Sixth post-Newtonian nonlocal-in-time dynamics of binary systems

Donato Bini,[1,2] Thibault Damour,[3] and Andrea Geralico[1]

[1]*Istituto per le Applicazioni del Calcolo "M. Picone," CNR, I-00185 Rome, Italy*
[2]*INFN, Sezione di Roma Tre, I-00146 Rome, Italy*
[3]*Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette, France*

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We complete our previous derivation, at the sixth post-Newtonian (6PN) accuracy, of the local-in-time dynamics of a gravitationally interacting two-body system by giving two gauge-invariant characterizations of its complementary nonlocal-in-time dynamics. On the one hand, we compute the nonlocal part of the scattering angle for hyberboliclike motions; and, on the other hand, we compute the nonlocal part of the averaged (Delaunay) Hamiltonian for ellipticlike motions. The former is computed as a large-angular-momentum expansion (given here to next-to-next-to-leading order), while the latter is given as a small-eccentricity expansion (given here to the tenth order). We note the appearance of $\zeta(3)$ in the nonlocal part of the scattering angle. The averaged Hamiltonian for ellipticlike motions then yields two more gauge-invariant observables: the energy and the periastron precession as functions of orbital frequencies. We point out the existence of a hidden simplicity in the mass-ratio dependence of the gravitational-wave energy loss of a two-body system. We include a Supplemental Material that gives the explicit analytic form of a scattering integral which we could only evaluate numerically.

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I. INTRODUCTION

A new strategy for deriving to higher post-Newtonian (PN) accuracy the conservative dynamics of gravitationally interacting two-body systems has been recently introduced [1]. This strategy combines, in a new way, various analytical approximation methods: post-Newtonian, post-Minkowskian (PM), multipolar-post-Minkowskian, effective-field-theory (EFT), gravitational self-force, effective one-body (EOB), and Delaunay averaging. In Ref. [2], we have shown how to use this new methodology to derive the two-body dynamics at the fifth post-Newtonian (5PN), and fifth-and-a-half post-Newtonian (5.5PN) levels. The latter results were then extended to the sixth post-Newtonian (6PN) level in Ref. [3].

A basic aspect of our new method is to split the Hamiltonian describing the dynamics of binary systems into two separate parts: a local-in-time Hamiltonian, $H_{\rm loc,f}$ (which starts at the Newtonian level), and a nonlocal-in-time one, $H_{\rm nonloc,f}$ (which starts at the fourth post-Newtonian, 4PN, level [4]). The total Hamiltonian,

$$H^{\rm tot} = H^{\rm loc,f} + H^{\rm nonloc,f}, \tag{1.1}$$

is independent of the choice of the flexibility factor $f(t)$. The latter enters the nonlocal Hamiltonian via a multiplicative renormalization of the time scale $\Delta t^f = f(t)\Delta t^h$ used as ultraviolet cutoff in the (external) nonlocal tail action, so that one has

$$H^{\rm nonloc,f}(t) = H^{\rm nonloc,h}(t) + \Delta^{\rm f-h} H(t), \tag{1.2}$$

where $H^{\rm nonloc,h}(t)$ is (uniquely[1]) defined by choosing the *harmonic-coordinate* cutoff $\Delta t^h = 2r^h_{12}/c$ (where r^h_{12} denotes the two-body radial separation in harmonic coordinates), while

$$\Delta^{\rm f-h} H(t) = +2\frac{GH}{c^5}\mathcal{F}^{\rm GW}(t)\ln\left(f(t)\right) \tag{1.3}$$

is an additional contribution which involves the gravitational-wave (GW) energy flux $\mathcal{F}^{\rm GW}(t)$, and which vanishes when $f(t) = 1$. An element of our new method is to choose a flexibility factor $f(t)$ such that decomposition (1.1) of the total Hamiltonian $H^{\rm tot}$ into local and nonlocal parts implies that the two corresponding parts of the total scattering angle, say,

$$\chi^{\rm tot}(E,J) = \chi^{\rm loc,f}(E,J) + \chi^{\rm nonloc,f}(E,J), \tag{1.4}$$

separately satisfy the simple mass-ratio dependence proven in Ref. [7] for $\chi^{\rm tot}$. [Here, χ is considered as a function of the center-of-mass (c.m.) energy, E, and c.m. angular momentum, J, of the binary system.]

[1]We work here at the second-post-Newtonian (2PN) fractional accuracy, where harmonic coordinates are uniquely defined and lead to a finite higher-order action [5,6].

In our previous work [3] we computed the *local-in-time* part of the Hamiltonian, $H^{\rm loc,f}$, at the 6PN accuracy. We gave two gauge-invariant characterizations of $H^{\rm loc,f}$. First, we explicitly derived the 6PN-accurate contribution to the scattering angle, say $\chi^{\rm loc,f}_{\rm 6PN}(E,J)$, coming from $H^{\rm loc,f}$. Second, we computed the 6PN-accurate radial action,

$$I_R^{\rm loc,f}(E,J)=\frac{1}{2\pi}\oint dRP_R, \tag{1.5}$$

along ellipticlike motions (with energy E and angular momentum J) described by $H^{\rm loc,f}$.

The aim of the present work is to complete the results of Ref. [3] by deriving the explicit 6PN-accurate values of the complementary contributions, both to χ and to I_R, coming from the *nonlocal-in-time* dynamics, $H^{\rm nonloc,f}$. More precisely, we shall compute here both $\chi^{\rm nonloc,f}_{\rm 6PN}(E,J)$ and $I^{\rm nonloc,f}_{R\,\rm 6PN}(E,J)$, such that the quantities

$$\chi^{\rm tot}_{\rm 6PN}(E,J)=\chi^{\rm loc,f}_{\rm 6PN}(E,J)+\chi^{\rm nonloc,f}_{\rm 6PN}(E,J) \tag{1.6}$$

and

$$I^{\rm tot}_{R\rm 6PN}(E,J)=I^{\rm loc,f}_{R\rm 6PN}(E,J)+I^{\rm nonloc,f}_{R\rm 6PN}(E,J) \tag{1.7}$$

give the scattering angle (for hyperboliclike motions), and the radial action (for ellipticlike motions) described by the total Hamiltonian (1.1), considered at the 6PN accuracy. Because of the nonlocal-in-time nature of $H^{\rm nonloc,f}$, it seems impossible to derive (for general motions) closed-form expressions for $\chi^{\rm nonloc,f}_{\rm 6PN}(E,J)$ and $I^{\rm nonloc,f}_{R\rm 6PN}(E,J)$. We will compute them in the form of expansions in a relevant small parameter. For hyperboliclike motions, the expansion parameter is the inverse eccentricity $\frac{1}{e}$, or equivalently the inverse impact parameter $\frac{1}{b}$, or the inverse angular momentum $\frac{1}{J}$. For ellipticlike motions, the expansion parameter is the (unperturbed) squared eccentricity $e^2_{\rm loc}(E,J)$, or, equivalently, the (unperturbed) radial action $I_R^{\rm loc}(E,J)$. We will also give the 6PN-accurate value of the energy, and of the periastron advance, along circular orbits.

Let us stress that both quantities Eqs. (1.6) and (1.7) are gauge-invariant characteristics of the (6PN-accurate) two-body dynamics. In addition, the left-hand sides of Eqs. (1.6) and (1.7) are completely independent of the choice of the flexibility factor f. It is only the decomposition into the two parts ($\chi^{\rm loc,f}$ versus $\chi^{\rm nonloc,f}$ and $I^{\rm loc,f}_{R\rm 6PN}$ versus $I^{\rm nonloc,f}_{R\rm 6PN}$) which depends on the choice of $f(t)$. Finally, we will derive below the explicit form of the constraints that must be satisfied by $f(t)$, so that the specific separability condition (between local and nonocal) that we assumed in our previous work [3] is satisfied. The gauge-invariant content of the corresponding Hamiltonian contribution $\Delta^{\rm f-h}H$ will be explicitly displayed.

The possibility of characterizing (in a gauge-invariant manner) the conservative dynamics of binary systems by means of the functional relation between the radial action, I_R, and the energy and angular momentum, E, J [or, equivalently, the functional relation $E(I_R,I_\phi)$, with $I_\phi=\frac{1}{2\pi}\oint P_\phi d\phi=J$] is well known in classical mechanics (particularly since the work of Delaunay on the averaging of action-angle Hamiltonians), and was emphasized many years ago in the general-relativistic context [8]. By contrast, the possibility of fully characterizing (in a gauge-invariant manner) the conservative dynamics of binary systems by means of the functional relation between the (c.m.) scattering angle χ and E and J has only been recently emphasized [9,10]. Many different aspects of the physics of classical and quantum scattering (and of the relation between the two) have been recently explored [7,11–52].

Let us summarize the current state of the art in the theoretical knowledge of the conservative dynamics of gravitationally interacting two-body systems. The PN-expanded dynamics is fully known at the 4PN level (corresponding to $1/c^8$ fractional corrections to the Newtonian description) [4,53–59]. At the 5PN level, our new method [1] has allowed us to derive, in a gauge-invariant way, the full dynamics modulo *two undetermined* numerical parameters, denoted $\bar d_5^{\nu^2}$ and $a_6^{\nu^2}$. These coefficients parametrize terms of the (sketchy) form

$$\Delta H^{\rm loc}_{\rm 5PN}\sim \bar d_5^{\nu^2}\frac{G^5m_1^3m_2^3}{c^{10}R^5}p_r^2+a_6^{\nu^2}\frac{G^6m_1^3m_2^3(m_1+m_2)}{c^{10}R^6}, \tag{1.8}$$

in the (c.m. frame) local 5PN Hamiltonian. Here m_1 and m_2 denote the two masses, $R=|\mathbf{x}_1-\mathbf{x}_2|$ their radial distance, while $p_r=P_R/\mu$ denotes the radial momentum $P_R=\mathbf{n}_{12}\cdot\mathbf{P}_1=-\mathbf{n}_{12}\cdot\mathbf{P}_2$, rescaled by the reduced mass of the system $\mu\equiv m_1m_2/(m_1+m_2)$. (Note that p_r has the dimension of a velocity, and, actually, is equal, in lowest approximation, to the relative radial velocity dR/dt.) Recent progress in the (EFT-based) computer-aided evaluation of the PN-expanded interaction potential of binary systems [59–62] gives hope that the two missing coefficients $\bar d_5^{\nu^2}$ and $a_6^{\nu^2}$ might be soon derived. This would lead to a complete knowledge of the 5PN dynamics.

The 5.5PN Hamiltonian is entirely nonlocal, and it is fully known [2]. At the 6PN level, our method has allowed us to derive [3], in a gauge-invariant way, the full 6PN dynamics modulo *four undetermined* numerical parameters, denoted $q_{45}^{\nu^2}$, $\bar d_6^{\nu^2}$, $a_7^{\nu^2}$, and $a_7^{\nu^3}$. These coefficients parametrize terms of the (sketchy) form

$$\Delta H^{loc}_{\rm 6PN}\sim q_{45}^{\nu^2}\frac{G^5m_1^3m_2^3}{c^{12}R^5}p_r^4+\bar d_6^{\nu^2}\frac{G^6m_1^3m_2^3(m_1+m_2)}{c^{12}R^6}p_r^2 \\ +a_7^{\nu^2}\frac{G^7m_1^3m_2^3(m_1+m_2)^2}{c^{12}R^7}+a_7^{\nu^3}\frac{G^7m_1^4m_2^4}{c^{12}R^7} \tag{1.9}$$

in the (c.m. frame) local 6PN Hamiltonian.

Besides this knowledge of the PN-expanded dynamics (i.e., its expansion in powers of $\frac{1}{c}$), one has also recently acquired the knowledge of the first three terms in the (conservative) PM-expanded dynamics, i.e., its expansion in powers of the gravitational coupling constant G (keeping the velocity dependence exact). Hamiltonian formulations of the first post-Minkowskian [1PM, i.e., $O(G)$] dynamics have been derived in various gauges [9,63]. The second post-Minkowskian [2PM, i.e., $O(G^2)$] dynamics, whose equations of motion had been known for many years [64–66], was expressed only recently in Hamiltonian form [10,18]. The third post-Minkowskian [3PM, i.e., $O(G^3)$] dynamics has been derived in Refs. [23,32] (see also Refs. [7,51] for its simpler EOB formulation). Confirmations of the 3PM dynamics of Refs. [23,32] have been obtained in Refs. [1] (5PN level), [2,45,60] (6PN level), and [50] (3PM level).

Equations (1.8) and (1.9) clearly display the fact that the parts of the 5PN and 6PN dynamics left undetermined by our new method belong to the fifth, sixth, and seventh post-Minkowskian (5PM, 6PM, 7PM) approximations. This shows, in particular, that our current work leads to a complete knowledge of the fourth post-Minkowskian [4PM; $O(G^4)$] dynamics up to the 6PN level included. However, in order to explicate this knowledge (in a gauge-invariant way) from our current results [2,3], one needs to explicitly derive the (f-route) *nonlocal* contribution, $\chi_{6\mathrm{PN}}^{\mathrm{nonloc,f}}(E,J)$, to the total scattering angle, $\chi_{6\mathrm{PN}}^{\mathrm{tot}}(E,J)$, Eq. (1.4), so as to complete the explicit expression for the (f-route) local contribution $\chi_{6\mathrm{PN}}^{\mathrm{loc,f}}(E,J)$ given in Ref. [3].

Our basic tool for deriving the nonlocal contribution to the scattering angle will be the general, simple formula, derived in Ref. [52], that computes the additional contribution $\delta\chi(E,J)$ to $\chi(E,J)=\chi_0(E,J)+\delta\chi(E,J)$ induced by an additional contribution δH to the Hamiltonian $[H(q,p)=H_0(q,p)+\delta H(q,p)]$, namely

$$\delta\chi(E,J)=\frac{\partial}{\partial J}W_{\mathrm{hyp}}(E,J)+O[(\delta H)^2], \qquad (1.10)$$

where

$$W_{\mathrm{hyp}}(E,J)\equiv\int_{-\infty}^{+\infty}dt\,\delta H \qquad (1.11)$$

is integrated along the unperturbed hyperboliclike motion (with energy E and angular momentum J) defined by the unperturbed Hamiltonian H_0. Note the important point that Refs. [4,52,67] have shown that the relation (1.10), which is easily derived for usual *local* Hamiltonians, holds also in the present case of a *nonlocal* Hamiltonian.

Similarly, it is easy to relate the elliptic-motion analog of (1.11), say

$$W_{\mathrm{ell}}(E,J)\equiv\oint dt\,\delta H, \qquad (1.12)$$

where, now, the integral is taken over one radial period of an ellipticlike motion, to the (first-order) perturbation $\delta I_R(E,J)$ of the radial action,

$$I_R(E,J)=I_R^0(E,J)+\delta I_R(E,J), \qquad (1.13)$$

corresponding to a general perturbation $H=H_0(q,p)+\delta H(q,p)$ of the Hamiltonian. Indeed, the fundamental property of Delaunay averaging (for ellipticlike motions) is that the perturbation $\delta\bar{H}(I_R,I_\phi)$ of the angle-averaged Delaunay Hamiltonian,

$$\bar{H}(I_R,I_\phi)=\frac{1}{\oint dt}\oint dtH=\bar{H}_0(I_R,I_\phi)+\delta\bar{H}(I_R,I_\phi), \qquad (1.14)$$

is simply given by averaging the perturbation of the Hamiltonian,[2] so that

$$\delta\bar{H}(I_R,I_\phi)=\frac{1}{\oint dt}\oint dt\delta H(q,p)=\frac{\Omega_R}{2\pi}\oint dt\delta H(q,p)$$
$$=\frac{\Omega_R}{2\pi}[W_{\mathrm{ell}}(E,J)]_{E\mapsto\bar{H}_0(I_R,I_\phi)}. \qquad (1.15)$$

Here, $\Omega_R=\frac{2\pi}{T_R}=\partial\bar{H}(I_R,I_\phi)/\partial I_R$ denotes the radial angular frequency ($T_R=\oint dt$ denoting the radial period). Note that in the last equation (1.15) one can use the leading-order replacement $E\mapsto\bar{H}_0(I_R,I_\phi)$ to express $\delta\bar{H}$ as a function of I_R, and I_ϕ, instead of the natural variables E, J entering the integrated action $W_{\mathrm{ell}}(E,J)$, (1.12). Writing that $I_R(E,J)$ is the inverse function of $\bar{H}(I_R,I_\phi)$, and using $\Omega_R=\partial\bar{H}(I_R,I_\phi)/\partial I_R$, also leads to the result that the perturbation $\delta I_R(E,J)$ of the radial action $I_R(E,J)=I_R^0(E,J)+\delta I_R(E,J)$ is simply given by

$$\delta I_R(E,J)=-\frac{1}{2\pi}W_{\mathrm{ell}}(E,J)+O[(\delta H)^2], \qquad (1.16)$$

where $W_{\mathrm{ell}}(E,J)$ is again the integrated elliptic-motion action defined in Eq. (1.12). Note in passing that by combining the result (1.16) with the standard general result for the periastron advance Φ (see, e.g., [8])

$$\frac{\Phi(E,J)}{2\pi}=-\frac{\partial I_R(E,J)}{\partial J}, \qquad (1.17)$$

one finds that the perturbation $\delta\Phi(E,J)$ of the periastron advance $\Phi(E,J)=\Phi_0(E,J)+\delta\Phi(E,J)$ is given by

[2]This fundamental result of classical mechanics played an important role in the development of quantum mechanics, where it got transmuted into the well-known Hellman-Feynman theorem.

$$\delta\Phi(E,J) = +\frac{\partial W_{\rm ell}(E,J)}{\partial J}. \qquad (1.18)$$

In the present paper we shall apply the general results of Eqs. (1.10)–(1.12), and (1.15), to the perturbed dynamics $H = H_0 + \delta H$ with

$$H_0 = H^{\rm loc,f},$$
$$\delta H = H^{\rm nonloc,f} = H^{\rm nonloc,h} + \Delta^{\rm f-h} H. \qquad (1.19)$$

As we have derived in Refs. [2,3] the contributions of $H_0 = H^{\rm loc,f}$ both to the scattering angle, $\chi^{\rm loc,f}_{\rm 6PN}(E,J)$ (see Sec. VIII in [3]), and to the Delaunay averaged Hamiltonian $H^{\rm loc,f}_{\rm 6PN}(I_R, I_\phi)$, or equivalently $I^{\rm loc,f}_{R6{\rm PN}}(E,J)$ (see Tables X and XI in [2] and Sec. IX in [3]), we only need now to compute the complementary contributions

$$\delta\chi(E,J) = \chi^{\rm nonloc,f}_{\rm 6PN}(E,J) = \frac{\partial}{\partial J} W^{\rm nonloc,f}_{\rm hyp}(E,J), \qquad (1.20)$$

and

$$\delta\bar{H}(I_R, I_\phi) = \bar{H}^{\rm nonloc,f}_{\rm 6PN}(I_R, I_\phi)$$
$$= \frac{\Omega_R}{2\pi}[W^{\rm nonloc,f}_{\rm ell}(E,J)]_{E\mapsto \bar{H}_0(I_R,I_\phi)}. \qquad (1.21)$$

From the latter result, we shall then be able to deduce the nonlocal contribution to the periastron advance

$$\delta^{\rm nonloc,f}\Phi(E,J) = +\frac{\partial W^{\rm nonloc,f}_{\rm ell}(E,J)}{\partial J}. \qquad (1.22)$$

Our first task will then be to compute the f-route, nonlocal perturbed action along hyperbolic motions, i.e.,

$$W^{\rm nonloc,f}_{\rm hyp}(E,J) = \int_{-\infty}^{+\infty} dt\, H^{\rm nonloc,f}(t). \qquad (1.23)$$

In view of the linear decomposition (1.2) of the f-route nonlocal Hamiltonian, $H^{\rm nonloc,f}$, we have a corresponding linear decomposition of $W^{\rm nonloc,f}_{\rm hyp}(E,J)$, namely

$$W^{\rm nonloc,f}_{\rm hyp}(E,J) = W^{\rm nonloc,h}_{\rm hyp}(E,J) + \Delta^{\rm f-h}_{\rm hyp} W(E,J), \qquad (1.24)$$

where

$$W^{\rm nonloc,h}_{\rm hyp}(E,J) = \int_{-\infty}^{+\infty} dt\, H^{\rm nonloc,h}(t), \qquad (1.25)$$

and

$$\Delta^{\rm f-h}_{\rm hyp} W(E,J) = \int_{-\infty}^{+\infty} dt \Delta^{\rm f-h} H(t), \qquad (1.26)$$

both integrals being evaluated along an hyperbolic motion of $H_0 = H^{\rm loc,f}_{\rm 6PN}$ with energy E and angular momentum J. Actually, as nonlocal effects start at the 4PN level, it is enough to use as H_0 in this calculation the 2PN-accurate Hamiltonian (whose Delaunay form was given in [8]; see Appendix A).

While $\Delta^{\rm f-h}_{\rm hyp} W(E,J)$ can be (and will be) computed in closed form, it does not seem possible to compute $W^{\rm nonloc,h}_{\rm hyp}(E,J)$ in closed form. But, it will be enough for our purposes to compute the first three terms in the large-J (or large eccentricity) expansion of the function, $W^{\rm nonloc,h}_{\rm hyp}(E,J)$, namely

$$W^{\rm nonloc,h}_{\rm hyp}(E,J) = W_4(E)\frac{(Gm_1m_2)^4}{J^3} + W_5(E)\frac{(Gm_1m_2)^5}{J^4}$$
$$+ W_6(E)\frac{(Gm_1m_2)^6}{J^5} + O\left(\frac{G^7}{J^6}\right). \qquad (1.27)$$

As displayed here, this expansion in powers of $\frac{1}{J}$ is also a PM expansion in powers of G. In view of Eq. (1.10), the corresponding expansion for the (h-route) nonlocal contribution to the scattering angle reads

$$\chi^{\rm nonloc,h}(E,J) = -3W_4(E)\frac{(Gm_1m_2)^4}{J^4} - 4W_5(E)\frac{(Gm_1m_2)^5}{J^5}$$
$$-5W_6(E)\frac{(Gm_1m_2)^6}{J^6} + O\left(\frac{G^7}{J^7}\right). \qquad (1.28)$$

While we will be able to analytically compute closed-form expressions for the first two expansion coefficients $W_4(E)$ and $W_5(E)$, we will only be able to write down integral expressions for the third expansion coefficient $W_6(E)$. We did not succeed in analytically computing the latter integral expressions, but we could estimate then numerically.

Our next task will be to use the mass-ratio dependence of the coefficients $W_4(E)$, $W_5(E)$, and $W_6(E)$ to constrain the choice of the flexibility factor $f(t)$. Indeed, as recalled above, the choice of $f(t)$ is constrained, within our method, by requiring that the two parts, $\chi^{\rm loc,f}$ and $\chi^{\rm nonloc,f} = \chi^{\rm nonloc,h} + \chi^{\rm f-h}$ of the total scattering angle $\chi^{\rm tot}$, Eq. (1.4), *separately* satisfy the simple mass-ratio dependence proven in Ref. [7] for $\chi^{\rm tot}$.

Finally, we will complete our 6PN-accurate description of the dynamics of elliptic like motions by computing the elliptic analog of Eq. (1.24), namely

$$W^{\rm nonloc,f}_{\rm ell}(E,J) = W^{\rm nonloc,h}_{\rm ell}(E,J) + \Delta^{\rm f-h}_{\rm ell} W(E,J), \qquad (1.29)$$

with

$$W^{\rm nonloc,h}_{\rm ell}(E,J) = \oint dt\, H^{\rm nonloc,h}(t), \qquad (1.30)$$

and

$$\Delta^{\mathrm{f-h}}_{\mathrm{ell}} W(E,J) = \oint dt \Delta^{\mathrm{f-h}} H(t), \tag{1.31}$$

both integrals being now evaluated along *one radial period* of an elliptic motion, with given energy E and angular momentum J of $H_0 = H^{\mathrm{loc,f}}_{\mathrm{6PN}}$. As before, it is enough to use $H_0 \approx H_{\mathrm{2PN}}$ in this calculation.

A. Notation

We use a mostly plus signature. We define the symmetric mass ratio ν as the ratio of the reduced mass $\mu \equiv m_1 m_2/(m_1+m_2)$ to the total mass $M = m_1 + m_2$:

$$\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}. \tag{1.32}$$

We use several different measures of the total energy $E_{\mathrm{tot}} = Mc^2 + \cdots$ of the binary system (considered in the c.m. frame). Of particular importance is the EOB effective energy, $\mathcal{E}_{\mathrm{eff}}$, which is defined by

$$\mathcal{E}_{\mathrm{eff}} = \frac{E^2_{\mathrm{tot}} - m_1^2 c^4 - m_2^2 c^4}{2(m_1+m_2)c^2}. \tag{1.33}$$

Equivalently, we have

$$\begin{aligned} E_{\mathrm{tot}} &= Mc^2 \sqrt{1 + 2\nu\left(\frac{\mathcal{E}_{\mathrm{eff}}}{\mu c^2} - 1\right)} \\ &\equiv Mc^2 \sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\mathrm{eff}} - 1)}, \end{aligned} \tag{1.34}$$

where

$$\hat{\mathcal{E}}_{\mathrm{eff}} \equiv \frac{\mathcal{E}_{\mathrm{eff}}}{\mu c^2}. \tag{1.35}$$

We also use the dimensionless specific binding energy

$$\bar{E} \equiv \frac{E_{\mathrm{tot}} - Mc^2}{\mu c^2}. \tag{1.36}$$

The total c.m. angular momentum J will often be measured by its dimensionless rescaled version

$$j \equiv \frac{cJ}{G m_1 m_2} = \frac{cJ}{GM\mu}. \tag{1.37}$$

(The definitions used in the present work for $\bar{E}$ and j differ by respective factors $\frac{1}{c^2}$ and c from those used in our last work [3].) The latter equation shows that one can formally consider that $j = O(\frac{c}{G})$, so that a term or order $\frac{1}{j^n}$ is of order $\frac{G^n}{c^n}$.

In the following, we shall often use the shorthand notations

$$\gamma \equiv \hat{\mathcal{E}}_{\mathrm{eff}}, \tag{1.38}$$

$$p_\infty \equiv \sqrt{\gamma^2 - 1}, \quad \text{so that } \gamma = \sqrt{1 + p_\infty^2}, \tag{1.39}$$

and

$$h(\gamma,\nu) \equiv \sqrt{1 + 2\nu(\gamma - 1)}. \tag{1.40}$$

We shall often find it convenient to work with dimensionless rescaled orbital parameters, such as $r_{12} \equiv c^2 r^{\mathrm{phys}}_{12}/(GM)$, or $a \equiv c^2 a^{\mathrm{phys}}/(GM)$. The context should make it clear whether we use physical or rescaled quantities.

Most of our final results will be expressed in terms of dimensionless quantities, such as $\bar{E}$, j, p_∞, and $a \equiv c^2 a^{\mathrm{phys}}/(GM)$. In other words, we essentially use units where c and G (and sometimes also GM) are set to unity. However, in some formulas we indicate the powers of G (or GM) that they originally contain. Concerning the powers of c, and the corresponding *absolute* PN order, we will not explicitly keep track of them. However, we will keep track of the *fractional* PN order of various contributions to PN-expanded quantities by using $\eta \sim \frac{1}{c}$ (to be set to one at the end) as a bookkeeping device for PN orders beyond the leading-order term in a quantity. For example, we will write $Q = Q^{\mathrm{LO}}(1 + \eta^2 q_2 + \eta^4 q_4)$ for a quantity Q which is expanded to fractional 2PN accuracy beyond its leading order PN contribution. To help the reader keep track of the absolute PN order of the quantities we shall compute, let us note that (i) nonlocal effects in the dynamics start at the absolute 4PN order, and (ii) one can use the formal scalings $\frac{1}{j} = O(\frac{G}{c})$, $\bar{E} = O(\frac{1}{c^2}) = \gamma - 1$, and $p_\infty = O(\frac{1}{c})$ to recover the powers of G and c.

II. BRIEF REMINDER ABOUT THE NONLOCAL PART OF THE ACTION

Let us consider in more detail the structure of the nonlocal part of the action, $S_{\mathrm{nonloc,f}}$. As discussed in Ref. [2], at the 6PN accuracy the nonlocal action can be linearly decomposed into its $4+5+6$PN piece, and its 5.5PN piece,

$$S^{\le 6\mathrm{PN}}_{\mathrm{nonloc,f}} = S^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}} + S^{5.5\mathrm{PN}}_{\mathrm{nonloc}}, \tag{2.1}$$

where each piece is a time-nonlocal functional of the two worldlines (considered in the center-of-mass frame)

$$S^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}[x_1(s_1), x_2(s_2)] = -\int dt\, H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t), \tag{2.2}$$

and

$$S^{5.5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)] = -\int dt\, H^{5.5\mathrm{PN}}_{\mathrm{nonloc}}(t). \tag{2.3}$$

The two nonlocal Hamiltonians $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t)$ and $H^{5.5\mathrm{PN}}_{\mathrm{nonloc}}(t)$ are given by integrals over a shifted time $t' \equiv t+\tau$. The τ integral entering $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t)$ is logarithmically divergent when $\tau \to 0$, and is defined by introducing a specific (Hadamard Partie finie, Pf) timescale $\Delta t_f = 2r^f_{12}(t)/c$. By contrast, the τ integral entering $H^{5.5\mathrm{PN}}_{\mathrm{nonloc}}(t)$ is convergent when $\tau \to 0$, and therefore involves no regularization scale.

More precisely, the $4+5+6$PN piece reads

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t) = \frac{G\mathcal{M}}{c^3}\mathrm{Pf}_{2r^f_{12}(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t'). \tag{2.4}$$

Here, $\mathcal{M}$ denotes the total Arnowitt-Deser-Misner conserved mass energy of the binary system;

$$r^f_{12}(t) = f(t) r^h_{12}(t), \tag{2.5}$$

is a flexed version of the radial distance between the two bodies ($r^h_{12}(t)$ denoting the harmonic-coordinate distance and $f(t)$ being a function of the instantaneous state of the system), while $\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t')$ is the time-split version of the fractionally 2PN-accurate gravitational-wave energy flux (absorbed and) emitted by the (conservative) system.

On the other hand, the 5.5 PN Hamiltonian is given by the following nonlocal (second-order tail) expression

$$H^{5.5\mathrm{PN}}_{\mathrm{nonloc}}(t) = \frac{B}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \int_{-\infty}^{\infty}\frac{d\tau}{\tau}[\mathcal{G}^{\mathrm{split}}(t,t+\tau) - \mathcal{G}^{\mathrm{split}}(t,t-\tau)], \tag{2.6}$$

with $B=-\frac{107}{105}$. Similarly to the first-order tail effect entering $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t)$, this action involves a time-split bilinear function of the multipole moments that is closely linked to the gravitational-wave flux, namely

$$\mathcal{G}^{\mathrm{split}}(t,t') = \frac{G}{5c^5} I^{(3)}_{ij}(t) I^{(4)}_{ij}(t') + \ldots. \tag{2.7}$$

At the present 6PN accuracy, it is enough to use the leading-order version of the time-split function $\mathcal{G}^{\mathrm{split}}(t,t')$, obtained by keeping only the quadrupolar contribution (neglecting higher multipole terms), and by evaluating $I_{ij}(t)$ at the Newtonian level.

Up to the 7PN-accuracy included, each piece of the nonlocal action can be treated as a first-order perturbation of the (local) 3PN dynamics, and their contributions to the scattering angle can be treated separately, and then linearly added together.

The $4+5+6$PN nonlocal Hamiltonian can be further decomposed into its purely harmonic, unflexed contribution $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}$ [defined by using $\Delta t_h = 2r^h_{12}(t)/c$ as Pf scale], and a contribution $\Delta^{\mathrm{f-h}}H(t)$ proportional to $\ln f(t)$:

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t) = H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}} + \Delta^{\mathrm{f-h}}_{5+6\mathrm{PN}}H(t). \tag{2.8}$$

Replacing $\mathcal{M} = \frac{E_{\mathrm{tot}}}{c^2} = \frac{H}{c^2}$ where H is the (2PN-accurate, as needed for the present computation) Hamiltonian, and introducing an intermediate length scale s, we have

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) = -\frac{GH}{c^5}\mathrm{Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t+\tau) + 2\frac{GH}{c^5}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right), \tag{2.9}$$

and

$$\Delta^{\mathrm{f-h}}_{5+6\mathrm{PN}}H(t) = +2\frac{GH}{c^5}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t)\ln(f(t)). \tag{2.10}$$

A. Scattering angle

As already mentioned the "f-route" local Hamiltonian $H_{\mathrm{loc,f}}$ is defined so that

$$H_{\mathrm{tot}} = H_{\mathrm{loc,f}} + H_{\mathrm{nonloc,f}}, \tag{2.11}$$

where $H_{\mathrm{nonloc,f}}$ is defined by Eqs. (2.8)–(2.10). References [2,3] have determined $H_{\mathrm{loc,f}}$ at the 6PN accuracy. In order to complete the derivation of the f-route 6PN dynamics we need to compute the h-route nonlocal part of the scattering angle, say $\chi^{\mathrm{nonloc,h}}$, at the 6PN accuracy, i.e., at order $\frac{1}{c^{\leq 12}}$, and at the 6PM accuracy, i.e., at order $G^{\leq 6}$. Indeed, it is the ν dependence of $\chi^{\mathrm{nonloc,h}}$ which constrains the additional, f-dependent contribution $\chi^{\mathrm{f-h}}$ needed to render $\chi^{\mathrm{nonloc,f}} = \chi^{\mathrm{nonloc,h}} + \chi^{\mathrm{f-h}}$ compatible with the particular ν dependence of χ^{tot} pointed out in Ref. [7].

The leading-order (LO) contribution to $\chi^{\mathrm{nonloc,f}}$ is at the 4PN and 4PM levels (i.e., of order $\frac{G^4}{c^8}$). In view of the PN and PM scalings of p_∞ and $\frac{1}{j}$ recalled above, this means that the LO contribution to $\chi^{\mathrm{nonloc,f}}$ starts by a contribution of order $\frac{p_\infty^4}{j^4}$. Beyond this order, we can [see Eq. (1.28), which concerned $\chi^{\mathrm{nonloc,h}}$] write an expansion for $\chi^{\mathrm{nonloc,f}}$ of the type

$$\frac{1}{2}\chi^{\mathrm{nonloc,f}}(p_\infty, j; \nu) = \nu\frac{p_\infty^4}{j^4}\left(A_0(p_\infty;\nu) + \frac{A_1(p_\infty;\nu)}{p_\infty j} + \frac{A_2(p_\infty;\nu)}{(p_\infty j)^2} + \cdots\right), \tag{2.12}$$

where $A_0(p_\infty;\nu)$, $A_1(p_\infty;\nu)$, $A_2(p_\infty;\nu)$, etc., are further PN-expanded in powers of p_∞. Namely,

$$\begin{aligned} A_0(p_\infty;\nu) &= A_0^{\mathrm{N}} + \eta^2 A_0^{1\mathrm{PN}} + \eta^4 A_0^{2\mathrm{PN}} + \cdots, \\ A_1(p_\infty;\nu) &= A_1^{\mathrm{N}} + \eta^2 A_1^{1\mathrm{PN}} + \eta^3 A_1^{1.5\mathrm{PN}} + \eta^4 A_1^{2\mathrm{PN}} + \cdots, \\ A_2(p_\infty;\nu) &= A_2^{\mathrm{N}} + \eta^2 A_2^{1\mathrm{PN}} + \eta^3 A_2^{1.5\mathrm{PN}} + \eta^4 A_2^{2\mathrm{PN}} + \cdots. \end{aligned} \tag{2.13}$$

Here the label N (standing for Newtonian) denotes a term of order p_∞^0, modulo a $\ln p_\infty$ correction, while the label 1PN (respectively, 1.5PN or 2PN) denotes a term of order p_∞^2 (respectively, p_∞^3 or p_∞^4), modulo $\ln p_\infty$ corrections. [As explained in the Introduction, $\eta(=1)$ is used as a bookkeeping parameter for counting the *fractional* PN orders.] As we shall see the 1.5PN fractional corrections come from the 5.5PN nonlocal action, and only contribute at orders $\frac{1}{j^{\geq 5}}$ (i.e., $G^{\geq 5}$). We recall that the powers of $\frac{1}{j}$ count the powers of G, i.e., the PM order. It should also be noted that the product $p_\infty j$ in the denominators entering Eq. (2.12) scales like c^0, i.e., is of Newtonian order. Actually, at the Newtonian level, the quantity

$$e_{\rm N} \equiv \sqrt{1+p_\infty^2 j^2} \tag{2.14}$$

measures the eccentricity of the hyperbolic trajectory of a scattering motion. The PM expansion in powers of $\frac{1}{j} \sim G$ used in Eq. (2.12) is also a large-eccentricity expansion.

As already explained in the Introduction, the combined PN and PM expansion of $\chi^{\rm nonloc,f}$, Eq. (2.12), will be obtained by computing the various contributions to the integrated nonlocal action,

$$W_{\rm hyp}^{\rm nonloc,f}(p_\infty, j;\nu) = \int_{-\infty}^{\infty} dt H_{\rm nonloc,f}(t), \tag{2.15}$$

and then by differentiating it with respect to j. We can rewrite Eq. (1.10) (setting $c=1$) as

$$\chi^{\rm nonloc,f}(p_\infty, j;\nu) = \frac{1}{GM^2\nu}\frac{\partial W_{\rm hyp}^{\rm nonloc,f}(p_\infty, j;\nu)}{\partial j}. \tag{2.16}$$

In the following, we shall use the shorthand notation[3]

$$\langle H_{\rm nonloc,X}\rangle_\infty \equiv \int_{-\infty}^{\infty} dt\, H_{\rm nonloc,X}(t) \tag{2.17}$$

for the various time-integrated contributions to the nonlocal Hamiltonian (where X is a label for these contributions).

The total nonlocal potential $W_{\rm hyp}^{\rm nonloc,f}(p_\infty, j;\nu) = \langle \sum_X H_{\rm nonloc,X}\rangle_\infty$ is then decomposed as

$$W_{\rm hyp}^{\rm nonloc,f} = W^{\rm tail,h} + W^{\rm tail,f-h} + W^{\rm 5.5PN}, \tag{2.18}$$

where

$$W^{\rm tail,h} \equiv \langle H_{\rm nonloc,h}^{4+5+6{\rm PN}}\rangle_\infty,$$
$$W^{\rm tail,f-h} \equiv \langle \Delta^{\rm f-h} H(t)\rangle_\infty,$$
$$W^{\rm 5.5PN} \equiv \langle H_{\rm nonloc}^{\rm 5.5PN}\rangle_\infty. \tag{2.19}$$

For brevity, we used the label "tail" to denote the $(4+5+6{\rm PN})$ first-order tail contribution of Eq. (2.4), which is proportional to $\frac{G\mathcal{M}}{c^3}$. The second-order tail contribution of Eq. (2.6) [which is proportional to $(\frac{G\mathcal{M}}{c^3})^2$] is simply denoted by the label 5.5PN because it will be evaluated at this accuracy.

In the following sections, we shall successively compute $W^{\rm tail,h}$, $W^{\rm tail,f-h}$, and $W^{\rm 5.5PN}$. Of particular importance will be to control the ν dependence of these quantities. As the split fluxes $\mathcal{F}^{\rm split}(t,t')$ and $\mathcal{G}^{\rm split}(t,t')$ contain an overall factor ν^2 (coming from $I_{ij} = \mu x^{\langle i}x^{j\rangle} + \cdots$, etc.), each contribution to $W^{\rm nonloc,f}$ will contain an overall factor ν^2. (This applies also to $W^{\rm tail,f-h}$ whose role is to compensate some terms in $W^{\rm tail,h}$.) We then see from Eq. (2.16) that $\chi^{\rm nonloc,f}$ contains an overall factor ν^1, which has been factored out in Eq. (2.12). The ν dependence of the coefficients $A_n(p_\infty;\nu)$ entering the large-eccentricity expansion (2.13) will then be generated by the ν dependence of the solution of the hyperbolic motion $x^i(t)$ inserted in the computation of the (ν-dependent) multipole moments $I_{ij}(t)$, etc.

III. COMPUTATION OF $W^{\rm tail,h} \equiv \langle H_{\rm nonloc,h}^{4+5+6{\rm PN}}\rangle_\infty$

Let us start with the computation of the time integral of $H_{\rm nonloc,h}^{4+5+6{\rm PN}}(t)$ along a 2PN-accurate hyperboliclike motion in harmonic coordinates. The time-split version of the fractionally 2PN-accurate gravitational-wave energy flux $\mathcal{F}_{\rm 2PN}^{\rm split}(t,t')$ emitted by the system can be written as

$$\mathcal{F}_{\rm 2PN}^{\rm split}(t,t') = \frac{G}{c^5}[F_{I_2}^{\rm split}(t,t') + \eta^2 F_{I_3,J_2}^{\rm split}(t,t') + \eta^4 F_{I_4,J_3}^{\rm split}(t,t')], \tag{3.1}$$

where

$$F_{I_2}^{\rm split}(t,t') = \frac{1}{5} I_{ab}^{(3)}(t) I_{ab}^{(3)}(t'),$$
$$F_{I_3,J_2}^{\rm split}(t,t') = \frac{1}{189} I_{abc}^{(4)}(t) I_{abc}^{(4)}(t') + \frac{16}{45} J_{ab}^{(3)}(t) J_{ab}^{(3)}(t'),$$
$$F_{I_4,J_3}^{\rm split}(t,t') = \frac{1}{9072} I_{abcd}^{(5)}(t) I_{abcd}^{(5)}(t') + \frac{1}{84} J_{abc}^{(4)}(t) J_{abc}^{(4)}(t'). \tag{3.2}$$

Here $\eta \equiv 1/c$ and the superscript in parenthesis denotes repeated time derivatives. The multipole moments I_L, J_L denote here the values of the canonical moments M_L, S_L

[3]Beware of distinguishing the use of the notation $\langle\cdots\rangle_\infty$ for a hyperbolic-motion *integral* from the use of $\langle\cdots\rangle$ for denoting an elliptic motion *average*.

entering the PN-matched [68–72] multipolar-post-Minkowskian formalism [73], when they are reexpressed as explicit functionals of the instantaneous state of the binary system. These multipole moments parametrize (in a minimal, gauge-fixed way) the exterior gravitational field (and therefore the relevant coupling between the system and a long-wavelength external radiation field).

A. The 2PN-accurate n-polar moments

At the 2PN accuracy, we need the 2PN-accurate value of the quadrupole moment expressed in terms of the material source [74,75]. The other moments (the electric octupole moment I_{ijk}, electric hexadecapole moment, I_{ijkl}, the magnetic quadrupole moment, J_{ij}, and the magnetic octupole moment, J_{ijk}) need only to be known at the 1PN fractional accuracy [69,70,76]. They have the following explicit expressions (in the c.m. harmonic coordinate frame) [77]:

$$\begin{aligned} I_{ij} &= C_1 x_{\langle ij\rangle} + C_2 v_{\langle ij\rangle} + C_3 x_{\langle i}v_{j\rangle}, \\ I_{ijk} &= B_1 x_{\langle ijk\rangle} + B_2 x_{\langle ij}v_{k\rangle} + B_3 x_{\langle i}v_{jk\rangle}, \\ I_{ijkl} &= \nu M(1-3\nu)x_{\langle ijkl\rangle}, \\ J_{ij} &= D_1 L_{\langle i}x_{j\rangle} + D_2 L_{\langle i}v_{j\rangle}, \\ J_{ijk} &= \nu M(1-3\nu)L_{\langle i}x_{jk\rangle}, \end{aligned} \tag{3.3}$$

where the various coefficients (as well as the notation) have been summarized in Table I. [See also Refs. [2,3]. A misprint (v^2 instead of v^4) in the third line of C_1 in Table I of Ref. [3] is corrected here.]

B. The harmonic-coordinate quasi-Keplerian parametrization of the hyperbolic motion

We need also to use the 2PN-accurate dynamics of a binary system in harmonic coordinates [5,6], and the corresponding quasi-Keplerian parametrization [78] of the hyperbolic motion [79] (which we checked against the 2PN equations of motion given in Ref. [80]):

$$\begin{aligned} r &= \bar{a}_r(e_r\cosh v - 1), \\ \ell &= \bar{n}(t-t_P) = e_t \sinh v - v + f_t V + g_t \sin V, \\ \bar{\phi} &= \frac{\phi-\phi_P}{K} = V + f_\phi \sin 2V + g_\phi \sin 3V. \end{aligned} \tag{3.4}$$

Here, we use adimensionalized variables (and $c=1$), notably $r = r^{\rm phys}/(GM)$, $t = r^{\rm phys}/(GM)$, while $V = V(v)$ is given by

$$V = 2\arctan\left[\Omega_{e_\phi}\tanh\frac{v}{2}\right], \tag{3.5}$$

with the notation

$$\Omega_{e_\phi} = \sqrt{\frac{e_\phi+1}{e_\phi-1}}. \tag{3.6}$$

The 2PN-accurate expressions of the orbital parameters $\bar{n}$, $\bar{a}_r$, K, e_t, e_r, e_ϕ, f_t, g_t, f_ϕ, g_ϕ are given in Appendix A as functions of the specific binding energy $\bar{E} \equiv (E_{\rm tot} - Mc^2)/(\mu c^2)$, Eq. (1.36), and of the dimensionless angular momentum $j = cJ/(GM\mu)$, Eq. (1.37), of the system, and in harmonic coordinates (modified harmonic coordinates, according to the notation of Ref. [77]). Note that, as discussed in Ref. [79], the analytic continuation from the ellipticlike to the hyperboliclike case (namely from $\bar{E} < 0$ to $\bar{E} > 0$) cannot be performed in as simple a way at 2PN than at 1PN [81]. As a consequence, the orbital parameters entering the hyperbolic-motion representation (3.4) (notably $\bar{n}$, e_t, f_t, and g_t) are not directly related to the analytic continuation in $\bar{E}$ of the orbital parameters, denoted in a similar way (namely n, e_t, f_t and g_t), entering the elliptic-motion quasi-Keplerian representation.

TABLE I. Coefficients entering the multipolar moments (3.3) used in the 2PN flux. Here, x^i and $v^i \equiv \frac{dx^i}{dt}$ denote the harmonic-coordinate relative center-of-mass position and velocity of a two-body system, whereas $L_i \equiv \epsilon_{ijk}x^j v^k$. We assume $m_1 \le m_2$.

C_1	$1+\eta^2[\frac{29}{42}(1-3\nu)v^2 - \frac{1}{7}(5-8\nu)\frac{GM}{r}]$ $+\eta^4[\frac{GM}{r}v^2(\frac{2021}{756}-\frac{5947}{756}\nu-\frac{4833}{756}\nu^2)+\frac{G^2M^2}{r^2}(\frac{355}{252}-\frac{953}{126}\nu+\frac{337}{252}\nu^2)$ $+v^2(\frac{253}{504}-\frac{1835}{504}\nu+\frac{3545}{504}\nu^2)+\frac{GM}{r}\dot{r}^2(-\frac{131}{756}+\frac{907}{756}\nu-\frac{1273}{756}\nu^2)]$
C_2	$\eta^2 r^2\{\frac{11}{21}(1-3\nu)+\eta^2[\frac{GM}{r}(\frac{106}{27}-\frac{335}{189}\nu-\frac{985}{189}\nu^2)+v^2(\frac{41}{126}-\frac{337}{126}\nu+\frac{733}{126}\nu^2)+\dot{r}^2(\frac{5}{63}-\frac{25}{63}\nu+\frac{25}{63}\nu^2)]\}$
C_3	$2\eta^2 r\dot{r}\{-\frac{2}{7}+\frac{6}{7}\nu+\eta^2[v^2(-\frac{13}{63}+\frac{101}{63}\nu-\frac{209}{63}\nu^2)+\frac{GM}{r}(-\frac{155}{108}+\frac{4057}{756}\nu+\frac{209}{108}\nu^2)]\}$
B_1	$\sqrt{1-4\nu}\{-1+\eta^2[\frac{GM}{r}(\frac{5}{6}-\frac{13}{6}\nu)+v^2(-\frac{5}{6}+\frac{19}{6}\nu)]\}$
B_2	$\sqrt{1-4\nu}(1-2\nu)\eta^2 r\dot{r}$
B_3	$-\sqrt{1-4\nu}(1-2\nu)\eta^2 r^2$
D_1	$\sqrt{1-4\nu}\{-1+\eta^2[\frac{GM}{r}(-\frac{27}{14}-\frac{15}{7}\nu)+v^2(-\frac{13}{28}+\frac{17}{7}\nu)]\}$
D_2	$\sqrt{1-4\nu}r\dot{r}(-\frac{5}{28}-\frac{5}{14}\nu)\eta^2$

C. $W^{\rm tail,h}$ along hyperbolic orbits: Computational details

Consider the h-route nonlocal Hamiltonian in units $G=1=c$

$$H^{4+5+6\rm PN}_{\rm nonloc,h}(t)=-H_{\rm tot}{\rm Pf}_{2s/c}\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')+2H_{\rm tot}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right),\qquad(3.7)$$

where $\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')$ was defined above. We need to compute the integral of $H^{4+5+6\rm PN}_{\rm nonloc,h}(t)$ along a 2PN-accurate hyperbolic motion:

$$W^{\rm tail,h}(E,j)=\int_{-\infty}^{\infty}H^{4+5+6\rm PN}_{\rm nonloc,h}(t)dt.\qquad(3.8)$$

Following the decomposition, displayed in Eq. (3.7), of $H^{4+5+6\rm PN}_{\rm nonloc,h}(t)$ in two terms, we correspondingly decompose $W^{\rm tail,h}(E,j)$ in two integrals, namely

$$W^{\rm tail,h}(E,j)=W^{\rm tail,h}_1(E,j)+W^{\rm tail,h}_2(E,j),\qquad(3.9)$$

where

$$W^{\rm tail,h}_1(E,j)\equiv-H_{\rm tot}\int_{-\infty}^{\infty}dt\,{\rm Pf}_{2s/c}\times\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t'),\qquad(3.10)$$

while

$$W^{\rm tail,h}_2(E,j)\equiv2H_{\rm tot}\int_{-\infty}^{\infty}dt\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right).\qquad(3.11)$$

Let us consider first the term $W^{\rm tail,h}_1$. A crucial role is played by the measure

$$d\mathcal{M}_{(t,t')}\equiv\frac{dtdt'}{|t-t'|}.\qquad(3.12)$$

In order to compute the double integral ${\rm Pf}\int d\mathcal{M}_{(t,t')}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')$, it is useful to replace the integral over t and t' by an integral over the variables

$$T\equiv\tanh\frac{v}{2};\qquad T'\equiv\tanh\frac{v'}{2},\qquad(3.13)$$

where v is the hyperbolic eccentric anomaly entering the quasi-Keplerian parametrization of the 2PN hyperbolic motion given above. This change of variables maps the original integration domain $(t,t')\in\mathbb{R}\times\mathbb{R}$ onto the compact domain $(T,T')\in[-1,1]\times[-1,1]$. It also transforms the singular line $t=t'$ into $T=T'$, together with a transformation of the constant cutoff $|t'-t|=2s/c$ implied by the Pf operation into a corresponding T-dependent cutoff (see below).

We succeeded in computing, with 2PN accuracy, the first three terms in the large-eccentricity expansion of $W^{\rm tail}$, i.e.,

$$W^{\rm tail,h}_1=W_1^{\rm tail,h\,LO}+W_1^{\rm tail,h\,NLO}+W_1^{\rm tail,h\,NNLO}+O(e_r^{-6}),\qquad(3.14)$$

where we used the fact that the leading order (LO) term $W_1^{\rm tail,h\,LO}$ starts at order $O(e_r^{-3})$ (see below). At the LO, and the next-to-leading order (NLO) in $\frac{1}{e_r}$ (and $\frac{1}{p_\infty j}$), both integrals in T' (with Pf) and in T can be analytically performed. At the next-to-next-to-leading order (NNLO) in $\frac{1}{e_r}$, we could explicitly write down the integrand to be integrated, but we could only analytically compute part of the integral, and we had to resort to numerical integration to evaluate the rest. During the various computational steps we keep as fundamental eccentricity e_r, but, at the end, we express the final result in terms of an expansion in powers of $\frac{1}{p_\infty j}$ [as in Eq. (2.13)]. Some details follow.

The 2PN-exact relation t vs T is given by

$$\frac{t^{\rm phys}}{M}\equiv t=\frac{2}{\bar n}\left[e_t\frac{T}{(1-T^2)}-{\rm arctanh}(T)+f_t\arctan(\Omega_{e_\phi}T)+g_t\frac{\Omega_{e_\phi}T}{1+\Omega^2_{e_\phi}T^2}\right],\qquad(3.15)$$

with a corresponding expression for t' vs T'. One then forms $|t-t'|$, whose eccentricity expansion reads

$$|t-t'|=|T-T'|\frac{1+TT'}{(1-T^2)(1-T'^2)}\bar a_r^{3/2}e_r\times\left[2-(1+2\nu)\frac{\eta^2}{\bar a_r}+\frac{8\nu^2-8\nu-1}{4}\frac{\eta^4}{\bar a_r^2}\right]\times\left[1+\frac{1}{e_r}\mathcal{P}_1+\frac{1}{e_r^2}\mathcal{P}_2+O\left(\frac{1}{e_r^3}\right)\right],\qquad(3.16)$$

with $\mathcal{P}_1$ and $\mathcal{P}_2$ of the form

$$\mathcal{P}_1=\mathcal{P}_{10}(T,T')+\mathcal{P}_{12}(T,T')\frac{\eta^2}{\bar a_r}+\mathcal{P}_{14}(T,T')\frac{\eta^4}{\bar a_r^2},$$
$$\mathcal{P}_2=\mathcal{P}_{24}(T,T')\frac{\eta^4}{\bar a_r^2}.\qquad(3.17)$$

The coefficients $\mathcal{P}_{nm}(T,T')$ entering $\mathcal{P}_1$ and $\mathcal{P}_2$ read

$$\mathcal{P}_{10}(T,T') = -\frac{(1-T'^2)(1-T^2)}{(TT'+1)(T-T')}K(T,T'),$$
$$\mathcal{P}_{12}(T,T') = -\frac{1}{8}\frac{(1-T'^2)(1-T^2)(12\nu-32)}{(TT'+1)(T-T')}K(T,T'),$$
$$\mathcal{P}_{14}(T,T') = -\frac{1}{8}\frac{(1-T'^2)(1-T^2)}{(TT'+1)(T-T')}(3\nu^2-29\nu)K(T,T') + \frac{1}{8}\frac{(-15+\nu)\nu(TT'-1)(1-T'^2)(1-T^2)}{(1+T'^2)(1+T^2)(TT'+1)},$$
$$\begin{aligned}\mathcal{P}_{24}(T,T') = &-\frac{3}{2}\frac{(1-T'^2)(1-T^2)(-5+2\nu)}{(TT'+1)(T-T')}\kappa(T,T')\\ &-\frac{1}{8(1+T'^2)^2(1+T^2)^2}(16-2\nu^2T^2+\nu^2T^4-2\nu^2T'^2+\nu^2T'^4-172\nu T^2T'^2\\ &-26\nu T^2T'^4-26\nu T^4T'^2-43\nu T^4T'^4-43\nu T^4-4\nu^2TT'-26\nu T'^2\\ &-43\nu T'^4-26\nu T^2+60\nu TT'+4\nu^2T^3T'+4\nu^2TT'^3-4\nu^2T^3T'^3+\nu^2-43\nu+64T^2T'^2\\ &+32T^2T'^4+32T^4T'^2+16T^4T'^4+32T^2+16T^4+32T'^2+16T'^4+4\nu^2T^2T'^2-2\nu^2T^2T'^4\\ &-2\nu^2T^4T'^2+\nu^2T^4T'^4-60\nu TT'^3-60\nu T^3T'+60\nu T^3T'^3).\end{aligned} \tag{3.18}$$

Here, we used the notation

$$\kappa(T,T') \equiv \arctan(T) - \arctan(T'),$$
$$K(T,T') \equiv \mathrm{arctanh}(T) - \mathrm{arctanh}(T'). \tag{3.19}$$

These relations imply for the reexpression of the measure, Eq. (3.12), in the $T-T'$ plane,[4]

$$d\mathcal{M}_{(T,T')} = \frac{1}{|t(T)-t'(T')|}\frac{dt}{dT}\frac{dt'}{dT'}dTdT', \tag{3.20}$$

the following (schematic) expression

$$\begin{aligned}d\mathcal{M}_{(T,T')} = &\,2e_r\bar{a}_r^{3/2}\left[1-\frac{1+2\nu}{2\bar{a}_r}\eta^2-\frac{1+8\nu-8\nu^2}{8\bar{a}_r^2}\eta^4\right]\\ &\times\frac{(1+T'^2)(1+T^2)dTdT'}{(1-T'^2)(1-T^2)(1+TT')|T-T'|}\\ &\times\left(1+\frac{\mathcal{M}_1}{e_r}+\frac{\mathcal{M}_2}{e_r^2}+O\left(\frac{1}{e_r^3}\right)\right),\end{aligned} \tag{3.21}$$

where we have explicitly shown only the LO contribution in the large-eccentricity expansion. The NLO and NNLO contributions [described by the coefficients $\mathcal{M}_1(T,T';\nu,\eta)$ and $\mathcal{M}_2(T,T';\nu,\eta)$] have large expressions that we do not explicitly display here. Let us simply note that $\mathcal{M}_1(T,T';\nu,\eta)$ involves the function $K(T,T')$ linearly, while $\mathcal{M}_2(T,T';\nu,\eta)$ involves $K(T,T')$, $K^2(T,T')$ and $\kappa(T,T')$ [defined in Eq. (3.19)].

Similarly to the measure $d\mathcal{M}_{(T,T')}$, we expand, in the following, many quantities in inverse powers of the eccentricity e_r. For instance, the first three terms of the large-eccentricity expansion of the 2PN-accurate split-flux integrand $\mathcal{F}^{\rm split}_{\rm 2PN}(T,T')$ will be denoted as

$$\mathcal{F}^{\rm split}_{\rm 2PN}(T,T') = \mathcal{F}^{\rm LO}_{\rm 2PN} + \mathcal{F}^{\rm NLO}_{\rm 2PN} + \mathcal{F}^{\rm NNLO}_{\rm 2PN} + \cdots. \tag{3.22}$$

In the following, we reserve the notation LO, NLO, NNLO to the first three terms in expansion in e_r^{-1}. Note that each term in this expansion is itself PN-expanded in powers of $\eta = \frac{1}{c}$ up to the 2PN fractional accuracy, so that we have (for $n = 0, 1, 2$)

$$\mathcal{F}^{\rm N^nLO}_{\rm 2PN} = \mathcal{F}^{\rm N^nLO}_0 + \eta^2\mathcal{F}^{\rm N^nLO}_2 + \eta^4\mathcal{F}^{\rm N^nLO}_4 + O(\eta^6). \tag{3.23}$$

The LO term in the eccentricity expansion of $\mathcal{F}^{\rm split}_{\rm 2PN}(T,T')$ is of order e_r^{-4}. Therefore, the full structure of the double expansion in $\eta = \frac{1}{c}$ and in e_r^{-1} of the split-flux reads

$$\mathcal{F}^{\rm split}_{\rm 2PN}(T,T') = \sum_{k=0}^{2}\sum_{m=4}^{6}\eta^{2k}e_r^{-m}\mathcal{F}_{(2k,-m)}, \tag{3.24}$$

where $k = 0, 1, 2$ counts the (fractional) PN order, while $m = 4, 5, 6$ indicates the eccentricity order. Let us note in passing that the 1PN terms $\mathcal{F}_{(2,-m)}$ are linear in ν, while the 2PN ones $\mathcal{F}_{(4,-m)}$ are quadratic in ν.

The structure of the expansion coefficients $\mathcal{F}_{(2k,-m)} \equiv \mathcal{F}_{(2k,-m)}(T,T')$ is described in Table II. The explicit expressions of the polynomials $P_N^{(n,-m)}(T,T')$, appearing as coefficients in Table II, are given in Table III for the Newtonian-level case ($n = 0$). All these polynomials are either symmetric in T, T', or antisymmetric when they appear multiplied by $\kappa(T,T')$.

[4] Note that the measure $d\mathcal{M}_{(T,T')}$ is a symmetric function of T and T'.

TABLE II. Structure of the terms $\mathcal{F}_{(n,-m)} \equiv \mathcal{F}_{(\eta^n, e_r^{-m})}$ entering the large-eccentricity expansion of the 2PN-accurate harmonic coordinate flux, Eq. (3.24). Here, $P_N^{(n,-m)}(T,T')$ denotes an N-degree polynomial in T and T' entering $\mathcal{F}_{(n,-m)}$.

$\mathcal{F}_{(0,-4)}$	$\frac{32}{15}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^5(1+T^2)^5(1+T'^2)^5} P_{12}^{(0,-4)}(T,T')$
$\mathcal{F}_{(0,-5)}$	$-\frac{64}{5}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^5(1+T^2)^6(1+T'^2)^6} [(TT')^2-1] P_{12}^{(0,-5)}(T,T')$
$\mathcal{F}_{(0,-6)}$	$\frac{32}{5}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^5(1+T^2)^7(1+T'^2)^7} P_{20}^{(0,-6)}(T,T')$
$\mathcal{F}_{(2,-4)}$	$-\frac{2}{105}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^6(1+T^2)^7(1+T'^2)^7} P_{20}^{(2,-4)}(T,T')$
$\mathcal{F}_{(2,-5)}$	$\frac{16}{105}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^6(1+T^2)^8(1+T'^2)^8} P_{24}^{(2,-5)}(T,T')$
$\mathcal{F}_{(2,-6)}$	$-\frac{4}{105}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^6(1+T^2)^9(1+T'^2)^9} [P_{27}^{(2,-6)}(T,T')\kappa(T,T') + P_{28}^{(2,-6)}(T,T')]$
$\mathcal{F}_{(4,-4)}$	$-\frac{4}{315}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^7(1+T^2)^9(1+T'^2)^9} P_{26}^{(4,-4)}(T,T')$
$\mathcal{F}_{(4,-5)}$	$\frac{8}{945}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^7(1+T'^2)^{10}(1+T^2)^{10}} P_{32}^{(4,-5)}(T,T')$
$\mathcal{F}_{(4,-6)}$	$\frac{2}{2835}\nu^2 \frac{(1-T^2)^2(1-T'^2)^2}{\bar{a}_r^7(1+T^2)^{11}(1+T'^2)^{11}} [P_{35}^{(4,-6)}(T,T')\kappa(T,T') + P_{36}^{(4,-6)}(T,T')]$

TABLE III. Explicit expressions of the various polynomials $P_N^{(n,-m)}(T,T')$ parametrizing the structure displayed in Table II in the Newtonian limit $n=0$. We have checked the symmetry property of these polynomials when exchanging T and T'.

$P_{12}^{(0,-4)}(T,T')$	sym	$(-3-15T'^4+3T'^6+15T'^2)T^6+(37T'^5+37T'-52T'^3)T^5+(-75T'^2-15T'^6+15+75T'^4)T^4$ $+(-52T'-52T'^5+76T'^3)T^3+(75T'^2-75T'^4-15+15T'^6)T^2+(37T'^5+37T'-52T'^3)T$ $+3-15T'^2+15T'^4-3T'^6$
$P_{12}^{(0,-5)}(T,T')$	sym	$(-10-22T'^4+3T'^6+25T'^2)T^6+(-90T'^3+51T'^5+69T')T^5+(-122T'^2-22T'^6+25+131T'^4)T^4$ $+(-90T'-90T'^5+120T'^3)T^3+(131T'^2-122T'^4-22+25T'^6)T^2+(69T'^5+51T'-90T'^3)T$ $+3-22T'^2+25T'^4-10T'^6$
$P_{20}^{(0,-6)}(T,T')$	sym	$(20T'^{10}-2+26T'^2-182T'^4+290T'^6-200T'^8)T^{10}$ $+(-128T'^3+21T'-1024T'^7+904T'^5+443T'^9)T^9$ $+(-910T'^6+262T'^2+1420T'^8-200T'^{10}+26-134T'^4)T^8$ $+(-1024T'^9-128T'+1120T'^5-1700T'^3+228T'^7)T^7$ $+(-134T'^2-910T'^8+2388T'^4-1740T'^6-182+290T'^{10})T^6$ $+(904T'^9+1120T'^7-1728T'^5+1120T'^3+904T')T^5$ $+(-182T'^{10}+2388T'^6-134T'^8-1740T'^4-910T'^2+290)T^4$ $+(-1024T'+1120T'^5-128T'^9+228T'^3-1700T'^7)T^3$ $+(-200-134T'^6+262T'^8+1420T'^2-910T'^4+26T'^{10})T^2$ $+(904T'^5+21T'^9-1024T'^3-128T'^7+443T')T$ $+20+290T'^4-182T'^6-200T'^2+26T'^8-2T'^{10}$

Multiplying $d\mathcal{M}_{(T,T')}$ and $\mathcal{F}^{\rm 2PN}(T,T')$ yields an integrand that we denote as

$$d\mathcal{M}_{(T,T')}\mathcal{F}_{\rm 2PN}^{\rm split}(T,T') = \mathcal{G}(T,T')\frac{dTdT'}{|T-T'|}, \qquad (3.25)$$

where the (2PN-accurate) function $\mathcal{G}(T,T')$ is expanded only up to the (fractional) second order in e_r^{-1}, say

$$\mathcal{G}(T,T') = \mathcal{G}^{\rm LO}(T,T') + \mathcal{G}^{\rm NLO}(T,T') + \mathcal{G}^{\rm NNLO}(T,T') + \cdots. \qquad (3.26)$$

As before, the notation LO, NLO, NNLO refers to the expansion in powers of e_r^{-1} (starting at order e_r^{-3} and extending up to order e_r^{-5}).

The function $\mathcal{G}(T,T')$ [which should not be confused with the time-split function entering Eq. (2.6)] is symmetric in T and T'. In addition, we recall that each term in Eq. (3.26) is itself PN-expanded with fractional 2PN accuracy, so that $\mathcal{G}^{\rm N^n\,LO} = \mathcal{G}_0^{\rm N^n\,LO} + \eta^2\mathcal{G}_2^{\rm N^n\,LO} + \eta^4\mathcal{G}_4^{\rm N^n\,LO} + O(\eta^6)$.

The original integral was singular at $t=t'$, i.e., along the bisecting line of the $t-t'$ plane. This singular line becomes the bisecting line in the plane $T-T'$, but endowed with a T-dependent slit, which is identified from the relation $dT = f(T)dt$. Here, the Jacobian $f(T) = dT/dt$ admits also an expansion in powers of e_r^{-1}, say $f(T) = f^{\rm LO}(T) + f^{\rm NLO}(T) + f^{\rm NNLO}(T) + O(e_r^{-4})$, where, for example, the LO term in the expansion in e_r^{-1} reads

$$f^{\rm LO}(T) = \frac{\bar{n}}{2e_r}\frac{(1-T^2)^2}{1+T^2}. \tag{3.27}$$

Note that, because of our use of the quasi-Keplerian representation, this Newtonian-looking $O(e_r^{-1})$ expression is PN exact. The fractional PN corrections only enter the higher-order corrections in e_r^{-1}.

As explained in Ref. [2], when considering the partie-finie integral giving the first part $W_1^{\rm tail,h}$ of $W^{\rm tail,h}$, Eq. (3.9), the width of the slit around the bisecting line $T = T'$ defined by the partie-finie procedure is $dT = f(T)\epsilon$ where ϵ is initially considered as being infinitesimal, before replacing it by the finite value $2s/c$ at the end. This leads to the following partie-finie integral

$$\begin{aligned}\mathcal{I}(T) &= {\rm Pf}_{2sf(T)/c}\int_{-1}^{1} dT' \frac{\mathcal{G}(T,T')}{|-T'+T|} \\ &= \int_{-1}^{1} dT' \frac{\mathcal{G}(T,T') - \mathcal{G}(T,T)}{|-T'+T|} \\ &\quad + \mathcal{G}(T,T)\left[\int_{-1}^{T-\epsilon f(T)} \frac{dT'}{T-T'} + \int_{T+\epsilon f(T)}^{1} \frac{dT'}{T'-T}\right]_{\epsilon\mapsto 2s/c}.\end{aligned} \tag{3.28}$$

Defining

$$q(T,T') = \mathcal{G}(T,T') - \mathcal{G}(T,T), \tag{3.29}$$

the above expression can be rewritten as

$$\begin{aligned}\mathcal{I}(T) &= \int_{-1}^{1} dT' \frac{q(T,T')}{|-T'+T|} \\ &\quad + \mathcal{G}(T,T)\ln\left(\frac{1-T^2}{\epsilon^2 f^2(T)}\right),\end{aligned} \tag{3.30}$$

where $\epsilon = 2s/c$. Note that $q(T,T')$ is not [contrary to $\mathcal{G}(T,T')$] a symmetric function of T and T'. As we are going to integrate $\mathcal{I}(T)$ over a (T,T')-symmetric domain, one could replace $q(T,T')$ by its symmetric part.

Further integration in T gives

$$\begin{aligned}\mathcal{J} &\equiv \int_{-1}^{1}\mathcal{I}(T)dT \\ &= \int_{-1}^{1} dT \int_{-1}^{1} dT' \frac{q(T,T')}{|-T'+T|} - 2\ln\left(\frac{2s}{c}\right)\int_{-1}^{1} dT\mathcal{G}(T,T) \\ &\quad + \int_{-1}^{1} dT\mathcal{G}(T,T)\ln\left(\frac{1-T^2}{f^2(T)}\right).\end{aligned} \tag{3.31}$$

So far, we only discussed the first term $W_1^{\rm tail,h}$ in $W^{\rm tail}$. The second term is easier to evaluate because it is given by a simple integral, namely (using as above $G = 1 = c$)

$$W_2^{\rm tail,h} = 2H_{\rm tot}\int_{-\infty}^{+\infty} dt\mathcal{F}_{\rm 2PN}^{\rm split}(t,t)\ln\left(\frac{r_{12}^h(t)}{s}\right). \tag{3.32}$$

Again, we use the quasi-Keplerian representation of the hyperbolic motion, and therefore replace the integration with respect to t by an integration with respect to T, using the explicit functional link $t = F(T)$ given in Eq. (3.15).

Because of the presence of the logarithm of $r_{12}^h(t)$, the integral $W_2^{\rm tail,h}$, Eq. (3.32), cannot be computed as an exact function of the energy and angular momentum. However, like $W_1^{\rm tail,h}$, one can compute the first three terms in its expansion in powers of e_r^{-1}, i.e.,

$$\begin{aligned}W_2^{\rm tail,h} &= W_2{}^{\rm tail,h\,LO} + W_2{}^{\rm tail,h\,NLO} \\ &\quad + W_2{}^{\rm tail,h\,NNLO} + O(e_r^{-6}).\end{aligned} \tag{3.33}$$

The LO term $W_2{}^{\rm tail,h\,LO}$ starts at order $O(e_r^{-3})$.

In the next subsection we illustrate the results of the computation of $W^{\rm tail,h}$.

D. Value of $W^{\rm tail,h}$ up to the NNLO in e_r^{-1} and at the fractional 2PN accuracy

Let us recap the methodology used in the present section. We have been considering the time integral of the first-order-tail harmonic, nonlocal Hamiltonian,

$$\begin{aligned}W^{\rm tail,h} &= \int_{-\infty}^{+\infty} dt H_{\rm nonloc,h}^{4+5+6{\rm PN}}(t) \\ &= W_1^{\rm tail,h} + W_2^{\rm tail,h},\end{aligned} \tag{3.34}$$

where the decomposition in the two contributions, $W_1^{\rm tail,h}$ and $W_2^{\rm tail,h}$ was defined in Eqs. (3.9)–(3.11).

Using the quasi-Keplerian representation of the hyperbolic motion, we could compute the first three terms in the large-e_r expansion of $W^{\rm tail,h}$, say

$$\begin{aligned}W^{\rm tail,h} &= W^{\rm tail,h\,LO} + W^{\rm tail,h\,NLO} \\ &\quad + W^{\rm tail,h\,NNLO} + O(e_r^{-6}),\end{aligned} \tag{3.35}$$

where each contribution, $W^{\rm tail,h\,N^n\,LO}$, is itself computed as a fractionally 2PN-accurate expression:

$$\begin{aligned}W^{\rm tail,h\,N^n\,LO} &= W_{\eta^0}^{\rm tail,h\,N^n\,LO} + \eta^2 W_{\eta^2}^{\rm tail,h\,N^n\,LO} \\ &\quad + W_{\eta^4}^{\rm tail,h\,N^n\,LO} + O(\eta^6).\end{aligned} \tag{3.36}$$

The 2PN-accurate values of the two contributions to $W^{\rm tail,h}$ at the LO in e_r^{-1} are easily computed. They read, respectively,

$$W_1^{\text{tail,h LO}} = \frac{2}{15}\frac{\pi M\nu^2}{e_r^3\bar{a}_r^{7/2}}H_{\text{tot}}\left\{100 + 37\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right) + \left[\frac{685}{4} - \frac{1017}{14}\nu + \left(\frac{3429}{56} - \frac{37}{2}\nu\right)\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]\frac{\eta^2}{\bar{a}_r}\right.$$
$$\left. + \left[\frac{3656939}{8064} - \frac{18181}{72}\nu + \frac{235453}{4032}\nu^2 + \left(\frac{114101}{672} - \frac{7055}{112}\nu + \frac{111}{8}\nu^2\right)\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]\frac{\eta^4}{\bar{a}_r^2}\right\}, \tag{3.37}$$

and

$$W_2^{\text{tail,h LO}} = \frac{2}{15}\frac{\pi M\nu^2}{e_r^3\bar{a}_r^{7/2}}H_{\text{tot}}\left\{-\frac{85}{4} - 37\ln\left(\frac{s}{2e_r\bar{a}_r}\right) + \left[-\frac{9679}{224} + \frac{981}{56}\nu + \left(-\frac{3429}{56} + \frac{37}{2}\nu\right)\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\right]\frac{\eta^2}{\bar{a}_r}\right.$$
$$\left. + \left[-\frac{1830565}{16128} + \frac{54899}{1152}\nu - \frac{29969}{4032}\nu^2 + \left(-\frac{114101}{672} + \frac{7055}{112}\nu - \frac{111}{8}\nu^2\right)\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\right]\frac{\eta^4}{\bar{a}_r^2}\right\}. \tag{3.38}$$

It is easily seen that the intermediate scale s cancels between the two contributions. The same will hold at the NLO and NNLO levels.

The extension of these results to the NLO level is significantly more involved (especially for the 1PN and 2PN corrections). In fact, though the first integration over T' in the split-flux integrals is found to be relatively straightforward [in spite of the presence of the transcendental functions $K(T,T')$ and $\kappa(T,T')$, Eq. (3.19)], the subsequent integration over T is significantly more difficult because of the presence of polylogarithmic functions. However, we have been able to analytically compute the 2PN-accurate NLO value of $W_1^{\text{tail,h}}$. The 2PN-accurate NLO values of $W_1^{\text{tail,h}}$ and $W_2^{\text{tail,h}}$ read, respectively,

$$W_1^{\text{tail,h NLO}} = \frac{2}{15}\frac{M\nu^2}{e_r^4\bar{a}_r^{7/2}}H_{\text{tot}}\left\{\frac{2224}{9} + \frac{1568}{3}\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right) + \left[-\frac{28072}{225} - \frac{38872}{63}\nu + \left(\frac{944}{105} - \frac{1136}{3}\nu\right)\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right)\right]\frac{\eta^2}{\bar{a}_r}\right.$$
$$\left. + \left[-\frac{67489874}{77175} - \frac{3115726}{3675}\nu + \frac{165086}{315}\nu^2 + \left(\frac{419036}{735} - \frac{3244}{7}\nu + \frac{764}{3}\nu^2\right)\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right)\right]\frac{\eta^4}{\bar{a}_r^2}\right\}, \tag{3.39}$$

and

$$W_2^{\text{tail,h NLO}} = \frac{2}{15}\frac{M\nu^2}{e_r^4\bar{a}_r^{7/2}}H_{\text{tot}}\left\{\frac{2768}{9} - \frac{1568}{3}\ln\left(\frac{2s}{e_r\bar{a}_r}\right) + \left[-\frac{64904}{225} - \frac{5992}{45}\nu + \left(-\frac{944}{105} + \frac{1136}{3}\nu\right)\ln\left(\frac{2s}{e_r\bar{a}_r}\right)\right]\frac{\eta^2}{\bar{a}_r}\right.$$
$$\left. + \left[-\frac{2925494}{77175} - \frac{542014}{2205}\nu + \frac{145498}{735}\nu^2 + \left(-\frac{419036}{735} + \frac{3244}{7}\nu - \frac{764}{3}\nu^2\right)\ln\left(\frac{2s}{e_r\bar{a}_r}\right)\right]\frac{\eta^4}{\bar{a}_r^2}\right\}. \tag{3.40}$$

By contrast, at the NNLO level, we encountered integrals of the type

$$\int_{-1}^{1} dT \int_{-1}^{1} \frac{dT'}{|T-T'|}[f_2(T,T')K^2(T,T') + f_1(T,T')K(T,T') + f_0(T,T')], \tag{3.41}$$

where $K(T,T')$ is defined in Eq. (3.19), and where the $f_j(T,T')$ are rather complicated rational functions of T and T'. For instance, even at the Newtonian level (i.e., the lowest order in η) we had to deal with the integrand $q^{\text{N}}(T,T') = f_2^{\text{N}}(T,T')K^2(T,T') + f_1^{\text{N}}(T,T')K(T,T') + f_0^{\text{N}}(T,T')$ with rational coefficients $f_2^{\text{N}}(T,T')$, $f_1^{\text{N}}(T,T')$ and $f_0^{\text{N}}(T,T')$ given by

$$f_2^{\text{N}}(T,T') = \frac{64}{15}\frac{(1-T'^2)^3(1-T^2)^3}{(1+T^2)^4(1+T'^2)^4(T-T')^2(TT'+1)^3}P_{10}^{f_2}(T,T'),$$
$$f_1^{\text{N}}(T,T') = -\frac{128}{15}\frac{(1-T^2)^2(1-T'^2)^2(TT'-1)}{(1+T'^2)^5(1+T^2)^5(T-T')(TT'+1)}P_{12}^{f_1}(T,T'),$$
$$f_0^{\text{N}}(T,T') = -\frac{64}{15}\frac{(1-T^2)}{(1+T'^2)^6(1+T^2)^7(TT'+1)}P_{24}^{f_0}(T,T'), \tag{3.42}$$

where the polynomials $P_{10}^{f_2}(T,T')$, $P_{12}^{f_1}(T,T')$, and $P_{24}^{f_0}(T,T')$ are displayed in Table IV. We had to resort to numerical integration to evaluate some terms.

Our final results for the 2PN-accurate NNLO values of $W_1^{\rm tail,h}$ and $W_2^{\rm tail,h}$ read, respectively,

$$
\begin{aligned}
W_1^{\rm tail,h\,NNLO} = {} & \frac{2}{15}\frac{\pi M\nu^2}{e_r^5\bar{a}_r^{7/2}}H_{\rm tot}\Bigg\{\frac{5997}{8}+\frac{2529}{4}\ln(2)-\frac{15}{2}c_{00}+\frac{843}{2}\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\\
& +\Bigg[-\frac{400845}{448}-\frac{200997}{224}\ln(2)-\frac{15}{2}c_{20}+\left(-\frac{51711}{112}-\frac{5481}{8}\ln(2)-\frac{15}{2}c_{21}\right)\nu\\
& +\left(-\frac{66999}{112}-\frac{1827}{4}\nu\right)\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\Bigg]\frac{\eta^2}{\bar{a}_r}\\
& +\Bigg[-\frac{165424487}{96768}-\frac{442237}{896}\ln(2)-\frac{15}{2}c_{40}+\left(\frac{22110289}{16128}+\frac{86205}{64}\ln(2)-\frac{15}{2}c_{41}\right)\nu\\
& +\left(-\frac{321757}{896}+\frac{13491}{32}\ln(2)-\frac{15}{2}c_{42}\right)\nu^2+\left(-\frac{442237}{1344}+\frac{28735}{32}\nu+\frac{4497}{16}\nu^2\right)\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\Bigg]\frac{\eta^4}{\bar{a}_r^2}\Bigg\}, \qquad (3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
W_2^{\rm tail,h\,NNLO} = {} & \frac{2}{15}\frac{\pi M\nu^2}{e_r^5\bar{a}_r^{7/2}}H_{\rm tot}\Bigg\{-\frac{3419}{8}-\frac{843}{2}\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\\
& +\left[\frac{103645}{448}+\frac{56559}{112}\nu+\left(\frac{66999}{112}+\frac{1827}{4}\nu\right)\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\right]\frac{\eta^2}{\bar{a}_r}\\
& +\left[\frac{2467109}{13824}-\frac{3706175}{5376}\nu-\frac{1577635}{8064}\nu^2+\left(\frac{442237}{1344}-\frac{28735}{32}\nu-\frac{4497}{16}\nu^2\right)\ln\left(\frac{s}{2e_r\bar{a}_r}\right)\right]\frac{\eta^4}{\bar{a}_r^2}\Bigg\}. \qquad (3.44)
\end{aligned}
$$

The coefficients c_{00}, c_{20}, c_{21}, c_{40}, c_{41}, and c_{42} entering $W_1^{\rm tail,h\,NNLO}$ have been numerically computed. Our estimates of their values are listed in Table V. From some numerical studies (increasing the working precision used in the computation), and by comparing with the exact values of c_{00} and c_{21} given below, we estimate that the latter values have an absolute numerical error of order 1×10^{-8}. We accordingly cite eight digits after the decimal point.

TABLE IV. Polynomial expressions entering the Newtonian level integral of Eq. (3.42).

$P_{10}^{f_2}(T,T')$	$3-15T^2+15T^4-3T^6-15T'^2+15T'^4-3T'^6+37TT'+75T^4T'^4-15T^6T'^4-75T'^4T^2+15T'^6T^2$ $-75T^4T'^2+75T'^2T^2-15T^4T'^6+3T^6T'^6+76T^3T'^3-52T^3T'-52T^5T'^3+15T^6T'^2+37T^5T'+37T^5T'^5$ $-52T'^3T-52T^3T'^5+37T'^5T]$
$P_{12}^{f_1}(T,T')$	$6-51T^2+60T^4-27T^6-51T'^2+60T'^4-27T'^6+116TT'+318T^4T'^4-51T^6T'^4-291T'^4T^2$ $+60T'^6T^2-291T^4T'^2+318T'^2T^2-51T^4T'^6+6T^6T'^6+284T^3T'^3-218T^3T'-218T^5T'^3+60T^6T'^2$ $+170T^5T'+116T^5T'^5-218T'^3T-218T^3T'^5+170T'^5T$
$P_{24}^{f_0}(T,T')$	$15-411T^7T'^{13}+411T^5T'^{13}-207T^3T'^{13}+42TT'^{13}+207T^9T'^{13}-42T^{11}T'^{13}+117T^2$ $+180T^4-579T^6+558T^8+630T'^2-303T'^4+1836T'^6+153T'^8-712TT'+12636T^4T'^4-9357T^6T'^4$ $+651T'^4T^2-4836T'^6T^2+1716T^4T'^2-4050T'^2T^2-1920T^4T'^6-8220T^6T'^6-4365T^2T'^8$ $+10356T'^8T^4-2973T'^8T^6-1086T'^4T^8+14280T'^6T^8+432T'^2T^8-3366T^8T'^8$ $+60T'^{12}T^4+438T^8T'^{12}+27T^{12}T'^{12}-102T^{10}+3T^{12}+318T'^{10}+39T'^{12}-147T^2T'^{12}-243T'^{12}T^6$ $-366T'^{12}T^{10}+630T^{10}T'^4+106T^{11}T'^3+4788T^9T'^3-68T^{11}T'+513T^9T'+272T'^9T-6157T^7T'^5$ $-12292T^7T'^7-876T'^2T^{10}-144T'^6T^{10}+2556T'^{10}T^{10}-4650T'^{10}T^6-618T'^{10}T^2+3056T'^7T+326T^7T'^3$ $-1963T^7T'-66T^{12}T'^2+477T^{12}T'^4-4911T^9T'^5-2846T^{11}T'^5-3181T^7T'^9-4386T^{10}T'^8+1992T^8T'^{10}$ $+2895T^9T'^9+9804T^9T'^7+933T^{12}T'^8-3428T^{11}T'^9+3088T^{11}T'^7-996T^{12}T'^6-2626T^7T'^{11}-48T^9T'^{11}$ $-378T^{12}T'^{10}+502T^{11}T'^{11}+278T'^{11}T+8212T^5T'^7+3876T^3T'^7+3276T^4T'^{10}+3373T^5T'^9$ $-6651T^3T'^9+4018T^5T'^{11}-1548T^3T'^{11}-1032T^3T'^3+1083T^3T'-518T^5T'^3-282T^6T'^2+571T^5T'$ $+10237T^5T'^5+3050T'^3T-8769T^3T'^5-3298T'^5T$

TABLE V. Detailed results from numerical integration. See, however, below for the exact values of c_{00} and c_{21}.

Coefficient	Numerical value
c_{00}	−49.20484109
c_{20}	+115.95128578
c_{21}	+161.90919858
c_{40}	+22.31105671
c_{41}	−116.85535736
c_{42}	−209.81006553

E. Computation of $W^{\rm tail,h}(E,j)$ in the frequency domain

So far, we have discussed the direct time-domain approach to the computation of the integrated tail action $W^{\rm tail,h\,NNLO}$. It was shown in Ref. [52] that $W^{\rm tail,h}$ has a simple expression in the frequency-domain. Let us now briefly discuss the method we used to tackle, in parallel, the computation of $W^{\rm tail,h}(E,j)$ in the frequency domain. The use of this method allowed us to go beyond the results obtained by the direct time-domain approach presented in the previous subsection. In particular, we succeeded in analytically computing two of the NNLO integrals entering $W_1^{\rm tail,h\,NNLO}$ (namely c_{00} and c_{21}) by working in the frequency domain, while we could not compute them in the time domain. There remain four other integrals (one at the 1PN level, and three at the 2PN level) in $W_1^{\rm tail,h\,NNLO}$ that we were still unable to compute analytically.

Let us start by presenting the analytical values we obtained for the two parameters c_{00} and c_{21}, which could only be numerically estimated in the time domain, but which could be obtained in the frequency domain. Namely,

$$\begin{aligned}c_{00}&=\frac{1039}{60}+\frac{843}{10}\ln(2)-\frac{2079}{20}\zeta(3)\\&=-49.20484109556976971676344738 34\ldots\end{aligned}\tag{3.45}$$

and

$$\begin{aligned}c_{21}&=-\frac{1827}{20}\ln(2)+\frac{21867}{280}+\frac{612}{5}\zeta(3)\\&=161.909198574011907946235225245\ldots.\end{aligned}\tag{3.46}$$

Note the remarkable presence of $\zeta(3)$ in these expressions. These analytical results are in agreement (within $\pm1\times10^{-8}$) with the numerical ones listed in Table V.

Let us now sketch our frequency-domain approach, relegating most details to Appendixes B and C. We recall that the tail potential $W^{\rm tail,h}(E,j)$, Eq. (3.8), computed along hyperbolic motion, can be split into two terms, Eq. (3.9), namely $W_1^{\rm tail,h}(E,j)$, Eq. (3.10), and $W_2^{\rm tail,h}(E,j)$, Eq. (3.11). Both $W_1^{\rm tail,h}(E,j)$ and $W_2^{\rm tail,h}(E,j)$ can be evaluated in the frequency domain, after Fourier-transforming the various multipolar moments. This frequency-domain approach turns out to be more convenient in the case of $W_1^{\rm tail,h}(E,j)$ because the logarithmic term in $W_2^{\rm tail,h}(E,j)$ complicates matters.

The first step is to Fourier transform[5] the multipolar moments. For example,

$$I_{ab}(t)=\int\frac{d\omega}{2\pi}e^{-i\omega t}\hat{I}_{ab}(\omega),\tag{3.47}$$

where

$$\hat{I}_{ab}(\omega)=\int\frac{dt}{dv}e^{i\omega t(v)}I_{ab}(t)|_{t=t(v)}dv,\tag{3.48}$$

with the associated PN expansion

$$\hat{I}_{ab}(\omega)=\hat{I}^{\rm N}_{ab}(\omega)+\eta^2\hat{I}^{\rm 1PN}_{ab}(\omega)+\eta^4\hat{I}^{\rm 2PN}_{ab}(\omega)+O(\eta^6).\tag{3.49}$$

The PN expansion[6] of the exponential term $e^{i\omega t(v)}$ gives

$$e^{i\omega t(v)}=e^{q\sinh v-pv}(1+b_2\eta^2+b_4\eta^4),\tag{3.50}$$

where

$$u\equiv\omega e_r\bar{a}_r^{3/2};\qquad q\equiv iu;\qquad p\equiv\frac{q}{e_r}=i\frac{u}{e_r},\tag{3.51}$$

and

$$\begin{aligned}b_2&=-\frac{i\omega\sqrt{\bar{a}_r}}{2}[(2\nu+1)e_r\sinh v+(\nu-9)v],\\b_4&=\frac{i\omega}{8(e_r^2-1)\sqrt{\bar{a}_r}}\\&\quad\times\left[b_{40}+b_{41}\arctan\left(\sqrt{\frac{e_r-1}{e_r+1}}\tanh\frac{v}{2}\right)\right],\end{aligned}\tag{3.52}$$

with

$$\begin{aligned}b_{40}&=(e_r^2-1)\left[-4i\frac{\sqrt{\bar{a}_r}}{\omega}b_2^2+v(\nu^2+11\nu-15)\right.\\&\quad\left.-\nu(\nu-15)e_r\frac{\sinh v}{e_r\cosh v-1}\right]+12v(4-7\nu)\\&\quad+8e_r\left[\left(\nu^2-\nu-\frac{1}{8}\right)e_r^2-\nu^2+\frac{9}{2}\nu-\frac{15}{8}\right]\sinh v,\\b_{41}&=-48\sqrt{e_r^2-1}\left(\nu-\frac{5}{2}\right).\end{aligned}\tag{3.53}$$

Moreover,

[5] In the following, we use $GM=1$, i.e., we work with GM-rescaled time and frequency variables.

[6] We take e_r and $\bar{a}_r$, defined in Table II, as fundamental variables.

$$\frac{dt}{dv} = (e_r \cosh v - 1)\bar{a}_r^{3/2} + c_2\eta^2 + c_4\eta^4, \tag{3.54}$$

with

$$c_2 = -\frac{1}{2}\sqrt{\bar{a}_r}[(2\nu+1)e_r\cosh v + \nu - 9],$$
$$c_4 = \frac{1}{8\sqrt{\bar{a}_r}}\Big\{\nu^2 + 11\nu - 15 - e_r\cosh v(1+8\nu-8\nu^2) - 4\frac{4-7\nu}{e_r^2-1}(e_r\cosh v - 3) + \frac{\nu^2-39\nu+60}{e_r\cosh v - 1} + \frac{\nu(15-\nu)(e_r^2-1)}{(e_r\cosh v-1)^2}\Big\}. \tag{3.55}$$

The computation is done by using the integral representation of the Hankel functions of the first kind of order p and argument q [with Eqs. (3.51)]

$$H_p^{(1)}(q) = \frac{1}{i\pi}\int_{-\infty}^{\infty} e^{q\sinh v - pv}dv. \tag{3.56}$$

As the argument $q = iu$ of the Hankel function is purely imaginary, the Hankel function becomes converted into a Bessel K function, according to the relation

$$H_p^{(1)}(iu) = \frac{2}{\pi}e^{-i\frac{\pi}{2}(p+1)}K_p(u). \tag{3.57}$$

Note that the order $p = iu/e_r$ of the Bessel functions is purely imaginary, and proportional to the (frequency-dependent) argument $u = \omega e_r \bar{a}_r^{3/2}$. However, the order p tends to zero when $e_r \to \infty$, which allows some integrals to be explicitly computed when performing a large-eccentricity expansion.

Actually, the computation gives rise to several Bessel functions having the same argument u but various orders differing by integers. However, standard identities valid for Bessel functions allows one to reduce the orders to either p or $p+1$. When taking the large-eccentricity expansion, one then expands with respect to the order of the Bessel functions. This gives rise, at LO, to $K_0(u)$, and $K_1(u)$, and at NLO and NNLO to derivatives of $K_0(u)$, and $K_1(u)$ with respect to their orders. Such an expansion is explicitly shown below in Eqs. (B12), while studying the Newtonian limit, and several useful relations are listed in Appendix B.

Using ($\gamma = 0.577215\ldots$)

$$\mathrm{Pf}_T\int_0^\infty d\tau\frac{\cos\omega\tau}{\tau} = -\ln(|\omega|Te^\gamma), \tag{3.58}$$

we find (see Sec. V of Ref. [52] for details)

$$W_1^{\rm tail,h}(E,j) = \frac{G^2 H_{\rm tot}}{\pi c^5}2\nu^2\int_0^\infty d\omega\mathcal{K}(\omega)\ln\left(\omega\frac{2s}{c}e^\gamma\right), \tag{3.59}$$

where

$$\mathcal{K}(\omega) = \frac{1}{5}\omega^6|\hat{I}_{ab}(\omega)|^2 + \eta^2\left[\frac{\omega^8}{189}|\hat{I}_{abc}(\omega)|^2 + \frac{16}{45}\omega^6|\hat{J}_{ab}(\omega)|^2\right] + \eta^4\left[\frac{\omega^{10}}{9072}|\hat{I}_{abcd}(\omega)|^2 + \frac{\omega^8}{84}|\hat{J}_{abc}(\omega)|^2\right]. \tag{3.60}$$

Here, the frequency-domain multipole moments are also given in a PN-expanded form, e.g.,

$$\hat{I}_{ab}(\omega) = \hat{I}^{\rm N}_{ab}(\omega) + \eta^2\hat{I}^{\rm 1PN}_{ab}(\omega) + \eta^4\hat{I}^{\rm 2PN}_{ab}(\omega) + O(\eta^6). \tag{3.61}$$

The expression, Eq. (3.59), for $W_1^{\rm tail,h}(E,j)$ is closely related to the total energy flux emitted during the scattering process

$$\Delta E_{\rm GW} = \frac{G^2 H_{\rm tot}}{\pi c^5}\nu^2\int_0^\infty d\omega\mathcal{K}(\omega). \tag{3.62}$$

However, it crucially differs from it by the presence of the logarithmic term $\ln(\omega\frac{2s}{c}e^\gamma)$, which is characteristic of the tail in the frequency domain [82].

It is convenient to replace the integration over the frequency ω by an integration over the variable u, using

$$\omega = \frac{u}{e_r\bar{a}_r^{3/2}}. \tag{3.63}$$

The result is the following

$$W_1^{\rm tail,h}(E,j) = \frac{G^2H_{\rm tot}}{\pi c^5}\frac{2\nu^2}{e_r\bar{a}_r^{3/2}}\int_0^\infty du\mathcal{K}(u)\ln(u\alpha), \tag{3.64}$$

with

$$\mathcal{K}(u) = \mathcal{K}(\omega)|_{\omega = u/(e_r\bar{a}_r^{3/2})}, \tag{3.65}$$

and

$$\alpha = \frac{2s}{ce_r\bar{a}_r^{3/2}}e^\gamma. \tag{3.66}$$

The integral in Eq. (3.64) requires special care to be performed, even in the Newtonian limit[7] where

[7]At the Newtonian level, $\eta \to 0$, all eccentricities agree: $e_t = e_r = e_\phi = e$. However, we will continue to denote the eccentricity as e_r to avoid confusion with the exponentials.

$$[\mathcal{K}(\omega)]^{\eta=0} = \frac{1}{5}\omega^6|\hat{I}^{\rm N}_{ab}(\omega)|^2. \tag{3.67}$$

In the Newtonian limit, $\eta \to 0$, we have (in units of $G = c = 1$, but putting back the appropriate power of M)

$$W_1^{\rm tail,h,N}(E,j) = \frac{2}{5}\frac{M^2}{\pi}\frac{\nu^2}{e_r^7\bar{a}_r^{21/2}}\int_0^\infty du \mathcal{I}_{\rm N}(u)\ln(u\alpha) \tag{3.68}$$

with

$$\mathcal{I}_{\rm N}(u) \equiv u^6|\hat{I}^{\rm N}_{ab}(\omega)|^2|_{\omega=u/(e_r\bar{a}_r^{3/2})}. \tag{3.69}$$

At this Newtonian level, $\mathcal{I}_{\rm N}(u)$ can be given a very compact form in terms of Bessel $K_\nu(u)$ functions

$$\mathcal{I}_{\rm N}(u) = \frac{64\nu^2}{p^4}\bar{a}_r^7 u^2 e^{-ip\pi}[AK^2_{p+1}(u) + BK_p(u)K_{p+1}(u) + CK_p^2(u)], \tag{3.70}$$

where $p = iu/e_r$ and

$$A = \frac{u^2}{2}(p^2+u^2)(p^2+u^2+1),$$
$$B = -u(p^2+u^2)\left[\left(p-\frac{3}{2}\right)u^2 + p(p-1)^2\right],$$
$$C = \frac{u^6}{2} + \left(2p^2 - \frac{3}{2}p + \frac{1}{6}\right)u^4 + \left(\frac{5}{2}p^2 - \frac{7}{2}p + 1\right)p^2u^2 + p^4(p-1)^2. \tag{3.71}$$

Indeed, Eq. (3.70) implies that $\mathcal{I}_{\rm N}(u)$ is quadratic in $K_p(u)$, and $K_{p+1}(u)$. Furthermore, p is purely imaginary and enters both the coefficients A, B, C and the order of the Bessel K functions.

However, even at this Newtonian order, the integration variable (u) appears both in the argument (u) and in the order ($p = iu/e_r$) of the Bessel functions. The computation proceeds then by expanding the integrand in the large eccentricity limit, with the useful consequence of removing, at leading order in $\frac{1}{e_r}$, the u dependence from the orders of the Bessel functions, reducing them either to 0 or to 1. At the NLO in $\frac{1}{e_r}$, there appears the first derivative of $K_\nu(u)$ with respect to the order ν, around the two values $\nu = 0$ and $\nu = 1$. Luckily, these first derivatives can be explicitly computed, namely [see Eqs. (9.1.66)–(9.1.68) of Ref. [83]]

$$\left.\frac{\partial K_\nu(u)}{\partial \nu}\right|_{\nu=0} = 0, \qquad \left.\frac{\partial K_\nu(u)}{\partial \nu}\right|_{\nu=1} = \frac{1}{u}K_0(u). \tag{3.72}$$

However, at the NNLO in the $\frac{1}{e_r}$ expansion, there appears the second derivative of $K_\nu(u)$ with respect to the order ν. Though there exist explicit representations for the latter (see Appendix B), they introduce a level of complexity which did not allow us to fully compute the NNLO expansion of $W_1^{\rm tail,h,N}(E,j)$, even at the presently discussed Newtonian level, $\eta \to 0$.

When going beyond the Newtonian level, the Fourier transforms

$$e^{q\sinh v-(p+k)v} \to 2e^{-i\frac{\pi}{2}(p+k)}K_{p+k}(u) \tag{3.73}$$

become replaced by Fourier transforms of $v^n e^{q\sinh v-(p+k)v}$ and $e^{q\sinh v-(p+k)v}V(v)$. The Fourier transforms of $v^n e^{q\sinh v-(p+k)v}$ lead to integrands involving

$$v^n e^{q\sinh v-(p+k)v} \to 2(-1)^n\frac{\partial^n}{\partial p^n}[e^{-i\frac{\pi}{2}(p+k)}K_{p+k}(u)], \tag{3.74}$$

while the Fourier transforms of the terms $e^{q\sinh v-(p+k)v}V(v)$ would require one to work with the large-e_r expansion of the V term [see Eq. (3.5)], i.e.,

$$V(v) = 2\arctan\left(\tanh\frac{v}{2}\right) + \frac{1}{e_r}\tanh v + \frac{\sinh v}{e_r^2\cosh^2 v} + O(e_r^{-3}). \tag{3.75}$$

Unfortunately, we did not find a way to replace the first, $\arctan\left(\tanh\frac{v}{2}\right)$, term by some uniform expansion in v that could be integrated term-by-term. Keeping it as is complicates matters. In some cases, we could overcome this new difficulty by exchanging the order of the v and u integrations, i.e., by integrating with respect to u first. This has allowed us to analytically compute some of the remaining integrals.

More details on our computations are given in Appendixes B and C. From the practical point of view, the main outcome of the frequency-domain approach has been the analytical results (3.45) and (3.46). In addition, this allowed us to analytically compute the first two terms in the large-eccentricity expansion of the 5.5PN integrated action $W_{\rm 5.5PN}$, as discussed below.

IV. FINAL RESULTS FOR THE H-ROUTE TAIL CONTRIBUTION TO THE SCATTERING ANGLE

The results discussed in the previous section are intermediate results towards our real aim which is to compute the nonlocal scattering angle as a function of energy and angular momentum: $\chi^{\rm nonloc,f}(p_\infty, j;\nu)$. In view of Eq. (2.16), the knowledge of $\chi^{\rm nonloc,f}(p_\infty, j;\nu)$ is equivalent to the knowledge of the integrated nonlocal Hamiltonian, $W^{\rm nonloc,f}_{\rm hyp}$ as a function of energy and angular momentum. The total nonlocal potential $W^{\rm nonloc}_{\rm hyp}(p_\infty, j;\nu)$ was decomposed in Eq. (2.18) into three terms,

$$W^{\rm nonloc}_{\rm hyp}(p_\infty, j;\nu) = W^{\rm tail,h} + W^{\rm tail,f-h} + W^{\rm 5.5PN}, \quad (4.1)$$

which were defined in Eq. (2.19). In the present section we finish the discussion of the first contribution, $W^{\rm tail,h} = W_1^{\rm tail,h} + W_2^{\rm tail,h}$.

The expression of $W^{\rm tail,h}$ as a function of energy and angular momentum is obtained (besides adding together $W_1^{\rm tail,h}$ and $W_2^{\rm tail,h}$) from the results described above (which were expressed in terms of the quasi-Keplerian elements $\bar a_r$ and e_r) by reexpressing $\bar a_r$ and e_r in terms of p_∞ and j, using the links given in Appendix A. Introducing [see Eq. (2.14)]

$$e_{\rm N} \equiv e_{\rm N}(p_\infty, j) \equiv \sqrt{1+p_\infty^2 j^2}, \quad (4.2)$$

we have

$$\begin{aligned} e_r &= e_{\rm N} + \frac{1}{2}\frac{p_\infty^2}{e_{\rm N}}[(p_\infty^2 j^2+1)\nu - 4p_\infty^2 j^2 - 6]\eta^2 \\ &\quad - \frac{p_\infty^2}{8j^2 e_{\rm N}^3}[p_\infty^2 j^2 (p_\infty^2 j^2+1)^2\nu^2 \\ &\quad + (p_\infty^2 j^2+1)(9p_\infty^4 j^4 - 91 p_\infty^2 j^2 - 112)\nu \\ &\quad - 32 p_\infty^6 j^6 - 36 p_\infty^4 j^4 + 64 p_\infty^2 j^2 + 64]\eta^4, \\ \bar a_r &= \frac{1}{p_\infty^2} + 2\eta^2 - \frac{(7\nu-4)}{j^2}\eta^4. \end{aligned} \quad (4.3)$$

The use of the new variables p_∞ and j (instead of $\bar a_r$ and e_r) leads to a reshuffling of the large-eccentricity expansion. Indeed, we are actually interested in expanding $W^{\rm tail,h}(p_\infty, j)$ in powers of $\frac{1}{j}$. This changes the meaning of the decomposition in LO, NLO, and NNLO terms. To clarify this change of meaning we write the expansion in powers of $\frac{1}{j}$ as

$$W^{\rm tail,h}(p_\infty, j) = W^{\rm tail,h\,LO_j}(p_\infty, j) + W^{\rm tail,h\,NLO_j}(p_\infty, j) + W^{\rm tail,h\,NNLO_j}(p_\infty, j) + O\left(\frac{1}{j^6}\right), \quad (4.4)$$

where we have added a subscript j to the superscripts N^{n}LO, because we are now referring to an expansion of $W^{\rm tail,h\,LO_j}(p_\infty, j)$ in powers of $\frac{1}{j}$.

For instance, at the leading order in $\frac{1}{j}$, the combination, and reexpression, of Eqs. (3.37) and (3.38) yields the result

$$\begin{aligned} W^{\rm tail,h\,LO_j}(p_\infty, j) &= \frac{2}{15}\frac{M^2\nu^2 p_\infty^4 \pi}{j^3}\left[\frac{315}{4} + 37\ln\left(\frac{p_\infty}{2}\right) + \left[\frac{2753}{224} - \frac{1071}{8}\nu + \left(\frac{1357}{56} - \frac{111}{2}\nu\right)\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^2\eta^2 \right. \\ &\quad \left. + \left[\frac{155473}{1792} - \frac{109559}{8064}\nu + \frac{186317}{1008}\nu^2 + \left(\frac{27953}{672} - \frac{2517}{112}\nu + \frac{555}{8}\nu^2\right)\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^4\eta^4\right]. \end{aligned} \quad (4.5)$$

As we see, the LO$_j$ term is $\propto \frac{1}{j^3}$. Correspondingly, the NLO$_j$ one will be $\propto \frac{1}{j^4}$, and the NNLO$_j$ one $\propto \frac{1}{j^5}$. As per Eq. (2.16), the corresponding contributions to the scattering angle will be, respectively, $\propto \frac{1}{j^4}$, $\propto \frac{1}{j^5}$ and $\propto \frac{1}{j^6}$. Remembering that $\frac{1}{j} = O(G)$ this will give us the value of the scattering angle up to the sixth order in G. We display below the explicit results for the 2PN-accurate scattering angles associated with the LO$_j$, NLO$_j$, and NNLO$_j$ values of $W^{\rm tail,h}$.

Inserting Eq. (4.4) in Eq. (2.16) yields

$$\chi^{\rm nonloc,f} = \chi^{\rm tail,h} + \chi^{\rm f-h} + \chi^{\rm 5.5PN}, \quad (4.6)$$

where the first (h-route) contribution is given by

$$\begin{aligned} \frac{1}{2}\chi^{\rm tail,h} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h} \\ &= \frac{\chi_4^{\rm tail,h}(\gamma,\nu)}{j^4} + \frac{\chi_5^{\rm tail,h}(\gamma,\nu)}{j^5} + \frac{\chi_6^{\rm tail,h}(\gamma,\nu)}{j^6} \\ &\quad + O\left(\frac{1}{j^6}\right), \end{aligned} \quad (4.7)$$

with

$$\begin{aligned} \frac{\chi_4^{\rm tail,h}}{j^4} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,LO_j}, \\ \frac{\chi_5^{\rm tail,h}}{j^5} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,NLO_j}, \\ \frac{\chi_6^{\rm tail,h}}{j^6} &= \frac{1}{2M^2\nu}\frac{\partial}{\partial j}W^{\rm tail,h\,NNLO_j}. \end{aligned} \quad (4.8)$$

As already announced, this yields results for $\chi^{\rm tail,h}$ that can be written as

$$\frac{1}{2}\chi^{\rm tail,h}(p_\infty,j;\nu)=\nu\frac{p_\infty^4}{j^4}\left(A_0^{\rm tail,h}(p_\infty;\nu)+\frac{A_1^{\rm tail,h}(p_\infty;\nu)}{p_\infty j}+\frac{A_2^{\rm tail,h}(p_\infty;\nu)}{(p_\infty j)^2}+\cdots\right), \quad (4.9)$$

where the $\frac{1}{j}$ coefficients $A_0^{\rm tail,h}(p_\infty;\nu)$, $A_1^{\rm tail,h}(p_\infty;\nu)$, $A_2^{\rm tail,h}(p_\infty;\nu)$ are themselves given by a 2PN-accurate expansion in powers of p_∞, say

$$\begin{aligned}
A_0(p_\infty;\nu)&=A_0^{\rm tail,h,N}+A_0^{\rm tail,h,1PN}+A_0^{\rm tail,h,2PN}+\cdots,\\
A_1(p_\infty;\nu)&=A_0^{\rm tail,h,N}+A_1^{\rm tail,h,1PN}+A_1^{\rm tail,h,2PN}+\cdots,\\
A_2(p_\infty;\nu)&=A_2^{\rm tail,h,N}+A_2^{\rm tail,h,1PN}+A_2^{\rm tail,h,2PN}+\cdots.
\end{aligned} \quad (4.10)$$

Note that, in absence of the 5.5PN contribution, we do not have here fractional 1.5PN contributions, as in Eq. (2.13).

The LO_j expressions given above yield

$$\begin{aligned}
A_0^{\rm tail,h,N}&=\pi\left[-\frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)-\frac{63}{4}\right],\\
A_0^{\rm tail,h,1PN}&=\pi\left[\left(-\frac{1357}{280}+\frac{111}{10}\nu\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{2753}{1120}+\frac{1071}{40}\nu\right]p_\infty^2,\\
A_0^{\rm tail,h,2PN}&=\pi\left[\left(-\frac{27953}{3360}+\frac{2517}{560}\nu-\frac{111}{8}\nu^2\right)\ln\left(\frac{p_\infty}{2}\right)-\frac{155473}{8960}+\frac{109559}{40320}\nu-\frac{186317}{5040}\nu^2\right]p_\infty^4.
\end{aligned} \quad (4.11)$$

Our final results for the coefficients of the NLO_j and NNLO_j contributions to the tail part of the scattering angle then read

$$\begin{aligned}
A_1^{\rm tail,h,N}&=-\frac{6656}{45}-\frac{6272}{45}\ln\left(4\frac{p_\infty}{2}\right),\\
A_1^{\rm tail,h,1PN}&=\left[\left(-\frac{74432}{525}+\frac{13952}{45}\nu\right)\ln\left(4\frac{p_\infty}{2}\right)+\frac{114368}{1125}+\frac{221504}{525}\nu\right]p_\infty^2,\\
A_1^{\rm tail,h,2PN}&=\left[\left(-\frac{881392}{11025}+\frac{288224}{1575}\nu-\frac{21632}{45}\nu^2\right)\ln\left(4\frac{p_\infty}{2}\right)+\frac{48497312}{231525}-\frac{5134816}{23625}\nu-\frac{25465952}{33075}\nu^2\right]p_\infty^4
\end{aligned} \quad (4.12)$$

and

$$\begin{aligned}
A_2^{\rm tail,h,N}&=\pi\left[-122\ln\left(\frac{p_\infty}{2}\right)-\frac{1633}{24}+\frac{5}{2}c_{00}-\frac{843}{4}\ln(2)\right],\\
A_2^{\rm tail,h,1PN}&=\pi\left[\left(\frac{44845}{336}+\frac{5199}{8}\ln(2)+\frac{5}{2}c_{21}+\frac{811}{2}\ln\left(\frac{p_\infty}{2}\right)-5c_{00}\right)\nu\right.\\
&\quad\left.-\frac{6379}{192}+\frac{5}{2}c_{20}-\frac{13831}{56}\ln\left(\frac{p_\infty}{2}\right)+\frac{15}{2}c_{00}-\frac{74625}{224}\ln(2)\right]p_\infty^2,\\
A_2^{\rm tail,h,2PN}&=\pi\left[\left(\frac{5}{2}c_{42}-785\ln\left(\frac{p_\infty}{2}\right)-\frac{97127}{6048}+\frac{15}{2}c_{00}-5c_{21}-\frac{39345}{32}\ln(2)\right)\nu^2\right.\\
&\quad+\left(-\frac{2989465}{48384}+\frac{5}{2}c_{41}+\frac{75595}{168}\ln\left(\frac{p_\infty}{2}\right)+\frac{5}{2}c_{21}+\frac{152459}{448}\ln(2)-5c_{20}-\frac{55}{4}c_{00}\right)\nu\\
&\quad\left.+\frac{5}{2}c_{40}+\frac{986233}{1536}+\frac{396481}{2688}\ln(2)+\frac{5}{2}c_{20}+\frac{15}{4}c_{00}+\frac{64579}{1008}\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^4,
\end{aligned} \quad (4.13)$$

respectively. Introducing the new set of parameters

$$\begin{aligned} d_{00} &= \frac{5}{2}c_{00} - \frac{1633}{24} - \frac{843}{4}\ln(2), \\ d_{20} &= \frac{5}{2}c_{20} + \frac{32813}{192} + \frac{66999}{224}\ln(2), \\ d_{21} &= \frac{5}{2}c_{21} - \frac{293}{112} + \frac{1827}{8}\ln(2), \\ d_{40} &= \frac{5}{2}c_{40} + \frac{293499}{512} + \frac{442237}{2688}\ln(2), \\ d_{41} &= \frac{5}{2}c_{41} - \frac{4431841}{48384} - \frac{28735}{64}\ln(2), \\ d_{42} &= \frac{5}{2}c_{42} + \frac{1105777}{6048} - \frac{4497}{32}\ln(2), \end{aligned} \tag{4.14}$$

with numerical values listed in Table VI, the latter expressions can be rewritten as

$$\begin{aligned} A_2^{\rm tail,h,N} &= \pi\left[-122\ln\left(\frac{p_\infty}{2}\right) + d_{00}\right], \\ A_2^{\rm tail,h,1PN} &= \pi\left[\left(d_{21} - 2d_{00} + \frac{811}{2}\ln\left(\frac{p_\infty}{2}\right)\right)\nu \right. \\ &\quad \left. + d_{20} + 3d_{00} - \frac{13831}{56}\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^2, \\ A_2^{\rm tail,h,2PN} &= \pi\left[\left(d_{42} + 3d_{00} - 2d_{21} - 785\ln\left(\frac{p_\infty}{2}\right)\right)\nu^2 \right. \\ &\quad + \left(d_{41} + d_{21} - 2d_{20} - \frac{11}{2}d_{00} \right. \\ &\quad \left. + \frac{75595}{168}\ln\left(\frac{p_\infty}{2}\right)\right)\nu \\ &\quad \left. + d_{40} + \frac{3}{2}d_{00} + d_{20} + \frac{64579}{1008}\ln\left(\frac{p_\infty}{2}\right)\right]p_\infty^4. \end{aligned} \tag{4.15}$$

Among the numerical coefficients entering the NNLO$_j$ quantity $A_2(p_\infty;\nu) = A_2^{\rm tail,h,N} + A_2^{\rm tail,h,1PN} + A_2^{\rm tail,h,2PN}$ two can be written down in analytical form (thanks to our frequency-domain computation), namely, using Eqs. (3.45) and (3.46),

TABLE VI. Numerical values of the coefficients (4.14).

Coefficient	Numerical value
d_{00}	−337.13453770
d_{20}	+668.10143447
d_{21}	+560.45441238
d_{40}	+743.05631726
d_{41}	−694.94788994
d_{42}	−439.10050487

$$d_{00} = -\frac{99}{4} - \frac{2079}{8}\zeta(3) \tag{4.16}$$

and

$$d_{21} = \frac{1541}{8} + 306\zeta(3), \tag{4.17}$$

while the other ones are (only) known numerically (see Table V). (See Ref. [84] for the analytical computation of d_{20}, d_{40}, d_{41}, and d_{42}.)

V. SECOND-ORDER TAIL CONTRIBUTION TO THE INTEGRATED ACTION AND TO THE SCATTERING ANGLE: $W^{5.5\rm PN}$ AND $\chi^{5.5\rm PN}$

Before finishing our discussion of the h-route tail contribution, $\chi^{\rm tail,h}(p_\infty, j;\nu)$, to the scattering angle, and of its ν dependence, let us recall that, at the 6PN accuracy where we are working, the total scattering angle is made of the following four contributions:

$$\chi^{\rm tot}(p_\infty, j;\nu) = \chi^{\rm loc,f} + \chi^{\rm tail,h} + \chi^{5.5\rm PN} + \chi^{\rm f-h}. \tag{5.1}$$

Among these contributions two of them are directly linked with nonlocal effects computed in harmonic coordinates: indeed, $\chi^{\rm tail,h}$ comes from the *first-order tail* (linear in $G\mathcal{M}$), while $\chi^{5.5\rm PN}$ comes from the *second-order tail* (quadratic in $G\mathcal{M}$). Before being able to discuss the constraint that must be satisfied by the flexibility factor $f(t)$ entering the last contribution, $\chi^{\rm f-h}$, we must control the structure (and, notably, the ν dependence) of the quadratic-tail contribution $\chi^{5.5\rm PN}$.

From Eq. (2.6), denoting $B = -\frac{107}{105}$ and

$$\mathcal{H}^{\rm split}(t,\tau) = \frac{G}{5c^5}[I_{ij}^{(3)}(t)I_{ij}^{(4)}(t+\tau) - I_{ij}^{(3)}(t)I_{ij}^{(4)}(t-\tau)], \tag{5.2}$$

the 5.5PN Hamiltonian reads

$$H_{5.5\rm PN}^{\rm nonloc} = \frac{B}{2}\left(\frac{G\mathcal{M}}{c^3}\right)^2 \int_{-\infty}^{\infty}\frac{d\tau}{\tau}\mathcal{H}^{\rm split}(t,\tau). \tag{5.3}$$

Note that the function $\mathcal{H}^{\rm split}(t,\tau)$ is odd in τ, so that $\tau^{-1}\mathcal{H}^{\rm split}(t,\tau)$ is even in τ (and regular at $\tau = 0$).

As usual the computation can be done either in the time domain or in the Fourier domain. Working in the Fourier domain we find

$$W_{5.5\rm PN}^{\rm nonloc} = -B\left(\frac{G\mathcal{M}}{c^3}\right)^2\frac{G}{5c^5}\int_0^\infty d\omega\omega^7|\hat{I}_{ij}(\omega)|^2. \tag{5.4}$$

At our present level of accuracy, it is enough to use the Newtonian approximation to the Fourier transform $\hat{I}_{ij}(\omega)$

of the quadrupole moment. Using the relations given in Appendix B we have then

$$W^{\rm nonloc}_{\rm 5.5PN} = -B\left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{G}{5c^5}\int_0^\infty d\omega\omega^7 |\hat{I}_{ij}(\omega)|^2 = \frac{107}{105}\left(\frac{\bar{n}}{e_r}\right)^8 \left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{G}{5c^5}\int_0^\infty du u \mathcal{I}_{\rm N}(u), \tag{5.5}$$

where the function $\mathcal{I}_{\rm N}(u)$ was defined in Eq. (3.70).

As explained above, though the function $\mathcal{I}_{\rm N}(u)$ is here evaluated at the Newtonian level, it is quadratic in Bessel functions $K_\nu(u)$ whose order is u dependent: either $\nu = p$ or $\nu = p+1$, with $p = iu/e_r$. This makes it impossible to compute $W^{\rm nonloc}_{\rm 5.5PN}$ in closed form. However, it is enough for our purpose to compute the first two terms in the large-eccentricity expansion of the integral (5.5). Thanks to the relations (3.72), this computation only involves integrals containing $K_0(u)$ and $K_1(u)$. We find

$$W^{\rm LO+NLO}_{\rm 5.5PN} = \frac{107}{105}\left(\frac{\bar{n}}{e_r}\right)^8 \left(\frac{G\mathcal{M}}{c^3}\right)^2 \frac{G}{5c^5}\nu^2 32\bar{a}_r^7 e_r^4 \times \int_0^\infty du u \mathcal{F}(u)\left(1+\frac{u}{e_r}+O\left(\frac{1}{e_r^2}\right)\right), \tag{5.6}$$

where LO and NLO refer to the large-eccentricity expansion, and where

$$\mathcal{F}(u) = \left(\frac{u^2}{3}+u^4\right)K_0^2(u) + 3u^3 K_0(u)K_1(u) + (u^2+u^4)K_1^2(u) \tag{5.7}$$

denotes (as in Ref. [52], and in Appendix B) the gravitational-wave energy spectrum in the Newtonian-level “splash” approximation [85,86], i.e., at Newtonian order, and at leading order in the large-eccentricity limit.

The NLO-accurate result $W^{\rm LO+NLO}_{\rm 5.5PN}$ involves the two nontrivial integrals $\mathsf{f}^u = \int_0^\infty du u\mathcal{F}(u)$, and $\mathsf{f}^{u^2} = \int_0^\infty du u^2 \mathcal{F}(u)$. These integrals are computed in Appendix B. This leads to the following explicit NLO result for $W^{\rm nonloc}_{\rm 5.5PN}$ (using $G=1=c$):

$$W^{\rm LO+NLO}_{\rm 5.5PN} = \frac{32}{5}\frac{107}{105}\frac{p_\infty^{10}}{e_r^4}M^2\nu^2\left(\frac{49}{9}+\frac{1}{e_r}\frac{297}{256}\pi^2 + O\left(\frac{1}{e_r^2}\right)\right). \tag{5.8}$$

Replacing the eccentricity in terms of j finally leads to the following explicit NLO$_j$ result for $W^{\rm nonloc}_{\rm 5.5PN}$

$$W^{\rm LO+NLO}_{\rm 5.5PN} = M^2\nu^2\left(\frac{23968}{675}\frac{p_\infty^6}{j^4}+\frac{10593}{1400}\pi^2\frac{p_\infty^5}{j^5} + O\left(\frac{1}{j^6}\right)\right). \tag{5.9}$$

Using the formula (2.16) we finally get

$$\chi^{\rm LO+NLO}_{\rm 5.5PN} = -\nu\left(\frac{95872}{675}\frac{p_\infty^6}{j^5}+\frac{10593}{280}\pi^2\frac{p_\infty^5}{j^6}+O\left(\frac{1}{j^7}\right)\right) = -\nu\frac{95872}{675}\frac{p_\infty^6}{j^5}\left(1+\frac{13365}{50176}\frac{\pi^2}{p_\infty j} + O\left(\frac{1}{j^2}\right)\right). \tag{5.10}$$

VI. ANALYSIS OF THE ν DEPENDENCE OF THE HARMONIC-COORDINATE NONLOCAL SCATTERING ANGLE $\chi^{\rm nonloc,h}$

Let us recall that a crucial tool of our method is to exploit the special ν dependence [7] satisfied by the total scattering angle $\chi^{\rm tot}(p_\infty, j;\nu)$. This structure is embodied in a restricted ν-polynomial dependence of the energy-rescaled PM-expansion coefficients of $\chi^{\rm tot}(p_\infty, j;\nu)$.

The total scattering angle $\chi^{\rm tot}(p_\infty, j;\nu)$ is obtained as a sum of partial contributions, namely

$$\chi^{\rm tot}(p_\infty, j;\nu) = \chi^{\rm loc,f} + \chi^{\rm tail,h} + \chi^{\rm 5.5PN} + \chi^{\rm f-h}. \tag{6.1}$$

Some of these contributions can fail to satisfy the special ν dependence satisfied by $\chi^{\rm tot}(p_\infty, j;\nu)$. One ingredient of our method is to assume that $\chi^{\rm loc,f}$ does satisfy the latter special ν dependence. We must, then, constrain the flexibility factor f in such a way that the complementary, nonlocal-related, contribution

$$\chi^{\rm tail,h} + \chi^{\rm 5.5PN} + \chi^{\rm f-h} \tag{6.2}$$

does satisfy the special ν dependence satisfied by $\chi^{\rm tot}(p_\infty, j;\nu)$. This will be the task of the present section. We will start by recalling what is the special ν structure we are talking about. Then we will measure the extent to which the sum of the two h-route nonlocal contributions, say $\chi^{\rm nonloc,h} \equiv \chi^{\rm tail,h} + \chi^{\rm 5.5PN}$ fails to satisfy the latter special ν dependence. This will finally allow us to constrain $f(t)$.

A. Reminder of the ν rule to be satisfied

Let us define precisely the ν rule to be satisfied. We expand in powers of $\frac{1}{j} = O(G)$ any partial contribution $\chi^X(p_\infty, j;\nu)$ to the total scattering angle, $\chi^{\rm tot} = \sum_X \chi^X$, say

$$\frac{1}{2}\chi^X(p_\infty, j;\nu) = \sum_n \frac{\chi_n^X(p_\infty;\nu)}{j^n}, \tag{6.3}$$

and then define the energy-rescaled coefficients

$$\tilde{\chi}_n^X(p_\infty;\nu) \equiv h^{n-1}(\gamma,\nu)\chi_n^X(p_\infty;\nu), \tag{6.4}$$

where we recall that

$$h(\gamma,\nu) = \sqrt{1+2\nu(\gamma-1)}, \qquad \gamma = \sqrt{1+p_\infty^2\eta^2}, \tag{6.5}$$

that is

$$h = 1 + \frac{1}{2}\nu p_\infty^2\eta^2 - \frac{1}{8}\nu(1+\nu)p_\infty^4\eta^4 + O(\eta^6). \tag{6.6}$$

The special ν structure says that

$$\tilde{\chi}_n^X(p_\infty;\nu) = P_{d(n)}^{X\gamma}(\nu), \tag{6.7}$$

where $P_{d(n)}^{X\gamma}(\nu)$ denotes a polynomial in ν, of degree

$$d(n) \equiv \left[\frac{n-1}{2}\right], \tag{6.8}$$

with γ-dependent coefficients. (Here, $[\cdots]$ denotes the integer part.) Let us analyze the ν structure Eq. (6.7) for the case where the label X is equal to nonloc,h,in the sense of the following definition of the sum of the two harmonic-coordinate nonlocal contributions

$$\chi^{\rm nonloc,h,\,tot} \equiv \chi^{\rm tail,h} + \chi^{\rm 5.5PN}. \tag{6.9}$$

Our results above have led to the determination of $\chi_4^{\rm nonloc,h\,tot}$, $\chi_5^{\rm nonloc,h\,tot}$ and $\chi_6^{\rm nonloc,h\,tot}$. More precisely, we must, according to the definition, Eq. (6.9), add the 5.5PN contribution Eq. (5.10) to the corresponding results for the $\frac{1}{j}$ expansion of $\chi^{\rm tail,h}$ given in Sec. IV.

Let us first remark that, in fact, the 5.5PN contribution $\chi^{\rm 5.5PN}$ *separately satisfies* the rule (6.7). Indeed, as we are at the 5.5PN level, we can use $h\approx 1$ so that $\tilde{\chi}_n^{\rm 5.5PN}(p_\infty;\nu) \approx \chi_n^{\rm 5.5PN}(p_\infty;\nu)$. Then, for the relevant exponents $n=5,6$ of $\frac{1}{j}$, the rule (6.7) says that $\tilde{\chi}_n^{\rm 5.5PN}(p_\infty;\nu)$ should be at most quadratic in ν. However, our explicit results Eq. (5.10) for the $\tilde{\chi}_n^{\rm 5.5PN}$'s show that they are actually *linear* in ν.

In view of this structure of $\chi^{\rm 5.5PN}$, we can henceforth focus only on the remaining h-route contribution to χ, namely $\chi^{\rm tail,h}$. In the following, we shall use the notation

$$\chi^{\rm nonloc,h} \equiv \chi^{\rm tail,h} \tag{6.10}$$

to emphasize that this is the crucial additional h-route contribution to the local piece $\chi^{\rm loc,f}$, to be eventually modified by a suitable f-dependent piece $\chi^{\rm f-h}$.

We transform the results given in Sec. IV for $\chi^{\rm tail,h} = \chi^{\rm nonloc,h}$ into corresponding results for their energy-rescaled versions $\tilde{\chi}_4^{\rm nonloc,h} = h^3\chi_4^{\rm tail,h}$, $\tilde{\chi}_5^{\rm nonloc,h} = h^4\chi_5^{\rm tail,h}$, and $\tilde{\chi}_6^{\rm nonloc,h} = h^5\chi_6^{\rm tail,h}$. We find

$$\begin{aligned}
\tilde{\chi}_4^{\rm nonloc,h} &= \left(-\frac{63}{4} - \frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)\right)\pi\nu p_\infty^4 + \left(-\frac{2753}{1120} - \frac{1357}{280}\ln\left(\frac{p_\infty}{2}\right) + \frac{63}{20}\nu\right)\pi\nu p_\infty^6\eta^2\\
&\quad + \left(-\frac{27331}{10080}\nu^2 + \frac{199037}{40320}\nu - \frac{155473}{8960} - \frac{27953}{3360}\ln\left(\frac{p_\infty}{2}\right)\right)\pi\nu p_\infty^8\eta^4,\\
\tilde{\chi}_5^{\rm nonloc,h} &= \left(-\frac{6656}{45} - \frac{12544}{45}\ln(2) - \frac{6272}{45}\ln\left(\frac{p_\infty}{2}\right)\right)\nu p_\infty^3\\
&\quad + \left[\left(-\frac{148864}{525}\ln(2) + \frac{114368}{1125} - \frac{74432}{525}\ln\left(\frac{p_\infty}{2}\right)\right) + \left(\frac{2816}{45}\ln(2) + \frac{198592}{1575} + \frac{1408}{45}\ln\left(\frac{p_\infty}{2}\right)\right)\nu\right]\nu p_\infty^5\eta^2\\
&\quad + \left[\left(-\frac{13888}{225}\ln(2) - \frac{6944}{225}\ln\left(\frac{p_\infty}{2}\right) + \frac{283168}{4725}\right)\nu - \frac{2448608}{33075}\nu^2\right.\\
&\quad \left. + \left(\frac{48497312}{231525} - \frac{881392}{11025}\ln\left(\frac{p_\infty}{2}\right) - \frac{1762784}{11025}\ln(2)\right)\right]\nu p_\infty^7\eta^4,\\
\tilde{\chi}_6^{\rm nonloc,h} &= \left(-122\ln\left(\frac{p_\infty}{2}\right) + d_{00}\right)\pi\nu p_\infty^2\\
&\quad + \left[\left(-\frac{13831}{56}\ln\left(\frac{p_\infty}{2}\right) + d_{20} + 3d_{00}\right) + \left(\frac{201}{2}\ln\left(\frac{p_\infty}{2}\right) + \frac{1}{2}d_{00} + d_{21}\right)\nu\right]\pi\nu p_\infty^4\eta^2\\
&\quad + \left[\left(-\frac{30655}{336}\ln\left(\frac{p_\infty}{2}\right) + d_{21} + \frac{11}{8}d_{00} + \frac{1}{2}d_{20} + d_{41}\right)\nu + \left(\frac{1}{2}d_{21} + d_{42} - \frac{1}{8}d_{00}\right)\nu^2\right.\\
&\quad \left. + \left(\frac{64579}{1008}\ln\left(\frac{p_\infty}{2}\right) + \frac{3}{2}d_{00} + d_{20} + d_{40}\right)\right]\pi\nu p_\infty^6\eta^4.
\end{aligned} \tag{6.11}$$

We recall that the powers of η in Eqs. (6.11) denote fractional (rather than absolute) PN corrections. Actually, the leading-order contributions to $\tilde{\chi}_4^{\rm nonloc,h}$, $\tilde{\chi}_5^{\rm nonloc,h}$, and $\tilde{\chi}_6^{\rm nonloc,h}$ are all at the 4PN level, so that the η^2 (and η^4) terms denote 5PN (and 6PN) corrections, respectively.

B. On the ν structure of the logarithmic contributions, and of the gravitational-wave energy loss

The rule Eq. (6.7) says that $\tilde{\chi}_4^X$ should be at most linear in ν, while $\tilde{\chi}_5^X$ and $\tilde{\chi}_6^X$ should be at most quadratic in ν.

Let us first note that this rule is satisfied by *all the logarithmic contributions*. This is a nontrivial check of the validity of this rule because all the logarithmic contributions have a genuinely nonlocal origin, and could not be compensated by additional (logarithmic-free) local terms. Let us also note that the simple ν-polynomial structure of the logarithmic (tail) contributions to $\tilde{\chi}_n^{\rm nonloc,h}$ is rather hidden in the structure of the multipole moments and, thereby, in the structure of the total gravitational-radiation energy loss. It is worth pausing a moment to comment more on this structure.

From Eq. (2.9), one sees that the logarithmic contributions to[8] $H^{\rm nonloc,h}$ are proportional to

$$\frac{GH_{\rm tot}}{c^5}\mathcal{F}^{\rm split}(t,t)=\frac{GH_{\rm tot}}{c^5}\mathcal{F}^{\rm GW}(t), \qquad (6.12)$$

where $\mathcal{F}^{\rm GW}(t)$ is the instantaneous flux of gravitational-wave energy. Therefore, the logarithmic contributions to $W^{\rm nonloc,h}=\int dtH^{\rm nonloc,h}$ are proportional to

$$\frac{GH_{\rm tot}}{c^5}\int dt\mathcal{F}^{\rm GW}(t)=\frac{GH_{\rm tot}}{c^5}\Delta E^{\rm GW}, \qquad (6.13)$$

where $\Delta E^{\rm GW}$ denotes the total energy radiated[9] during an hyperbolic encounter. Let us consider the functional dependence of $\Delta E^{\rm GW}$ on γ (or, equivalently, p_∞), j and ν, and the expansion of $\Delta E^{\rm GW}(\gamma,j,\nu)$ in powers of $\frac{1}{j}$:

$$\Delta E^{\rm GW}(\gamma,j;\nu)=\sum_{n\geq3}\frac{\Delta E_n^{\rm GW}(\gamma;\nu)}{j^n}. \qquad (6.14)$$

The logarithmic contributions to $W^{\rm nonloc,h}$ have a $\frac{1}{j}$ expansion proportional to $h\sum_{n\geq3}\frac{\Delta E_n^{\rm GW}(\gamma;\nu)}{j^n}$, so that the logarithmic contributions to $\chi^{\rm nonloc,h}=\frac{1}{M^2\nu}\partial W^{\rm nonloc,h}/\partial j$ are proportional to

$$-\frac{h}{\nu}\sum_{n\geq3}n\frac{\Delta E_n^{\rm GW}(\gamma;\nu)}{j^{n+1}}. \qquad (6.15)$$

In order for the rule Eq. (6.7) to be separately satisfied by the logarithmic contributions to the scattering angle, and taking into account both the factor $\frac{1}{\nu}$ in the previous equation, and the fact that $\Delta E^{\rm GW}\propto\nu^2$, we conclude that the coefficient of $\frac{1}{j^n}$ in the gravitational-radiation loss should satisfy the nontrivial rule

$$h^{n+1}(\gamma,\nu)\frac{\Delta E_n^{\rm GW}(\gamma;\nu)}{\nu^2}=P^\gamma_{[\frac{n-2}{2}]}(\nu)\quad \text{for } n\geq3. \qquad (6.16)$$

We have confirmed the validity of this rule in two different ways.

First, we note that the rule (6.16) states that the leading-order contribution to $\Delta E^{\rm GW}(\gamma,j;\nu)$ in its expansion in powers of $\frac{1}{j}$, i.e., its leading-order PM contribution $\propto\frac{1}{j^3}=O(G^3)$ must depend on ν as $\propto\nu^2/h^4(\gamma,\nu)$. In view of the relation [10]

$$\frac{GM}{b}=\frac{p_\infty}{hj}, \qquad (6.17)$$

between j and the impact parameter b, this is equivalent to saying that

$$h(\gamma,\nu)\Delta E^{\rm GW}(\gamma,b,m_1,m_2)=(m_1m_2)^2\left(\frac{G}{b}\right)^3\mathcal{E}(\gamma)+O(G^4), \qquad (6.18)$$

where the dimensionless factor $\mathcal{E}(\gamma)$ depends only on γ and not on the mass ratio. The validity of this statement to all orders in the PN expansion is a nontrivial fact which follows from the structure of the LO post-Minkowskian gravitational Bremsstrahlung results of Refs. [87,88]. Indeed, the latter references have proven that the LO PM gravitational wave form has three properties: (i) it depends on the masses only through an overall factor $G^2m_1m_2$; (ii) it depends on time through two separate timescales of the form $bf_A(\gamma)$, $bf_B(\gamma)$; and, (iii) it enjoys a forward-backward symmetry in the *center-of-velocity frame* $\tilde{\mathcal{S}}$. These properties imply that the four-momentum $P^\mu_{\rm GW}$ radiated as gravitational waves[10] is of the form

[8]As explained in the previous subsection, we henceforth label as "nonloc, h" the crucial $4+5+6$PN nonlocal contribution, because the second-order tail contribution separately satisfies the constraint we are studying.

[9]Actually, as we are considering a time-symmetric interaction, *à la* Fokker-Wheeler-Feynman, this energy is first absorbed by the system in the form of advanced waves, before being radiated in the form of retarded waves.

[10]As we are discussing the time-symmetric dynamics, the system "emits" both advanced and retarded waves and therefore absorbs $-P^\mu_{\rm GW}$ before emitting $+P^\mu_{\rm GW}$.

TABLE VII. Functions $\hat{E}_n(p_\infty, \bar{\epsilon})$ (for $n = 3, 4, 6$), Eq. (6.24), in terms of $\bar{\epsilon}$ and $B \equiv B(\bar{\epsilon})$, Eqs. (6.23).

$\hat{E}_3(p_\infty, \bar{\epsilon})$	$[(-\frac{64579}{2520}\bar{\epsilon}^2 + \frac{11947909}{3240}\bar{\epsilon}^6 + \frac{19319}{189}\bar{\epsilon}^4 + \frac{5839651}{1008}\bar{\epsilon}^8 + \frac{27953}{5040})B + \frac{5839651}{1008}\bar{\epsilon}^7 + \frac{79675961}{45360}\bar{\epsilon}^5 + \frac{64}{5}\frac{\bar{\epsilon}}{(1+\bar{\epsilon}^2)^2} + \frac{4309531}{136080}\bar{\epsilon}^3 - \frac{8807569}{58800}\bar{\epsilon} + \frac{1060}{7}\frac{\bar{\epsilon}}{(1+\bar{\epsilon}^2)}]p_\infty^{11} + [(\frac{13831}{140}\bar{\epsilon}^2 + \frac{13447}{20}\bar{\epsilon}^6 + \frac{2259}{4}\bar{\epsilon}^4 + \frac{1357}{420})B + \frac{31509}{700}\bar{\epsilon} - \frac{64}{5}\frac{\bar{\epsilon}}{(1+\bar{\epsilon}^2)} + \frac{13447}{20}\bar{\epsilon}^5 + \frac{10219}{30}\bar{\epsilon}^3]p_\infty^9 + [(\frac{244}{5}\bar{\epsilon}^2 + \frac{74}{15} + \frac{170}{3}\bar{\epsilon}^4)B + \frac{1346}{45}\bar{\epsilon} + \frac{170}{3}\bar{\epsilon}^3]p_\infty^7$
$\hat{E}_4(p_\infty, \bar{\epsilon})$	$[(-\frac{62813}{168}\bar{\epsilon}^3 - \frac{1628347}{360}\bar{\epsilon}^5 - \frac{258051}{40}\bar{\epsilon}^7 + \frac{6131}{168}\bar{\epsilon})B - \frac{16546}{105(1+\bar{\epsilon}^2)} - \frac{427097}{180}\bar{\epsilon}^4 - \frac{5912419}{37800}\bar{\epsilon}^2 + \frac{280502}{1575} - \frac{258051}{40}\bar{\epsilon}^6 - \frac{64}{5(1+\bar{\epsilon}^2)^2}]p_\infty^{11} + [(-\frac{1127}{3}\bar{\epsilon}^5 - \frac{910}{3}\bar{\epsilon}^3 - \frac{201}{5}\bar{\epsilon})B - \frac{128}{9} + \frac{32}{5(1+\bar{\epsilon}^2)} - \frac{1127}{3}\bar{\epsilon}^4 - \frac{1603}{9}\bar{\epsilon}^2]p_\infty^9$
$\hat{E}_6(p_\infty, \bar{\epsilon})$	$[(\frac{5929}{6}\bar{\epsilon}^3 + \frac{485}{4}\bar{\epsilon} + \frac{5481}{4}\bar{\epsilon}^5)B + \frac{2966}{45} + \frac{5481}{4}\bar{\epsilon}^4 - \frac{578}{15(1+\bar{\epsilon}^2)} + \frac{6377}{12}\bar{\epsilon}^2 - \frac{16}{5(1+\bar{\epsilon}^2)^2}]p_\infty^{11}$

$$P^\mu_{\rm GW} = (m_1 m_2)^2 \left(\frac{G}{b}\right)^3 \mathcal{E}(\gamma)\frac{u_1^\mu + u_2^\mu}{\gamma + 1} + O(G^4), \quad (6.19)$$

where u_1^μ, u_2^μ denote the incoming 4-velocities. Computing from Eq. (6.19) the *center-of-mass* energy loss $\Delta E^{\rm GW} = -P^\mu_{\rm GW}(p_{1\mu} + p_{2\mu})/|p_1 + p_2|$ [where $p_{a\mu} = m_a u_{a\mu}$ and $|p_1 + p_2| = Mh(\gamma, \nu)$] leads to Eq. (6.18).

Second, we have computed $\Delta E^{\rm GW}$ to the 2PN accuracy (thereby generalizing the 1PN-accurate result of Blanchet and Schäfer [89]). We give in Appendix D, Eqs. (D2) and (D3), the 2PN-level contribution to $\Delta E^{\rm GW}$ when (following Ref. [89]) it is expressed in terms of $e_r = e_r^h$ and j. However, expressing $\Delta E_{\rm GW}$ in terms of e_r and j does not help to reveal its hidden simple ν dependence because e_r^h is itself a rather involved function of $\bar{E} \equiv (E_{\rm tot} - Mc^2)/\mu = (h(\gamma, \nu) - 1)/\nu$, j and ν given by

$$e_r^h = \sqrt{1 + 2\bar{E}j^2}(1 + e_2\eta^2 + e_4\eta^4), \quad (6.20)$$

where

$$e_2 = \frac{\bar{E}}{2(2\bar{E}j^2 + 1)}[(5\bar{E}j^2 + 2)\nu - 15\bar{E}j^2 - 12],$$
$$e_4 = \frac{\bar{E}}{8(2\bar{E}j^2 + 1)^2 j^2}[\bar{E}^2 j^4(7\bar{E}j^2 + 4)\nu^2 + (-210\bar{E}^3 j^6 + 224 + 792\bar{E}j^2 + 592\bar{E}^2 j^4)\nu + 415\bar{E}^3 j^6 + 200\bar{E}^2 j^4 - 280\bar{E}j^2 - 128]. \quad (6.21)$$

It is better to reexpress $\Delta E_{\rm GW}$ in terms of γ (or equivalently p_∞) and of the (gauge-invariant) eccentricity-like variable[11]

$$e_{hj}^2 \equiv 1 + (\gamma^2 - 1)h^2 j^2 = 1 + p_\infty^2 h^2 j^2, \quad (6.22)$$

and of the related quantities

$$\bar{\epsilon} \equiv \frac{1}{p_\infty hj} = \frac{1}{\sqrt{e_{hj}^2 - 1}},$$
$$B(\bar{\epsilon}) \equiv \frac{\pi}{2} + \arctan\bar{\epsilon} = \arccos\left(-\frac{1}{e_{hj}}\right). \quad (6.23)$$

Note that j enters these quantities only in the combination $h(\gamma, \nu)j$. This leads to a 2PN-accurate result of the form

$$h(\gamma, \nu)\frac{\Delta E^{\rm GW}(p_\infty, \bar{\epsilon}; \nu)}{M\nu^2} = \bar{\epsilon}^3 \hat{E}_3(p_\infty, \bar{\epsilon}) + \nu\bar{\epsilon}^4 \hat{E}_4(p_\infty, \bar{\epsilon}) + \nu^2\bar{\epsilon}^6 \hat{E}_6(p_\infty, \bar{\epsilon}) + O(p_\infty^{13}), \quad (6.24)$$

where the functions $\hat{E}_n(p_\infty, \bar{\epsilon})$, with $n = 3, 4, 6$, are given in Table VII.

These functions have a smooth limit as $\bar{\epsilon} \to 0$ (equivalent to $e_{hj} \to \infty$, or $j \to \infty$), i.e.,

$$\hat{E}_n(p_\infty, \bar{\epsilon}) = \hat{E}_{n0}(p_\infty) + \bar{\epsilon}\hat{E}_{n1}(p_\infty) + \bar{\epsilon}^2 \hat{E}_{n2}(p_\infty) + \cdots. \quad (6.25)$$

The error term $O(p_\infty^{13})$ in (6.24) indicates a fractional 3PN error level. Indeed, the leading PN contribution to $\hat{E}_3(p_\infty, \bar{\epsilon})$ is $O(p_\infty^7)$ (corresponding to the large-eccentricity Newtonian-level energy loss $\sim\bar{\epsilon}^3 p_\infty^7 \sim e_{hj}^4 j^{-7}$). It is then easily checked that the properties embodied in the expansion of the expression (6.24) in powers of $\bar{\epsilon} = 1/(p_\infty hj)$ implies that the expansion coefficients of $h\Delta E^{\rm GW}(\gamma, j; \nu)/\nu^2$ in powers of $\frac{1}{j}$ satisfy the rule Eq. (6.16).

Let us also note that there is a simple link between the total gravitational-wave energy loss along a hyperbolic motion, and the gravitational-wave energy loss during one radial period of an elliptic motion, namely

$$\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j) = \Delta E_{\rm GW}^{\rm hyperbolic}(\gamma, j) - \Delta E_{\rm GW}^{\rm hyperbolic}(\gamma, -j). \quad (6.26)$$

This result is obtained by analytically continuing the quasi-Keplerian representation of the hyperbolic motion used above [79] back to the elliptic-motion case (expressing all

[11] The quantity e_{hj} is a PN-acceptable measure of the eccentricity in the range $0 < e_{hj} < \infty$, and the value $e_{hj} = 1$ does describe parabolic motions (with zero binding energy). However, e_{hj}^2 does not vanish along circular orbits.

quantities in terms of γ and j and analytically continuing γ from $\gamma^{\rm hyperbolic} > 1$ to $\gamma^{\rm elliptic} < 1$). The result (6.26) is consistent with the analytic-continuation link between the scattering angle and the periastron precession [37], as is easily seen in view of the link [52] used above between the tail contribution to the scattering angle and the time integral of the gravitational-wave energy loss. The functional structure of $\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j)$ is much simpler than that of $\Delta E_{\rm GW}^{\rm hyperbolic}(\gamma, j)$. In particular the arccos, or arctan, factors present in $\Delta E_{\rm GW}^{\rm hyperbolic}(\gamma, j)$ are simply replaced by π in $\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j)$. Finally, $\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j)$ has a polynomial structure in p_∞ and $\bar{\epsilon}$.

Note also that our rule (6.16) about the special ν dependence of the hyperbolic gravitational-wave energy loss implies, via the link (6.26), that the same property should be satisfied by the elliptic gravitational-wave energy loss. In view of the odd dependence of $\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j)$ on j (and therefore $\bar{\epsilon}$) displayed in Eq. (6.26), this transforms the result (6.24) into

$$\begin{aligned}\frac{h(\gamma,\nu)\Delta E_{\rm GW}^{\rm elliptic}(p_\infty,\bar{\epsilon};\nu)}{M\nu^2} &= \bar{\epsilon}^3 p_\infty^7[P_2(\bar{\epsilon}^2) + p_\infty^2 P_3(\bar{\epsilon}^2)\\ &\quad + p_\infty^4 P_4(\bar{\epsilon}^2)]\\ &\quad + \nu\bar{\epsilon}^5 p_\infty^9[P_2(\bar{\epsilon}^2) + p_\infty^2 P_3(\bar{\epsilon}^2)]\\ &\quad + \nu^2\bar{\epsilon}^7 p_\infty^{11} P_2(\bar{\epsilon}^2) + O(p_\infty^{13}),\end{aligned} \tag{6.27}$$

where each $P_n(\bar{\epsilon}^2)$ denotes a different polynomial of order n in $\bar{\epsilon}^2$.

We have checked (using the elliptic 3PN results of Refs. [77,90]) that the remarkable constraint on the ν dependence of $\Delta E_{\rm GW}^{\rm elliptic}(\gamma, j; \nu)$ displayed in Eqs. (6.16) and (6.27) is satisfied at the 3PN level [with the evident generalization of the structure (6.27)]. In particular, the $O(\nu^3)$ contribution is of the form $\nu^3\bar{\epsilon}^9 p_\infty^{13} P_2(\bar{\epsilon}^2)$.

C. Contributions to $\tilde{\chi}_n^{\rm nonloc,h}$ violating the special ν structure

Let us now highlight the relatively small number of contributions to $\tilde{\chi}_n^{\rm nonloc,h} = h^{n-1}\chi_n^{\rm nonloc,h}$ that do not satisfy the rule (6.7) by separating them from those that satisfy the rule:

$$\begin{aligned}\tilde{\chi}_4^{\rm nonloc,h} &= [\tilde{\chi}_4^{\rm nonloc,h}]^{\nu} + \frac{63}{20}\nu^2\pi p_\infty^6\eta^2\\ &\quad + \left(\frac{199037}{40320}\nu^2 - \frac{27331}{10080}\nu^3\right)\pi p_\infty^8\eta^4,\\ \tilde{\chi}_5^{\rm nonloc,h} &= [\tilde{\chi}_5^{\rm nonloc,h}]^{\nu+\nu^2} - \frac{2448608}{33075}\nu^3 p_\infty^7\eta^4,\\ \tilde{\chi}_6^{\rm nonloc,h} &= [\tilde{\chi}_6^{\rm nonloc,h}]^{\nu+\nu^2} + D\nu^3\pi p_\infty^6\eta^4.\end{aligned} \tag{6.28}$$

In other words, there are *only five terms* violating the rule (6.7) in $\tilde{\chi}_n^{\rm nonloc,h}$ ($n = 4$, 5, 6): (i) one term of fractional order η^2, i.e., at 5PN [$O(\nu^2)$ term in $\tilde{\chi}_4^{\rm nonloc,h}$]; and (ii) four terms of fractional order η^4, i.e., at 6PN [$O(\nu^2)$ term and $O(\nu^3)$ term in $\tilde{\chi}_4^{\rm nonloc,h}$, and $O(\nu^3)$ terms in $\tilde{\chi}_5^{\rm nonloc,h}$ and $\tilde{\chi}_6^{\rm nonloc,h}$]. The coefficients of all those terms have been analytically derived, apart from the last one which has been only partially analytically derived. However, we have evaluated it numerically:

$$D \equiv \frac{1}{2}d_{21} + d_{42} - \frac{1}{8}d_{00} \approx -116.73148147. \tag{6.29}$$

Note that the contributions d_{21} and d_{00} to the coefficient D are known analytically and contain $\zeta(3)$ [see Eqs. (4.16) and (4.17)]. The only integral we could not analytically compute is $d_{42} \approx -439.10050487$ (see Table VI). For completeness, we give the explicit integral form of c_{42} [equivalent to d_{42}, see Eq. (4.14)] in the Supplemental Material [91] of this paper.

VII. DETERMINATION OF THE FLEXIBILITY FACTOR $f(t)$

Let us recall again the logic behind the introduction of the flexibility factor $f(t)$. The total (local-plus-nonlocal) scattering angle $\chi^{\rm tot}(p_\infty, j; \nu)$ has been shown [7] to have a special ν dependence at each PM order, i.e., at each order in $\frac{1}{j}$. However, if we were to decompose $\chi^{\rm tot}(p_\infty, j; \nu)$ in its harmonic-coordinate nonlocal contribution $\chi^{\rm nonloc,h}(p_\infty, j; \nu)$ and the complementary harmonic-coordinate local contribution $\chi^{\rm loc,h}(p_\infty, j; \nu)$, each contribution would not separately satisfy the special ν dependence of their sum $\chi^{\rm tot}(p_\infty, j; \nu) = \chi^{\rm nonloc,h}(p_\infty, j; \nu) + \chi^{\rm loc,h}(p_\infty, j; \nu)$. This situation is improved by slightly modifying the (conventional) definition of the nonlocal Hamiltonian, and thereby the separation of $\chi^{\rm tot}(p_\infty, j; \nu)$ into a flexed nonlocal piece, $\chi^{\rm nonloc,f}(p_\infty, j; \nu)$, and a complementary flexed local piece, $\chi^{\rm loc,f}(p_\infty, j; \nu)$, such that each contribution to $\chi^{\rm tot}(p_\infty, j; \nu) = \chi^{\rm nonloc,f}(p_\infty, j; \nu) + \chi^{\rm loc,f}(p_\infty, j; \nu)$ *separately* satisfies the simple ν dependence satisfied by $\chi^{\rm tot}(p_\infty, j; \nu)$. We have already determined in Ref. [3] the structure of the flexed 6PN-accurate local Hamiltonian by using the condition that its corresponding local scattering angle $\chi^{\rm loc,f}(p_\infty, j; \nu)$ satisfies the special ν dependence of Ref. [7]. In the present section, we shall use the results derived in the previous section for the harmonic-coordinate nonlocal contribution $\chi^{\rm nonloc,h}(p_\infty, j; \nu)$ as a tool for determining the value of the flexibility factor f. Specifically, by writing that the sum

$$\tilde{\chi}^{\rm nonloc,f}(p_\infty,j;\nu) = \tilde{\chi}^{\rm nonloc,h}(p_\infty,j;\nu) + \tilde{\chi}^{\rm f-h}(p_\infty,j;\nu) \tag{7.1}$$

satisfies the special ν dependence satisfied by $\chi^{\rm tot}(p_\infty, j;\nu)$ we are going to get some constraints on the value of f.

The determination of $f(t)$ is done by going through three successive steps: (i) explicit computation of the few coefficients measuring to what extent the harmonic-coordinate angle $\chi^{\rm nonloc,h}(p_\infty, j;\nu)$ fails to satisfy the special ν dependence; (ii) computation of the $f-h$ additional contribution, $\chi^{\rm f-h}(p_\infty, j;\nu)$, to the nonlocal scattering angle; and (iii) determination of $f(t)$ by the condition that $\chi^{\rm f-h}(p_\infty, j;\nu)$ compensates the rule-violating contributions, Eq. (6.28), present in $\tilde\chi^{\rm nonloc,h}(p_\infty, j;\nu)$. The step (i) was already accomplished in the previous section. We now go through steps (ii) and (iii).

A. Determination of $W^{\rm f-h}$ and $\tilde\chi^{\rm f-h}(p_\infty, j;\nu)$

The f-induced additional contribution $W^{\rm f-h}$ to $W^{\rm nonloc} = \int dt H^{\rm nonloc}$ is defined as

$$W^{\rm f-h} = +2\frac{GH_{\rm tot}}{c^5}\int dt \mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln(f(t)). \quad (7.2)$$

A simplification is that, as it is enough to look for a flexibility factor of the type

$$f(t) = 1 + \eta^2 f_1(t) + \eta^4 f_2(t) + O(\eta^6), \quad (7.3)$$

we have

$$\ln(f(t)) = \eta^2 f_1 + \eta^4\left(f_2 - \frac{1}{2}f_1^2\right) + O(\eta^6), \quad (7.4)$$

so that it is enough to work at the 1PN fractional accuracy. [Indeed, the factor $2\frac{GH_{\rm tot}}{c^5}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)$ in Eq. (7.2) starts at the 4PN order, while $\ln(f(t)) = O(\frac{1}{c^2})$, so that $W^{\rm f-h}$ starts at the 5PN order, as appropriate to cancel the 5PN + 6PN rule-violating terms delineated in Eqs. (6.28).] Namely, we can use in Eq. (7.2) the 1PN-accurate gravitational-wave flux $\mathcal{F}^{\rm GW}_{\rm 1PN}(t) = \mathcal{F}^{\rm split}_{\rm 1PN}(t,t)$, and we can compute the integral by using the 1PN-accurate quasi-Keplerian dynamics.

There are several possible ways to parametrize a general 1PN expression for the flexibility function $f(t)$. One could use a direct parametrization in terms of harmonic-coordinate positions and velocities. Here, we shall follow our previous (Newtonian-accurate) determination of the 1PN term $\eta^2 f_1$ [2] by parametrizing $f(t)$ in terms of (1PN-accurate) harmonic-coordinate relative positions $\mathbf{x}$ and momenta $\mathbf{p}$. (As in [2], we work with rescaled, dimensionless positions and momenta.) We write

$$f_1 = \nu\left(c_1 p_r^2 + c_2 p^2 + c_3\frac{1}{r}\right),$$
$$f_2 = \nu\left(d_1 p_r^4 + d_2 p^4 + d_3\frac{1}{r^2} + d_4 p^2 p_r^2 + d_5\frac{p_r^2}{r} + d_6\frac{p^2}{r}\right). \quad (7.5)$$

The coefficients c_i used here differ from the corresponding quantities in Ref. [2] by an overall factor ν: $\nu c_i^{\rm here} = c_i^{\rm there}$. As a consequence, the $c_i^{\rm here}$'s can be chosen to be pure numbers (independent of the value of ν). On the other hand, in spite of a similar ν overall rescaling, the coefficients d_i will be found to be linear functions of ν:

$$d_i = d_i^0 + \nu d_i^1. \quad (7.6)$$

Equation (7.2) becomes

$$W^{\rm f-h} = 2GH_{\rm tot}\int dt \mathcal{F}^{\rm GW}_{\rm 1PN}(t)\left[f_1 + \eta^2\left(f_2 - \frac{1}{2}f_1^2\right)\right]. \quad (7.7)$$

One should insert in Eq. (7.7) the expression of $\mathcal{F}^{\rm GW}_{\rm 1PN}(t)$ [89] in terms of the 1PN-accurate momenta. It is derived in Appendix A using classic results on the 1PN Lagrangian for the (harmonic-coordinate) relative motion (see, e.g., Ref. [81]). See Eqs. (A1) and (A5).

It is straightforward to compute the integral (7.7), and then to differentiate it with respect to j to obtain the corresponding contribution to the scattering angle:

$$\frac{1}{2}\chi^{\rm f-h} = \frac{1}{2M^2\nu}\frac{\partial W^{\rm f-h}(p_\infty, j;\nu)}{\partial j}. \quad (7.8)$$

The result of this computation is a contribution that starts at the 5PN level, and that is fractionally 1PN accurate, say

$$\frac{1}{2}\chi^{\rm f-h} = \frac{1}{2}\chi_0^{\rm f-h} + \frac{1}{2}\eta^2\chi_2^{\rm f-h}. \quad (7.9)$$

The large-j expansion of the 5PN-level contribution, $\frac{1}{2}\chi_0^{\rm f-h}$, reads

$$\frac{1}{2}\chi_0^{\rm f-h} = -\frac{1}{10}\nu^2\frac{p_\infty^6}{j^4}\pi(74c_2 + 13c_1) - \frac{128}{225}\nu^2\frac{p_\infty^5}{j^5}(343c_2 + 49c_3 + 51c_1) - \frac{1}{2}\nu^2\frac{p_\infty^4}{j^6}\pi(63c_1 + 488c_2 + 122c_3) + O\left(\frac{1}{j^7}\right), \quad (7.10)$$

while that of the 6PN-level one, $\frac{1}{2}\chi_2^{\rm f-h}$, reads

$$\frac{1}{2}\chi_2^{\mathrm{f-h}} = \chi_{4,2}^{\mathrm{f-h}}\frac{p_\infty^8}{j^4} + \chi_{5,2}^{\mathrm{f-h}}\frac{p_\infty^7}{j^5} + \chi_{6,2}^{\mathrm{f-h}}\frac{p_\infty^6}{j^6} + O\left(\frac{1}{j^7}\right), \tag{7.11}$$

with

$$\begin{aligned}
\chi_{4,2}^{\mathrm{f-h}} &= \frac{1}{2240}\nu^2\pi(-10856c_2 + 41440\nu c_2 + 2383c_1 + 3572c_1\nu - 2912d_4 + 2912\nu c_2 c_1\\
&\quad - 16576d_2 + 574\nu c_1^2 + 8288\nu c_2^2 - 1148d_1),\\
\chi_{5,2}^{\mathrm{f-h}} &= +\frac{64}{11025}\nu^2(107024\nu c_2 - 51669c_2 + 7864c_1\nu + 439c_1 + 10357\nu c_3 - 4019c_3\\
&\quad + 6426\nu c_2 c_1 - 2226d_1 + 714\nu c_1 c_3 + 21609\nu c_2^2 - 4802d_6 + 4802\nu c_2 c_3\\
&\quad - 714d_5 + 1113\nu c_1^2 - 6426d_4 - 43218d_2),\\
\chi_{6,2}^{\mathrm{f-h}} &= +\frac{1}{336}\nu^2\pi(-12139c_1 - 5460d_1 + 17640\nu c_2 c_1 + 3528\nu c_1 c_3 - 136640d_2 + 2730\nu c_1^2 + 21580c_1\nu\\
&\quad + 58740\nu c_3 - 246374c_2 + 1708\nu c_3^2 + 68320\nu c_2^2 + 335600\nu c_2 - 30454c_3 - 3416d_3 - 17640d_4\\
&\quad - 3528d_5 - 27328d_6 + 27328\nu c_2 c_3).
\end{aligned} \tag{7.12}$$

The quantities of most interest are the corresponding energy-rescaled coefficients of $\frac{1}{j^n}$ in the scattering angle $\frac{1}{2}\chi^{\mathrm{f-h}} = \sum_n \chi_n^{\mathrm{f-h}}/j^n$, i.e.,

$$\tilde\chi_n^{\mathrm{f-h}} = h^{n-1}\chi_n^{\mathrm{f-h}} = \tilde\chi_{n,0}^{\mathrm{f-h}} + \eta^2\tilde\chi_{n,2}^{\mathrm{f-h}} + \cdots. \tag{7.13}$$

They read (at the fractional 1PN accuracy, and setting $\eta = 1$)

$$\begin{aligned}
\pi^{-1}\tilde\chi_4^{\mathrm{f-h}} &= \left(-\frac{13}{10}c_1 - \frac{37}{5}c_2\right)\nu^2 p_\infty^6\\
&\quad + \left[\frac{2383}{2240}c_1 - \frac{1357}{280}c_2 - \frac{41}{80}d_1 - \frac{37}{5}d_2 - \frac{13}{10}d_4\right.\\
&\quad \left. + \left(-\frac{199}{560}c_1 + \frac{37}{5}c_2 + \frac{13}{10}c_2 c_1 + \frac{41}{160}c_1^2 + \frac{37}{10}c_2^2\right)\nu\right]\nu^2 p_\infty^8,\\
\tilde\chi_5^{\mathrm{f-h}} &= \left(-\frac{2176}{75}c_1 - \frac{43904}{225}c_2 - \frac{6272}{225}c_3\right)\nu^2 p_\infty^5\\
&\quad + \left[\frac{28096}{11025}c_1 - \frac{367424}{1225}c_2 - \frac{257216}{11025}c_3 - \frac{6784}{525}d_1 - \frac{6272}{25}d_2 - \frac{6528}{175}d_4 - \frac{2176}{525}d_5 - \frac{6272}{225}d_6\right.\\
&\quad + \left(-\frac{136448}{11025}c_1 + \frac{2546944}{11025}c_2 + \frac{16064}{3675}c_3 + \frac{6528}{175}c_2 c_1 + \frac{2176}{525}c_1 c_3 + \frac{3136}{25}c_2^2\right.\\
&\quad \left.\left. + \frac{6272}{225}c_2 c_3 + \frac{3392}{525}c_1^2\right)\nu\right]\nu^2 p_\infty^7,\\
\pi^{-1}\tilde\chi_6^{\mathrm{f-h}} &= \left(-\frac{63}{2}c_1 - 244c_2 - 61c_3\right)\nu^2 p_\infty^4\\
&\quad + \left[-\frac{12139}{336}c_1 - \frac{123187}{168}c_2 - \frac{15227}{168}c_3 - \frac{65}{4}d_1 - \frac{1220}{3}d_2 - \frac{61}{6}d_3 - \frac{105}{2}d_4 - \frac{21}{2}d_5 - \frac{244}{3}d_6\right.\\
&\quad \left. + \left(-\frac{305}{21}c_1 + \frac{8165}{21}c_2 + \frac{625}{28}c_3 + \frac{21}{2}c_1 c_3 + \frac{244}{3}c_2 c_3 + \frac{105}{2}c_2 c_1 + \frac{65}{8}c_1^2 + \frac{61}{12}c_3^2 + \frac{610}{3}c_2^2\right)\nu\right]\nu^2 p_\infty^6.
\end{aligned} \tag{7.14}$$

B. Reparametrization of the flexibility factor $f(t)$, and constraints on its parameters

Combining the results (6.28) and (7.14), we can now write the condition that the sums $\tilde{\chi}_n^{\rm nonloc,f} = \tilde{\chi}_n^{\rm nonloc,h} + \tilde{\chi}_n^{\rm f-h}$ satisfy the ν dependence of $\tilde{\chi}_n^{\rm tot}$, i.e.,

$$[\tilde{\chi}_4^{\rm nonloc,f}] \sim \nu, \qquad [\tilde{\chi}_5^{\rm nonloc,f}] \sim \nu + \nu^2, \qquad [\tilde{\chi}_6^{\rm nonloc,f}] \sim \nu + \nu^2. \tag{7.15}$$

Remembering the ν independence of the c_i's, and the ν linearity of the d_i's, Eq. (7.6), these conditions yield five equations. One equation (already discussed in Ref. [2]) comes from the 5PN level and reads

$$\frac{13}{10}c_1 + \frac{37}{5}c_2 = \frac{63}{20}. \tag{7.16}$$

The 6PN level yields four additional constraints, namely

$$\begin{aligned}
(a)\quad & 0 = -\frac{1357}{56}c_2 + \frac{2383}{448}c_1 - 37d_2^0 - \frac{13}{2}d_4^0 - \frac{41}{16}d_1^0 + \frac{199037}{8064},\\
(b)\quad & 0 = 37c_2 - \frac{13}{2}d_4^1 + \frac{13}{2}c_2c_1 - 37d_2^1 + \frac{41}{32}c_1^2 + \frac{37}{2}c_2^2 - \frac{199}{112}c_1 - \frac{41}{16}d_1^1 - \frac{27331}{2016},\\
(c)\quad & 0 = \frac{2546944}{441}c_2 - \frac{6784}{21}d_1^1 + \frac{2176}{21}c_1c_3 - \frac{136448}{441}c_1 - \frac{6272}{9}d_6^1 + \frac{6272}{9}c_2c_3 + 3136c_2^2 - \frac{2176}{21}d_5^1 + \frac{3392}{21}c_1^2\\
& \quad + \frac{16064}{147}c_3 - \frac{6528}{7}d_4^1 + \frac{6528}{7}c_2c_1 - 6272d_2^1 - \frac{2448608}{1323},\\
(d)\quad & 0 = \frac{65}{8}c_1^2 + \frac{610}{3}c_2^2 + \frac{625}{28}c_3 + \frac{105}{2}c_2c_1 - \frac{61}{6}d_3^1 + \frac{21}{2}c_1c_3 - \frac{65}{4}d_1^1 + \frac{244}{3}c_2c_3 - \frac{1220}{3}d_2^1 - \frac{21}{2}d_5^1 + \frac{8165}{21}c_2\\
& \quad - \frac{244}{3}d_6^1 + \frac{61}{12}c_3^2 - \frac{105}{2}d_4^1 - \frac{305}{21}c_1 + D,
\end{aligned} \tag{7.17}$$

where the constant $D = \frac{1}{2}d_{21} + d_{42} - \frac{1}{8}d_{00} \approx -116.73148147$, was already discussed above, see Eq. (6.29).

There are many ways to satisfy these constraints. Indeed, at the 5PN level, we have one constraint, Eq. (7.16), for three coefficients, c_1, c_2, c_3, while at the 6PN level we have 4 constraints, Eqs. (7.17), for the 12 coefficients d_1^0, d_1^1, d_2^0, d_2^1, d_3^0, d_3^1, d_4^0, d_4^1, d_5^0, d_5^1, d_6^0, d_6^1. We can, however, streamline the discussion of these constraints by defining a convenient reparametrization of the gauge-invariant content of the Hamiltonian contribution associated with the flexibility factor $f(t)$, namely

$$\begin{aligned}
\Delta^{\rm f-h}H_{5+6{\rm PN}} &= 2H_{\rm tot}\mathcal{F}^{\rm GW}_{1{\rm PN}}\ln(f)\\
&= 2H_{\rm tot}\mathcal{F}^{\rm GW}_{1{\rm PN}}(p,p_r,r)\left[f_1 + \eta^2\left(f_2 - \frac{1}{2}f_1^2\right)\right].
\end{aligned} \tag{7.18}$$

The latter flexibility-related Hamiltonian contains the three 5PN parameters c_i, and the four 6PN parameters d_i entering the flexibility factor $f(t)$, Eqs. (7.5). [Here, we count for simplicity each d_i, $i = 1, \ldots, 4$, as one parameter, though one must remember that each $d_i(\nu) = d_i^0 + \nu d_i^1$ actually contains two numerical parameters.] Let us, however, show that the flexibility described by $f(t)$ can be parametrized by three other 5PN parameters, C_1, C_2, C_3, and *only four* 6PN parameters D_1, D_2, D_3, D_4. [Each new 6PN parameter D_i will be again a linear function of ν, $D_i(\nu) = D_i^0 + \nu D_i^1$, and actually contain two numerical parameters.]

Indeed, it is shown in Appendix E that the 6PN flexibility contribution to the Hamiltonian, Eq. (7.18), is canonically equivalent to the following (p_r-gauge-type) Hamiltonian:

$$\begin{aligned}
\Delta^{\rm f-h}H'_{5+6{\rm PN}} = \frac{M\nu^3}{r^4}\Bigg[&C_1p_r^4 + C_2\frac{p_r^2}{r} + C_3\frac{1}{r^2}\\
&+ \eta^2\left(D_1p_r^6 + D_2\frac{p_r^4}{r} + D_3\frac{p_r^2}{r^2} + D_4\frac{1}{r^3}\right)\Bigg].
\end{aligned} \tag{7.19}$$

The seven new parameters C_1, C_2, C_3 and D_1, D_2, D_3, D_4 entering Eq. (7.19) are defined by the following explicit functions of the original nine parameters c_i, d_i:

$$\begin{aligned}
C_1 &= \frac{16}{15}(13c_1 + 74c_2),\\
C_2 &= \frac{16}{15}(49c_3 + 121c_2 + 12c_1),\\
C_3 &= \frac{64}{5}(c_2 + c_3),
\end{aligned} \tag{7.20}$$

and

$$D_1 = \left(-\frac{328c_1^2}{75} - \frac{1664c_1c_2}{75} + \frac{38128c_1}{525} - \frac{4736c_2^2}{75} + \frac{18944c_2}{75}\right)\nu - \frac{15356c_1}{525} + \frac{3424c_2}{175} + \frac{656d_1}{75} + \frac{9472d_2}{75} + \frac{1664d_4}{75},$$

$$D_2 = \left(-\frac{32c_1^2}{5} - \frac{784c_1c_2}{15} - \frac{272c_1c_3}{15} + \frac{1576c_1}{45} - \frac{3496c_2^2}{15} - \frac{5488c_2c_3}{45} - \frac{5584c_2}{105} + \frac{70808c_3}{315}\right)\nu$$
$$- \frac{11212c_1}{63} - \frac{28496c_2}{45} + \frac{12944c_3}{315} + \frac{64d_1}{5} + \frac{6992d_2}{15} + \frac{784d_4}{15} + \frac{272d_5}{15} + \frac{5488d_6}{45},$$

$$D_3 = \left(-\frac{64c_1c_2}{5} - \frac{64c_1c_3}{5} - \frac{112c_1}{5} - \frac{1928c_2^2}{15} - \frac{464c_2c_3}{3} - \frac{8440c_2}{21} - \frac{488c_3^2}{15} - \frac{3048c_3}{35}\right)\nu$$
$$- \frac{11708c_1}{105} - \frac{18884c_2}{21} - \frac{1724c_3}{3} + \frac{3856d_2}{15} + \frac{976d_3}{15} + \frac{64d_4}{5} + \frac{64d_5}{5} + \frac{464d_6}{3},$$

$$D_4 = \left(-\frac{32c_2^2}{5} - \frac{64c_2c_3}{5} - \frac{112c_2}{5} - \frac{32c_3^2}{5} - \frac{112c_3}{5}\right)\nu - \frac{6332c_2}{105} - \frac{11708c_3}{105} + \frac{64d_2}{5} + \frac{64d_3}{5} + \frac{64d_6}{5}. \tag{7.21}$$

The three C_i's are in one-to-one correspondence with the three c_i's, with the inverse relations $c_i = f_i(C_j)$ given in Eqs. (E6). On the other hand, the four D_i's capture the full gauge-invariant content of the six d_i's. [Two of the d_i's being pure gauge parameters; see Eqs. (E7).]

The five constraints discussed in the previous subsection can be entirely reexpressed in terms of the parameters C_i ($i = 1…3$), and $D_i = D_i^0 + \nu D_i^1$ ($i = 1…4$). Indeed, the scattering angle only depends on the time-integral (along a hyperbolic motion) of $\Delta^{\mathrm{f-h}} H_{6\mathrm{PN}}$, which is equal to the time integral of $\Delta^{\mathrm{f-h}} H'_{6\mathrm{PN}}$. This ensures that the scattering angle only depends on the C_i's and D_i's. Alternatively, using Eqs. (E6) and (E7), we could reexpress the energy-rescaled scattering-angle coefficients (7.14) in terms the C_i's and D_i's. The results read

$$\pi^{-1}\tilde{\chi}_4^{\mathrm{f-h}} = -\frac{3}{32}C_1\nu^2 p_\infty^6 - \frac{3}{32}\left[\left(\frac{1}{2} - 3\nu\right)C_1 + \frac{5}{8}D_1\right]\nu^2 p_\infty^8,$$
$$\tilde{\chi}_5^{\mathrm{f-h}} = -\frac{8}{5}\left(C_1 + \frac{1}{3}C_2\right)\nu^2 p_\infty^5 - \frac{8}{5}\left[\left(\frac{43}{14} - \frac{41}{14}\nu\right)C_1 + \left(\frac{1}{6} - \frac{2}{3}\nu\right)C_2 + \frac{5}{7}D_1 + \frac{1}{7}D_2\right]\nu^2 p_\infty^7,$$
$$\pi^{-1}\tilde{\chi}_6^{\mathrm{f-h}} = -\frac{15}{16}\left(\frac{3}{2}C_1 + C_2 + C_3\right)\nu^2 p_\infty^4$$
$$-\frac{15}{16}\left[\left(\frac{41}{4} - \frac{9}{2}\nu\right)C_1 + \left(\frac{19}{6} - \frac{25}{12}\nu\right)C_2 + \left(\frac{1}{2} - \nu\right)C_3 + \frac{5}{4}D_1 + \frac{1}{2}D_2 + \frac{1}{6}D_3\right]\nu^2 p_\infty^6. \tag{7.22}$$

Comparing these (simplified) expressions with the five contributions to $\tilde{\chi}_n^{\mathrm{nonloc,h}}$ that do not satisfy the rule (6.7) { which were written down in Eqs. (6.28)], we now get the following simplified versions of the five constraints (7.16) and (7.17).

At 5PN we have only one constraint, Eq. (7.16), which now reads

$$C_1 = \frac{168}{5}. \tag{7.23}$$

At 6PN, the four constraints, Eq. (7.17), now imply

$$D_1^0 = \frac{398074}{4725} - \frac{4}{5}C_1 = \frac{271066}{4725},$$
$$D_1^1 = -\frac{218648}{4725} + \frac{24}{5}C_1 = \frac{21736}{189},$$
$$D_2^1 = -\frac{87428}{945} - \frac{7}{2}C_1 + \frac{14}{3}C_2 = -\frac{39712}{189} + \frac{14}{3}C_2,$$
$$D_3^1 = \frac{65584}{105} + \frac{3}{2}C_1 - \frac{3}{2}C_2 + 6C_3 + \frac{32}{5}D$$
$$= \frac{70876}{105} - \frac{3}{2}C_2 + 6C_3 + \frac{32}{5}D. \tag{7.24}$$

At the 5PN level, we have three flexibility parameters, C_1, C_2, C_3, and only one of them is determined, namely C_1, Eq. (7.23). It was pointed out in Ref. [2] that the presence of two unconstrained 5PN flexibility parameters (namely C_2 and C_3) is in one-to-one correspondence with the existence of two 5PN-level undetermined coefficients in the local Hamiltonian (namely $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$). More precisely, changing the values of C_2 and C_3 was shown to be equivalent to shifting the values of $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$ [see Eqs. (8.21)–(8.22) of Ref. [2]]. Alternatively, one could uniquely fix C_2 and C_3, i.e., uniquely fix the flexibility factor f, so as to reduce $\Delta^{\rm f-h}H$ to be *minimal*, in a p_r-type gauge, i.e., to contain the minimum number of terms needed to satisfy the scattering constraints. This was formulated there in terms of the EOB parametrization of the Hamiltonian. The result was that by choosing [see Eqs. (8.24) of Ref. [2], here rescaled by ν as we recall]

$$\begin{aligned} c_1^{\rm min} &= \frac{189}{4}, \\ c_2^{\rm min} &= -\frac{63}{8}, \\ c_3^{\rm min} &= \frac{63}{8}, \end{aligned} \tag{7.25}$$

the $f-h$ piece of the EOB effective Hamiltonian was reduced to be fully contained in the following specific (minimal) Q term:

$$\Delta^{\rm f} Q^{\rm min} = \frac{336}{5}\nu^2 \frac{p_r^4}{r^4}. \tag{7.26}$$

Let us now show how these results can be generalized to the 6PN level.[12] Let us first note that, when transcribing the 5PN-level minimal constraints (7.25) in terms of the new parameters C_i, they are easily seen to simply correspond to completing the constraint (7.23) by the additional simple constraints

$$\begin{aligned} C_2^{\rm min} &= 0, \\ C_3^{\rm min} &= 0. \end{aligned} \tag{7.27}$$

If we then insert the latter results in the four 6PN-level constraints (7.24), we find that, among the eight 6PN coefficients D_i^0, D_i^1, $i = 1, \ldots, 4$, four of them, namely D_1^0, D_1^1, D_2^1, and D_3^1 are completely fixed by combining the 5PN minimal choice (7.27) with the general 6PN constraints. This lead us to define the following *minimal* solution of the $5+6$PN constraints:

$$\begin{aligned} C_1^{\rm min} &= \frac{168}{5}, \\ C_2^{\rm min} &= 0, \\ C_3^{\rm min} &= 0, \\ D_1^{\rm min} &= \frac{271066}{4725} + \frac{21736}{189}\nu, \\ D_2^{\rm min} &= -\frac{39712}{189}\nu, \\ D_3^{\rm min} &= \left(\frac{70876}{105} + \frac{32}{5}D\right)\nu, \\ D_4^{\rm min} &= 0. \end{aligned} \tag{7.28}$$

Starting from this minimal solution of the flexibility constraints, we can decompose $\Delta^{\rm f-h}H'_{5+6\rm PN}$ into two parts, say

$$\Delta^{\rm f-h}H'_{5+6\rm PN} = \Delta^{\rm f-h}H'^{\rm min}_{5+6\rm PN} + \Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}. \tag{7.29}$$

Here, $\Delta^{\rm f-h}H'^{\rm min}_{5+6\rm PN}$ denotes the part that is built with the minimal solution (7.28), namely

$$\begin{aligned} \frac{\Delta^{\rm f-h}H'^{\rm min}_{5+6\rm PN}}{M} &= \nu^3\frac{168}{5}\frac{p_r^4}{r^4} + \nu^3\left(\frac{271066}{4725} + \frac{21736}{189}\nu\right)\frac{p_r^6}{r^4} \\ &\quad - \nu^4\frac{39712}{189}\frac{p_r^4}{r^5} + \nu^4\left(\frac{70876}{105} + \frac{32}{5}D\right)\frac{p_r^2}{r^6}. \end{aligned} \tag{7.30}$$

On the other hand, $\Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}$ denotes the part that involves the six flexibility parameters that are left unconstrained by the general constraints (7.23) and (7.24), namely: C_2, C_3, D_2^0, D_3^0, and $D_4 = D_4^0 + \nu D_4^1$. Explicitly, we have

$$\begin{aligned} \frac{\Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}}{M} &= C_2\frac{\nu^3 p_r^2}{r^5} + C_3\frac{\nu^3}{r^6} + \left(D_2^0 + \frac{14}{3}\nu C_2\right)\frac{\nu^3 p_r^4}{r^5} \\ &\quad + \left[D_3^0 + \nu\left(-\frac{3}{2}C_2 + 6C_3\right)\right]\frac{\nu^3 p_r^2}{r^6} \\ &\quad + (D_4^0 + \nu D_4^1)\frac{\nu^3}{r^7}. \end{aligned} \tag{7.31}$$

By using a suitable canonical transformation to transform into standard EOB gauge the harmonic-type gauge to which $\Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}$ belongs,[13] we can then transcribe the unconstrained f-dependent Hamiltonian contribution $\Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}$ in EOB format, i.e., in terms of the potentials A, $\bar{D}$, and Q parametrizing a general effective Hamiltonian in p_r gauge, as in Eqs. (4.1) and (4.2) of Ref. [3]). One then finds that

[12] It can be shown that a similar result holds at higher PN orders.

[13] Indeed, $\Delta^{\rm f-h}H'^{CD}_{5+6\rm PN}$ is a contribution to the total nonlocal Hamiltonian $H^{\rm nonloc,f}$ which is expressed in terms of harmonic coordinates.

adding the Hamiltonian contribution $\Delta^{\mathrm{f-h}} H'^{CD}_{5+6\mathrm{PN}}$, Eq. (7.31), is equivalent to adding to the EOB potentials entering the f-route local Hamiltonian $H^{\mathrm{nonloc,f}}$ the following supplementary (5PN and 6PN) contributions:

$$A^{CD} = a_6^{CD} u^6 + a_7^{CD} u^7,$$
$$\bar{D}^{CD} = \bar{d}_5^{CD} u^5 + \bar{d}_6^{CD} u^6,$$
$$\hat{Q}^{CD} = q_{45}^{CD} p_r^4 u^5, \tag{7.32}$$

with 5PN-level terms,

$$a_6^{CD} = 2\nu^2 C_3,$$
$$\bar{d}_5^{CD} = 2\nu^2 C_2, \tag{7.33}$$

and 6PN-level ones:

$$a_7^{CD} = 2\nu^2(D_4^0 + \nu D_4^1) + \nu^2(9-\nu)C_3,$$
$$\bar{d}_6^{CD} = \nu^2(2D_3^0 + 17C_2 - 8C_3) - \nu^3(2C_2 + 30C_3),$$
$$q_{45}^{CD} = \nu^2\left(2D_2^0 + \frac{7}{3}C_2\right) - \frac{28}{3}\nu^3 C_2. \tag{7.34}$$

By comparing the expressions (7.33) and (7.34) to the explicit form of the EOB potentials of the 6PN f-route local Hamiltonian $H^{\mathrm{loc,f}}$, as displayed in Table X of [3], it is easily checked that the addition of the contributions (7.33) and (7.34) [including their explicit $O(\nu^3)$ terms] to $H^{\mathrm{loc,f}}$ is equivalent to replacing the undetermined EOB coefficients $a_6^{\nu^2 \mathrm{loc,f}}, \bar{d}_5^{\nu^2 \mathrm{loc,f}}, \ldots$ appearing in $H^{\mathrm{loc,f}}(a_6^{\nu^2 \mathrm{loc,f}}, \bar{d}_5^{\nu^2 \mathrm{loc,f}}, \ldots)$ by the following shifted values:

$$a_6^{\nu^2 \mathrm{shifted}} = a_6^{\nu^2 \mathrm{loc,f}} + 2C_3,$$
$$\bar{d}_5^{\nu^2 \mathrm{shifted}} = \bar{d}_5^{\nu^2 \mathrm{loc,f}} + 2C_2,$$
$$a_7^{\nu^2 \mathrm{shifted}} = a_7^{\nu^2 \mathrm{loc,f}} + 2D_4^0 + 9C_3,$$
$$a_7^{\nu^3 \mathrm{shifted}} = a_7^{\nu^3 \mathrm{loc,f}} + 2D_4^1 - C_3,$$
$$\bar{d}_6^{\nu^2 \mathrm{shifted}} = \bar{d}_6^{\nu^2 \mathrm{loc,f}} + 2D_3^0 + 17C_2 - 8C_3,$$
$$q_{45}^{\nu^2 \mathrm{shifted}} = q_{45}^{\nu^2 \mathrm{loc,f}} + 2D_2^0 + \frac{7}{3}C_2. \tag{7.35}$$

The first two (5PN-level) equations are equivalent to Eqs. (8.20)–(8.21) of Ref. [2] (taking into account the fact that we separated here the term $\nu^3 \frac{168}{5}\frac{p_r^4}{r^4}$).

In Eqs. (7.35) the undetermined parameters $a_6^{\nu^2 \mathrm{loc,f}}, \ldots$, appearing on the right-hand sides of the definitions of the various shifted parameters depend on the choice of f (i.e., on the choice of the unconstrained C_i's and D_i's), while the shifted parameters $a_6^{\nu^2 \mathrm{shifted}}, \ldots$, on the left-hand sides do not depend on the choice of f (because they parametrize the Hamiltonian $H^{\mathrm{tot}} - H^{\mathrm{loc,h}} - \Delta^{\mathrm{f-h}} H'^{\mathrm{min}}_{5+6\mathrm{PN}}$). Therefore, the choice of the values of the unconstrained flexibility parameters $C_2, C_3, D_2^0, \ldots$ is a kind of gauge freedom that has no effect on the physical consequences of the total Hamiltonian [which only depends on the gauge-invariant shifted parameters defined in Eqs. (7.35)]. In other words, imposing the simple additional constraints

$$C_2 = 0,$$
$$C_3 = 0,$$
$$D_2^0 = 0,$$
$$D_3^0 = 0,$$
$$D_4^0 = 0,$$
$$D_4^1 = 0, \tag{7.36}$$

which leads to the *minimal* values (7.28) of the flexibility parameters, is a "gauge choice" such that the corresponding *minimal* values of the undetermined EOB parameters, say $a_6^{\nu^2 \mathrm{min}}, \ldots$, simply coincide with the general gauge-invariant shifted values defined in Eqs. (7.35):

$$a_6^{\nu^2 \mathrm{min}} = a_6^{\nu^2 \mathrm{shifted}},$$
$$\bar{d}_5^{\nu^2 \mathrm{min}} = \bar{d}_5^{\nu^2 \mathrm{shifted}},$$
$$a_7^{\nu^2 \mathrm{min}} = a_7^{\nu^2 \mathrm{shifted}},$$
$$a_7^{\nu^3 \mathrm{min}} = a_7^{\nu^3 \mathrm{shifted}},$$
$$\bar{d}_6^{\nu^2 \mathrm{min}} = \bar{d}_6^{\nu^2 \mathrm{shifted}},$$
$$q_{45}^{\nu^2 \mathrm{min}} = q_{45}^{\nu^2 \mathrm{shifted}}. \tag{7.37}$$

In the following, we shall often use by default the minimal fixing of the flexibility factor, and of the associated Hamiltonians, defined by using Eqs. (7.28) [i.e., satisfying Eqs. (7.36)]. This leads, in particular, to the specific value of $\Delta^{\mathrm{f-h}} H'_{5+6\mathrm{PN}}$ given by Eq. (7.30). The corresponding specific values of the original flexibility parameters c_i, d_i defining the flexibility factor $f(t)$ are discussed in Appendix E.

VIII. NONLOCAL DELAUNAY HAMILTONIAN, $\bar{H}^{\mathrm{nonloc,f}}_{4+5+6\mathrm{PN}}(I_R, I_\phi)$, RADIAL ACTION, $I^{\mathrm{nonloc,f}}_{R4+5+6\mathrm{PN}}(E,J)$, AND PERIASTRON PRECESSION

As said in the Introduction, besides the scattering angle, a second gauge-invariant characterization of the f-route nonlocal dynamics can be given. It consists in presenting the explicit form of the f-route nonlocal contribution to the averaged (Delaunay) Hamiltonian, $\bar{H}^{\mathrm{nonloc,f}}(I_R, I_\phi)$, or equivalently the corresponding contribution, $I^{\mathrm{nonloc,f}}_{R4+5+6\mathrm{PN}}(E, J)$, to the radial action. The (gauge-invariant) information contained in $\bar{H}^{\mathrm{nonloc,f}}(I_R, I_\phi)$ or $I^{\mathrm{nonloc,f}}_{R4+5+6\mathrm{PN}}(E, J)$ is also nearly fully encoded in the corresponding contribution to the periastron advance. Indeed, we have the general identity

$$d\bar{H}(I_R, I_\phi) = \Omega_R dI_R + \Omega_\phi dI_\phi = \Omega_R dI_R + K\Omega_R dI_\phi, \quad (8.1)$$

where

$$\Omega_R = \frac{\partial \bar{H}(I_R, I_\phi)}{\partial I_R} = \left[\frac{\partial I_R(E,J)}{\partial E}\right]^{-1} \quad (8.2)$$

denotes the radial frequency $2\pi/T_R$, while

$$K \equiv \frac{\Phi(E,J)}{2\pi} = \frac{\Omega_\phi}{\Omega_R} = -\frac{\partial I_R(E,J)}{\partial J} = +\frac{1}{\Omega_R}\frac{\partial \bar{H}(I_R, I_\phi)}{\partial I_\phi} \quad (8.3)$$

denotes the periastron advance $K = 1 + k$ (where the value 1 would correspond to the absence of periastron advance).

We have given in Table XI of Ref. [2] the explicit, 5PN-accurate, expression of the f-route *local* Delaunay Hamiltonian, $\bar{H}^{\rm loc,f}(I_R, I_\phi)$. We gave also the explicit value of the function $I_R^{\rm loc,f}(E,J)$ at the 5PN accuracy in Ref. [2]. Concerning the 6PN-accurate f-route *local* dynamics, we gave in Ref. [3] the explicit expression of the radial action as a function of the EOB effective energy $I_R^{\rm loc,f}(E_{\rm eff}, J)$. We proved there that it had a remarkably simple structure. Namely, it reads

$$\frac{I_r^{\rm loc,f}(\gamma,j)}{GM\mu} = -j + I_0^S(\gamma) + \frac{I_1^S(\gamma)}{hj} + \frac{I_3(\gamma;\nu)}{(hj)^3} + \frac{I_5(\gamma;\nu)}{(hj)^5} + \frac{I_7(\gamma;\nu)}{(hj)^7} + \frac{I_9(\gamma;\nu)}{(hj)^9} + \frac{I_{11}(\gamma;\nu)}{(hj)^{11}}, \quad (8.4)$$

where $h = h(\gamma,\nu) = E^{\rm tot}/M$ as above; where the first two coefficients, $I_0^S(\gamma)$, $I_1^S(\gamma)$, only depend on γ and have the following very simple exact expressions:

$$I_0^S(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{1-\gamma^2}}, \qquad I_1^S(\gamma) = \frac{3}{4}(5\gamma^2 - 1), \quad (8.5)$$

and where all the other coefficients $I_{2n+1}(\gamma;\nu)$ are polynomials in ν of order n:

$$I_{2n+1}(\gamma;\nu) = I_{2n+1}^S(\gamma) + \sum_{k=1}^{n} I_{2n+1}^{\nu^k}(\gamma)\nu^k. \quad (8.6)$$

The explicit values of the coefficients $I_{2n+1}(\gamma;\nu)$ were given (at the 6PN accuracy) in Table XIV of Ref. [3], while the exact ("Schwarzschild") values, $I_{2n+1}^S(\gamma)$, of their test-mass limit, $\nu \to 0$, were given in Eq. (9.5) there.

In view of the existence of efficient algebraic-manipulation programs, there is no need to write down here the 6PN-accurate f-route local effective Delaunay Hamiltonian, $\bar{H}_{\rm eff}^{\rm loc,f}(I_R, I_\phi)$ corresponding to the inversion of the explicit expression for $I_R^{\rm loc,f}(E_{\rm eff}, J)$ given in Ref. [3]. It might, however, be useful to emphasize again the relation between the effective energy $E_{\rm eff} = \mu c^2 + \cdots$ and the total energy $E_{\rm tot} = Mc^2 + \cdots$ [see Eq. (1.34)]:

$$E_{\rm tot} = Mc^2\sqrt{1 + 2\nu\left(\frac{\mathcal{E}_{\rm eff}}{\mu c^2} - 1\right)} \equiv Mc^2\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\rm eff} - 1)} \equiv Mc^2 h(\gamma,\nu), \quad (8.7)$$

where

$$\hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu c^2} \equiv \gamma. \quad (8.8)$$

Let us now complete the results of Ref. [3] by explaining in detail how the results derived above allow one to explicitly write down the complementary nonlocal contribution $\bar{H}_{4+5+6\rm PN}^{\rm nonloc,f}(I_R, I_\phi)$ to the total Delaunay Hamiltonian

$$\bar{H}_{6\rm PN}^{\rm tot}(I_R, I_\phi) = \bar{H}_{6\rm PN}^{\rm loc,f}(I_R, I_\phi) + \bar{H}_{4+5+6\rm PN}^{\rm nonloc,f}(I_R, I_\phi). \quad (8.9)$$

It is the sum of three contributions

$$\bar{H}_{4+5+6\rm PN}^{\rm nonloc,f}(I_R, I_\phi) = \bar{H}_{4+5+6\rm PN}^{\rm nonloc,h}(I_R, I_\phi) + \bar{H}_{5.5\rm PN}^{\rm nonloc,h}(I_R, I_\phi) + \Delta^{\rm f-h}\bar{H}_{5+6\rm PN}(I_R, I_\phi). \quad (8.10)$$

The first contribution was computed in Ref. [3] [see Eq. (3.31) there] in terms of the harmonic coordinate semimajor axis a_r^h and eccentricity[14] e_t^h [as a power series expansion up to the order $O((e_t^h)^{10})$ included] and reads

$$\frac{\bar{H}_{4+5+6\rm PN}^{\rm nonloc,h}}{M} = \frac{\nu^2}{(a_r^h)^5}[\mathcal{A}^{4\rm PN}(e_t^h) + \mathcal{B}^{4\rm PN}(e_t^h)\ln a_r^h] + \frac{\nu^2}{(a_r^h)^6}[\mathcal{A}^{5\rm PN}(e_t^h) + \mathcal{B}^{5\rm PN}(e_t^h)\ln a_r^h] + \frac{\nu^2}{(a_r^h)^7}[\mathcal{A}^{6\rm PN}(e_t^h) + \mathcal{B}^{6\rm PN}(e_t^h)\ln a_r^h]. \quad (8.11)$$

The explicit expressions of the 4PN and 5PN coefficients $\mathcal{A}^{4\rm PN}$, $\mathcal{B}^{4\rm PN}$, $\mathcal{A}^{5\rm PN}$, $\mathcal{B}^{5\rm PN}$ are written down in Table I of Ref. [2], while the explicit expressions of the 6PN coefficients $\mathcal{A}^{6\rm PN}$, $\mathcal{B}^{6\rm PN}$ have been written down in Table V or Ref. [3].

[14] Here, we are talking about ellipticlike orbital elements.

The second contribution was computed in Ref. [2] and reads[15]

$$\bar{H}^{\rm nonloc,h}_{\rm 5.5PN} = +\frac{\mu^2}{M}c^2\frac{6848}{525}\frac{\pi}{(a_r^h)^{13/2}}\varphi(e_t^h), \tag{8.12}$$

where the expansion of the function $\varphi(e)$ in powers of e (up to the 16th order) is given in Eq. (12.7) there.

Let us clarify that the intermediate (ellipticlike) orbital elements a_r^h and e_t^h used as arguments in these expressions acquire a gauge-invariant meaning when they are reexpressed as functions of $\bar{E} \equiv \frac{E_{\rm tot}-Mc^2}{\mu}$ and $j \equiv \frac{J}{GM\mu}$. The corresponding expressions are given in Eqs. (A7) (see also Table III in Ref. [3]).

Note that the replacement of the latter functions[16] $a_r(E,J)$, $e_t(E,J)$ in the expressions (8.11) and (8.12) would be appropriate for computing the corresponding values of the radial action, namely

$$\begin{aligned} I^{\rm nonloc,h}_{R4+5+6\rm PN}(E,J) &= -\frac{1}{\Omega_R}\bar{H}^{\rm nonloc,h}_{4+5+6\rm PN}(I_R,I_\phi), \\ I^{\rm nonloc}_{R5.5\rm PN}(E,J) &= -\frac{1}{\Omega_R}\bar{H}^{\rm nonloc}_{5.5\rm PN}(I_R,I_\phi), \end{aligned} \tag{8.13}$$

where $\Omega_R = 2\pi/T_R$ denotes the radial frequency. The 2PN-accurate expression of $n \equiv GM\Omega_R$ in terms of $\bar{E}$ and j is given in Eq. (A12).

Indeed, E and J are the natural arguments for the radial action. On the other hand, the natural variables for the Delaunay Hamiltonian are, by definition, I_R and $I_\phi \equiv J$. Therefore we must use the (2PN-accurate) transformation between E, J and I_R, I_ϕ. This transformation (first derived in [8]) is given (in both directions), at the 2PN accuracy, in Appendix A in terms of the rescaled action variables

$$\begin{aligned} i_r &\equiv \frac{I_R}{GM\mu}, \\ i_\phi &\equiv \frac{I_\phi}{GM\mu} \equiv j, \\ i_{r\phi} &\equiv i_r + i_\phi \equiv i_r + j. \end{aligned} \tag{8.14}$$

Note the important point that the function $e_t^2(i_r,i_\phi)$, given in Eq. (A11), contains i_r as an overall factor. In other words, e_t^2 vanishes like i_r when $i_r \to 0$, keeping fixed i_ϕ. This expresses the fact that the ellipticlike eccentricity[17] e_t is a good quasi-Keplerian eccentricity that vanishes along circular motions (the latter being intrinsically defined by the property $i_r = 0$). This property also ensures that the expression we computed for the nonlocal Delaunay Hamiltonian as a truncated expansion in powers of e_t (up to e_t^{10} included) becomes transformed, when expressed as a function of i_r and $i_\phi = j$, as a truncated expansion in powers of i_r (up to i_r^5 included). In turn, this ensures that, for example, the corresponding contribution to the periastron advance is obtained as an expansion in powers of i_r (up to i_r^5 included).

So far we have discussed the explicit expressions of the first two contributions to the nonlocal Delaunay Hamiltonian, Eq. (8.10). It remains to discuss the third contribution, namely $\Delta^{\rm f-h}\bar{H}(I_R,I_\phi)$.

In view of Eq. (7.2), it is given by

$$\Delta^{\rm f-h}\bar{H}(I_R,I_\phi) = \frac{2\pi}{\Omega_R}W^{\rm f-h}_{\rm ell}, \tag{8.15}$$

where

$$\begin{aligned} W^{\rm f-h}_{\rm ell} &= +2\frac{GH_{\rm tot}}{c^3}\oint dt\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln(f(t)) \\ &= 2\frac{GH_{\rm tot}}{c^5}\oint dt\mathcal{F}^{\rm GW}_{\rm 1PN}(t)\left[f_1 + \eta^2\left(f_2 - \frac{1}{2}f_1^2\right)\right] \\ &= \oint dt\Delta^{\rm f-h}H'_{5+6\rm PN}, \end{aligned} \tag{8.16}$$

where $\Delta^{\rm f-h}H'_{5+6\rm PN}$ is given by Eq. (7.19). Using the 2PN-accurate quasi-Keplerian representation of *elliptic* motions in harmonic coordinates (see, e.g., Sec. III of [3]), it is a straightforward matter to compute the elliptic integral $W^{\rm f-h}_{\rm ell}$. Its *exact expression* in terms of a_r and e_t reads

$$W^{\rm f-h}_{\rm ell}(a_r,e_t) = W^{\rm f-h}_{\rm ell\,0} + \eta^2 W^{\rm f-h}_{\rm ell\,2}, \tag{8.17}$$

where

$$\begin{aligned} W^{\rm f-h}_{\rm ell\,0} &= 2\pi M^2\nu^3\frac{w_0}{[a_r(1-e_t^2)]^{9/2}}, \\ W^{\rm f-h}_{\rm ell\,2} &= 2\pi M^2\nu^3\frac{w_2^{\nu^0}+\nu w_2^{\nu^1}}{[a_r(1-e_t^2)]^{11/2}}, \end{aligned} \tag{8.18}$$

with

[15] After correcting a sign error on the right-hand side of Eq. (12.6) in Ref. [2].

[16] For brevity, we henceforth omit the superscript h on a_r, and e_t.

[17] Beware that it does not coincide with the analytic continuation of its hyperboliclike counterpart.

$$
\begin{aligned}
w_0 &= C_3 + \left(3C_3 + \frac{1}{2}C_2\right)e_t^2 + \frac{3}{8}(C_3 + C_2 + C_1)e_t^4 + \frac{1}{16}C_1 e_t^6,\\
w_2^{\nu^0} &= \frac{9}{2}C_3 + D_4^0 + \left(\frac{1}{2}D_3^0 + 81C_3 + \frac{21}{4}C_2 + 5D_4^0\right)e_t^2 + \left(\frac{371}{16}C_2 + \frac{3}{4}D_3^0 + \frac{3}{8}D_2^0 + \frac{99}{16}C_1 + \frac{1539}{16}C_3 + \frac{15}{8}D_4^0\right)e_t^4\\
&\quad + \left(\frac{117}{16}C_3 + \frac{133}{16}C_2 + \frac{199037}{7560} + \frac{339}{32}C_1 + \frac{1}{16}D_3^0 + \frac{3}{16}D_2^0\right)e_t^6 + \left(\frac{15}{16}C_1 + \frac{199037}{60480}\right)e_t^8,\\
w_2^{\nu^1} &= -\frac{1}{2}C_3 + D_4^1 + \left(-\frac{3}{2}C_2 + \frac{32792}{105} + 5D_4^1 + \frac{16}{5}D + \frac{3}{4}C_1 - 25C_3\right)e_t^2\\
&\quad + \left(-\frac{555}{16}C_3 - \frac{15}{2}C_2 - \frac{9}{8}C_1 + \frac{15}{8}D_4^1 + \frac{24}{5}D + \frac{273271}{630}\right)e_t^4\\
&\quad + \left(-\frac{105}{32}C_1 - \frac{45}{16}C_3 + \frac{27331}{3780} - \frac{45}{16}C_2 + \frac{2}{5}D\right)e_t^6 + \left(-\frac{27331}{15120} - \frac{9}{32}C_1\right)e_t^8. \qquad (8.19)
\end{aligned}
$$

When using the minimal values, Eqs. (7.28), of the flexibility parameters, this result takes the following explicit form:

$$
\begin{aligned}
w_{0\,\mathrm{min}} &= \frac{63}{5}e_t^4 + \frac{21}{10}e_t^6,\\
w_{2\,\mathrm{min}}^{\nu^0} &= \frac{2079}{10}e_t^4 + \frac{2890019}{7560}e_t^6 + \frac{2104157}{60480}e_t^8,\\
w_{2\,\mathrm{min}}^{\nu^1} &= \left(\frac{35438}{105} + \frac{16}{5}D\right)e_t^2 + \left(\frac{249457}{630} + \frac{24}{5}D\right)e_t^4\\
&\quad + \left(-\frac{194707}{1890} + \frac{2}{5}D\right)e_t^6 - \frac{34043}{3024}e_t^8. \qquad (8.20)
\end{aligned}
$$

Similarly to the treatment above of $\bar{H}^{\mathrm{nonloc,h}}_{4+5+6\mathrm{PN}}$ and $\bar{H}^{\mathrm{nonloc,h}}_{5.5\mathrm{PN}}$ we can then reexpress $W^{\mathrm{f-h}}_{\mathrm{ell}}$ as a function of E and J, and $\Delta^{\mathrm{f-h}}\bar{H}$ as a function of I_R, and I_ϕ, by using the 2PN-accurate transformations explicitly given above.

As already mentioned, in view of the existence of efficient algebraic-manipulation programmes there is no need to write down here the long expressions obtained after these transformations. Let us, instead, cite the explicit forms of two of the simplest gauge-invariant quantities one can derive from our results: the value of the nonlocal contribution to the total energy *along circular orbits*, and the value of the nonlocal contribution to the periastron advance, also computed *along circular orbits*. They are both obtained by taking the limit $I_R \to 0$, namely

$$
\begin{aligned}
E^{\mathrm{nonloc,X,circ}}(J) &= [\bar{H}^{\mathrm{nonloc,X}}(I_R, I_\phi)]_{I_R=0},\\
K^{\mathrm{nonloc,X,circ}}(J) &= \left[\frac{1}{\Omega_R}\frac{\partial \bar{H}^{\mathrm{nonloc,X}}(I_R, I_\phi)}{\partial I_\phi}\right]_{I_R=0}. \qquad (8.21)
\end{aligned}
$$

Here, X, is a label distinguishing the various contributions to the nonlocal action. Following the decomposition (8.10) we have

$$
\begin{aligned}
E^{\mathrm{nonloc,f,circ}}(J) &= E^{\mathrm{nonloc,h,circ}}_{4+5+6\mathrm{PN}}(J) + E^{\mathrm{nonloc,h,circ}}_{5.5\mathrm{PN}}(J)\\
&\quad + E^{\mathrm{f-h,circ}}_{5+6PN}(J). \qquad (8.22)
\end{aligned}
$$

These three nonlocal contributions must be added to the f-route local contribution, $E^{\mathrm{loc,f,circ}}(J)$, to obtain the total circular energy

$$
E^{\mathrm{tot,circ}}(J) = E^{\mathrm{loc,f,circ}}(J) + E^{\mathrm{nonloc,f,circ}}(J). \qquad (8.23)
$$

Similarly, the total periastron advance along circular orbits can be decomposed as

$$
K^{\mathrm{tot,circ}}(J) = K^{\mathrm{loc,f,circ}}(J) + K^{\mathrm{nonloc,f,circ}}(J), \qquad (8.24)
$$

where

$$
\begin{aligned}
K^{\mathrm{nonloc,f,circ}}(J) &= K^{\mathrm{nonloc,h,circ}}_{4+5+6\mathrm{PN}}(J) + K^{\mathrm{nonloc,h,circ}}_{5.5\mathrm{PN}}(J)\\
&\quad + K^{\mathrm{f-h,circ}}(J). \qquad (8.25)
\end{aligned}
$$

Using rescaled variables, we find the following results for these quantities:

$$
\begin{aligned}
\frac{E^{\mathrm{loc,f,circ}}_{\leq 6\mathrm{PN}}(j)}{M} ={}& 1-\frac{\nu}{2}\frac{\eta^2}{j^2}+\left(-\frac{\nu^2}{8}-\frac{9\nu}{8}\right)\frac{\eta^4}{j^4}+\left(-\frac{\nu^3}{16}+\frac{7\nu^2}{16}-\frac{81\nu}{16}\right)\frac{\eta^6}{j^6}\\
&+\left[-\frac{5\nu^4}{128}+\frac{5\nu^3}{64}+\left(\frac{8833}{384}-\frac{41\pi^2}{64}\right)\nu^2-\frac{3861\nu}{128}\right]\frac{\eta^8}{j^8}\\
&+\left[-\frac{7\nu^5}{256}+\frac{3\nu^4}{128}+\left(\frac{41\pi^2}{128}-\frac{8875}{768}\right)\nu^3+\left(\frac{989911}{3840}-\frac{6581\pi^2}{1024}\right)\nu^2-\frac{53703\nu}{256}\right]\frac{\eta^{10}}{j^{10}}\\
&+\left[\left(\frac{a_6^{\nu^2}}{2}+\frac{29335\pi^2}{2048}-\frac{1679647}{3840}\right)\nu^3-\frac{21\nu^6}{1024}+\frac{5\nu^5}{1024}+\left(\frac{41\pi^2}{512}-\frac{3769}{3072}\right)\nu^4\right.\\
&\left.+\left(\frac{3747183493}{1612800}-\frac{31547\pi^2}{1536}\right)\nu^2-\frac{1648269\nu}{1024}\right]\frac{\eta^{12}}{j^{12}}\\
&+\left[\nu^3\left(\frac{39a_6^{\nu^2}}{4}+\frac{a_7^{\nu^2}}{2}-\frac{1681\pi^4}{512}+\frac{10605841\pi^2}{24576}-\frac{10727952929}{1075200}\right)\right.\\
&+\nu^4\left(\frac{a_6^{\nu^2}}{4}+\frac{a_7^{\nu^3}}{2}-\frac{21383\pi^2}{8192}+\frac{1007737}{7680}\right)-\frac{33\nu^7}{2048}-\frac{7\nu^6}{2048}+\left(\frac{41\pi^2}{1024}-\frac{2537}{3072}\right)\nu^5\\
&\left.+\left(\frac{576215112401}{29030400}+\frac{1322752463\pi^2}{3538944}-\frac{2800873\pi^4}{524288}\right)\nu^2-\frac{27078705\nu}{2048}\right]\frac{\eta^{14}}{j^{14}},
\end{aligned} \tag{8.26}
$$

$$
\begin{aligned}
\frac{E^{\mathrm{nonloc,h,circ}}_{4+5+6\mathrm{PN}}(j)}{M} ={}& \frac{64}{5}\nu^2\frac{\eta^{10}}{j^{10}}\left\{\ln\left(4\frac{e^\gamma}{j}\right)+\left[\frac{1}{2}+\frac{3793}{336}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{155}{12}\ln(2)+\frac{1215}{896}\ln(3)\right.\right.\\
&\left.+\left(\frac{1}{2}-\frac{7}{4}\ln\left(4\frac{e^\gamma}{j}\right)+\frac{155}{28}\ln(2)-\frac{1215}{224}\ln(3)\right)\nu\right]\frac{\eta^2}{j^2}\\
&+\left[\frac{982207}{9072}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{106783}{9072}\ln(2)+\frac{6075}{448}\ln(3)+\frac{5977}{672}\right.\\
&+\left(-\frac{79727}{2016}\ln\left(4\frac{e^\gamma}{j}\right)+\frac{211849}{6048}\ln(2)+\frac{5977}{672}-\frac{83835}{1792}\ln(3)\right)\nu\\
&\left.\left.+\left(\frac{76319}{1512}\ln(2)-\frac{5}{8}+\frac{1}{2}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{13365}{448}\ln(3)\right)\nu^2\right]\frac{\eta^4}{j^4}\right\},\\
\frac{E^{\mathrm{nonloc,h,circ}}_{5.5\mathrm{PN}}(j)}{M} ={}& \frac{6848}{525}\nu^2\pi\frac{\eta^{13}}{j^{13}},\\
\frac{E^{\mathrm{f-h,circ}}_{5+6\mathrm{PN}}(j)}{M} ={}& \nu^3\frac{\eta^{12}}{j^{12}}\left[C_3+(24C_3+D_4)\frac{\eta^2}{j^2}\right].
\end{aligned} \tag{8.27}
$$

Note that the minimal version of $\Delta^{\mathrm{f-h}}H'_{5+6\mathrm{PN}}$, Eq. (7.30), leads to a vanishing value of $E^{\mathrm{f-h,circ}}_{5+6\mathrm{PN}}(j)$

$$
E^{\mathrm{f-h,circ}}_{\mathrm{min}}(j)=0. \tag{8.28}
$$

On the other hand, if one does not use the minimal version of $\Delta^{\mathrm{f-h}}H'_{5+6\mathrm{PN}}$, the total energy is easily checked to depend only on the shifted versions of the undetermined parameters $a_6^{\nu^2}$, $a_7^{\nu^2}$, and $a_7^{\nu^3}$ defined in Eqs. (7.33)–(7.35).

Similarly for the periastron advance

$$\begin{aligned}
K^{\rm loc,f,circ}_{\le 6\rm PN}(j) &= 1+3\frac{\eta^2}{j^2}+\left(\frac{45}{2}-6\nu\right)\frac{\eta^4}{j^4}+\left[\frac{405}{2}+\left(-202+\frac{123}{32}\pi^2\right)\nu+3\nu^2\right]\frac{\eta^6}{j^6}\\
&+\left[\frac{15795}{8}+\left(\frac{185767}{3072}\pi^2-\frac{105991}{36}\right)\nu+\left(-\frac{41}{4}\pi^2+\frac{2479}{6}\right)\nu^2\right]\frac{\eta^8}{j^8}\\
&+\left[\frac{161109}{8}+\left(-\frac{18144676}{525}+\frac{488373}{2048}\pi^2\right)\nu+\left(-\frac{1}{2}\bar{d}_5^{\nu^2}-\frac{15}{2}a_6^{\nu^2}-\frac{9225}{32}\pi^2+\frac{21399}{2}\right)\nu^2\right.\\
&\left.+\left(-\frac{1627}{6}+\frac{205}{32}\pi^2\right)\nu^3\right]\frac{\eta^{10}}{j^{10}}\\
&+\left[\frac{3383289}{16}+\left(-\frac{2299413173213}{6350400}-\frac{10107671003}{1179648}\pi^2+\frac{7335303}{65536}\pi^4\right)\nu\right.\\
&+\left(-\frac{361}{2}a_6^{\nu^2}-\frac{21}{2}a_7^{\nu^2}-9\bar{d}_5^{\nu^2}-\frac{1}{2}\bar{d}_6^{\nu^2}+\frac{85731}{2048}\pi^4-\frac{8043499}{1024}\pi^2+\frac{1859633}{8}\right)\nu^2\\
&\left.+\left(\frac{15}{2}a_6^{\nu^2}+\frac{1}{2}\bar{d}_5^{\nu^2}-\frac{21}{2}a_7^{\nu^3}+\frac{1290233}{3072}\pi^2-\frac{2190437}{144}\right)\nu^3+\frac{75}{2}\nu^4\right]\frac{\eta^{12}}{j^{12}},\\
K^{\rm nonloc,h,circ}_{4+5+6\rm PN}(j) &= -\frac{64}{10}\nu\frac{\eta^8}{j^8}\left\{-11+\frac{157}{6}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{277}{6}\ln(2)+\frac{729}{16}\ln(3)\right.\\
&+\left[-\frac{59723}{336}+\frac{9421}{28}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{11237}{28}\ln(2)+\frac{112995}{224}\ln(3)\right.\\
&\left.+\left(\frac{2227}{42}-\frac{617}{6}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{1957}{2}\ln(2)+\frac{54675}{112}\ln(3)\right)\nu\right]\frac{\eta^2}{j^2}\\
&+\left[-\frac{4446899}{2016}+\frac{11076725}{3024}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{5347151}{1008}\ln(2)+\frac{10528947}{1792}\ln(3)+\frac{48828125}{145152}\ln(5)\right.\\
&+\left(\frac{358987}{252}-\frac{363851}{168}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{10931765}{1512}\ln(2)+\frac{4626963}{896}\ln(3)-\frac{48828125}{24192}\ln(5)\right)\nu\\
&\left.\left.+\left(-\frac{136369}{1512}+\frac{775}{6}\ln\left(4\frac{e^\gamma}{j}\right)-\frac{1315051}{126}\ln(2)+\frac{4333905}{1792}\ln(3)+\frac{48828125}{16128}\ln(5)\right)\nu^2\right]\frac{\eta^4}{j^4}\right\},\\
K^{\rm nonloc,h,circ}_{5.5\rm PN}(j) &= -\frac{99938}{315}\nu\pi\frac{\eta^{11}}{j^{11}},\\
K^{\rm f-h,circ}_{5+6\rm PN}(j) &= -\nu^2\frac{\eta^{10}}{j^{10}}\left\{15C_3+C_2+\left[\frac{903}{2}C_3+\frac{53}{2}C_2+21D_4^0+D_3^0\right.\right.\\
&\left.\left.+\left(-\frac{51}{2}C_3+\frac{70876}{105}-C_2+21D_4^1+\frac{32}{5}D\right)\nu\right]\frac{\eta^2}{j^2}\right\}.
\end{aligned}\tag{8.29}$$

The minimal version of $\Delta^{\rm f-h}H'_{5+6\rm PN}$, (7.30), leads to the following simple value for $K^{\rm f-h,circ}_{5+6\rm pn}(j)$:

$$K^{\rm f-h,circ}_{\rm min}(j) = -\frac{32}{5}\nu^3\frac{\eta^{12}}{j^{12}}\left(\frac{17719}{168}+D\right).\tag{8.30}$$

Again, if one does not use the minimal version of $\Delta^{\rm f-h}H'_{5+6\rm PN}$, the total periastron advance is easily checked to depend only on the shifted versions of the undetermined parameters $\bar{d}_5^{\nu^2}, \ldots$ defined in Eqs. (7.33)–(7.35).

It is useful to express both the binding energy and the periastron advance along circular orbits in terms of the dimensionless frequency variable $x=(GM\Omega_\phi/c^3)^{2/3}$ by replacing j as a function of x. For simplicity, we henceforth use the minimal version, Eqs. (7.28), of the flexibility factor [corresponding to the explicit minimal Hamiltonian contribution (7.30)]. (Accordingly, we replace the undetermined parameters by their minimal values.)

We then find the following explicit relation between j and x:

$$
\begin{aligned}
j = \frac{1}{\sqrt{x}}\Bigg\{ & 1 + \left(\frac{1}{6}\nu + \frac{3}{2}\right)x + \left(\frac{1}{24}\nu^2 - \frac{19}{8}\nu + \frac{27}{8}\right)x^2 \\
& + \left[\frac{135}{16} + \frac{7}{1296}\nu^3 + \frac{31}{24}\nu^2 + \left(\frac{41}{24}\pi^2 - \frac{6889}{144}\right)\nu\right]x^3 \\
& + \Bigg[\frac{2835}{128} - \frac{55}{31104}\nu^4 - \frac{215}{1728}\nu^3 + \left(\frac{356035}{3456} - \frac{2255}{576}\pi^2\right)\nu^2 \\
& + \left(-\frac{128}{3}\gamma - \frac{6455}{1536}\pi^2 - \frac{256}{3}\ln(2) - \frac{64}{3}\ln(x) + \frac{98869}{5760}\right)\nu\Bigg]x^4 \\
& + \Bigg[\frac{15309}{256} - \frac{1}{768}\nu^5 - \frac{55}{768}\nu^4 + \left(\frac{451}{128}\pi^2 - \frac{25189}{256}\right)\nu^3 \\
& + \left(\frac{1312}{15}\ln(x) + \frac{1944}{7}\ln(3) + \frac{21337}{1536}\pi^2 - 2a_6^{\nu^2\,\mathrm{min}} + \frac{6976}{105}\ln(2) + \frac{2624}{15}\gamma - \frac{341671}{1440}\right)\nu^2 \\
& + \left(\frac{59112343}{44800} + \frac{9976}{105}\ln(x) - \frac{486}{7}\ln(3) + \frac{47344}{105}\ln(2) + \frac{19952}{105}\gamma - \frac{126779}{768}\pi^2\right)\nu\Bigg]x^5 - \frac{89024}{1575}\pi\nu x^{11/2} \\
& + \Bigg[\frac{168399}{1024} - \frac{1729}{6718464}\nu^6 - \frac{3283}{248832}\nu^5 + \left(\frac{18298567}{373248} - \frac{173635}{124416}\pi^2\right)\nu^4 \\
& + \Bigg(-\frac{41216}{135}\gamma - \frac{7}{3}a_7^{\nu^3\,\mathrm{min}} - \frac{4221791}{110592}\pi^2 + \frac{7}{2}a_6^{\nu^2\,\mathrm{min}} + 1134\ln(3) \\
& - \frac{20608}{135}\ln(x) + \frac{49890383}{51840} - \frac{240112}{81}\ln(2)\Bigg)\nu^3 \\
& + \Bigg(\frac{99652}{81}\ln(2) - \frac{7}{3}a_7^{\nu^2\,\mathrm{min}} - \frac{76581497731}{14515200} - \frac{11767}{2304}\pi^4 \\
& + \frac{54738593}{110592}\pi^2 - \frac{166324}{135}\gamma - \frac{5751}{2}\ln(3) - \frac{7}{2}a_6^{\nu^2\,\mathrm{min}} - \frac{83162}{135}\ln(x)\Bigg)\nu^2 \\
& + \Bigg(\frac{247758680837}{43545600} + \frac{178844}{1215}\ln(x) + \frac{357688}{1215}\gamma - \frac{47656}{243}\ln(2) \\
& + \frac{19606111}{786432}\pi^4 + 648\ln(3) - \frac{5802762665}{5308416}\pi^2\Bigg)\nu\Bigg]x^6\Bigg\}.
\end{aligned}
\tag{8.31}
$$

Using the latter relation, the binding energy as a function of x reads

$$
E^{\mathrm{tot,circ}}_{\leq 6\mathrm{PN}}(x) = Mc^2 + E^{\mathrm{tot,circ}}_{\leq 4\mathrm{PN}}(x) + E^{\mathrm{tot,circ}}_{5+5.5+6\mathrm{PN}}(x), \tag{8.32}
$$

where $E^{\mathrm{tot,circ}}_{\leq 4\mathrm{PN}}(x)$ is given by Eq. (5.5) of Ref. [4], and where

$$
\begin{aligned}
E^{\rm tot,circ}_{5+5.5+6\rm PN}(x) = -\frac{\mu}{2}x\Bigg\{\Bigg[&-\frac{45927}{512}+\left(-\frac{228916843}{115200}-\frac{23672}{35}\ln(2)-\frac{9976}{35}\gamma+\frac{729}{7}\ln(3)-\frac{4988}{35}\ln(x)+\frac{126779}{512}\pi^2\right)\nu\\
&\left(\frac{1004021}{2880}+3a_6^{\nu^2\,\rm min}-\frac{2916}{7}\ln(3)-\frac{3488}{35}\ln(2)-\frac{21337}{1024}\pi^2-\frac{656}{5}\ln(x)-\frac{1312}{5}\gamma\right)\nu^2\\
&+\left(\frac{75567}{512}-\frac{1353\pi^2}{256}\right)\nu^3+\frac{55}{512}\nu^4+\frac{1}{512}\nu^5\Bigg]x^5+\frac{178048}{1575}\pi\nu x^{11/2}\\
&+\Bigg[-\frac{264627}{1024}+\Bigg(\frac{9118627045}{5308416}\pi^2-\frac{389727504721}{43545600}-\frac{7128}{7}\ln(3)-\frac{30809603}{786432}\pi^4-\frac{3934568}{8505}\gamma\\
&-\frac{1967284}{8505}\ln(x)+\frac{74888}{243}\ln(2)\Bigg)\nu\\
&+\Bigg(\frac{18491}{2304}\pi^4+\frac{11}{2}a_6^{\nu^2\,\rm min}-\frac{86017789}{110592}\pi^2+\frac{914782}{945}\ln(x)+\frac{63261}{14}\ln(3)\\
&+\frac{120889797143}{14515200}+\frac{11}{3}a_7^{\nu^2\,\rm min}-\frac{156596}{81}\ln(2)+\frac{1829564}{945}\gamma\Bigg)\nu^2\\
&+\Bigg(\frac{64768}{135}\gamma+\frac{32384}{135}\ln(x)-\frac{11}{2}a_6^{\nu^2\,\rm min}-1782\ln(3)+\frac{6634243}{110592}\pi^2\\
&+\frac{2641232}{567}\ln(2)-\frac{15582935}{10368}+\frac{11}{3}a_7^{\nu^3\,\rm min}\Bigg)\nu^3+\left(\frac{272855\pi^2}{124416}-\frac{28754891}{373248}\right)\nu^4+\frac{5159}{248832}\nu^5+\frac{2717}{6718464}\nu^6\Bigg]x^6\Bigg\}.
\end{aligned}
\tag{8.33}
$$

Similarly, when using the minimal version of the flexibility factor, the periastron advance expressed in terms of x reads

$$
\begin{aligned}
K^{\rm tot,circ}_{\le 6\rm PN}(x) = 1 + 3x + &\left(\frac{27}{2}-7\nu\right)x^2+\left[\frac{135}{2}+7\nu^2+\left(-\frac{649}{4}+\frac{123}{32}\pi^2\right)\nu\right]x^3\\
&+\Bigg[\frac{2835}{8}+\left(\frac{48007}{3072}\pi^2-\frac{1256}{15}\ln(x)-\frac{275941}{360}-\frac{1458}{5}\ln(3)-\frac{2512}{15}\gamma-\frac{592}{15}\ln(2)\right)\nu\\
&+\left(-\frac{451}{32}\pi^2+\frac{5861}{12}\right)\nu^2-\frac{98}{27}\nu^3\Bigg]x^4\\
&+\Bigg[\frac{15309}{8}+\left(\frac{9477}{35}\ln(3)+\frac{3928}{35}\gamma+\frac{1964}{35}\ln(x)-\frac{1056789}{1600}-\frac{953995}{2048}\pi^2-\frac{26344}{35}\ln(2)\right)\nu\\
&+\Bigg(\frac{19832}{45}\ln(x)-\frac{1}{2}\bar{d}_5^{\nu^2\,\rm min}-\frac{186997}{2304}\pi^2+\frac{42385559}{15120}-\frac{15}{2}a_6^{\nu^2\,\rm min}\\
&+\frac{343408}{45}\ln(2)-\frac{95742}{35}\ln(3)+\frac{39664}{45}\gamma\Bigg)\nu^2+\left(-\frac{6512}{9}+\frac{1025}{48}\pi^2\right)\nu^3+\frac{70}{81}\nu^4\Bigg]x^5-\frac{99938}{315}\pi\nu x^{11/2}\\
&+\Bigg[\frac{168399}{16}+\Bigg(\frac{343173600941}{12700800}-\frac{1261899}{280}\ln(3)-\frac{9765625}{4536}\ln(5)+\frac{12925966}{945}\ln(2)\\
&-\frac{10245894299}{1179648}\pi^2+\frac{7335303}{65536}\pi^4+\frac{1287518}{945}\gamma+\frac{643759}{945}\ln(x)\Bigg)\nu\\
&+\Bigg(-\frac{185881}{6144}\pi^2-\frac{3}{2}\bar{d}_5^{\nu^2\,\rm min}-\frac{21}{2}a_7^{\nu^2\,\rm min}-56a_6^{\nu^2\,\rm min}+\frac{388640863}{20160}\\
&+\frac{943569}{140}\ln(3)+\frac{5043}{2048}\pi^4+\frac{9765625}{756}\ln(5)-\frac{38699404}{945}\ln(2)-\frac{86092}{105}\gamma-\frac{43046}{105}\ln(x)-\frac{1}{2}\bar{d}_6^{\nu^2\,\rm min}\Bigg)\nu^2\\
&+\Bigg(-\frac{10176}{5}\gamma-\frac{88248109}{15120}-\frac{9765625}{504}\ln(5)-\frac{2930337}{280}\ln(3)+\frac{1499825}{9216}\pi^2+20a_6^{\nu^2\,\rm min}\\
&+\frac{1836992}{35}\ln(2)+\frac{4}{3}\bar{d}_5^{\nu^2\,\rm min}-\frac{21}{2}a_7^{\nu^3\,\rm min}-\frac{5088}{5}\ln(x)-\frac{32}{5}D\Bigg)\nu^3+\left(-\frac{1681}{96}\pi^2+\frac{22991}{36}\right)\nu^4\Bigg]x^6.
\end{aligned}
\tag{8.34}
$$

The 4PN-level periastron advance (along circular orbits) was first obtained in Refs. [55,67], and later rederived by a different approach in Ref. [92]. Reference [67] also derived the 5.5PN periastron advance. The terms $O(x^5)$ and $O(x^6)$ corresponding to the 5PN and 6PN orders, respectively, are computed here for the first time, modulo the undetermined parameters $\bar{d}_5^{\nu^2\,\mathrm{min}}$, $a_6^{\nu^2\,\mathrm{min}}, \ldots$, that enter the minimal version defined in Eqs. (7.28). We recall that, when using nonzero values of the unconstrained flexibility parameters, any physical quantity will be given by the same expression as the minimal one, with the qualification that the parameters $a_6^{\nu^2\,\mathrm{min}}, \ldots$, would be replaced by $a_6^{\nu^2\mathrm{shifted}}$, etc., as defined in Eqs. (7.35). By contrast, the linear-in-ν part of these coefficients is fully determined, reproducing the corresponding known terms [93] in the EOB function $\rho(x)$ such that $K^{-2}(x) = 1\text{–}6x + \nu\rho(x) + O(\nu^2)$.

IX. DISCUSSION

The recent renewed interest in the gravitational scattering of a two-body system has led to further improvements in the associated analytical modeling within PN-PM theory. In this work we have raised the present knowledge of the nonlocal-in-time part of the scattering angle at the 6PN level, and at the next-to-next-to-leading order in the large eccentricity of the orbital dynamics. The intricacy of the NNLO level in the scattering angle shows up in the appearance of $\zeta(3)$ in some of the integrals making up the final result, see Eqs. (3.45) and (3.46). It also shows up in the fact that we could not compute analytically a third integral [namely c_{42} or equivalently d_{42}, Eq. (4.14), Table VI] entering the final result, though we did evaluate it numerically. Going beyond the NNLO in the large eccentricity expansion remains a challenge for future calculations. By considering the mass-ratio dependence of the scattering angle, we discovered in passing a hidden simplicity in the mass-ratio dependence of the gravitational-wave energy loss of a two-body system (see Sec. VII B). The mass-ratio dependence of the nonlocal scattering angle allowed us to determine (in Sec. VII) the contribution to the Hamiltonian linked to the flexibility factor $f(t)$. In particular, we discussed a minimal way to fix the residual gauge freedom present in the choice of $f(t)$, see Eqs. (7.28) and (7.30).

Besides our results on the scattering angle at the 6PN level, we gave several other gauge-invariant characterizations of the nonlocal-in-time dynamics. We computed the nonlocal part of the averaged (Delaunay) Hamiltonian for ellipticlike motions up to the tenth order in eccentricity, see Sec. IX and Appendix F. We then extracted from the latter results two (partial but useful) physical observables: the energy and the periastron precession along circular orbits. We expressed the latter quantities both in terms of the angular momentum and in terms of the orbital frequency. Additional results and details are presented in several appendixes. In particular, (i) the details of our frequency-domain computations are presented in Appendixes B and C, (ii) Appendix G completes the information about the h-route nonlocal dynamics by giving the explicit value of the $O(p_r^8)$ part of the corresponding EOB Q potential, while (iii) Appendix H gives the elliptic-motion average of the $\ln(r_{12}^h/s)$ part of the Hamiltonian.

Though our results for the nonlocal dynamics are complete, our method has allowed us to compute the complementary local dynamics only modulo a small number of undetermined numerical parameters. Namely, two parameters at the 5PN level, and four at the 6PN level. Recent progress in the computer-aided evaluation of the 5PN-level dynamics of binary systems [59–62] gives hope that it might become soon possible to extract the two missing 5PN coefficients (denoted $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$) by comparing the observables deducible from a 5PN-accurate Hamiltonian computed in (say) harmonic coordinates with the gauge-invariant functions we presented above, thereby completing the knowledge of the 5PN dynamics. However several of the subtleties we had to cope with at 5PN might stand in the way.

We have particularly in mind the fact that our method uniquely determines all the terms quadratic in one mass in the action by a matching between the near zone (potential modes) and the wave zone (soft radiation modes) based on the use of a global Green's function (computed by means of black-hole perturbation theory). This well-defined nearzone-wavezone matching is similar to the one that was used, at the 4PN level, in Ref. [4] by combining the globally matched 4PN-level self-force result of Ref. [94], with the 4PN near-zone computation of Ref. [53]. By contrast, the EFT-based derivations of the full (local-plus-nonlocal) 4PN dynamics in Refs. [58,59] have combined the results of two different EFT-like computations (namely a wave-zone EFT computation [95,96], and a near-zone EFT one [57,59]) without showing in detail how this combination comes out automatically by applying the "strategy of regions" [97] to the original point-particle action (i.e., by decomposing the original, PM-expanded, but not PN-expanded, point-particle action into complementary contributions coming from two different regions of momentum space). This absence of a detailed, *ab initio* application of the strategy of regions at the 4PN level makes us expect that it will be difficult for a direct EFT computation of the action to unambiguously apply, in a technically complete way, the strategy of regions at the more intricate 5PN level.

Having this potential difficulty in mind, we therefore suggest to use, within the EFT approach, an analog of the strategy used in [4] at the 4PN level. Indeed, the basic fact underlying the success (and completeness) of this strategy is that the sole possible ambiguity in combining the near-zone Hamiltonian with the wave-zone one comes from combining the *logarithmic* infrared divergence entering the former computation, with the *logarithmic* ultraviolet divergence enter the latter one. In other words, if we introduce (like in the old-style computations of the Lamb shift) an intermediate scale s (with $r_{12} \ll s \ll c/\Omega_\phi$), the former computation contains a term $2\frac{GH}{c^5}\mathcal{F}^{\rm GW}(t)\ln(\frac{r_{12}^h(t)}{s_{\rm NZ}})$ while the latter one contains a term $2\frac{GH}{c^5}\mathcal{F}^{\rm GW}(t)\ln(\Omega_\phi s_{\rm WZ}/c)$. Here, $s_{\rm NZ}$ denotes the intermediate scale s when it is used as an infrared cutoff in a near-zone computation (involving potential modes), while $s_{\rm WZ}$ denotes the intermediate scale s when it is used as an ultraviolet cutoff in a wave-zone computation (involving radiation modes). In summing the results of these two regions, the intermediate scale s should disappear, but any ambiguity in the identification between $s_{\rm NZ}$ and $s_{\rm WZ}$ will introduce an ambiguity in the total Hamiltonian equal to

$$H^C = 2C\frac{GH}{c^5}\mathcal{F}^{\rm GW}(t), \tag{9.1}$$

where

$$C = \ln\left(\frac{s_{\rm WZ}}{s_{\rm NZ}}\right), \tag{9.2}$$

is some pure number. We emphasize here that the same result [presence of the single-parameter ambiguity (9.2)] holds also at the 5PN and 6PN levels because the only delicate divergences[18] entering the near-zone and wave-zone computations are logarithmic, and have both the same, known coefficient $2\frac{GH}{c^5}\mathcal{F}^{\rm GW}(t)$.

At the 4PN level, Ref. [4] had introduced such a single logarithmic ambiguity constant and had shown how the sum of the near-zone (local[19]) Hamiltonian and the wave-zone (tail-related) one, together with the use of the globally matched self-force 4PN Hamiltonian [94], led to a unique answer for the full (local-plus-nonlocal) Hamiltonian. The advantage of this strategy is that it is enough to know three partial results to apply it, namely, (i) a knowledge of the near-zone (potential-modes) Hamiltonian restricted to the scales $r < s_{\rm NZ}$, (ii) a knowledge of the wave-zone Hamiltonian, restricted to the scales $r > s_{\rm WZ}$, and (iii) a knowledge of the globally matched self-force result [which unambiguously determines the $O(\nu^2)$ part of the total Hamiltonian]. Our method provides explicit (and complete) results for the items (ii) and (iii), while it needs to be completed by a near-zone computation for determining the undetermined parameters $a_6^{\nu^2}$, etc., entering our local Hamiltonian.

[18]We assume here that the (unphysical [98]) ultraviolet divergences due to the use of a point-mass description have been separately regularized; e.g., by using dimensional regularization.

[19]Note that in Ref. [4] and in the present discussion the meaning of "local" is different from the one used in our method.

From the practical point of view, we are therefore suggesting to compare (say at the 6PN level) the gauge-invariant content of

$$H^{\rm EFT,tot}_{\rm 6PN} = H^{\rm EFT,loc,s}_{\rm 6PN} + H^C - \frac{GH}{c^5}{\rm Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t+\tau), \tag{9.3}$$

to that of our full Hamiltonian

$$H^{\rm our,tot}_{\rm 6PN} = H^{\rm loc,f}_{\rm 6PN} + H^{\rm nonloc,f}_{\rm 6PN} = H^{\rm loc,f}_{\rm 6PN} + H^{\rm nonloc,h}_{\rm 4+5+6PN} + \Delta^{\rm f-h}H_{\rm 5+6PN}. \tag{9.4}$$

As

$$H^{\rm nonloc,h}_{\rm 4+5+6PN}(t) = -\frac{GH_{\rm tot}}{c^5}{\rm Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t+\tau) + 2\frac{GH_{\rm tot}}{c^5}\mathcal{F}^{\rm GW}_{\rm 2PN}(t)\ln\left(\frac{r_{12}^h(t)}{s}\right), \tag{9.5}$$

we see that the identification between the two Hamiltonians boils down to identifying what one can call their *near-zone parts*, namely, on the one hand,

$$H^{\rm EFT,NZ}_{\rm 6PN} = H^{\rm EFT,loc,s}_{\rm 6PN} + H^C, \tag{9.6}$$

where s denotes any scale used to regularize the infrared divergence of $H^{\rm EFT,loc}$, and, on the other hand,

$$H^{\rm our,NZ}_{\rm 6PN} = H^{\rm loc,f}_{\rm 6PN} + \Delta^{\rm f-h}H_{\rm 5+6PN} + 2\frac{GH}{c^5}\mathcal{F}^{\rm GW}_{\rm 2PN}(t)\ln\left(\frac{r_{12}^h(t)}{s}\right). \tag{9.7}$$

There are various ways to identify (in a gauge-invariant manner) these two near-zone Hamiltonians. One can look for a canonical transformation mapping on into the other one, or one can identify gauge-invariant observables. We have provided above (and in our previous papers [2,3]) several gauge-invariant functions that can be used in this respect. However, as the last term on the right-hand side of Eq. (9.7) has been incorporated in our recent developments into the nonlocal part of the Hamiltonian, and was not separately studied (in a gauge-invariant way), we decided to complete our gauge-invariant characterization of the near-zone dynamics by giving the value of its Delaunay average, namely

$$\langle H^{\rm 4+5+6PN}_{\rm nonloc,ln,h}\rangle = \frac{1}{\oint dt}\oint 2\frac{GH_{\rm tot}}{c^5}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln\left(\frac{r_{12}^h(t)}{s}\right)dt. \tag{9.8}$$

The explicit value of the latter 6PN-accurate Delaunay average will be found in Appendix H as a function of a_r^h and e_t^h (up to the tenth order in e_t^h).

Summarizing, the identification between Eqs. (9.6) and (9.7) yields, in our opinion, an efficient way (avoiding a full use of the strategy of regions) to determine at once the values of our undetermined parameters $a_6^{\nu^2}, \ldots,$ and the value of the single near-zone–wave-zone separation ambiguity constant C [which we have incorporated here in Eq. (9.6)]. Our determination of most of the ν dependence of the Hamiltonian will also provide many checks of the computation of $H_{6\mathrm{PN}}^{\mathrm{EFT,NZ}}$.

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APPENDIX A: COMPENDIUM OF USEFUL PN RESULTS

We collect in this appendix some known results in PN theory. When working at the 6PN level we often need only fractionally 2PN-accurate results on the dynamics. In some parts, we only need 1PN-level results such as the Einstein-Infeld-Hoffmann-Fichtenholz 1PN Lagrangian for the relative motion (see, e.g., Ref. [81])

$$\frac{\mathcal{L}^h_{1\mathrm{PN}}}{\mu} = \frac{1}{2}v^2 + \frac{GM}{r} + \eta^2\left\{\frac{1}{8}(1-3\nu)v^4 + \frac{GM}{2r}\left[(3+\nu)v^2 + \nu(\mathbf{n}\cdot\mathbf{v})^2 - \frac{GM}{r}\right]\right\}, \tag{A1}$$

where $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$. This determines the corresponding momenta

$$p_r = \frac{\partial \mathcal{L}^h_{1\mathrm{PN}}}{\partial \dot{r}} = C_r \dot{r}, \qquad p_\phi = \frac{\partial \mathcal{L}^h_{1\mathrm{PN}}}{\partial \dot{\phi}} = C_\phi r^2 \dot{\phi}, \tag{A2}$$

with

$$C_r = 1 + \eta^2\left(\frac{(1-3\nu)}{2}v^2 + \frac{GM}{r}(3+2\nu)\right),$$
$$C_\phi = 1 + \eta^2\left(\frac{1-3\nu}{2}v^2 + (3+\nu)\frac{GM}{r}\right), \tag{A3}$$

so that $p^2 = p_r^2 + \frac{p_\phi^2}{r^2} = C_r^2\dot{r}^2 + C_\phi^2 r^2\dot{\phi}^2$.

The corresponding 1PN-accurate Hamiltonian (expressed in terms of $p = p^{\mathrm{phys}}/\mu$; and using $c = 1$) reads

$$\frac{H_{1\mathrm{PN}}(r,p_r,j) - M}{\mu} = \left(\frac{1}{2}p^2 - \frac{GM}{r}\right) + \eta^2\left[\frac{1}{8}(3\nu-1)p^4 - \frac{GM}{2r}(\nu+3)p^2 - \frac{GM}{2r}\nu p_r^2 + \frac{(GM)^2}{2r^2}\right]. \tag{A4}$$

We often rescale r according to $r^{\mathrm{phys}} = GMr$.

We will also need the expression of the 1PN-accurate gravitational-wave energy flux [89] in terms of $r = r^{\mathrm{phys}}/GM$ and p:

$$\mathcal{F}^{\mathrm{GW}}_{1\mathrm{PN}}(p,p_r,r) = \frac{8}{15}\nu^2\frac{(12p^2 - 11p_r^2)}{r^4} + \eta^2\nu^2\left\{\frac{1}{r^4}\left[\left(\frac{1374}{35} - \frac{248}{7}\nu\right)p_r^4 + \left(-\frac{5332}{105} + \frac{248}{7}\nu\right)p^2p_r^2 + \left(\frac{898}{105} + \frac{104}{35}\nu\right)p^4\right] + \frac{1}{r^5}\left[\left(\frac{176}{21}\nu + \frac{9568}{105}\right)p_r^2 + \left(-\frac{9472}{105} - \frac{1024}{105}\nu\right)p^2\right] + \frac{1}{r^6}\left(\frac{32}{105} - \frac{128}{105}\nu\right)\right\}. \tag{A5}$$

The parameters entering the quasi-Keplerian parametrization, Eq. (3.4), of the hyperbolic motion (in harmonic coordinates) are listed in Table VIII, as functions of the variables

$$\bar{E} \equiv \frac{E_{\mathrm{tot}} - Mc^2}{\mu c^2}; \qquad j \equiv \frac{cJ}{GM\mu}. \tag{A6}$$

Let us also recall (from Table II in Ref. [3]) the expressions of the (harmonic-coordinates) rescaled semimajor axis and time eccentricity entering the 2PN-accurate quasi-Keplerian representation of elliptic motion in terms of $\bar{E}$ and j:

$$a_r = \frac{1}{(-2\bar{E})}\left\{1 + \frac{(-2\bar{E})}{4}(-7+\nu)\eta^2 + \frac{(-2\bar{E})^2}{16}\left[1+\nu^2+\frac{16}{(-2\bar{E})j^2}(7\nu-4)\right]\eta^4\right\},$$

$$e_t^2 = 1 + 2\bar{E}j^2 + \frac{(-2\bar{E})}{4}[-8(1-\nu) - (-2\bar{E})j^2(-17+7\nu)]\eta^2 + \frac{(-2\bar{E})^2}{8}\left[4(3+18\nu+5\nu^2) - (-2\bar{E})j^2(112-47\nu+16\nu^2) - \frac{16}{(-2\bar{E})j^2}(-4+7\nu) - 24\sqrt{-2\bar{E}}j(-5+2\nu) + \frac{24}{\sqrt{-2\bar{E}}j}(-5+2\nu)\right]\eta^4. \quad \text{(A7)}$$

Beware that the latter elliptic definition of e_t^2 is not equal to the analytic continuation in $\bar{E}$ of its hyperbolic counterpart, listed in Table VIII (while $\bar{a}_r$ is the analytic continuation of $-a_r$). Using the rescaled action variables

$$i_r \equiv \frac{cI_R}{GM\mu}, \quad i_\varphi \equiv \frac{cI_\varphi}{GM\mu} \equiv j, \quad i_{r\varphi} \equiv i_r + i_\varphi \equiv i_r + j, \quad \text{(A8)}$$

with

$$i_r = -j - \frac{1}{\sqrt{-2\bar{E}}} + \left[\frac{3}{j} - \frac{1}{8}(\nu-15)\sqrt{-2\bar{E}}\right]\eta^2 + \left[-\frac{5(2\nu-7)}{4j^3} + \frac{3(2\nu-5)(-2\bar{E})}{4j} - \frac{1}{128}(3\nu^2+30\nu+35)(-2\bar{E})^{3/2}\right]\eta^4, \quad \text{(A9)}$$

the 2PN-accurate Delaunay Hamiltonian reads [8]

$$\bar{E}(i_r, i_\phi) = -\frac{1}{2i_{r\phi}^2}\left[1 + \frac{1}{4}\frac{24i_r + (9+\nu)i_\phi}{i_\phi i_{r\phi}^2}\eta^2 - \frac{1}{8}\frac{20i_r^3(2\nu-7) + 12i_r^2 i_\phi(10\nu-53) + 72i_r i_\phi^2(\nu-6) + i_\phi^3(-\nu^2+7\nu-81)}{i_\phi^3 i_{r\phi}^4}\eta^4\right]. \quad \text{(A10)}$$

Using this transformation, we get the following explicit (2PN-accurate) expressions for the ellipticlike parameters a_r and e_r as functions of the action variables i_r and i_ϕ:

TABLE VIII. Quasi-Keplerian representation of the hyperbolic 2PN motion (in harmonic coordinates). We use the variables $\nu = m_1 m_2/(m_1+m_2)^2$, $\bar{E}$, Eq. (1.36), and j, Eq. (1.37).

$\bar{n}$	$(2\bar{E})^{3/2}[1 + \frac{\bar{E}}{4}(15-\nu)\eta^2 + \frac{\bar{E}^2}{32}(555+30\nu+11\nu^2)\eta^4]$
$\bar{a}_r$	$\frac{1}{2\bar{E}}\{1 + \frac{\bar{E}}{2}(7-\nu)\eta^2 + \frac{\bar{E}^2}{4}[1+\nu^2-\frac{8}{\bar{E}j^2}(7\nu-4)]\eta^4\}$
e_t^2	$1 + 2\bar{E}j^2 + \bar{E}[-\bar{E}j^2(-17+7\nu) + 4(1-\nu)]\eta^2 + \bar{E}^2[2(3+18\nu+5\nu^2) + \bar{E}j^2(112-47\nu+16\nu^2) + \frac{4}{\bar{E}j^2}(-4+7\nu)]\eta^4$
e_r^2	$1 + 2\bar{E}j^2 + \bar{E}[-5\bar{E}j^2(3-\nu) + 2(-6+\nu)]\eta^2 + \bar{E}^2[30+74\nu+\nu^2 + \bar{E}j^2(80-45\nu+4\nu^2) + \frac{8}{\bar{E}j^2}(-4+7\nu)]\eta^4$
e_ϕ^2	$1 + 2\bar{E}j^2 + \bar{E}[-\bar{E}j^2(15-\nu) - 12]\eta^2 + \frac{\bar{E}^2}{4}[\frac{-416+91\nu+15\nu^2}{2\bar{E}j^2} + 2(-20+17\nu+9\nu^2) + 2\bar{E}j^2(160-31\nu+3\nu^2)]\eta^4$
f_t	$\frac{3}{2}\frac{(2\bar{E})^{3/2}}{j}(5-2\nu)\eta^4$
g_t	$-\frac{(2\bar{E})^{3/2}}{8j}\sqrt{1+2\bar{E}j^2}\nu(-15+\nu)\eta^4$
f_ϕ	$\frac{1+2\bar{E}j^2}{8j^4}(1+19\nu-3\nu^2)\eta^4$
g_ϕ	$\frac{1}{32}\frac{(1+2\bar{E}j^2)^{3/2}}{j^4}\nu(1-3\nu)\eta^4$
K	$1 + \frac{3}{j^2}\eta^2 + \frac{3}{4j^4}[-2\bar{E}j^2(-5+2\nu) + 5(7-2\nu)]\eta^4$

$$a_r = i_{r\phi}^2 - 2\frac{3i_r + 2i_\phi}{i_\phi}\eta^2 + \frac{1}{2}\frac{5i_r^3(2\nu - 7) + i_r^2 i_\phi(44\nu - 95) + 2i_r i_\phi^2(26\nu - 35) + 18i_\phi^3(\nu - 1)}{i_{r\phi}^2 i_\phi^3}\eta^4,$$

$$e_t^2 = \frac{i_r}{i_{r\phi}^2}\left[i_r + 2i_\phi + 2\frac{i_r(\nu - 1) + i_\phi(2\nu - 5)}{i_{r\phi}^2}\eta^2 - \frac{1}{2}\frac{4i_r^3(7\nu - 4) + i_r^2 i_\phi(66\nu + 25) - i_r i_\phi^2(6\nu^2 - 28\nu - 207) - 2i_\phi^3(6\nu^2 - 18\nu - 19)}{i_\phi^2 i_{r\phi}^4}\eta^4\right]. \quad \text{(A11)}$$

Another useful 2PN-accurate quantity is the (adimensionalized) radial frequency. It reads

$$\begin{aligned} n &= \frac{GM\Omega_R}{c^3} \\ &= (-2\bar{E})^{3/2}\left[1 + \frac{(-2\bar{E})}{8}(-15 + \nu)\eta^2 + \frac{(-2\bar{E})^2}{128}\left(555 + 30\nu + 11\nu^2 + \frac{192(-5 + 2\nu)}{\sqrt{-2\bar{E}}j}\right)\eta^4\right] \\ &= \frac{1}{i_{r\phi}^3}\left[1 + \frac{1}{2}\frac{(3 + \nu)i_\phi + 18i_r}{i_\phi i_{r\phi}^2}\eta^2 - \frac{3}{8}\frac{-(9 + 5\nu + \nu^2)i_\phi^3 + 4i_r i_\phi^2(-37 + 5\nu) + 6i_r^2 i_\phi(-59 + 10\nu) + 10i_r^3(2\nu - 7)}{i_\phi^3 i_{r\phi}^4}\eta^4\right]. \end{aligned} \quad \text{(A12)}$$

APPENDIX B: LARGE-ECCENTRICITY EXPANSIONS OF THE FREQUENCY-DOMAIN, NEWTONIAN-LEVEL ENERGY FLUX AND INTEGRATED TAIL ACTION

This Appendix discusses the frequency-domain computation of the Newtonian-level energy flux, Eq. (3.62), and the related integrated action, Eqs. (3.59) and (3.64). The frequency-domain integrand (3.70) is of lowest (Newtonian) order with respect to the PN expansion but is exact in its eccentricity dependence. Let us consider its expansion in inverse powers of the eccentricity at successive levels: LO, NLO, NNLO, etc. For simplicity, we use $1 = GM = G = c$ in the following.

1. Newtonian flux at the LO in the large-eccentricity expansion

The expression (3.70) can be easily evaluated at the LO in the large eccentricity expansion where $p = i\frac{u}{e_r} \to 0$ [see Eq. (3.51)]. This limit entails a big simplification (already studied in the literature, see e.g., [52,85,86]) which leads to the following expression:

$$\mathcal{I}_{\rm N}^{\rm LO}(u) = 32e_r^4\nu^2\bar{a}_r^7 u^2[(u^2 + 1)K_1^2(u) + 3uK_0(u)K_1(u) + \frac{1}{3}(3u^2 + 1)K_0^2(u)]. \quad \text{(B1)}$$

Using the notation introduced in Ref. [52]

$$\mathcal{F}(u) = \left(\frac{u^2}{3} + u^4\right)K_0^2(u) + 3u^3K_0(u)K_1(u) + (u^2 + u^4)K_1^2(u), \quad \text{(B2)}$$

we find

$$\mathcal{I}_{\rm N}^{\rm LO}(u) = 32\nu^2 e_r^4\bar{a}_r^7\mathcal{F}(u), \quad \text{(B3)}$$

so that

$$\Delta E_{\rm GW}^{\rm LO} = \frac{32}{5\pi}\frac{\bar{n}^7\bar{a}_r^7}{e_r^3}\nu^2\int_0^\infty du\mathcal{F}(u). \quad \text{(B4)}$$

When using the Newtonian-level relations

$$\bar{n} = (\bar{a}_r)^{-3/2} = p_\infty^3, \bar{a}_r = p_\infty^{-2}, \quad \text{(B5)}$$

as well as

$$e_r = \sqrt{1 + p_\infty^2 j^2} \to e_r^{\rm LO} = p_\infty j, \quad \text{(B6)}$$

one recovers the known result for the LO gravitational-wave energy, or "splash radiation,"

$$\begin{aligned} \Delta E_{\rm GW\,N}^{\rm LO} &= \frac{32}{5\pi}\frac{p_\infty^4}{j^3}\nu^2\int_0^\infty du\mathcal{F}(u) \\ &= \frac{37}{15}\pi\frac{p_\infty^4}{j^3}\nu^2. \end{aligned} \quad \text{(B7)}$$

The tail potential $W_{1\mathrm{N}}^{(\mathrm{tail})\mathrm{LO}}$ instead turns out to be

$$W_{1\mathrm{N}}^{(\mathrm{tail})\mathrm{LO}} = \frac{64}{5\pi}\frac{p_\infty^4}{j^3}\nu^2\int_0^\infty du\mathcal{F}(u)\ln(\bar{\alpha}u) = \frac{2}{15}\frac{\pi\nu^2}{e_r^3\bar{a}_r^{7/2}}\left[100+37\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]. \quad \text{(B8)}$$

Let us then pass to the extension of these results at the higher N^{n}LO levels of approximation in the large-eccentricity expansion.

2. Working at the NNLO accuracy in $\frac{1}{e_r}$

Expanding the quantity (3.70) for large e_r up to the NNLO leads to

$$\mathcal{I}_\mathrm{N}(u) = \mathcal{I}_\mathrm{N}^\mathrm{LO} + \mathcal{I}_\mathrm{N}^\mathrm{NLO} + \mathcal{I}_\mathrm{N}^\mathrm{NNLO} + O\left(\frac{\mathcal{I}_\mathrm{N}^\mathrm{LO}}{e_r^3}\right), \quad \text{(B9)}$$

where

$$\frac{\mathcal{I}_\mathrm{N}^\mathrm{LO}}{32\bar{a}_r^7 e_r^4\nu^2} = \mathcal{F}(u), \qquad \frac{\mathcal{I}_\mathrm{N}^\mathrm{NLO}}{32\bar{a}_r^7 e_r^3\nu^2\pi} = u\mathcal{F}(u),$$
$$\frac{\mathcal{I}_\mathrm{N}^\mathrm{NNLO}}{16\bar{a}_r^7 e_r^2\nu^2} = C^{00}(u) + C^{20}(u)\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0} + C^{21}(u)\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1}, \quad \text{(B10)}$$

where

$$C^{00}(u) = -2u^2[(3u^2+1)K_0^2(u) + 7uK_0(u)K_1(u) + (1+2u^2)K_1^2(u)] + \frac{\pi^2}{u^4}\mathcal{F}(u),$$
$$C^{20}(u) = -\frac{u^4}{3}[2(3u^2+1)K_0(u) + 9uK_1(u)],$$
$$C^{21}(u) = -\frac{u^4}{3}[3uK_0(u) + 2(u^2+1)K_1(u)], \quad \text{(B11)}$$

and where the Bessel functions $K_p(u)$ and $K_{p+1}(u)$ have been Taylor-expanded around $p=0$ to second order in p,

$$\begin{aligned} K_p(u) &= K_0(u) + p\frac{\partial K_\nu(u)}{\partial\nu}\bigg|_{\nu=0} + \frac{1}{2}p^2\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0} + O(p^3) \\ &= K_0(u) + \frac{1}{2}p^2\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0} + O(p^3), \\ K_{p+1}(u) &= K_1(u) + p\frac{\partial K_\nu(u)}{\partial\nu}\bigg|_{\nu=1} + \frac{1}{2}p^2\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1} + O(p^3) \\ &= K_1(u) + \frac{p}{u}K_0(u) + \frac{1}{2}p^2\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1} + O(p^3). \end{aligned} \quad \text{(B12)}$$

In Eqs. (B12) above we have used the known results [see Eqs. (9.1.66)–(9.1.68) of Ref. [83]]

$$\frac{\partial K_\nu(u)}{\partial\nu}\bigg|_{\nu=0} = 0, \qquad \frac{\partial K_\nu(u)}{\partial\nu}\bigg|_{\nu=1} = \frac{1}{u}K_0(u). \quad \text{(B13)}$$

Moreover, in what follows the derivatives of K_ν with respect to the order will only enter integrals of the type

$$F(a,\mu) = \int_0^\infty duu^a K_\mu(u)\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0,1}. \quad \text{(B14)}$$

These integrals can be evaluated by considering the master integral

$$G(a,\mu,\nu) = \int_0^\infty duu^a K_\mu(u)K_\nu(u) = \frac{2^{a-2}}{\Gamma(a+1)}\Gamma_1\Gamma_2\Gamma_3\Gamma_4, \quad \text{(B15)}$$

where

$$\begin{aligned} \Gamma_1 &= \Gamma\left(\frac{1}{2}(a-\mu-\nu+1)\right), \\ \Gamma_2 &= \Gamma\left(\frac{1}{2}(a+\mu-\nu+1)\right), \\ \Gamma_3 &= \Gamma\left(\frac{1}{2}(a-\mu+\nu+1)\right), \\ \Gamma_4 &= \Gamma\left(\frac{1}{2}(a+\mu+\nu+1)\right), \end{aligned} \quad \text{(B16)}$$

the resulting expression being valid when the four conditions $\mathrm{Re}[a\pm\mu\pm\nu] > -1$ are all satisfied. Taking two derivatives of (B15) with respect to ν and evaluating the result at $\nu=0,1$ allows one to compute the integral (B14).

One can rewrite Eq. (B9) in various ways. For example,

$$\frac{\mathcal{I}_{\rm N}(u)}{32\bar{a}_r^7 e_r^4 \nu^2} = \mathcal{F}(u)\left[1+\frac{\pi u}{e_r}+\frac{1}{2}\left(\frac{\pi u}{e_r}\right)^2\right] - \frac{\mathcal{G}(u)+\mathcal{H}(u)}{e_r^2}, \tag{B17}$$

where $\mathcal{F}(u)$ is defined in Eq. (B2), and where we defined

$$\mathcal{G}(u) \equiv u^2[(3u^2+1)K_0^2(u)+7uK_0(u)K_1(u) +(1+2u^2)K_1^2(u)],$$
$$\mathcal{H}(u) \equiv \mathcal{H}_0(u)\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0}+\mathcal{H}_1(u)\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1}, \tag{B18}$$

with

$$\mathcal{H}_0(u) \equiv u^4\left[\left(u^2+\frac{1}{3}\right)K_0(u)+\frac{3}{2}uK_1(u)\right], = -\frac{C^{20}(u)}{2},$$
$$\mathcal{H}_1(u) \equiv u^4\left[\frac{3}{2}uK_0(u)+(u^2+1)K_1(u)\right] = -\frac{C^{21}(u)}{2}. \tag{B19}$$

The functions $\mathcal{G}(u), \mathcal{H}_0(u), \mathcal{H}_1(u)$ are such that

$$\mathcal{G}(u) = 3\mathcal{F}(u) - u^2[2uK_0(u)+(2+u^2)K_1(u)]K_1(u) \tag{B20}$$

and

$$\mathcal{H}_0(u)K_0(u)+\mathcal{H}_1(u)K_1(u) = u^2\mathcal{F}(u). \tag{B21}$$

We list in Table IX the integrals needed in order to compute both the gravitational-wave energy $\Delta E_{\rm GW}$ and the tail potential $W_1^{\rm tail}$ at the NNLO level in e_r^{-1} (but still at the Newtonian level, η^0).

The integral of $\mathcal{H}(u)$ can be written as the sum of the two pieces

$$\mathsf{h} = \int_0^\infty \mathcal{H}(u)du = \frac{\partial^2 \mathsf{h}_0(\nu)}{\partial\nu^2}\bigg|_{\nu=0}+\frac{\partial^2 \mathsf{h}_1(\nu)}{\partial\nu^2}\bigg|_{\nu=1}, \tag{B22}$$

where

$$\mathsf{h}_0(\nu) = \int_0^\infty du\mathcal{H}_0(u)K_\nu(u),$$
$$\mathsf{h}_1(\nu) = \int_0^\infty du\mathcal{H}_1(u)K_\nu(u). \tag{B23}$$

TABLE IX. Integrals needed for the Newtonian-level gravitational-wave energy $\Delta E_{\rm GW}$, and the tail potential $W_1^{\rm tail}$, at the NNLO level in e_r^{-1}.

	Expression	Value
f	$\int_0^\infty du\mathcal{F}(u)$	$\frac{37}{96}\pi^2$
$\mathsf{f}^{\ln}$	$\int_0^\infty du\mathcal{F}(u)\ln u$	$\frac{1}{96}\pi^2[100-37\gamma-111\ln(2)]$
f^u	$\int_0^\infty duu\mathcal{F}(u)$	$\frac{49}{9}$
$\mathsf{f}^{u\ln}$	$\int_0^\infty duu\mathcal{F}(u)\ln u$	$\frac{1}{54}[139-294\gamma+294\ln(2)]$
f^{u^2}	$\int_0^\infty duu^2\mathcal{F}(u)$	$\frac{297}{256}\pi^2$
$\mathsf{f}^{u^2\ln}$	$\int_0^\infty duu^2\mathcal{F}(u)\ln u$	$\frac{3}{256}\pi^2[350-99\gamma-297\ln(2)]$
g	$\int_0^\infty du\mathcal{G}(u)$	$\frac{403}{512}\pi^2$
$\mathsf{g}^{\ln}$	$\int_0^\infty du\mathcal{G}(u)\ln u$	$\frac{1}{2048}\pi^2[4591-1612\gamma-4836\ln(2)]$

We find

$$\frac{\partial^2 \mathsf{h}_0(\nu)}{\partial\nu^2}\bigg|_{\nu=0} = \frac{2007\pi^4}{8192}-\frac{179\pi^2}{80},$$
$$\frac{\partial^2 \mathsf{h}_1(\nu)}{\partial\nu^2}\bigg|_{\nu=1} = \frac{2745\pi^4}{8192}-\frac{7527\pi^2}{2560}, \tag{B24}$$

so that

$$\mathsf{h} = \frac{297\pi^4}{512}-\frac{2651\pi^2}{512}. \tag{B25}$$

The integral of $\mathcal{H}(u)\ln u$ can be computed in the same way

$$\mathsf{h}^{\ln} = \int_0^\infty \mathcal{H}(u)\ln u du = -\frac{\pi^2}{6144}[49896\zeta(3)+53437+132\gamma(27\pi^2-241) - 95436\log(2)+36\pi^2(297\log(2)-350)]. \tag{B26}$$

Finally, from

$$\int_0^\infty \mathcal{I}_{\rm N}(u)du = 32\bar{a}_r^7 e_r^4\nu^2\left\{\left[\mathsf{f}+\frac{\pi}{e_r}\mathsf{f}^u+\frac{1}{2}\left(\frac{\pi}{e_r}\right)^2\mathsf{f}^{u^2}\right] - \frac{1}{e_r^2}\mathsf{g}-\frac{1}{e_r^2}\mathsf{h}\right\}+O\left(\frac{1}{e_r^3}\right), \tag{B27}$$

we have that the first- and second-order eccentricity corrections to the (Newtonian-level) splash radiation energy (B7) read

$$\Delta E_{\rm GWN}^{\rm LO+NLO+NNLO} = \frac{\nu^2}{e_r^3\bar{a}_r^{7/2}}\left(\frac{37\pi}{15}+\frac{1568}{45e_r}+\frac{281\pi}{10e_r^2}+O\left(\frac{1}{e_r^3}\right)\right). \tag{B28}$$

Similarly,

$$W_{1\mathrm{N}}^{(\mathrm{tail})\mathrm{LO+NLO+NNLO}} = \frac{2}{15}\frac{\nu^2}{e_r^3\bar{a}_r^{7/2}}\left\{\pi\left[100+37\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]+\frac{1}{e_r}\left[\frac{2224}{9}+\frac{1568}{3}\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right)\right]\right.$$
$$\left.+\frac{\pi}{e_r^2}\left[\frac{2479}{4}+\frac{6237}{8}\zeta(3)+\frac{843}{2}\ln\left(\frac{s}{4e_r\bar{a}_r^{3/2}}\right)\right]+O\left(\frac{1}{e_r^3}\right)\right\}. \tag{B29}$$

This result allows one to fix the previously defined parameter c_{00} [see Eq. (3.45)] which could not be computed in the time domain.

3. Going at N^{3}LO in the energy flux

The next term in the e_r^{-1} expansion of Eq. (B17) is the following:

$$\frac{\mathcal{I}_\mathrm{N}(u)}{32\bar{a}_r^7e_r^4\nu^2u^6} = \ldots+\frac{1}{e_r^3}\left[-\frac{\pi}{u^5}\mathcal{H}_0(u)\frac{\partial^2K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=0}-\frac{\pi}{u^5}\mathcal{H}_1(u)\frac{\partial^2K_\nu(u)}{\partial\nu^2}\bigg|_{\nu=1}+\frac{\pi^3}{6u^3}\mathcal{F}(u)-\frac{\pi}{u^5}\mathcal{G}(u)\right]$$
$$=\ldots+\frac{1}{e_r^3}\left[-\frac{\pi}{u^5}\mathcal{H}(u)+\frac{\pi^3}{6u^3}\mathcal{F}(u)-\frac{\pi}{u^5}\mathcal{G}(u)\right]. \tag{B30}$$

Multiplying both sides by u^6 one has then

$$\frac{\mathcal{I}_\mathrm{N}(u)}{32\bar{a}_r^7e_r^4\nu^2} = \mathcal{F}(u)\left[1+\frac{\pi u}{e_r}+\frac{1}{2}\left(\frac{\pi u}{e_r}\right)^2+\frac{1}{6}\left(\frac{\pi u}{e_r}\right)^3\right]-\frac{\mathcal{G}(u)+\mathcal{H}(u)}{e_r^2}\left(1+\frac{\pi u}{e_r}\right), \tag{B31}$$

that is the $O(\frac{1}{e_r^3})$-accurate truncation of the compact expression

$$\frac{\mathcal{I}_\mathrm{N}(u)}{32\bar{a}_r^7e_r^4\nu^2} = e^{\pi u/e_r}\left[\mathcal{F}(u)-\frac{\mathcal{G}(u)+\mathcal{H}(u)}{e_r^2}\right]\bigg|_{O(\frac{1}{e_r^3})}. \tag{B32}$$

Equation (B28) is then extended as

$$\Delta E_{\mathrm{GW\,N}}^{\mathrm{LO}+\cdots+\mathrm{N^3LO}}(\bar{a}_r,e_r) = \frac{\nu^2}{e_r^3\bar{a}_r^{7/2}}\left(\frac{37\pi}{15}+\frac{1568}{45e_r}+\frac{281\pi}{10e_r^2}+\frac{7808}{45e_r^3}+O\left(\frac{1}{e_r^4}\right)\right). \tag{B33}$$

Expressing e_r and a_r in terms of p_∞ and j the above (Newtonian-level) expression becomes

$$\Delta E_{\mathrm{GW\,N}}^{\mathrm{LO}+\cdots+\mathrm{N^3LO}}(p_\infty,j) = \nu^2\left[\frac{37}{15}\pi\frac{p_\infty^4}{j^3}+\frac{1568}{45}\frac{p_\infty^3}{j^4}+\frac{122}{5}\pi\frac{p_\infty^2}{j^5}+\frac{4672}{45}\frac{p_\infty}{j^6}+O\left(\frac{1}{j^7}\right)\right]. \tag{B34}$$

No special, additional difficulties arise for the Newtonian energy flux when going up to higher orders in the large eccentricity expansion in the frequency domain.

APPENDIX C: ECCENTRICITY EXPANSION OF THE 1PN-ACCURATE FREQUENCY-DOMAIN ENERGY FLUX AND TAIL ACTION

Let us now consider the 1PN corrections ($\propto\eta^2$) to the (frequency-domain) energy flux and tail action. The corresponding integrands are linear in ν (after factoring out an overall factor). In this case Eq. (3.75) should be used, and complications arise already at the NLO as we are going to show. We have been able to compute the NNLO level too, but the associated expressions are very long and will not be displayed below.

Let us write ($1=GM=G=c$) the 1PN contribution to the energy flux as follows:

$$\Delta E^{1\mathrm{PN}} = \frac{1}{\pi e_r\bar{a}_r^{3/2}}\int_0^\infty du\mathcal{F}_\mathrm{GW}^{1\mathrm{PN}}(u), \tag{C1}$$

with a subsequent large-eccentricity expansion:

$$\mathcal{F}_\mathrm{GW}^{1\mathrm{PN}} = \frac{\nu^2}{e_r^2\bar{a}_r^3}\left[\mathcal{F}_\mathrm{GW}^{1\mathrm{PN,LO}}+\frac{1}{e_r}\mathcal{F}_\mathrm{GW}^{1\mathrm{PN,NLO}}+\frac{1}{e_r^2}\mathcal{F}_\mathrm{GW}^{1\mathrm{PN,NNLO}}+O\left(\frac{1}{e_r^3}\right)\right]. \tag{C2}$$

We recall that, at 1PN, each term $\mathcal{F}_\mathrm{GW}^{1\mathrm{PN,N^nLO}}$ depends on ν linearly, i.e.,

$$\mathcal{F}_\mathrm{GW}^{1\mathrm{PN,N^nLO}} = \mathcal{F}_{\mathrm{GW},\nu^0}^{1\mathrm{PN,N^nLO}}+\nu\mathcal{F}_{\mathrm{GW},\nu^1}^{1\mathrm{PN,N^nLO}}. \tag{C3}$$

We then find the explicit expressions

$$\begin{aligned}
\mathcal{F}^{\rm 1PN,LO}_{{\rm GW},\nu^0} &= -\frac{8}{105}u^2[(-10u^4+82u^2+156)K_0(u)^2\\
&\quad -4u(-43+61u^2)K_0(u)K_1(u)\\
&\quad +(-10u^4-45u^2-300)K_1^2(u)],\\
\mathcal{F}^{\rm 1PN,LO}_{{\rm GW},\nu^1} &= -\frac{8}{105}u^2[(-16+40u^4)K_0^2(u)\\
&\quad +(-48u^3-64u)K_0(u)K_1(u)\\
&\quad +(-4u^2+40u^4)K_1^2],\\
\mathcal{F}^{\rm 1PN,NLO}_{{\rm GW},\nu^0} &= -\frac{64}{21}u^3\left(-\frac{u^4}{4}+\frac{23}{2}u^2+\frac{141}{20}\right)K_0^2(u)\\
&\quad +\frac{128}{35}u^4\left(\frac{61}{12}u^2-\frac{653}{24}\right)K_0(u)K_1(u)\\
&\quad -\frac{64}{21}u^3\left(-\frac{u^4}{4}+\frac{333}{40}u^2+\frac{39}{20}\right)K_1^2(u)\\
&\quad -\frac{24}{5}iu^4[(A(u)+uB(u))K_0(u)\\
&\quad +(B(u)+2uA(u))K_1(u)],\\
\mathcal{F}^{\rm 1PN,NLO}_{{\rm GW},\nu^1} &= \pi\left[-\frac{64}{21}u^3\left(u^4-\frac{21}{20}u^2-\frac{3}{4}\right)K_0^2(u)\right.\\
&\quad +\frac{128}{35}u^4\left(u^2+\frac{95}{24}\right)K_0(u)K_1(u)\\
&\quad \left.-\frac{64}{21}u^3\left(u^4-\frac{23}{20}u^2-\frac{21}{20}\right)K_1^2(u)\right], \qquad \text{(C4)}
\end{aligned}$$

where

$$\begin{aligned}
A &= -\frac{i}{2}(G_2(u)+G_2^*(u))+\frac{i}{2}(G_{-2}(u)+G_{-2}^*(u))\\
&= -i[G_2^{\rm S}(u)-G_{-2}^{\rm S}(u)]\\
&= -2iG_{[2]}^{\rm S}(u),\\
B &= -\frac{1}{4}(G_3(u)-G_3^*(u))+\frac{1}{4}(G_{-3}(u)-G_{-3}^*(u))\\
&\quad +\frac{5}{4}(G_1(u)-G_1^*(u))+\frac{5}{4}(G_{-1}(u)-G_{-1}^*(u))\\
&= -\frac{1}{2}G_3^{\rm A}(u)-\frac{1}{2}G_{-3}^{\rm A}(u)+\frac{5}{2}G_1^{\rm A}(u)+\frac{5}{2}G_{-1}^{\rm A}(u)\\
&= -G_{(3)}^{\rm A}(u)+5G_{(1)}^{\rm A}(u). \qquad \text{(C5)}
\end{aligned}$$

Here we introduced the notation

$$G_n(u)\equiv\int_{-\infty}^{\infty}dv\arctan\left(\tanh\frac{v}{2}\right)e^{iu\sinh v-nv}, \qquad \text{(C6)}$$

as well as $G_n^{\rm S}(u)=\frac{1}{2}(G_n(u)+G_n^*(u))$, $G_n^{\rm A}(u)=\frac{1}{2}(G_n(u)-G_n^*(u))$ and the symmetry-related expressions $G_{(n)}^{\rm S}(u)=\frac{1}{2}(G_n^{\rm S}(u)+G_{-n}^{\rm S}(u))$, $G_{[n]}^{\rm S}(u)=\frac{1}{2}(G_n^{\rm S}(u)-G_{-n}^{\rm S}(u))$, $G_{(n)}^{\rm A}(u)=\frac{1}{2}(G_n^{\rm A}(u)+G_{-n}^{\rm A}(u))$, and $G_{[n]}^{\rm A}(u)=\frac{1}{2}(G_n^{\rm A}(u)-G_{-n}^{\rm A}(u))$. Due to parity reasons

$$G_{(n)}^{\rm S}(u)\equiv 0\equiv G_{[n]}^{\rm A}(u). \qquad \text{(C7)}$$

This, however, has no effect on A and B which only contain $G_{[n]}^{\rm S}(u)$ and $G_{(n)}^{\rm A}(u)$.

Going to the NNLO, the energy flux also contains terms involving the derivatives of the Bessel K functions with respect to the order. Furthermore, integrals over v enter the term $\mathcal{F}^{\rm 1PN,NNLO}_{{\rm GW},\nu^0}$. All integrations can be done analytically, leading to

$$\begin{aligned}
\Delta E_{\rm GW}^{\rm 1PN} &= \frac{\nu^2}{e_r^3\bar{a}_r^{9/2}}\left[\left(\frac{1143}{280}-\frac{37}{30}\nu\right)\pi+\frac{1}{e_r}\left(\frac{944}{1575}-\frac{1136}{45}\nu\right)\right.\\
&\quad \left.+\frac{1}{e_r^2}\left(-\frac{22333}{560}-\frac{609}{20}\nu\right)\pi\right]. \qquad \text{(C8)}
\end{aligned}$$

By contrast, W_1 can be analytically computed (in the frequency domain) only at the LO. Indeed, consider for instance the NLO term $W_1^{\rm 1PN,NLO}$, which we have succeeded to compute in the time domain [see Eq. (3.39)], with the result

$$\begin{aligned}
W_1^{\rm 1PN,NLO} &= \frac{2}{15}\frac{\nu^2}{e_r^4\bar{a}_r^{9/2}}H_{\rm tot}\left[-\frac{28072}{225}-\frac{38872}{63}\nu\right.\\
&\quad \left.+\left(\frac{944}{105}-\frac{1136}{3}\nu\right)\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right)\right]. \qquad \text{(C9)}
\end{aligned}$$

By contrast, the computation of $W_1^{\rm 1PN,NLO}$ in the frequency domain yields the expression

$$\begin{aligned}
W_1^{\rm 1PN,NLO} &= \frac{2}{15}\frac{\nu^2}{e_r^4\bar{a}_r^{9/2}}H_{\rm tot}\left[-\frac{1768}{9}-\frac{38872}{63}\nu\right.\\
&\quad +\frac{6144}{5}\gamma-\frac{6144}{5}\ln(2)+\frac{15}{2}X\\
&\quad \left.+\left(\frac{944}{105}-\frac{1136}{3}\nu\right)\ln\left(\frac{4s}{e_r\bar{a}_r^{3/2}}\right)\right], \qquad \text{(C10)}
\end{aligned}$$

where the quantity X denotes the following double integral:

$$X=\frac{48}{\pi}\int_0^{\infty}du\int_{-\infty}^{\infty}dvu^4\mathcal{X}(u,v)\arctan\left(\tanh\frac{v}{2}\right)\ln(u), \qquad \text{(C11)}$$

with

$$\begin{aligned}
\mathcal{X}(u,v) &= \alpha(u)\left(-\frac{1}{5}\cosh(3v)+\cosh(v)\right)S(u,v)\\
&\quad -\frac{4}{5}\beta(u)\sinh(2v)C(u,v). \qquad \text{(C12)}
\end{aligned}$$

Here, to shorten the expression, we denoted $\alpha(u) \equiv uK_0(u) + K_1(u)$, $\beta(u) \equiv \frac{1}{2}K_0(u) + uK_1(u)$, as well as $[S(u,v), C(u,v)] = [\sin(u\sinh(v)), \cos(u\sinh(v))]$.

When attempting to compute X, one can first integrate over u by replacing $\ln(u) \to u^a$, taking then a derivative with respect to a, before finally setting $a \to 0$. Unfortunately, this method of integration generates derivatives of hypergeometric functions with respect to the parameters, which did not allow us to compute the integral over v in closed form. However, direct comparison with the time-domain result (C9) yields the following simple result for X:

$$X = \frac{3584}{375} - \frac{4096}{25}\gamma + \frac{4096}{25}\ln(2). \tag{C13}$$

Going to the NNLO, one can similarly extract a Fourier space representation for the missing coefficients c_{20} and c_{21}. A straightforward calculation shows that

$$c_{20} = -\frac{599223}{560}\ln(2) + \frac{1637641}{3360} + \frac{99837}{160}\zeta(3) - \frac{1584}{5}\gamma - \frac{1}{\pi^2}(Y_1 + Y_2), \tag{C14}$$

where

$$Y_1 = 48\int_0^\infty du \int_{-\infty}^\infty dv u^4 \left[\frac{\pi}{2}u\mathcal{X}(u,v) + \mathcal{Y}(u,v)\right] \times \arctan\left(\tanh\frac{v}{2}\right)\ln(u),$$
$$Y_2 = 24\int_0^\infty du \int_{-\infty}^\infty dv u^4 \mathcal{X}(u,v)\tanh v \ln(u), \tag{C15}$$

with

$$\mathcal{Y}(u,v) = A(u,v)C(u,v) + B(u,v)S(u,v), \tag{C16}$$

and

$$A(u,v) = -\alpha(u)uv\left(-\frac{1}{5}\cosh(3v) + \cosh(v)\right) + 2\beta(u)\left(\frac{1}{5}\sinh(3v) + \sinh(v)\right),$$
$$B(u,v) = -2\alpha(u)\left(1 + \frac{1}{5}\cosh(2v)\right) - \frac{4}{5}\beta(u)uv\sinh(2v). \tag{C17}$$

These Fourier-domain expressions did not allow us to compute c_{20}. By contrast, we could analytically compute their analogs for the 1PN coefficient c_{21} [see Eq. (3.46)].

APPENDIX D: GRAVITATIONAL WAVE ENERGY EMITTED DURING A SCATTERING PROCESS AT THE 2PN ACCURACY

The total gravitational-wave energy emitted during a scattering process

$$\Delta E_{\rm GW} = \Delta E^{\rm N}_{\rm GW} + \Delta E^{\rm 1PN}_{\rm GW} + \Delta E^{\rm 2PN}_{\rm GW} + \cdots \tag{D1}$$

was computed long ago at the 1PN accuracy by Blanchet and Schäfer [see Eq. (5.7) of Ref. [89]]. Let us extend their result by giving here the 2PN term, $\Delta E^{\rm 2PN}_{\rm GW}$, when $\Delta E_{\rm GW}$ is expressed in terms of $e_r = e_r^h$ and j, as in Ref. [89]:

$$\Delta E^{\rm 2PN}_{\rm GW}(e_r, j) = \frac{2}{15}\frac{\nu^2}{j^{11}}\left[\mathcal{E}_1 \arccos\left(-\frac{1}{e_r}\right) + \mathcal{E}_2\sqrt{e_r^2 - 1}\right], \tag{D2}$$

with

$$\mathcal{E}_1 = \frac{1636769}{189} + \frac{2380852}{189}e_r^2 + \frac{596996}{63}e_r^4 + \frac{494977}{48}e_r^6 + \frac{1615745}{672}e_r^8 + \nu\left(-\frac{74435}{21} - \frac{23953}{3}e_r^2 - \frac{527659}{28}e_r^4 - \frac{1775713}{112}e_r^6 - \frac{120745}{56}e_r^8\right) + \nu^2\left(48 + \frac{1463}{2}e_r^2 + \frac{31215}{8}e_r^4 + \frac{10155}{2}e_r^6 + 518e_r^8\right)$$
$$\mathcal{E}_2 = \frac{307844062}{19845} + \frac{1280690597}{158760}e_r^2 + \frac{1596923303}{158760}e_r^4 + \frac{76924511}{7840}e_r^6 + \nu\left(-\frac{281551}{45} - \frac{25157339}{2520}e_r^2 - \frac{104242423}{5040}e_r^4 - \frac{3209299}{280}e_r^6\right) + \nu^2\left(\frac{453}{4} + \frac{10777}{8}e_r^2 + \frac{10765}{2}e_r^4 + 3434e_r^6\right). \tag{D3}$$

In the parabolic orbit limit $e_r \to 1$ we find (see Ref. [89])

$$\Delta E_{\rm GW}^{\rm 2PN\ term\ e_r\to 1} = \frac{2\pi}{15}\frac{\nu^2}{j^{11}}\left[\frac{29198255}{672} - \frac{774153}{16}\nu + \frac{82215}{8}\nu^2\right]. \qquad \text{(D4)}$$

In the main text we study the 2PN-accurate expression of $\Delta E_{\rm GW}$ when it is expressed in terms of the energy and the angular momentum (and more precisely in terms of p_∞ and hj, where $h = E_{\rm tot}/M$).

APPENDIX E: REPARAMETRIZATION AND MINIMAL VALUE OF THE FLEXIBILITY FACTOR

The proof of the canonical equivalence of the two flexibility-related Hamiltonians $\Delta^{\rm f-h}H_{\rm 6PN}(r, p_r, j)$, Eq. (7.18), and $\Delta^{\rm f-h}H'_{\rm 6PN}(r', p'_r, j)$, Eq. (7.19), [with $p'_\phi = j = p_\phi$)], i.e.,

$$\Delta^{\rm f-h}H'_{5+6\rm PN} = \Delta^{\rm f-h}H_{5+6\rm PN} - \{g, H_{\rm 1PN}\}, \qquad \text{(E1)}$$

is obtained by a direct construction of the generating function $g(r, p'_r, j)$ of the canonical transformation

$$r' = r + \frac{\partial g(r, p'_r, j)}{\partial p'_r}, \qquad p_r = p'_r + \frac{\partial g(r, p'_r, j)}{\partial r}, \qquad \text{(E2)}$$

with the 1PN (harmonic-coordinate) Hamiltonian [recalled in Eq. (A4)]. Using the 1PN-accurate gravitational-wave energy flux given in Eq. (A5), one can solve for all the unknowns, i.e., the C_i's and the D_i's as functions of the c_i's and d_i's [see Eqs. (7.20) and (7.21)], as well as the coefficients g_i, n_i entering $g(r, p_r, j)$:

$$\begin{aligned} g(r, p_r, j) = \frac{\nu^3 p_r}{r^3}\bigg[& g_1\frac{1}{r} + g_2\frac{j^2}{r^2} + g_3 p_r^2 \\ & + \eta^2\left(n_1\frac{1}{r^2} + n_2\frac{j^4}{r^4} + n_3 p_r^4 \right. \\ & \left. + n_4\frac{j^2}{r^3} + n_5\frac{p_r^2}{r} + n_6\frac{j^2 p_r^2}{r^2}\right)\bigg]. \end{aligned} \qquad \text{(E3)}$$

We found the explicit results

$$\begin{aligned} g_1 &= \frac{64}{5}(c_2 + c_3), \\ g_2 &= \frac{64}{5}c_2, \\ g_3 &= \frac{16}{45}(12c_1 + 73c_2), \end{aligned} \qquad \text{(E4)}$$

and

$$\begin{aligned} n_1 &= \left(-\frac{32c_2^2}{5} - \frac{64c_2c_3}{5} - \frac{112c_2}{5} - \frac{32c_3^2}{5} - \frac{2096c_3}{105}\right)\nu - \frac{4988c_2}{105} - \frac{3476c_3}{35} + \frac{64d_2}{5} + \frac{64d_3}{5} + \frac{64d_6}{5}, \\ n_2 &= \left(-\frac{32c_2^2}{5} - \frac{48c_2}{7}\right)\nu + \frac{2468c_2}{105} + \frac{64d_2}{5}, \\ n_3 &= \left(-\frac{32c_1^2}{25} - \frac{176c_1c_2}{25} + \frac{8216c_1}{525} - \frac{4696c_2^2}{225} + \frac{67888c_2}{1575}\right)\nu - \frac{1268c_1}{225} + \frac{33832c_2}{1575} + \frac{64d_1}{25} + \frac{9392d_2}{225} + \frac{176d_4}{25}, \\ n_4 &= \left(-\frac{32c_2^2}{5} - \frac{64c_2c_3}{5} - \frac{4112c_2}{105} - \frac{48c_3}{7}\right)\nu - \frac{3252c_2}{35} + \frac{2468c_3}{105} + \frac{64d_2}{5} + \frac{64d_6}{5}, \\ n_5 &= \left(-\frac{64c_1c_2}{15} - \frac{64c_1c_3}{15} - \frac{2096c_1}{315} - \frac{1448c_2^2}{45} - \frac{272c_2c_3}{9} - \frac{2968c_2}{45} + \frac{1592c_3}{45}\right)\nu \\ &\quad - \frac{3476c_1}{105} - \frac{15752c_2}{105} + \frac{1144c_3}{63} + \frac{2896d_2}{45} + \frac{64d_4}{15} + \frac{64d_5}{15} + \frac{272d_6}{9}, \\ n_6 &= \left(-\frac{64c_1c_2}{15} - \frac{16c_1}{7} - \frac{872c_2^2}{45} + \frac{1472c_2}{315}\right)\nu + \frac{2468c_1}{315} + \frac{2680c_2}{63} + \frac{1744d_2}{45} + \frac{64d_4}{15}. \end{aligned} \qquad \text{(E5)}$$

Let us note that, while the three C_i's are in one-to-one correspondence with the three c_i's, with the inverse relations

$$c_1 = \frac{456}{32}C_1 + \frac{90656}{1536}C_3 - \frac{1856}{128}C_2,$$
$$c_2 = -\frac{31856}{3072}C_3 + \frac{656}{256}C_2 - \frac{156}{64}C_1,$$
$$c_3 = -\frac{656}{256}C_2 + \frac{34256}{3072}C_3 + \frac{156}{64}C_1, \tag{E6}$$

one cannot express the six d_i's in terms of the four D_i's. However, Eqs. (7.21) can be inverted to express the first four d_i's, namely $d_1, \ldots, d_4$, in terms of d_5, d_6, and of the new parameters C_i and D_i:

$$d_1 = \left(\frac{356325C_1^2}{616448} - \frac{1409225C_2C_1}{1232896} + \frac{67026425C_3C_1}{14794752} - \frac{146435C_1}{8428} + \frac{16657175C_2^2}{29589504} + \frac{5345155025C_3^2}{608698368} + \frac{390545C_2}{28896} - \frac{789897575C_2C_3}{177537024} - \frac{161067325C_3}{4854528}\right)\nu + \frac{1235525C_1}{134848} - \frac{3774345C_2}{539392} + \frac{37456235C_3}{1078784} - \frac{181445}{4816} + \frac{11175D_1}{4816} - \frac{1815D_2}{1204} + \frac{10275D_3}{4816} - \frac{208925D_4}{19264} + \frac{8d_5}{301} - \frac{944d_6}{129},$$

$$d_2 = \left(\frac{10275C_1^2}{2465792} - \frac{34925C_2C_1}{4931584} + \frac{1251725C_3C_1}{59179008} - \frac{476265C_1}{539392} + \frac{329225C_2^2}{118358016} + \frac{22825775C_3^2}{2434793472} + \frac{715625C_2}{924672} - \frac{9696425C_2C_3}{710148096} - \frac{174415025C_3}{77672448}\right)\nu + \frac{716985C_1}{2157568} - \frac{1500925C_2}{8630272} + \frac{303139985C_3}{310689792} + \frac{225D_1}{2408} - \frac{615D_2}{9632} + \frac{3805D_3}{38528} - \frac{232105D_4}{462336} - \frac{32d_5}{301} - \frac{137d_6}{129},$$

$$d_3 = \left(-\frac{10275C_1^2}{2465792} + \frac{34925C_2C_1}{4931584} - \frac{1251725C_3C_1}{59179008} + \frac{476265C_1}{539392} - \frac{329225C_2^2}{118358016} - \frac{15395375C_3^2}{2434793472} - \frac{715625C_2}{924672} + \frac{9696425C_2C_3}{710148096} + \frac{185034305C_3}{77672448}\right)\nu + \frac{1305735C_1}{2157568} - \frac{7264195C_2}{8630272} + \frac{1196779135C_3}{310689792} - \frac{225D_1}{2408} + \frac{615D_2}{9632} - \frac{3805D_3}{38528} + \frac{268225D_4}{462336} + \frac{32d_5}{301} + \frac{8d_6}{129},$$

$$d_4 = \left(-\frac{675C_1^2}{19264} + \frac{225C_2C_1}{4816} - \frac{57075C_3C_1}{616448} + \frac{5362055C_1}{539392} - \frac{2925C_2^2}{308224} + \frac{1229725C_3^2}{4227072} - \frac{7298675C_2}{924672} - \frac{80125C_2C_3}{2465792} + \frac{1425084095C_3}{77672448}\right)\nu - \frac{7428725C_1}{2157568} + \frac{13974505C_2}{8630272} - \frac{3276924925C_3}{310689792} - \frac{3375D_1}{2408} + \frac{9225D_2}{9632} - \frac{54065D_3}{38528} + \frac{3297965D_4}{462336} + \frac{179d_5}{301} + \frac{384d_6}{43}. \tag{E7}$$

We introduced in the text a minimal way, namely Eqs. (7.28), of fixing the values of the gauge-invariant parameters C_i and D_i associated with some flexibility factor $f(t)$. However, this unique choice of the C_i and D_i still leaves some gauge freedom in the choice of the flexibility factor $f(t)$ itself. If ever one wants to have also a specific value for the flexibility factor $f(t)$ itself, [i.e., specific values of the original flexibility parameters c_i, d_i entering Eq. (7.5)] one needs, in addition to the explicit values (7.25), to insert the minimal values Eqs. (7.28) in the relations (E7) expressing the d_i's in terms of the C_i's, the D_i's and of d_5 and d_6. This yields

$$d_1^{\rm min} = \left(\frac{4110D}{301} + \frac{1269775907}{606816}\right)\nu + \frac{33448631}{75852} + \frac{8d_5}{301} - \frac{944d_6}{129},$$
$$d_2^{\rm min} = \left(\frac{761D}{1204} + \frac{159864493}{2427264}\right)\nu + \frac{13371067}{809088} - \frac{32d_5}{301} - \frac{137d_6}{129},$$
$$d_3^{\rm min} = \left(-\frac{761D}{1204} - \frac{159864493}{2427264}\right)\nu + \frac{12115205}{809088} + \frac{32d_5}{301} + \frac{8d_6}{129},$$
$$d_4^{\rm min} = \left(-\frac{10813D}{1204} - \frac{34223993}{33712}\right)\nu - \frac{52885925}{269696} + \frac{179d_5}{301} + \frac{384d_6}{43}. \qquad \text{(E8)}$$

In these expressions, d_5 and d_6 can be given arbitrary values.

APPENDIX F: 6PN-ACCURATE F-ROUTE LOCAL DELAUNAY HAMILTONIAN

The 6PN-accurate f-route local effective Delaunay Hamiltonian [expressed in terms of $I_2 \equiv j$ and $I_3 \equiv i_r + j \equiv i_{r\phi}$, see Eqs. (A8)] is given by

$$\frac{H_{\rm eff}^{\rm 6PN,loc,f}(I_2, I_3; \nu)}{\mu c^2} = \eta^{-2} + \sum_{k=0}^{6} \eta^{2k} \bar{E}_{\rm eff}^{2k}(I_2, I_3; \nu) + O(\eta^{14}). \qquad \text{(F1)}$$

The coefficients up to the 5PN order [i.e., $O(\eta^{10})$] are listed in Table XI of Ref. [3]. We complete this result by adding the 6PN coefficient $\bar{E}_{\rm eff}^{12}$ (see Table X below).

TABLE X. 6PN coefficient $\bar{E}_{\rm eff}^{12}$ entering the PN expansion of the Delaunay effective Hamiltonian (F1).

$\bar{E}_{\rm eff}^{12}$	$[\nu^2(\frac{1911a_6^{\nu^2}}{32} + \frac{63a_7^{\nu^2}}{16} + \frac{273\bar{d}_5^{\nu^2}}{32} + \frac{21\bar{d}_6^{\nu^2}}{32} + \frac{9q_{45}^{\nu^2}}{32} - \frac{529515\pi^4}{65536} + \frac{179354853\pi^2}{65536} - \frac{2062272503}{22400}) + \nu^3(-\frac{315a_6^{\nu^2}}{32} - \frac{63\bar{d}_5^{\nu^2}}{32} + \frac{63a_7^{\nu^3}}{16} - \frac{24980025\pi^2}{65536} + \frac{978061}{64}) + \frac{819\nu^5}{256} + (\frac{38745\pi^2}{2048} - \frac{428085}{512})\nu^4 + (\frac{3236467169}{30240} + \frac{188085303629\pi^2}{50331648} - \frac{350055909\pi^4}{8388608})\nu - \frac{14196819}{256}]\frac{1}{I_2^{11}I_3^3}$ $+[\nu^2(\frac{315a_6^{\nu^2}}{16} + \frac{63\bar{d}_5^{\nu^2}}{16} - \frac{45387\pi^4}{32768} + \frac{5648989\pi^2}{4096} - \frac{2795413}{48}) - \frac{117\nu^4}{2} + (\frac{48355}{8} - \frac{16113\pi^2}{128})\nu^3 + (\frac{255513551}{1920} - \frac{55414387\pi^2}{32768})\nu - \frac{16298667}{256}]\frac{1}{I_2^{10}I_3^4}$ $+[\nu^2(-\frac{1225a_6^{\nu^2}}{16} - \frac{35a_7^{\nu^2}}{8} - \frac{245\bar{d}_5^{\nu^2}}{16} - \frac{35\bar{d}_6^{\nu^2}}{32} - \frac{21q_{45}^{\nu^2}}{32} + \frac{176505\pi^4}{32768} - \frac{346823785\pi^2}{98304} + \frac{13914839443}{115200}) + \nu^3(\frac{525a_6^{\nu^2}}{32} + \frac{133\bar{d}_5^{\nu^2}}{32} - \frac{35a_7^{\nu^3}}{8} + \frac{1057777\pi^2}{1536} - \frac{16220123}{576}) - \frac{3465\nu^5}{256} + (\frac{3398185}{1536} - \frac{50225\pi^2}{1024})\nu^4 + (-\frac{34107960371}{345600} - \frac{590940624077\pi^2}{113246208} + \frac{387365405\pi^4}{8388608})\nu + \frac{9066235}{256}]\frac{1}{I_2^9I_3^5}$ $+[\nu^2(-\frac{225a_6^{\nu^2}}{8} - \frac{75\bar{d}_5^{\nu^2}}{8} + \frac{25215\pi^4}{16384} - \frac{12455065\pi^2}{4096} + \frac{3386395}{24}) + 300\nu^4 + (\frac{22755\pi^2}{64} - \frac{75595}{4})\nu^3 + (\frac{170705335\pi^2}{32768} - \frac{822324589}{2688})\nu + \frac{33326145}{256}]\frac{1}{I_2^8I_3^6}$ $+[\nu^2(\frac{1095a_6^{\nu^2}}{32} + \frac{15a_7^{\nu^2}}{16} + \frac{365\bar{d}_5^{\nu^2}}{32} + \frac{15\bar{d}_6^{\nu^2}}{32} + \frac{15q_{45}^{\nu^2}}{32} - \frac{25215\pi^4}{65536} + \frac{14759815\pi^2}{8192} - \frac{50590683}{1120}) + \nu^3(-\frac{225a_6^{\nu^2}}{32} - \frac{85\bar{d}_5^{\nu^2}}{32} + \frac{15a_7^{\nu^3}}{16} - \frac{49631575\pi^2}{98304} + \frac{23365741}{1152}) + \frac{2835\nu^5}{128} + (\frac{21525\pi^2}{512} - \frac{532105}{256})\nu^4 + (-\frac{22549379339}{423360} + \frac{161199909365\pi^2}{75497472} - \frac{81987555\pi^4}{8388608})\nu + \frac{41670225}{512}]\frac{1}{I_2^7I_3^7}$ $+[\nu^2(\frac{63a_6^{\nu^2}}{16} + \frac{63\bar{d}_5^{\nu^2}}{16} - \frac{11767\pi^4}{32768} + \frac{8802269\pi^2}{4096} - \frac{21668549}{180}) - 462\nu^4 + (\frac{67123}{4} - \frac{14637\pi^2}{64})\nu^3 + (\frac{8093748209}{28800} - \frac{590358503\pi^2}{98304})\nu - \frac{11382315}{128}]\frac{1}{I_2^6I_3^8}$ $+[\nu^2(-\frac{25a_6^{\nu^2}}{8} - \frac{25\bar{d}_5^{\nu^2}}{8} - \frac{\bar{d}_6^{\nu^2}}{32} - \frac{3q_{45}^{\nu^2}}{32} - \frac{26480553\pi^2}{32768} + \frac{130268403}{44800}) + \nu^3(\frac{15a_6^{\nu^2}}{32} + \frac{15\bar{d}_5^{\nu^2}}{32} + \frac{1432335\pi^2}{8192} - \frac{438383}{64}) - \frac{2205\nu^5}{128} + (\frac{219555}{256} - \frac{12915\pi^2}{1024})\nu^4 + (\frac{2754048127363}{25401600} + \frac{12817445435\pi^2}{12582912} - \frac{135909\pi^4}{8388608})\nu - \frac{11393277}{64}]\frac{1}{I_2^5I_3^9}$ $+[270\nu^4 + (\frac{3321\pi^2}{128} - \frac{12231}{2})\nu^3 + (\frac{188409}{4} - \frac{1471401\pi^2}{4096})\nu^2 + (\frac{57005721\pi^2}{32768} - \frac{3445375221}{22400})\nu + \frac{3686445}{256}]\frac{1}{I_2^4I_3^{10}}$ $+[\frac{1575\nu^5}{256} + (\frac{1435\pi^2}{2048} - \frac{268555}{1536})\nu^4 + (\frac{449845}{384} - \frac{915845\pi^2}{65536})\nu^3 + (\frac{51899359}{67200} + \frac{7988539\pi^2}{65536})\nu^2 + (-\frac{8318335583}{176400} - \frac{27753064819\pi^2}{50331648})\nu + \frac{76277895}{256}]\frac{1}{I_2^3I_3^{11}}$ $+[-\frac{99\nu^4}{2} + \frac{8415\nu^3}{8} - 10395\nu^2 + \frac{4058505\nu}{64} - \frac{34218855}{128}]\frac{1}{I_2^2I_3^{12}}$ $+[-\frac{189\nu^5}{256} + \frac{9975\nu^4}{512} - \frac{15825\nu^3}{64} + \frac{1047627\nu^2}{512} - \frac{6699213\nu}{512} + \frac{47435571}{512}]\frac{1}{I_2I_3^{13}} - \frac{24188177}{2048}\frac{1}{I_3^{14}}$

In the case of circular motions, the f-route, local contribution to the 6PN-accurate effective energy $\bar{E}_{\rm eff} \equiv (E_{\rm eff} - \mu)/\mu$ is found to be

$$\begin{aligned}\bar{E}_{\rm eff}^{\rm circ} &= -\frac{1}{2j^2} - \frac{9}{8}\frac{\eta^2}{j^4} + \left(-\frac{81}{16} + \nu\right)\frac{\eta^4}{j^6} + \left[-\frac{3861}{128} + \left(-\frac{41}{64}\pi^2 + \frac{157}{6}\right)\nu\right]\frac{\eta^6}{j^8} \\ &\quad + \left[-\frac{53703}{256} + \left(-\frac{6581}{1024}\pi^2 + \frac{8357}{30}\right)\nu + \left(-\frac{275}{12} + \frac{41}{64}\pi^2\right)\nu^2\right]\frac{\eta^8}{j^{10}} \\ &\quad + \left[-\frac{1648269}{1024} + \left(\frac{15592753}{6300} - \frac{31547}{1536}\pi^2\right)\nu + \left(-\frac{4725}{8} + \frac{1}{2}a_6^{\nu^2} + \frac{2337}{128}\pi^2\right)\nu^2 + 2\nu^3\right]\frac{\eta^{10}}{j^{12}} \\ &\quad + \left[-\frac{27078705}{2048} + \left(-\frac{2800873}{524288}\pi^4 + \frac{298273237}{14175} + \frac{1322752463}{3538944}\pi^2\right)\nu\right. \\ &\quad + \left(-\frac{1681}{512}\pi^4 + \frac{39}{4}a_6^{\nu^2} + \frac{1389451}{3072}\pi^2 - \frac{3321439}{288} + \frac{1}{2}a_7^{\nu^2}\right)\nu^2 \\ &\quad \left. + \left(-\frac{615}{64}\pi^2 + \frac{1369}{4} + \frac{1}{2}a_7^{\nu^3}\right)\nu^3\right]\frac{\eta^{12}}{j^{14}}. \end{aligned} \tag{F2}$$

The relation between $\bar{E}_{\rm eff}$ and the specific binding energy $\bar{E}$, Eq. (1.36), is given by

$$\bar{E} = \frac{\sqrt{1 + 2\nu\bar{E}_{\rm eff}} - 1}{\nu}, \tag{F3}$$

so that in the circular case we get

$$\begin{aligned}\bar{E}_{\rm loc,f}^{\rm circ} &= -\frac{1}{2j^2} + \left(-\frac{\nu}{8} - \frac{9}{8}\right)\frac{\eta^2}{j^4} + \left(-\frac{\nu^2}{16} + \frac{7\nu}{16} - \frac{81}{16}\right)\frac{\eta^4}{j^6} + \left[-\frac{5\nu^3}{128} + \frac{5\nu^2}{64} + \left(\frac{8833}{384} - \frac{41\pi^2}{64}\right)\nu - \frac{3861}{128}\right]\frac{\eta^6}{j^8} \\ &\quad + \left[-\frac{7\nu^4}{256} + \frac{3\nu^3}{128} + \left(\frac{41\pi^2}{128} - \frac{8875}{768}\right)\nu^2 + \left(\frac{989911}{3840} - \frac{6581\pi^2}{1024}\right)\nu - \frac{53703}{256}\right]\frac{\eta^8}{j^{10}} \\ &\quad + \left[\left(\frac{a_6^{\nu^2}}{2} + \frac{29335\pi^2}{2048} - \frac{1679647}{3840}\right)\nu^2 - \frac{21\nu^5}{1024} + \frac{5\nu^4}{1024} + \left(\frac{41\pi^2}{512} - \frac{3769}{3072}\right)\nu^3\right. \\ &\quad \left. + \left(\frac{3747183493}{1612800} - \frac{31547\pi^2}{1536}\right)\nu - \frac{1648269}{1024}\right]\frac{\eta^{10}}{j^{12}} \\ &\quad + \left[\nu^2\left(\frac{39a_6^{\nu^2}}{4} + \frac{a_7^{\nu^2}}{2} - \frac{1681\pi^4}{512} + \frac{10605841\pi^2}{24576} - \frac{10727952929}{1075200}\right) + \nu^3\left(\frac{a_6^{\nu^2}}{4} + \frac{a_7^{\nu^3}}{2} - \frac{21383\pi^2}{8192} + \frac{1007737}{7680}\right)\right. \\ &\quad \left. - \frac{33\nu^6}{2048} - \frac{7\nu^5}{2048} + \left(\frac{41\pi^2}{1024} - \frac{2537}{3072}\right)\nu^4 + \left(\frac{576215112401}{29030400} + \frac{1322752463\pi^2}{3538944} - \frac{2800873\pi^4}{524288}\right)\nu - \frac{27078705}{2048}\right]\frac{\eta^{12}}{j^{14}}. \end{aligned} \tag{F4}$$

The f-route, local 6PN-accurate periastron advance (along arbitrary eccentric orbits), expressed in terms of $\bar{E}$ and j, reads

$$
\begin{aligned}
K(\bar{E},j)_{\rm loc,f} = {} & 1+\frac{3}{j^2}\eta^2+\left[\left(\frac{15}{2}-3\nu\right)\frac{\bar{E}}{j^2}+\left(\frac{105}{4}-\frac{15\nu}{2}\right)\frac{1}{j^4}\right]\eta^4 \\
& +\left[\left(3\nu^2-\frac{15\nu}{4}+\frac{15}{4}\right)\frac{\bar{E}^2}{j^2}+\left(\frac{45\nu^2}{2}+\left(\frac{123\pi^2}{64}-218\right)\nu+\frac{315}{2}\right)\frac{\bar{E}}{j^4}\right. \\
& \left.+\left(\frac{105\nu^2}{8}+\left(\frac{615\pi^2}{128}-\frac{625}{2}\right)\nu+\frac{1155}{4}\right)\frac{1}{j^6}\right]\eta^6 \\
& +\left[\left(\frac{15\nu^2}{4}-3\nu^3\right)\frac{\bar{E}^3}{j^2}+\left(-45\nu^3+\left(\frac{4045}{8}-\frac{615\pi^2}{128}\right)\nu^2+\left(\frac{35569\pi^2}{2048}-\frac{20323}{24}\right)\nu+\frac{4725}{16}\right)\frac{\bar{E}^2}{j^4}\right. \\
& +\left(-\frac{525\nu^3}{8}+\left(\frac{35065}{16}-\frac{615\pi^2}{16}\right)\nu^2+\left(\frac{257195\pi^2}{2048}-\frac{293413}{48}\right)\nu+\frac{45045}{16}\right)\frac{\bar{E}}{j^6} \\
& \left.+\left(-\frac{315\nu^3}{16}+\left(\frac{132475}{96}-\frac{7175\pi^2}{256}\right)\nu^2+\left(\frac{2975735\pi^2}{24576}-\frac{1736399}{288}\right)\nu+\frac{225225}{64}\right)\frac{1}{j^8}\right]\eta^8 \\
& +\left[\left(\nu^2\left(-\frac{15a_6^{\nu^2}}{4}-\frac{15\bar{d}_5^{\nu^2}}{4}-\frac{1203065\pi^2}{2048}+\frac{310189}{12}\right)+\frac{1575\nu^4}{8}+\left(\frac{35055\pi^2}{256}-\frac{240585}{32}\right)\nu^3\right.\right. \\
& \left.+\left(\frac{4899565\pi^2}{4096}-\frac{33023719}{840}\right)\nu+\frac{315315}{32}\right)\frac{\bar{E}^2}{j^6} \\
& +\left(\nu^2\left(-\frac{105a_6^{\nu^2}}{4}-\frac{35\bar{d}_5^{\nu^2}}{4}-\frac{12964665\pi^2}{8192}+\frac{549451}{8}\right)+\frac{2205\nu^4}{16}+\left(\frac{121975\pi^2}{512}-\frac{271705}{24}\right)\nu^3\right. \\
& \left.+\left(\frac{16173395\pi^2}{8192}-\frac{30690127}{240}\right)\nu+\frac{765765}{16}\right)\frac{\bar{E}}{j^8}+\left(3\nu^4-\frac{15\nu^3}{4}+\frac{15\nu^2}{16}\right)\frac{\bar{E}^4}{j^2} \\
& +\left(75\nu^4+\left(\frac{1107\pi^2}{128}-\frac{7113}{8}\right)\nu^3+\left(\frac{9689}{6}-\frac{35569\pi^2}{1024}\right)\nu^2+\left(\frac{15829\pi^2}{256}-\frac{12160657}{8400}\right)\nu+\frac{3465}{16}\right)\frac{\bar{E}^3}{j^4} \\
& +\left(\nu^2\left(-\frac{315a_6^{\nu^2}}{16}-\frac{63\bar{d}_5^{\nu^2}}{16}-\frac{15796431\pi^2}{16384}+\frac{5156991}{128}\right)+\frac{3465\nu^4}{128}+\left(\frac{90405\pi^2}{1024}-\frac{127995}{32}\right)\nu^3\right. \\
& \left.\left.+\left(\frac{1096263\pi^2}{1024}-\frac{61358067}{640}\right)\nu+\frac{2909907}{64}\right)\frac{1}{j^{10}}\right]\eta^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\left(-3\nu^5+\frac{15\nu^4}{4}-\frac{15\nu^3}{16}\right)\frac{\bar{E}^5}{j^2}+\Bigg(-\frac{225\nu^5}{2}+\left(\frac{2737}{2}-\frac{861\pi^2}{64}\right)\nu^4+\left(\frac{462397\pi^2}{8192}-\frac{247189}{96}\right)\nu^3\\
&+\left(\frac{25148189}{11200}-\frac{47487\pi^2}{512}\right)\nu^2+\left(\frac{104950259\pi^2}{1048576}-\frac{25669261}{29400}\right)\nu+\frac{3465}{64}\Bigg)\frac{\bar{E}^4}{j^4}\\
&+\Bigg(\nu^2\left(-\frac{25a_6^{\nu^2}}{2}-\frac{25\bar{d}_5^{\nu^2}}{2}-\frac{5\bar{d}_6^{\nu^2}}{4}-\frac{15q_{45}^{\nu^2}}{4}-\frac{8648125\pi^2}{2048}+\frac{58338869}{480}\right)\\
&+\nu^3\left(15a_6^{\nu^2}+15\bar{d}_5^{\nu^2}+\frac{6795375\pi^2}{4096}-\frac{2255935}{32}\right)-\frac{3675\nu^5}{8}+\left(\frac{601625}{32}-\frac{89175\pi^2}{256}\right)\nu^4\\
&+\left(-\frac{39123984017}{635040}-\frac{3128795225\pi^2}{1572864}-\frac{679545\pi^4}{1048576}\right)\nu+\frac{525525}{32}\Bigg)\frac{\bar{E}^3}{j^6}\\
&+\Bigg(\nu^2\left(-\frac{735a_6^{\nu^2}}{2}-\frac{105a_7^{\nu^2}}{4}-\frac{245\bar{d}_5^{\nu^2}}{2}-\frac{105\bar{d}_6^{\nu^2}}{8}-\frac{105q_{45}^{\nu^2}}{8}+\frac{176505\pi^4}{16384}-\frac{28607145\pi^2}{1024}+\frac{890209513}{960}\right)\\
&+\nu^3\left(\frac{735a_6^{\nu^2}}{4}+70\bar{d}_5^{\nu^2}-\frac{105a_7^{\nu^3}}{4}+\frac{392482055\pi^2}{49152}-\frac{48508187}{144}\right)-\frac{2205\nu^5}{4}+\left(\frac{9492035}{192}-\frac{1083425\pi^2}{1024}\right)\nu^4\\
&+\left(-\frac{36266340619}{60480}-\frac{749028566195\pi^2}{18874368}+\frac{573912885\pi^4}{2097152}\right)\nu+\frac{16081065}{64}\Bigg)\frac{\bar{E}^2}{j^8}\\
&+\Bigg(\nu^2\left(-\frac{17325a_6^{\nu^2}}{16}-\frac{315a_7^{\nu^2}}{4}-\frac{3465\bar{d}_5^{\nu^2}}{16}-\frac{315\bar{d}_6^{\nu^2}}{16}-\frac{189q_{45}^{\nu^2}}{16}+\frac{1588545\pi^4}{16384}-\frac{446396685\pi^2}{8192}+\frac{12054492193}{6400}\right)\\
&+\nu^3\left(\frac{4725a_6^{\nu^2}}{16}+\frac{1197\bar{d}_5^{\nu^2}}{16}-\frac{315a_7^{\nu^3}}{4}+\frac{11337249\pi^2}{1024}-\frac{7270879}{16}\right)-\frac{31185\nu^5}{128}+\left(\frac{10090605}{256}-\frac{452025\pi^2}{512}\right)\nu^4\\
&+\left(-\frac{31568079821}{19200}-\frac{551913398477\pi^2}{6291456}+\frac{3486288645\pi^4}{4194304}\right)\nu+\frac{101846745}{128}\Bigg)\frac{\bar{E}}{j^{10}}\\
&+\Bigg(\nu^2\left(-\frac{21021a_6^{\nu^2}}{32}-\frac{693a_7^{\nu^2}}{16}-\frac{3003\bar{d}_5^{\nu^2}}{32}-\frac{231\bar{d}_6^{\nu^2}}{32}-\frac{99q_{45}^{\nu^2}}{32}+\frac{5824665\pi^4}{65536}-\frac{1972903383\pi^2}{65536}+\frac{22684997533}{22400}\right)\\
&+\nu^3\left(\frac{3465a_6^{\nu^2}}{32}+\frac{693\bar{d}_5^{\nu^2}}{32}-\frac{693a_7^{\nu^3}}{16}+\frac{274780275\pi^2}{65536}-\frac{10758671}{64}\right)-\frac{9009\nu^5}{256}+\left(\frac{4708935}{512}-\frac{426195\pi^2}{2048}\right)\nu^4\\
&+\left(-\frac{35601138859}{30240}-\frac{2068938339919\pi^2}{50331648}+\frac{3850614999\pi^4}{8388608}\right)\nu+\frac{156165009}{256}\Bigg)\frac{1}{j^{12}}\Bigg]\eta^{12}.
\end{aligned}
$$

In the circular case this reduces to Eq. (8.29).

APPENDIX G: COMPLETING THE INFORMATION ON THE H-ROUTE, NONLOCAL q_8 EOB POTENTIAL

One of the intermediate steps of our analysis is to transform the h-route (i.e., r_{12}^h-scaled) nonlocal Hamiltonian, $H_{\rm nonloc,h}^{4+5+6\rm PN}(t)$, defined in Eq. (2.9), into its gauge-equivalent EOB potentials, $A^{\rm nonloc,h}(r)$, $\bar{D}^{\rm nonloc,h}(r)$, and $\hat{Q}^{\rm nonloc,h}(r,p_r)$. We have listed the PN-expansion coefficients of these potentials in Table IV of [2] (for the $4+5$PN-level contributions), and in Table VI of [3] (for the 6PN-level contributions). However, we did not include in Table IV of [2] the values of the $4+5$PN-level coefficients entering the q_8 EOB potential, i.e., the coefficients denoted $q_{81}^{\rm nonloc}$ and $q_{82}^{\rm nonloc}$ in the last line of Eq. (2.22) in [2]. The aim of this appendix is to remedy this gap by giving the 6PN-accurate values of the PN expansion coefficients of the $O(p_r^8)$ part of $\hat{Q}_{4+5+6\rm PN}^{\rm nonloc,h}(r,p_r;\nu)$, namely

$$
[\hat{Q}_{4+5+6\rm PN}^{\rm nonloc,h}(r,p_r;\nu)]_{p_r^8}=p_r^8q_8(u;\nu)=p_r^8(q_{81}^{\rm nonloc,h}(\nu)u+q_{82}^{\rm nonloc,h}(\nu)u^2+q_{83}^{\rm nonloc,h}(\nu)u^3). \tag{G1}
$$

As indicated here, the coefficients $q_{81}^{\rm nonloc,h}$ (4PN level), $q_{82}^{\rm nonloc,h}$ (5PN level), and $q_{83}^{\rm nonloc,h}$ (6PN level) depend only on ν, i.e., they do not involve any $\ln u$ contribution. [The logarithmic contributions come from the $2\frac{GH}{c^5}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln(\frac{r^h_{12}(t)}{s})$ term in Eq. (2.9) and start contributing to q_8 at the 7PN level.]

Although we have already given $q_{83}^{\rm nonloc,h}(\nu)$ in Table VI of [3], let us, for clarity, list here all the PN coefficients of q_8

$$
\begin{aligned}
q_{81}^{\rm nonloc,h}(\nu) &= \left(\frac{21668992}{45}\ln(2)+\frac{6591861}{350}\ln(3)-\frac{27734375}{126}\ln(5)-\frac{35772}{175}\right)\nu\\
q_{82}^{\rm nonloc,h}(\nu) &= \left(\frac{703189497728}{33075}\ln(2)+\frac{869626}{525}+\frac{332067403089}{39200}\ln(3)-\frac{468490234375}{42336}\ln(5)-\frac{13841287201}{4320}\ln(7)\right)\nu^2\\
&\quad+\left(\frac{5788281}{2450}-\frac{16175693888}{1575}\ln(2)-\frac{393786545409}{156800}\ln(3)+\frac{875090984375}{169344}\ln(5)+\frac{13841287201}{17280}\ln(7)\right)\nu\\
q_{83}^{\rm nonloc,h}(\nu) &= \left(-\frac{154862}{21}+\frac{57604236136064}{99225}\ln(2)+\frac{10467583300341}{39200}\ln(3)-\frac{73366198046875}{381024}\ln(5)\right.\\
&\quad\left.-\frac{7709596970957}{38880}\ln(7)\right)\nu^3\\
&\quad+\left(-\frac{1746293}{70}-\frac{177055674739808}{297675}\ln(2)-\frac{43719724468071}{156800}\ln(3)+\frac{366449151015625}{1524096}\ln(5)\right.\\
&\quad\left.+\frac{26506549233199}{155520}\ln(7)\right)\nu^2\\
&\quad+\left(-\frac{709195549}{132300}+\frac{5196312336176}{35721}\ln(2)+\frac{17515638027261}{313600}\ln(3)-\frac{63886617280625}{1016064}\ln(5)\right.\\
&\quad\left.-\frac{29247366220639}{933120}\ln(7)\right)\nu.
\end{aligned}
\tag{G2}
$$

For completeness, let us also mention that our self-force computation of the full (local-plus-nonlocal) q_8 potential has given the result

$$
q_{8,\le 6\rm PN}^{\rm loc+nonloc}=\nu(B_1u+B_2u^2+B_3u^3)+O(\nu^2),\tag{G3}
$$

where

$$
\begin{aligned}
B_1 &= -\frac{27734375}{126}\ln(5)+\frac{6591861}{350}\ln(3)+\frac{21668992}{45}\ln(2)-\frac{35772}{175},\\
B_2 &= \frac{13841287201}{17280}\ln(7)-\frac{393786545409}{156800}\ln(3)-\frac{16175693888}{1575}\ln(2)+\frac{875090984375}{169344}\ln(5)+\frac{5790381}{2450},\\
B_3 &= -\frac{29247366220639}{933120}\ln(7)-\frac{63886617280625}{1016064}\ln(5)+\frac{5196312336176}{35721}\ln(2)\\
&\quad+\frac{17515638027261}{313600}\ln(3)-\frac{2843819611}{529200}.
\end{aligned}
\tag{G4}
$$

The difference,

$$
\Delta q_{8,\le 6\rm PN}=q_{8,\le 6PN}^{\rm loc+nonloc}-q_{8,\le 6\rm PN}^{\rm nonloc,h},\tag{G5}
$$

was one of our sources of information for deriving the local part of the Hamiltonian, and is equal to

$$
\Delta q_{8,\le 6\rm PN}=\nu\left(\frac{6}{7}u^2-\frac{7447}{560}u^3\right)+O(\nu^2).\tag{G6}
$$

APPENDIX H: COMPUTING THE DELAUNAY NEAR-ZONE NONLOCAL HAMILTONIAN ASSOCIATED WITH THE $\ln(r_{12}^h/s)$ TERM ALONG ELLIPTICLIKE MOTION

Let us consider the $4+5+6$PN nonlocal, h-route (unflexed) Hamiltonian (2.9). We compute here the Delaunay-average (along an ellipticlike motion) of the $\ln(r_{12}^h/s)$ contribution to $H^{\rm nonloc,h}$, i.e.,

$$\langle H^{\rm nonloc,ln,h}_{4+5+6{\rm PN}}\rangle = \frac{1}{\oint dt}\oint 2\frac{GH}{c^5}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t)\ln\left(\frac{r_{12}^h(t)}{s}\right)dt, \tag{H1}$$

where

$$H = Mc^2(1+\nu\bar{E}\eta^2) \tag{H2}$$

and

$$\bar{E} = -\frac{1}{2a_r^h} - \frac{1}{2}\left(-\frac{7}{4}+\frac{1}{4}\nu\right)\frac{\eta^2}{(a_r^h)^2} + O(\eta^4). \tag{H3}$$

It can be written as

$$\begin{aligned}\langle H^{4+5+6{\rm PN}}_{\rm nonloc,ln,h}\rangle &= \frac{\nu^2}{(a_r^h)^5}\left[\mathcal{A}_{\rm ln}^{\rm 4PN}(e_t^h)+\mathcal{B}_{\rm ln}^{\rm 4PN}(e_t^h)\ln\left(\frac{a_r^h}{s}\right)\right]\\ &+\frac{\nu^2}{(a_r^h)^6}\left[\mathcal{A}_{\rm ln}^{\rm 5PN}(e_t^h)+\mathcal{B}_{\rm ln}^{\rm 5PN}(e_t^h)\ln\left(\frac{a_r^h}{s}\right)\right]\\ &+\frac{\nu^2}{(a_r^h)^7}\left[\mathcal{A}_{\rm ln}^{\rm 6PN}(e_t^h)+\mathcal{B}_{\rm ln}^{\rm 6PN}(e_t^h)\ln\left(\frac{a_r^h}{s}\right)\right].\end{aligned} \tag{H4}$$

Here the nonlogarithmic coefficients, $\mathcal{A}_{\rm ln}^{\rm nPN}$, were obtained as expansions in powers of e_t^h up to the order $O((e_t^h)^{10})$ included,

$$\begin{aligned}\mathcal{A}_{\rm ln}^{\rm 4PN}(e_t^h) &= -\frac{176}{5}(e_t^h)^2-\frac{2681}{15}(e_t^h)^4-\frac{90017}{180}(e_t^h)^6-\frac{306433}{288}(e_t^h)^8-\frac{18541327}{9600}(e_t^h)^{10},\\
\mathcal{A}_{\rm ln}^{\rm 5PN}(e_t^h) &= \left(\frac{18964}{105}\nu+\frac{2539}{35}\right)(e_t^h)^2+\left(\frac{55521}{35}\nu-\frac{524087}{840}\right)(e_t^h)^4+\left(\frac{456341}{72}\nu-\frac{11468869}{2520}\right)(e_t^h)^6\\
&+\left(\frac{2526889}{144}\nu-\frac{140341413}{8960}\right)(e_t^h)^8+\left(\frac{251185649}{6400}\nu-\frac{1320019027}{33600}\right)(e_t^h)^{10},\\
\mathcal{A}_{\rm ln}^{\rm 6PN}(e_t^h) &= \left(-\frac{9448}{27}\nu^2-\frac{907927}{630}\nu+\frac{2489}{45}\right)(e_t^h)^2+\left(-\frac{44830903}{7560}\nu^2-\frac{3709639}{1680}\nu+\frac{460759}{4536}\right)(e_t^h)^4\\
&+\left(-\frac{1067440939}{30240}\nu^2+\frac{56364713}{2016}\nu-\frac{1114216909}{68040}\right)(e_t^h)^6\\
&+\left(-\frac{699238489}{5376}\nu^2+\frac{28209572539}{161280}\nu-\frac{76207852937}{725760}\right)(e_t^h)^8\\
&+\left(-\frac{586193581933}{1612800}\nu^2+\frac{325106833717}{537600}\nu-\frac{76717484827}{201600}\right)(e_t^h)^{10},\end{aligned} \tag{H5}$$

whereas the logarithmic coefficients, $\mathcal{B}_{\rm ln}^{\rm nPN}$, are given by the following closed-form expressions:

$$\begin{aligned}\mathcal{B}_{\rm ln}^{\rm 4PN}(e_t^h) &= \frac{1}{(1-e_t^2)^{7/2}}\left[\frac{64}{5}+\frac{584}{15}(e_t^h)^2+\frac{74}{15}(e_t^h)^4\right],\\
\mathcal{B}_{\rm ln}^{\rm 5PN}(e_t^h) &= \frac{1}{(1-e_t^2)^{9/2}}\left[-\frac{11708}{105}-\frac{112}{5}\nu+\left(-\frac{5308}{15}\nu+\frac{1378}{7}\right)(e_t^h)^2+\left(-\frac{1857}{5}\nu+\frac{8941}{10}\right)(e_t^h)^4+\left(-\frac{74}{3}\nu+\frac{12539}{140}\right)(e_t^h)^6\right],\\
\mathcal{B}_{\rm ln}^{\rm 6PN}(e_t^h) &= \frac{1}{(1-e_t^2)^{11/2}}\left[\frac{32}{5}\nu^2+\frac{179234}{315}\nu+\frac{1445692}{2835}+\left(\frac{13547}{15}\nu^2+\frac{821056}{315}\nu-\frac{10378222}{2835}\right)(e_t^h)^2\right.\\
&+\left(\frac{220447}{60}\nu^2+\frac{3723539}{1890}-\frac{1062751}{105}\nu\right)(e_t^h)^4+\left(\frac{9393}{5}\nu^2+\frac{15416687}{1260}-\frac{7764587}{840}\nu\right)(e_t^h)^6\\
&\left.+\left(\frac{4979519}{5040}-\frac{204661}{420}\nu+74\nu^2\right)(e_t^h)^8+(-96+1060(e_t^h)^2+1863(e_t^h)^4+148(e_t^h)^6)\left(1-\frac{2}{5}\nu\right)(1-e_t^2)^{1/2}\right].\end{aligned} \tag{H6}$$

The latter coefficients are related via $\mathcal{B}_{\rm ln}^{\rm nPN}(e_t^h) = -2\mathcal{B}^{\rm nPN}(e_t^h) + O((e_t^h)^{11})$ to those entering the full Delaunay-averaged h-route nonlocal Hamiltonian (2.9) [see Eq. (8.11)].

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