and (80) amount to further constraints on the physical viability of Horndeski and beyond Horndeski theories, that already satisfy $c_{\rm gw}^2=1$.

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APPENDIX: DERIVATION OF THE MOTION EQUATION OF THE GALILEON IN HORNDESKI AND BEYOND HORNDESKI THEORIES

1. Horndeski theory

We rewrite the equations of motion (26) and (27) derived from the Horndeski Lagrangian as

$$2G_4G_{\mu\nu} = T^{\rm mat}_{\mu\nu} + 2G_4\sum_{i=2}^{4} T^{(i)}_{\mu\nu}. \tag{A1}$$

Then we take the divergence of (A1),

$$2\nabla_\nu(G_4G^{\mu\nu}) = 2G_{4,\phi}R^{\mu\nu}\nabla_\nu\phi - G_{4,\phi}R\nabla^\mu\phi = 2\sum_{i=2}^{4}\nabla_\nu[G_4T^{\mu\nu}_{(i)}], \tag{A2}$$

where we have taken into account the Bianchi identity and the continuity equation for the matter degrees of freedom, so that $\nabla_\nu G^{\mu\nu} = 0$ and $\nabla_\nu T^{\mu\nu}_{\rm mat} = 0$. In order to compute the divergence in (A2) we shall treat each term in the sum separately. From (27), by means of a straightforward calculation for $i = 2$, we obtain

$$2\nabla_\nu[G_4T^{\mu\nu}_{(2)}] \equiv \nabla_\nu[K_{,X}(\nabla^\mu\phi)\nabla^\nu\phi + g^{\mu\nu}K] = [K_{,\phi} + K_{,X}(\nabla^2\phi) + K_{,\phi X}(\nabla\phi)^2 + K_{,XX}(\nabla\phi\cdot\nabla X)](\nabla^\mu\phi). \tag{A3}$$

Meanwhile, if we take into account (27), for $i = 3$ we have

$$\begin{aligned}\nabla_\nu[2G_4T^{\mu\nu}_{(3)}] = -(\nabla^\mu\phi)\{&2G_{3,\phi}(\nabla^2\phi) + G_{3,X}[(\nabla^2\phi)^2 - R_{\nu\beta}(\nabla^\nu\phi)\nabla^\beta\phi - (\nabla_\nu\nabla_\beta\phi)^2] + G_{3,\phi\phi}(\nabla\phi)^2\\ &+G_{3,\phi X}[2(\nabla\phi)\cdot(\nabla X) + (\nabla\phi)^2(\nabla^2\phi)] + G_{3,XX}[(\nabla\phi)\cdot(\nabla X)(\nabla^2\phi) + (\nabla X)\cdot(\nabla X)]\},\end{aligned} \tag{A4}$$

where we have used the following identities,

$$(\nabla^2X) = -(\nabla_\beta\nabla^\beta\nabla_\alpha\phi)\nabla^\alpha\phi - (\nabla_\beta\nabla_\alpha\phi)^2, \qquad \nabla_\mu(\nabla^2\phi) = (\nabla_\beta\nabla^\beta\nabla_\mu\phi) - R^\beta_\mu\nabla_\beta\phi. \tag{A5}$$

Finally, for $i = 4$ we obtain

$$\begin{aligned}\nabla_\nu[G_4T^{\mu\nu}_{(4)}] &\equiv \nabla_\nu\{G_{4,\phi}(\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla^2\phi) + G_{4,\phi\phi}[\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}(\nabla\phi)^2]\},\\ &= G_{4,\phi}\nabla_\nu(\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla^2\phi) = G_{4,\phi}R^{\mu\nu}\nabla_\nu\phi,\end{aligned} \tag{A6}$$

where we have applied the identities,

$$\begin{aligned}(\nabla_\nu G_{4,\phi})(\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla^2\phi) + G_{4,\phi\phi}\nabla_\nu[\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}(\nabla\phi)^2] = 0,\\ (\nabla_\nu G_{4,\phi\phi})[\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}(\nabla\phi)^2] = 0.\end{aligned} \tag{A7}$$

Using (A6) we eliminate the term proportional to the Ricci tensor in (A2), then we have

$$2\sum_{i=2}^{3}\nabla_\nu[G_4T^{\mu\nu}_{(i)}] + G_{4,\phi}R\nabla^\mu\phi = 0. \tag{A8}$$

If we substitute (A3) and (A4) in (A8) we get,

$$(\partial^\mu\phi)\Phi = 0, \tag{A9}$$

where the function Φ is defined as

$$\begin{aligned}\Phi \equiv\ & K_{,\phi} + [K_{,X} - 2G_{3,\phi}](\nabla^2\phi) + [K_{,\phi X} - G_{3,\phi\phi}](\nabla\phi)^2 + K_{,XX}(\nabla\phi\cdot\nabla X)\\ &- G_{3,X}[(\nabla^2\phi)^2 - R_{\mu\nu}(\nabla^\mu\phi)(\nabla^\nu\phi) - (\nabla_\mu\nabla_\nu\phi)^2] - G_{3,XX}[(\nabla\phi\cdot\nabla X)(\nabla^2\phi) + \nabla X\cdot\nabla X]\\ &- G_{3,\phi X}[(\nabla\phi)^2(\nabla^2\phi) + 2(\nabla\phi\cdot\nabla X)] + G_{4,\phi}R.\end{aligned} \tag{A10}$$

Since, in general, the vector field is $\partial^\mu\phi \neq 0$, the motion equation of the galileon in Horndeski theory is written as

$$\Phi = 0. \tag{A11}$$

This coincides with the modified Klein-Gordon equation that is obtained by taking variations of the action (25) with respect to the galileon ϕ.

2. Beyond Horndeski theories

Following the procedure applied above we can rewrite the equations of motion derived from the beyond Horndeski Lagrangian (9) as

$$fG_{\mu\nu} = T^{\mathrm{mat}}_{\mu\nu} + f\sum_{i=1}^{2} T^{\mathrm{vbhorn}(i)}_{\mu\nu}. \tag{A12}$$

Then we compute the divergence of (A12)—recall that $\nabla_\nu G^{\mu\nu} = 0$ and $\nabla_\nu T^{\mu\nu}_{\mathrm{mat}} = 0$—to obtain

$$\nabla_\nu(fG^{\mu\nu}) = R^{\mu\nu}\nabla_\nu f - \frac{1}{2}R\nabla^\mu f = \sum_{i=1}^{2}\nabla_\nu[fT^{\mu\nu}_{\mathrm{vbhorn}(i)}], \tag{A13}$$

where we have defined

$$fT^{\mathrm{vbhorn}(1)}_{\mu\nu} \equiv \frac{1}{2}f_X R(\nabla_\mu\phi)(\nabla_\nu\phi) + \nabla_\mu\nabla_\nu f - g_{\mu\nu}(\nabla^2 f), \tag{A14}$$

$$fT^{\mathrm{vbhorn}(2)}_{\mu\nu} \equiv -\frac{1}{2}[A_{,X}(\nabla X)^2 + 2A_{,\phi}(\nabla\phi\cdot\nabla X) + 2A\nabla^2 X]\nabla_\mu\phi\nabla_\nu\phi - A\left[\nabla_\mu X\nabla_\nu X - \frac{1}{2}g_{\mu\nu}(\nabla X)^2\right], \tag{A15}$$

and the following expression,

$$\begin{aligned}\nabla_\mu\nabla_\nu f - g_{\mu\nu}(\nabla^2 f) = {}& f_{,\phi\phi}[(\nabla_\mu\phi)\nabla_\nu\phi + 2Xg_{\mu\nu}] + f_{,\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) + f_{,X}(\nabla_\mu\nabla_\nu X - g_{\mu\nu}\nabla^2 X)\\ &+ 2f_{,\phi X}[\nabla_{(\mu}\phi\nabla_{\nu)}X - g_{\mu\nu}(\nabla\phi\cdot\nabla X)] + f_{,XX}[(\nabla_\mu X)\nabla_\nu X - g_{\mu\nu}(\nabla X)^2],\end{aligned} \tag{A16}$$

has been taken into account. Note that the divergence of (A16) can be written as the compact expression:

$$\nabla^\nu[\nabla_\mu\nabla_\nu f - g_{\mu\nu}(\nabla^2 f)] \equiv R_{\mu\nu}\nabla^\nu f. \tag{A17}$$

As before we calculate separately each term in the sum in (A13). From (A14), and using (A17), a straightforward calculation for $i = 1$ leads to

$$\nabla_\nu[fT^{\mu\nu}_{\mathrm{vbhorn}(1)}] = \frac{1}{2}(\nabla^\mu\phi)[\{f_{,X}\nabla^2\phi + \nabla\phi\cdot\nabla(f_{,X})\}R + f_{,X}\nabla\phi\cdot\nabla R] - \frac{1}{2}f_{,X}R\nabla^\mu X + R^{\mu\nu}\nabla_\nu f, \tag{A18}$$

meanwhile, starting from (A15) a lengthy calculation for $i = 2$ yields

$$\begin{aligned}\nabla_\nu[fT^{\mu\nu}_{\mathrm{vbhorn}(2)}] = -\nabla^\mu\phi\Big[&\frac{1}{2}(\nabla X)^2\{A_{,X}\nabla^2\phi + \nabla\phi\cdot\nabla(A_{,X}) - A_{,\phi}\} + \frac{1}{2}A_{,X}\nabla\phi\cdot\nabla\{(\nabla X)^2\}\\ &+A\{(\nabla^2 X)\nabla^2\phi + \nabla\phi\cdot\nabla(\nabla^2 X)\} + A_{,\phi}\{\nabla\phi\cdot\nabla X\nabla^2\phi + (\nabla\phi)\cdot\nabla(\nabla\phi\cdot\nabla X)\}\\ &+\nabla\phi\cdot\nabla A(\nabla^2 X) + \{\nabla\phi\cdot\nabla(A_{,\phi})\}(\nabla\phi\cdot\nabla X)\Big].\end{aligned} \tag{A19}$$

Now we substitute (A18) and (A19) in (A13) to obtain

$$(\partial^\mu\phi)\Psi = 0, \tag{A20}$$

where the function Ψ is defined as follows:

$$\begin{aligned}\Psi \equiv &\{f_{,\phi} + f_{,X}\nabla^2\phi + \nabla\phi\cdot\nabla(f_{,X})\}R + f_{,X}\nabla\phi\cdot\nabla R\\ &-2\bigg[\frac{1}{2}(\nabla X)^2\{A_{,X}\nabla^2\phi + \nabla\phi\cdot\nabla(A_{,X}) - A_{,\phi}\} + \frac{1}{2}A_{,X}\nabla\phi\cdot\nabla\{(\nabla X)^2\}\\ &+A\{(\nabla^2 X)\nabla^2\phi + \nabla\phi\cdot\nabla(\nabla^2 X)\} + (\nabla\phi)\cdot(\nabla A)(\nabla^2 X)\\ &+A_{,\phi}\{(\nabla\phi)\cdot(\nabla X)\nabla^2\phi + (\nabla\phi)\cdot\nabla(\nabla\phi\cdot\nabla X)\}\\ &+\{\nabla\phi\cdot\nabla(A_{,\phi})\}(\nabla\phi\cdot\nabla X)\bigg].\end{aligned} \tag{A21}$$

Because, in general, $\partial^\mu\phi \neq 0$, the motion equation of the scalar field in beyond Horndeski theories is written as

$$\Psi = 0. \tag{A22}$$

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