Weyl gauge theories of gravity do not predict a second clock effect

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We consider Weyl gauge theories of gravity (WGTs), which are invariant both under local Poincaré transformations and local changes of scale. Such theories may be interpreted as gauge theories in Minkowski spacetime, but their gravitational interactions are most often reinterpreted geometrically in terms of a Weyl-Cartan spacetime, in which any matter fields then reside. Such a spacetime is a straightforward generalization of Weyl spacetime to include torsion. As first suggested by Einstein, Weyl spacetime is believed to exhibit a so-called second clock effect, which prevents the existence of experimentally observed sharp spectral lines, since the rates of (atomic) clocks depend on their past history. The prevailing view in the literature is that this rules out WGTs as unphysical. Contrary to this viewpoint, we show that if one adopts the natural covariant derivative identified in the geometric interpretation of WGTs, properly takes into account the scaling dimension of physical quantities, and recognizes that Einstein's original objection requires the presence of massive matter fields to represent atoms, observers and clocks, then WGTs do not predict a second clock effect.

I. INTRODUCTION

In 1918, Weyl proposed a unified theory of gravity and electromagnetism [1], which was based on a generalization of the Riemannian spacetime geometry assumed in Einstein's theory of general relativity. In particular, in Weyl's spacetime, the principle of relativity applies not only to the choice of reference frames, but also to the choice of local standards of length. This invariance under local changes of the unit of length (gauge) was realized by the introduction of an additional "compensating" vector field, that we shall denote by B_μ, which Weyl attempted to interpret as the electromagnetic four-potential.

In spite of the elegance and beauty of Weyl's theory, it did not achieve its original goal. It was soon recognized as being unable to accommodate well-known properties of electromagnetism, since the Weyl potential B_μ is not coupled to the electric current, but to the dilation current of matter. Indeed, one may easily show that B_μ interacts in the same manner with both particles and antiparticles, contrary to all experimental evidence about electromagnetic interactions. It was only later realized [2] that electromagnetism was related to localization of invariance under change of quantum-mechanical phase and, much later, that B_μ might instead be interpreted as mediating an additional *gravitational* interaction, within a theory of gravity that is locally scale invariant.

The first objection to Weyl's theory was, however, made by Einstein in a note published as an addendum to Weyl's original paper and applies irrespective of whether B_μ is interpreted as mediating the electromagnetic or gravitational interaction. Einstein claimed that Weyl's theory predicts a so-called "second clock effect," which is not experimentally observed. This phenomenon is in addition to the usual "first clock effect," which also occurs both in special and general relativity and has been experimentally verified to high precision.

As is well known, the latter refers to the fact that if two identical clocks, initially synchronized, coincident, and at relative rest, follow different (timelike) worldlines in spacetime before being brought back together, they will in general measure different elapsed (proper) time intervals. Nonetheless, provided the two clocks then remain coincident, they will thereafter continue to "tick" at the *same* rate. By contrast, in Weyl spacetime, if the field strength $H_{\mu\nu} \equiv 2\partial_{[\mu}B_{\nu]}$ of the Weyl potential does not vanish throughout the spacetime interior of the two-clock worldlines during their separation, then the clocks in this scenario will tick at *different* rates even after they are reunited, which is known as the second clock effect (SCE). An immediate physical consequence is that the existence of sharp spectral lines would not be possible in the presence of a nonzero field strength $H_{\mu\nu}$, since the rate of atomic clocks, as measured by some periodic physical process, would depend on their past history.

The original discussions of the SCE, which subsequently involved Weyl, Einstein, Eddington, and Pauli, among others [3–7], were based on the fact that in a Weyl

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spacetime, the "norms" of parallel transported vectors change in a manner that depends on the path taken (although the angle between two vectors remains the same; this changes only in general affine spaces for which the metric and connection are fully independent quantities). It was then argued that the norm of a timelike vector that is parallel transported along a timelike worldline can represent the tick rate of a clock, which hence leads to an SCE (equally, if the parallel transported vector is spacelike, then one may physically interpret the effect as the length of a rod being dependent on its past history, which is again contrary to experimental evidence).

The association of the clock rate with the norm of a parallel-transported vector is not trivial, particularly given that length is not a well-defined concept in Weyl's spacetime, but one may come to the same conclusion by defining a physically sensible notion of proper time along (timelike) worldlines in Weyl spacetime, which generalizes the concept of proper time used in Riemannian spacetimes [8–10]. By reconsidering the two-clock thought experiment outlined above, and computing the elapsed proper time measured by each clock between their reunion and some subsequent event, one again concludes that a Weyl spacetime does indeed exhibit an SCE, unless the Weyl potential can be expressed as the gradient of some smooth scalar field $B_\mu = \partial_\mu \phi$; this corresponds to a so-called Weyl integrable spacetime, in which the field strength $H_{\mu\nu}$ vanishes identically.

In this paper, we reconsider the issue of the SCE in the context of Weyl gauge theories of gravity (WGTs) [11–14]. These theories are derived by gauging the Weyl group, where one begins with some Minkowski spacetime matter action that is invariant under global Weyl transformations, which consist of Poincaré tranformations and dilations, and then demands that the action be invariant under local Weyl transformations, where the group parameters become independent arbitrary functions of position. This requires the introduction of gauge fields, which are interpreted as mediating gravitational interactions. Although WGTs are most naturally interpreted as gauge field theories in Minkowski spacetime, it is usual for them to be reinterpreted geometrically, whereby the gravitational interactions are considered in terms of the geometry of a Weyl-Cartan spacetime, in which any matter fields then reside [14,15]. Weyl-Cartan spacetime is a straightforward generalization of Weyl spacetime to include nonzero torsion and reduces to Weyl spacetime on imposing the properly covariant condition that the torsion vanishes. Since, as we will confirm, the presence of torsion is irrelevant to considerations of the SCE, it has thus previously been argued that Einstein's objection to Weyl spacetime rules out WGTs as unphysical, unless the Weyl potential is pure gauge [16–19].

Contrary to this prevailing view, we demonstrate that WGTs do not require this condition in order to avoid the presence of the SCE. In particular, we show that the geometric interpretation of WGTs leads to the identification of the Weyl covariant derivative as the natural derivative operator, which differs from the covariant derivative usually assumed in Weyl-Cartan spacetimes when applied to quantities having nonzero scaling dimension (or Weyl weight) w. This is especially important when differentiating the tangent vector $u^\mu(\lambda) = dx^\mu/d\lambda$ along an observer's worldline, which we show must have Weyl weight $w = -1$, rather than being invariant ($w = 0$) as is usually assumed. Finally, we point out that, since Einstein's objection to Weyl's theory is based on the observation of sharp spectral lines, one requires the presence of matter fields to represent the atoms, observers, and clocks; it is thus meaningless to consider the SCE in an empty Weyl-Cartan geometry. Moreover, such "ordinary" matter is most appropriately represented by a massive Dirac field, but in order to obey local Weyl invariance this field must acquire a mass dynamically through the introduction of a scalar compensator field, which we show is key to defining an interval of proper time as measured by a clock along an observer's worldline. On taking these considerations into account, WGTs do not predict an SCE, even when the Weyl potential is not pure gauge.

The outline of our argument is as follows. The geometric interpretation of WGTs identifies the (inverse) translational gauge field as the vierbein components $e^a{}_\mu$, which have Weyl weight $w = 1$ and relate the orthonormal tetrad frame vectors $\hat{e}_a(x)$ and the coordinate frame vectors $e_\mu(x)$ at any point x in a Weyl-Cartan spacetime. The vectors $\hat{e}_a(x)$ constitute a local Lorentz frame at each point, which defines a family of ideal observers whose worldlines are the integral curves of the timelike unit vector field $\hat{\mathbf{e}}_0$. Along a given worldline, the three spacelike unit vector fields $\hat{\mathbf{e}}_i$ ($i = 1, 2, 3$) specify the spatial triad carried by the corresponding observer, which may be thought of as defining the orthogonal spatial coordinate axes of a local laboratory frame that is valid near the observer's worldline. In general, the worldlines need not be timelike geodesics, and hence observers may be accelerating. For some test particle (or other observer) moving along some timelike worldline $\mathcal{C}$ given by $x^\mu = x^\mu(\lambda)$, where λ is some arbitrary parameter, the components of the tangent vector to this worldline, as measured by one of the above observers, will be $u^a(\lambda) = e^a{}_\mu u^\mu(\lambda)$, which are physically observable quantities in WGTs and so should be invariant ($w = 0$) under Weyl scale gauge transformations. Since the vierbein $e^a{}_\mu$ has weight $w = 1$, the weight of the components $u^\mu(\lambda)$ must thus be $w = -1$.[1] The length of the tangent vector is then invariant under Weyl scale gauge transformations.

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[1]As we will discuss in Sec. II, one may reach the same conclusion by demanding that the physical distance, as opposed to the coordinate distance, along the curve $\mathcal{C}$ is traced out at the same rate before and after a Weyl scale gauge transformation.

Moreover, by working in terms of the Weyl covariant derivative, we show that one may always find a parametrization $\xi = \xi(\lambda)$ for which the length of the tangent vector remains equal to unity under parallel transport along its worldline (and so $u^\mu(\xi) = dx^\mu/d\xi$ may be interpreted as the particle four-velocity). Consequently, the original argument for suggesting the existence of the SCE is removed.

Since $d/d\xi$ still has weight $w = -1$, however, the parameter ξ cannot be interpreted as the proper time of a particle moving along the worldline. To resolve this issue, we note that in order for WGTs to include ordinary matter, which is usually modeled by a Dirac field, one must introduce a scalar compensator field ϕ with Weyl weight $w = -1$ and make the replacement $m\bar\psi\psi \to \mu\phi\bar\psi\psi$ in the Dirac action, where μ is a dimensionless parameter but $\mu\phi$ has the dimensions of mass in natural units. In particular, the functioning of any form of practical atomic clock is based on the spacing of the energy levels in atoms, which is then characterized in the clock's local Lorentz frame by the Rydberg energy $E_R = \frac{1}{2}\mu\phi\alpha^2$ (in natural units), where α is the (dimensionless) fine structure constant. As a result, an interval of proper time measured by the clock along its worldline is in fact given by $d\tau = \phi d\xi$, which is invariant ($w = 0$) under Weyl scale gauge transformations, as required for a physically observable quantity. Applying this proper time definition to the two-clock thought experiment discussed above, one finds immediately that the clock rates are the same after their reunion, and so WGTs do not predict an SCE.

The remainder of this paper is arranged as follows. In Sec. II, we outline the basic properties a Weyl-Cartan spacetime, review the existing arguments for the presence of an SCE, and discuss an alternative approach to determining the scaling dimension of the tangent vector to an observer's worldline. In Sec. III, we then discuss Weyl gauge theories of gravity and describe their geometric interpretation in terms of Weyl-Cartan spacetime in Sec. IV, highlighting in particular the identification of the Weyl covariant derivative. We then reconsider the SCE in the context of the geometric interpretation of WGTs in Sec. V, both in terms of the norms of parallel transported vectors and in terms of an appropriately defined proper time. Finally, we present our conclusions in Sec. VI.

II. WEYL-CARTAN SPACETIME

A. Mathematical background

A Weyl-Cartan spacetime Y_4 is a differentiable manifold endowed with a metric and affine connection, whose components in some arbitrary holonomic coordinate system we denote by $g_{\mu\nu}$ and $\Gamma^\mu{}_{\rho\sigma}$, respectively. The latter defines a covariant derivative operator $\nabla_\mu = \partial_\mu + \Gamma^\sigma{}_{\rho\mu}X^\rho{}_\sigma$, where $X^\rho{}_\sigma$ are the $GL(4, R)$ generator matrices appropriate to the tensor character under general coordinate transformations (GCT) of the quantity to which ∇_μ is applied.

In particular, a Y_4 spacetime is defined by the requirement that this derivative operator satisfies the semimetricity condition

$$\nabla_\sigma g_{\mu\nu} = -2B_\sigma g_{\mu\nu}, \tag{1}$$

where B_μ is the Weyl potential (and we have included a factor of -2 for later convenience and to be consistent with the notation typically used in WGT). As Weyl originally showed, on performing the simultaneous (gauge) transformations

$$\bar g_{\mu\nu} = e^{2\rho}g_{\mu\nu}, \tag{2a}$$

$$\bar B_\mu = B_\mu - \partial_\mu\rho, \tag{2b}$$

where $\rho = \rho(x)$ is an arbitrary scalar function, the condition (1) is preserved, that is, one has $\nabla_\sigma \bar g_{\mu\nu} = -2\bar B_\sigma \bar g_{\mu\nu}$. Thus, these transformations define an equivalence class of Weyl-Cartan manifolds, all of which share the same connection. In this sense, the only geometrical quantities of real physical significance in Weyl-Cartan spacetime are those that transform covariantly under the transformations (2), which may be interpreted as a change in the length scale at every point of the manifold. It is worth noting that if the Weyl potential is a pure gradient $B_\mu = \partial_\mu\rho$, then the gauge transformations (2) reduce Weyl-Cartan spacetime Y_4 into a Riemann-Cartan U_4 spacetime, since $\bar B_\mu = 0$. More generally, if and only if the field strength $H_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}$ vanishes, a Y_4 spacetime can be reduced to U_4 by a suitable transformation of the form (2).

From (1), one finds that the connection is given by

$$\Gamma^\lambda{}_{\mu\nu} = {}^0\Gamma^{*\lambda}{}_{\mu\nu} + K^{*\lambda}{}_{\mu\nu}, \tag{3}$$

where the first term on the rhs in symmetric in (μ, ν) and reads

$${}^0\Gamma^{*\lambda}{}_{\mu\nu} = {}^0\Gamma^\lambda{}_{\mu\nu} + \delta^\lambda_\nu B_\mu + \delta^\lambda_\mu B_\nu - g_{\mu\nu}B^\lambda, \tag{4}$$

in which ${}^0\Gamma^\lambda{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$ is the standard metric (Christoffel) connection. The term $K^{*\lambda}{}_{\mu\nu}$ in (4) is the Y_4 contortion tensor, which is given in terms of (minus) the Y_4 torsion $T^{*\lambda}{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\nu\mu]}$ by

$$K^{*\lambda}{}_{\mu\nu} = -\frac{1}{2}(T^{*\lambda}{}_{\mu\nu} - T^*{}_\nu{}^\lambda{}_\mu + T^*{}_{\mu\nu}{}^\lambda) \tag{5}$$

and has the antisymmetry property $K^*_{\lambda\mu\nu} = -K^*_{\mu\lambda\nu}$ (we have placed asterisks on several quantities above and included a minus sign in the definition of the torsion to be consistent with the usual notation adopted in WGT). It is clear from (3)–(5) that setting the torsion to zero, which is a properly invariant condition under the gauge transformations (2),

then Weyl-Cartan spacetime Y_4 reduces to Weyl spacetime W_4.

B. Physical motivation

Aside from Weyl's original paper, much of the interest in Weyl(-Cartan) spacetimes stems from the work of Ehlers, Pirani, and Schild (EPS) [8], who proposed an axiomatic approach to determining a suitable mathematical model of spacetime using only basic assumptions about the behavior of freely falling massive and massless particles. This led to the conclusion that massless particles determine a conformal structure on spacetime, while the massive particles determine a projective structure on spacetime. By imposing a compatibility condition on these two structures, basically postulating that massless particle trajectories can be approximated arbitrarily well by massive particle ones, EPS arrived at Weyl spacetime as the appropriate mathematical model. In their approach, EPS assumed the connection to be symmetric (torsionless) from the outset, but their conclusions rely only on the semimetricity condition (1) and so it is reasonable to consider the more general Weyl-Cartan spacetime, which allows for nonzero torsion.

C. Parallel transported vectors and the SCE

Using the semimetricity condition (1), which holds irrespective of the presence of torsion, the evolution of the inner product of any two vectors v^μ and w^μ parallel transported along some curve $\mathcal{C}$ is given by

$$
\begin{aligned}
\frac{d}{d\lambda}[g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda)] &= \frac{D}{D\lambda}[g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda)], \\
&= (u^\sigma(\lambda)\nabla_\sigma g_{\mu\nu})v^\mu(\lambda)w^\nu(\lambda) \\
&= -2B_\sigma u^\sigma(\lambda)g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda), \quad (6)
\end{aligned}
$$

where $u^\mu(\lambda) = dx^\mu/d\lambda$ is the tangent vector to $\mathcal{C}$ at the parameter value λ and we have used the parallel transport conditions $Dv^\mu/D\lambda = 0 = Dw^\mu/D\lambda$. Hence, on integrating, the inner product as a function of the parameter λ along the curve is given by

$$
g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda) = g_{\mu\nu}v^\mu(\lambda_0)w^\nu(\lambda_0)e^{-2\int_{\lambda_0}^{\lambda} B_\sigma u^\sigma(\lambda')d\lambda'}. \quad (7)
$$

By setting $v^\mu = w^\mu$ and considering parallel transport around a closed curve $\mathcal{C}$, the length ℓ of a vector on completing a loop is related to its original length ℓ_0 by

$$
\ell = \ell_0 e^{-\oint_{\mathcal{C}} B_\mu dx^\mu}. \quad (8)
$$

Thus, using Stokes' theorem, the condition $\ell = \ell_0$ holds if and only if $H_{\mu\nu}$ vanishes throughout the region interior to $\mathcal{C}$. The above result forms the basis of the original discussions

of the SCE in Weyl's theory by Einstein and others, as described in the Introduction.

D. Proper time and the SCE

As also mentioned in the Introduction, however, the intuitive argument above is not rigorous, and a more careful approach is based on defining a consistent notion of proper time along (timelike) worldlines in Weyl spacetimes [8–10]. The simplest construction is based on the requirement that for a timelike curve $x^\mu(\tau)$ to be parametrized by proper time τ, one requires the tangent vector $u^\mu(\tau) = dx^\mu/d\tau$, which is thus the particle four-velocity, to be orthogonal to its four-acceleration $a^\mu(\tau) = Du^\mu/D\tau$, such that

$$
g_{\mu\nu}u^\mu(\tau)a^\nu(\tau) = 0. \quad (9)
$$

This derivation of the proper time τ is unaffected by the presence of nonzero torsion, since it also depends only on the semimetricity condition (1) and is hence applicable in Weyl-Cartan spacetimes. It is useful first to note that, for two *arbitrary* parametrizations λ and ξ of a worldline, the corresponding tangent vectors and their absolute derivatives (which should no longer strictly be interpreted as the particle four-velocity and acceleration) are related by [10]

$$
u^\mu(\lambda) = \frac{d\xi}{d\lambda}u^\mu(\xi), \quad (10a)
$$

$$
a^\mu(\lambda) = \frac{d^2\xi}{d\lambda^2}u^\mu(\xi) + \left(\frac{d\xi}{d\lambda}\right)^2 a^\mu(\xi). \quad (10b)
$$

By considering the quantity $g_{\mu\nu}u^\mu(\lambda)a^\nu(\lambda)$ and making the identification $\xi = \tau$, for which we require (9) to hold, one finds

$$
\frac{d^2\tau}{d\lambda^2} - \frac{g_{\mu\nu}u^\mu(\lambda)a^\nu(\lambda)}{g_{\mu\nu}u^\mu(\lambda)u^\nu(\lambda)}\frac{d\tau}{d\lambda} = 0. \quad (11)
$$

Since this a linear differential equation, if τ is a solution, then so too is $a\tau + b$, where a and b are constants that represent merely the scaling and zero point of the proper time variable, respectively. To proceed further, it is convenient to consider the quantity

$$
\begin{aligned}
\frac{d}{d\lambda}[g_{\mu\nu}u^\mu(\lambda)u^\nu(\lambda)] &= \frac{D}{D\lambda}[g_{\mu\nu}u^\mu(\lambda)u^\nu(\lambda)], \\
&= (u^\sigma\nabla_\sigma g_{\mu\nu})u^\mu u^\nu + 2g_{\mu\nu}u^\mu a^\nu, \\
&= -2B_\sigma u^\sigma g_{\mu\nu}u^\mu u^\nu + 2g_{\mu\nu}u^\mu a^\nu, \quad (12)
\end{aligned}
$$

where we use the semimetricity condition (1), and in the last two lines (and the remainder of this section), we drop the explicit dependence of quantities on the arbitrary parameter λ for brevity. Thus, (11) becomes

$$\frac{d^2\tau}{d\lambda^2} - \left[\frac{1}{2}\frac{d}{d\lambda}(g_{\mu\nu}u^\mu u^\nu) + g_{\mu\nu}B^\mu u^\nu\right]\frac{d\tau}{d\lambda} = 0, \quad (13)$$

which is straightforwardly solved to obtain the proper time interval $\Delta\tau$ between two events corresponding to the parameter values λ_0 and λ along the worldline,

$$\Delta\tau(\lambda) = \frac{d\tau/d\lambda}{\sqrt{g_{\mu\nu}u^\mu u^\nu}}\bigg|_{\lambda_0} \int_{\lambda_0}^\lambda e^{\int_{\lambda_0}^{\lambda'} B_\mu u^\mu d\lambda''} \sqrt{g_{\mu\nu}u^\mu u^\nu}\, d\lambda'. \quad (14)$$

The application of this result to the two-clock thought experiment discussed in the Introduction is straightforward. Suppose the two clocks are separated at some event and thereafter follow the worldlines $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, before being reunited at some other event. The ratio of the elapsed proper time measured by each clock between their reunion and some subsequent event along their joint worldline is given by

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \exp\left(\int_{\mathcal{C}_2} B_\mu dx^\mu - \int_{\mathcal{C}_1} B_\mu dx^\mu\right). \quad (15)$$

Thus, in general, the clock rates differ after their reunion and so Weyl-Cartan spacetime exhibits an SCE.

E. Weyl weight of worldline tangent vector

The above derivation of the proper time τ is based on the condition (9), which is assumed to be consistent across the whole equivalence class defined by the Weyl gauge transformations (2). This consistency holds, however, only if the quantities $u^\mu(\tau)$ transform covariantly with Weyl weight $w = 0$ (i.e., they are invariant) under these transformations. As discussed in the Introduction, however, we argue that these quantities in fact have weight $w = -1$, based on consideration of their corresponding components in the tetrad basis. One may, however, obtain further insight into this conclusion without the use of tetrads, as outlined below.

One may in fact work more generally in terms of an arbitrary parameter λ, such that $u^\mu(\lambda) = dx^\mu/d\lambda$ is the tangent vector at the parameter value λ to the worldline $\mathcal{C}$ given by $x^\mu = x^\mu(\lambda)$. Since the coordinates are unaffected by the gauge transformations (2), they have weights $w = 0$, so it remains only to determine the weight of the operator $d/d\lambda$. One may achieve this by first writing

$$\frac{d}{d\lambda} = \frac{ds}{d\lambda}\frac{d}{ds}, \quad (16)$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Similarly, after the gauge transformations (2), one has

$$\frac{d}{d\bar\lambda} = \frac{d\bar s}{d\bar\lambda}\frac{d}{d\bar s}. \quad (17)$$

By requiring that $d\bar s/d\bar\lambda = ds/d\lambda$, so that the physical distance, as opposed to the coordinate distance, along the curve $\mathcal{C}$ is traced out at the same rate before and after the gauge transformations (2), and using the fact that $d\bar s = e^\rho ds$, then

$$\frac{d}{d\bar\lambda} = e^{-\rho}\frac{ds}{d\lambda}\frac{d}{ds} = e^{-\rho}\frac{d}{d\lambda}. \quad (18)$$

Thus, the tangent vector $u^\mu(\lambda) = dx^\mu/d\lambda$ has Weyl weight $w = -1$. It is worth noting that this conclusion does not mean that the worldline $\mathcal{C}$ changes under the gauge transformation, but only that the coordinates along it are traced out at a different rate before and after the transformation. For example, if two points O and A on the curve with coordinates x_0^μ and x_A^μ correspond to parameter values $\lambda = 0$ and $\lambda = \lambda_A$, respectively, before the gauge transformation, then these points will have the same coordinates but parameter values $\bar\lambda = 0$ and $\bar\lambda \neq \lambda_A$ after the transformation. The most important consequence of $u^\mu(\lambda)$ having weight $w = -1$ is, however, that the length of the tangent vector is invariant under the gauge transformations (2), which follows immediately since

$$g_{\mu\nu}u^\mu u^\nu \to \bar g_{\mu\nu}\bar u^\mu \bar u^\nu = g_{\mu\nu}u^\mu u^\nu, \quad (19)$$

whereas the condition (9) (but with τ replaced by λ) is not invariant, since one may show that

$$\bar g_{\mu\nu}\bar u^\mu \bar a^\nu = e^{-\rho}g_{\mu\nu}u^\mu[a^\nu - u^\sigma(\partial_\sigma\rho)u^\nu]. \quad (20)$$

This lack of invariance of the condition (9) under Weyl gauge transformations undermines the physical significance of the proper time variable derived above. As we show below, however, by considering the geometric interpretation of Weyl gauge theories of gravity, one may identify an alternative form of covariant derivative which both leaves the length of a vector unchanged after parallel transport around a closed loop and allows one to define an analogous condition to (9) which is invariant under Weyl gauge transformations.

III. WEYL GAUGE THEORIES OF GRAVITY

It was the gauging of the Poincaré group $\mathcal{P}$ by Kibble [20] that first revealed how to achieve a meaningful gauging of groups that act on the points of spacetime as well as on the components of physical fields. The essence of Kibble's approach was to note that when the parameters of the Poincaré group become independent arbitrary functions of position, this leads to a complete decoupling of the translational parts from the rest of the group, and the former is then interpreted as arising from a GCT (or spacetime diffeomorphisms, if interpreted actively). Thus, the action of the gauged Poincaré group is considered as a GCT $x^\mu \to x'^\mu$, together with the local action of its Lorentz

subgroup on the orthonormal tetrad basis vectors $\hat{\boldsymbol{e}}_a(x)$ that define local Lorentz reference frames, where we adopt the common convention that Latin indices (from the start of the alphabet) refer to anholonomic local tetrad frames, while Greek indices refer to holonomic coordinate frames. This approach leads to Poincaré gauge theories (PGT) of gravity, but can be straightforwardly extended to more general spacetime symmetry groups [14,21,22].

A natural extension of PGT is also to demand local scale invariance, which is most directly achieved by gauging the Weyl group $\mathcal{W}$ [11–13]. This may be formulated in a number of ways, for example, by considering the Weyl transformations as active or passive, infinitesimal or finite, but they are all essentially equivalent. As in PGT, the physical model is an underlying Minkowski spacetime in which a set of matter fields φ_i is distributed continuously (these fields may include a scalar compensator field, which we occasionally also denote by ϕ). Since the spacetime is Minkowski, one may adopt a global Cartesian inertial coordinate system x^μ, which greatly simplifies calculations, but more general coordinate systems may be straightforwardly accommodated, if required [15]. The field dynamics are described by a matter action

$$S_{\mathrm{M}} = \int L_{\mathrm{M}}(\varphi_i, \partial_\mu \varphi_i) d^4 x, \tag{21}$$

which is invariant under the global action of the Weyl group. One then gauges the Weyl group $\mathcal{W}$ by demanding that the matter action be invariant with respect to (infinitesimal, passively interpreted) GCT and the local action of the subgroup $\mathcal{H}$ (the homogeneous Weyl group), obtained by setting the translation parameters of $\mathcal{W}$ to zero (which leaves the origin $x^\mu = 0$ invariant), and allowing the remaining group parameters to become independent arbitrary functions of position.

In this way, one is led to the introduction of new field variables $h_a{}^\mu$, $A^{ab}{}_\mu$, and B_μ, corresponding to the translational, rotational, and dilational parts of the Weyl group, respectively. These fields are interpreted as gravitational gauge fields and are used to assemble the covariant derivative (adopting the common notation in WGT [14,15])

$$\mathcal{D}_a^* \varphi_i = h_a{}^\mu D_\mu^* \varphi_i = h_a{}^\mu \left(\partial_\mu + \frac{1}{2} A^{ab}{}_\mu \Sigma_{ab} + w_i B_\mu \right) \varphi_i, \tag{22}$$

where the field φ_i is assumed to have Weyl weight w_i and $\Sigma_{ab} = -\Sigma_{ba}$ are the generators matrices of the $\mathrm{SL}(2, C)$ representation to which φ_i belongs. Since $\mathcal{D}_a^* \varphi_i$ is constructed to transform in the same way under the action of the gauged group $\mathcal{W}$ as $\partial_\mu \varphi_i$ does under the global action of $\mathcal{W}$, the matter action in the presence of gravity is then typically obtained by the minimal coupling procedure of replacing partial derivatives in the special-relativistic matter Lagrangian by covariant ones to obtain

$$S_{\mathrm{M}} = \int h^{-1} L_{\mathrm{M}}(\varphi_i, \mathcal{D}_a \varphi_i) d^4 x, \tag{23}$$

where the factor containing $h \equiv \det(h_a{}^\mu)$ is required to make the integrand a scalar density rather than a scalar. It should be noted that the requirement of local scale invariance imposes tight constraints on the precise form of L_{M}. In particular, since h^{-1} has a Weyl (or conformal) weight $w(h^{-1}) = 4$, the Lagrangian L_{M} must have a weight $w(L_{\mathrm{M}}) = -4$.

In addition to the matter action, the total action must also contain terms describing the dynamics of the free gravitational gauge fields. The latter is constructed from the field strength tensors $\mathcal{R}_{abcd}$, $\mathcal{T}^*_{abc}$, $\mathcal{H}_{ab}$ of the rotational, translational, and dilational gauge fields, respectively, which are defined through the action of the commutator of two covariant derivatives on some field φ of weight w by

$$[\mathcal{D}_c^*, \mathcal{D}_d^*]\varphi = \frac{1}{2}\mathcal{R}^{ab}{}_{cd}\Sigma_{ab}\varphi + w\mathcal{H}_{cd}\varphi - \mathcal{T}^{*a}{}_{cd}\mathcal{D}_a^*\varphi. \tag{24}$$

It is straightforward to show that the fields strengths have the forms $\mathcal{R}^{ab}{}_{cd} = h_a{}^\mu h_b{}^\nu R^{ab}{}_{\mu\nu}$, $\mathcal{H}_{cd} = h_c{}^\mu h_d{}^\nu H_{\mu\nu}$, and $\mathcal{T}^{*a}{}_{bc} = h_b{}^\mu h_c{}^\nu T^{*a}{}_{\mu\nu}$, where

$$R^{ab}{}_{\mu\nu} = 2(\partial_{[\mu} A^{ab}{}_{\nu]} + \eta_{cd} A^{ac}{}_{[\mu} A^{db}{}_{\nu]}), \tag{25}$$

$$H_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}, \tag{26}$$

$$T^{*a}{}_{\mu\nu} = 2D_{[\mu}^* b^a{}_{\nu]}, \tag{27}$$

and $b^a{}_\mu$ is the inverse of $h_a{}^\mu$. It is worth noting that $\mathcal{R}^{ab}{}_{cd}$ has the same functional form as the rotational field strength in PGT (which we thus denote by the same symbol), but that $\mathcal{T}^{*a}{}_{bc} = \mathcal{T}^a{}_{bc} + \delta_c^a \mathcal{B}_b - \delta_b^a \mathcal{B}_c$, where $\mathcal{T}^a{}_{bc}$ is the translational field strength in PGT and $\mathcal{B}_a = h_a{}^\mu B_\mu$. The free gravitational action then has the general form

$$S_{\mathrm{G}} = \int h^{-1} L_{\mathrm{G}}(\mathcal{R}^{ab}{}_{cd}, \mathcal{T}^{*a}{}_{bc}, \mathcal{H}_{ab}) d^4 x, \tag{28}$$

where L_{G} must also have a Weyl (conformal) weight $w(L_{\mathrm{G}}) = -4$, which places tight constraints on its form. It is easily shown that $w(\mathcal{R}^{ab}{}_{cd}) = w(\mathcal{H}_{ab}) = -2$ and $w(\mathcal{T}^{*a}{}_{bc}) = -1$, which means that L_{G} can be quadratic in $\mathcal{R}^{ab}{}_{cd}$ and $\mathcal{H}_{ab}$, while terms linear in $\mathcal{R} \equiv \mathcal{R}^{ab}{}_{ab}$ or quadratic in $\mathcal{T}^{*a}{}_{bc}$ are not allowed, despite them transforming covariantly under local Weyl transformations.

One can, however, construct further Weyl-covariant terms with the appropriate weight for inclusion in the total Lagrangian by introducing an additional massless scalar field (or fields) ϕ with Weyl weight $w(\phi) = -1$, often termed the compensator(s) [14]. This opens up possibilities for the inclusion of further action terms in which the scalar field is nonminimally (conformally) coupled to the field

strength tensors of the gravitational gauge fields, combined (usually) with an additional free kinetic term for ϕ. For example, terms proportional to $\phi^2 \mathcal{R}$ or $\phi^2 L_{\mathcal{T}^{*2}}$, where $L_{\mathcal{T}^{*2}}$ consists of terms quadratic in $\mathcal{T}^{*a}{}_{bc}$, are Weyl covariant with weight $w = -4$ and so may be added to the total Lagrangian [23–26].

The inclusion of scalar fields also allows for more flexibility in the allowed forms of the matter. An important example is a free Dirac field ψ, which has Weyl weight $w(\psi) = w(\bar{\psi}) = -\frac{3}{2}$, and for which the Lagrangian reads

$$L_{\mathrm{D}} = \frac{1}{2} i \bar{\psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m \bar{\psi} \psi. \qquad (29)$$

The corresponding action is not scale invariant owing to the mass term $m \bar{\psi} \psi$. It thus appears that one requires the field to be massless, which clearly cannot describe ordinary matter. This difficulty can be circumvented, however, by making the replacement $m \bar{\psi} \psi \rightarrow \mu \phi \bar{\psi} \psi$, where μ is a dimensionless parameter but $\mu \phi$ has the dimensions of mass in natural units. The action is then invariant under the global Weyl group, and one may also add kinetic and self-interaction terms of weight $w = -4$ for the scalar field ϕ. After gauging the Weyl group as outlined above, the resulting WGT matter Lagrangian for the Dirac and compensator scalar field is given by

$$L_{\mathrm{M}} = \frac{1}{2} i \bar{\psi} \gamma^a \overleftrightarrow{\mathcal{D}}_a \psi - \mu \phi \bar{\psi} \psi + \frac{1}{2} \nu (\mathcal{D}^*_a \phi)(\mathcal{D}^{*a} \phi) - \lambda \phi^4, \qquad (30)$$

where μ, ν, and λ are dimensionless constants (usually positive). In this way, although the trace of the total energy momentum tensor of the ψ and ϕ fields must vanish, the energy-momentum tensor of the Dirac matter field ψ itself need not be traceless, thereby allowing it to be massive. Indeed, this approach to the construction of gauge theories of gravity that are scale invariant but, at the same time, are able to accommodate ordinary matter was first explored by Dirac [23].

More generally, the introduction of scalar fields in WGT is also important since they provide a natural means for spontaneously breaking the scale symmetry. The approach most commonly adopted is to use local scale invariance to set the compensator scalar field ϕ to a constant value in the resulting field equations, which is known as the Einstein gauge. Setting $\phi = \phi_0$ in the equation of motion for the Dirac field ψ, for example, leads to its interpretation as a massive field with $m = \mu \phi_0$. It is usually considered that setting $\phi = \phi_0$ represents the choice of some definite scale in the theory, thereby breaking scale invariance. Indeed, it is often given the physical interpretation of corresponding to some spontaneous breaking of the scale symmetry (where nature chooses the gauge). This interpretation is questionable, however, since the equations of motion in the Einstein

gauge are identical in form to those obtained when working in scale-invariant variables, where the latter involves no breaking of the scale symmetry [15].

In any case, the total action is taken as the sum of the matter and gravitational actions, and variation of the total action with respect to the gauge fields $h_a{}^\mu$, $A^{ab}{}_\mu$, and B_μ leads to three coupled gravitational field equations in which the energy-momentum $\tau^k{}_\mu \equiv \delta \mathcal{L}_{\mathrm{M}} / \delta h_a{}^\mu$, spin-angular-momentum $\sigma_{ab}{}^\mu \equiv \delta \mathcal{L}_{\mathrm{M}} / \delta A^{ab}{}_\mu$, and dilation current $\zeta^\mu \equiv \delta \mathcal{L}_{\mathrm{M}} / \delta B_\mu$ of the matter field act as sources, where $\mathcal{L}_{\mathrm{M}} \equiv h^{-1} L_M$.

IV. GEOMETRIC INTERPRETATION OF WGT

Kibble's gauge approach to gravitation is most naturally interpreted as a field theory in Minkowski spacetime [15,27,28], in the same way as the gauge field theories describing the other fundamental interactions. It is more common, however, to reinterpret the mathematical structure of gravitational gauge theories geometrically [14].

At the heart of the geometric interpretation is the identification of $h_a{}^\mu$ as the components of a vierbein system in a more general spacetime. Thus, at any point x in the spacetime, one demands that the orthonormal tetrad frame vectors $\hat{\boldsymbol{e}}_a(x)$ and the coordinate frame vectors $\boldsymbol{e}_\mu(x)$ are related by

$$\hat{\boldsymbol{e}}_a = h_a{}^\mu \boldsymbol{e}_\mu, \qquad \boldsymbol{e}_\mu = b^a{}_\mu \hat{\boldsymbol{e}}_a, \qquad (31)$$

with similar relationships holding between the dual basis vectors $\hat{\boldsymbol{e}}^a(x)$ and $\boldsymbol{e}^\mu(x)$ in each set. For any other vector $\boldsymbol{V}$, written in the coordinate basis as (say) $V_\mu \boldsymbol{e}^\mu$, one then identifies the quantities $V_a = h_a{}^\mu V_\mu$, for example, as the components of the same vector, but in the tetrad basis. This is a fundamental difference from the Minkowski spacetime viewpoint presented earlier, in which $\mathcal{V}_a = h_a{}^\mu V_\mu$ would be regarded as the components in the tetrad basis of a *new* vector field $\mathcal{V}$.

The identification of $h_a{}^\mu$ as the components of a vierbein system has a number of far-reaching consequences. First, the index-conversion properties of $h_a{}^\mu$ and $b^a{}_\mu$ are extended. It is straightforward to show, for example, that $h_a{}^\mu V^a = V^\mu$ and $b^a{}_\mu V^\mu = V^a$. Moreover, any contraction over Latin (Greek) indices can be replaced by one over Greek (Latin) indices. None of these operations is admissible when the h, A, and B fields are viewed purely as gauge fields in Minkowski spacetime.

Perhaps the most important consequence of identifying $h_a{}^\mu$ as the components of a vierbein system is that the inner product of the coordinate basis vectors becomes

$$\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \eta_{ab} b^a{}_\mu b^b{}_\nu \equiv g_{\mu\nu}. \qquad (32)$$

Thus, in this geometric interpretation, one must work in a more general spacetime with metric $g_{\mu\nu}$. Conversely, since

the tetrad basis vectors still form an orthonormal set, one has

$$\hat{e}_a \cdot \hat{e}_b = \eta_{ab} = g_{\mu\nu} h_a{}^\mu h_b{}^\nu. \tag{33}$$

From (32), one also finds that $h^{-1} = \sqrt{-g}$ (where we are working with a metric signature of -2). Under a (local, physical) dilation, the spacetime metric and h-field have Weyl weights $w(g_{\mu\nu}) = 2$ and $w(h_a{}^\mu) = -1$, respectively, and so (32) and (33) imply that $w(\eta_{ab}) = 0$, as expected. From (31) to (33), one immediately finds that the h-field and its inverse are directly related by index raising/lowering, so there no need to distinguish between them by using different kernel letters. Consequently, the standard practice, which we will follow here, is to notate $h_a{}^\mu$ and $b^a{}_\mu$ as $e_a{}^\mu$ and $e^a{}_\mu$, respectively. One should also note that, if the components V_μ and V^μ have Weyl weights w and $\tilde{w} = w - 2$, respectively, then the components V_a and V^a have weights $w - 1$ and $\tilde{w} + 1 = w - 1$.

One is also led naturally to the interpretation of $A^{ab}{}_\mu$ as the components of the "spin-connection" that encodes the rotation of the local tetrad frame between points x and $x + \delta x$, which is accompanied by a local change in the standard of length between the two points, which is encoded by B_μ. Thus, the operation of parallel transport for some vector V^a of weight w is defined as

$$\delta V^a = -(A^a{}_{b\mu} + wB_\mu \delta^a_b) V^b \delta x^\mu, \tag{34}$$

which is required to compare vectors $V^a(x)$ and $V^a(x + \delta x)$ at points x and $x + \delta x$, determined with respect to the tetrad frames $\hat{e}_a(x)$ and $\hat{e}_a(x + \delta x)$, respectively. Hence, in general, a vector not only changes its direction on parallel transport around a closed loop, but also its length. The expression (34) establishes the correct form for the related (Λ, ρ)-covariant derivative, for example,

$$D^*_\mu V^a = \partial_\mu V^a + wB_\mu V^a + A^a{}_{b\mu} V^b = \partial^*_\mu V^a + A^a{}_{b\mu} V^b, \tag{35}$$

where $\partial^*_\mu \equiv \partial_\mu + wB_\mu$. Moreover, the existence of tetrad frames at each point of the spacetime implies the existence of the Lorentz metric η_{ab} at each point. Then demanding that η_{ab} is invariant under parallel transport, and recalling that $w(\eta_{ab}) = 0$, requires the spin connection to be antisymmetric, that is, $A^{ab}{}_\mu = -A^{ba}{}_\mu$, as previously.

Further differences between the Minkowski spacetime gauge field viewpoint and the geometric interpretation occur when generalizing the (Λ, ρ)-covariant derivative to apply to fields with definite GCT tensor behavior. First, in the geometric interpretation, one can in general no longer construct a global inertial Cartesian coordinate system in the more general spacetime. Thus, one must rely on arbitrary coordinates and so define the "total" covariant derivative

$$\Delta^*_\mu \equiv \partial^*_\mu + \Gamma^\sigma{}_{\rho\mu} \mathsf{X}^\rho{}_\sigma + \frac{1}{2} A^{ab}{}_\mu \Sigma_{ab} = \nabla^*_\mu + D^*_\mu - \partial^*_\mu, \tag{36}$$

where $\nabla^*_\mu = \partial^*_\mu + \Gamma^\sigma{}_{\rho\mu} \mathsf{X}^\rho{}_\sigma = \nabla_\mu + wB_\mu$, in which $\mathsf{X}^\rho{}_\sigma$ are the $GL(4, R)$ generator matrices appropriate to the GCT tensor character of the field to which Δ^*_μ is applied and w is the Weyl weight of the field. If a field ψ carries only Latin indices, then $\nabla^*_\mu \psi = \partial^*_\mu \psi$ and so $\Delta^*_\mu \psi = D^*_\mu \psi$; conversely, if a field ψ carries only Greek indices, then $D^*_\mu \psi = \partial^*_\mu \psi$ and so $\Delta^*_\mu \psi = \nabla^*_\mu \psi$. When acting on an object of weight w, for all these derivative operators the resulting object also transforms covariantly with the same weight w.

Most importantly, in a dynamical spacetime, the affine connection coefficients $\Gamma^\sigma{}_{\rho\mu}$ are themselves dynamical variables, no longer fixed by our choice of coordinate system. They are, however, necessarily related to the spin-connection and dilation vector since the tetrad components V^a of a vector with coordinate components V^μ should, when parallel transported from x to $x + \delta x$, be equal to $V^a + \delta V^a$, that is,

$$V^a + \delta V^a = (V^\mu + \delta V^\mu) e^a{}_\mu(x + \delta x). \tag{37}$$

If the vector components V^μ have Weyl weight w, we substitute for δV^a using (34), but with $w \to w - 1$, and denote parallel transport of the coordinate basis components in an analogous fashion by defining

$$\delta V^\mu = -(\Gamma^\mu{}_{\rho\sigma} + wB_\sigma \delta^\mu_\rho) V^\rho \delta x^\sigma. \tag{38}$$

In other words, the quantities $\Gamma^\sigma{}_{\rho\mu}$ contain the same geometrical information as $e^a{}_\mu$ and $A^{ab}{}_\mu$, but in a different frame. This information corresponds to 40 independent gravitational field variables, and there are a further four variables contained in B_μ.

From (37), we obtain the relation

$$\Delta^*_\mu e^a{}_\nu \equiv \partial^*_\mu e^a{}_\nu - \Gamma^\sigma{}_{\nu\mu} e^a{}_\sigma + A^a{}_{b\mu} e^a{}_\nu = 0, \tag{39}$$

which relates A and Γ (and B); in particular, we note that $w(\Gamma^\sigma{}_{\nu\mu}) = 0$. The relation (39) is sometimes known as the "tetrad postulate," but note that it always holds. It is straightforward to show that A or Γ may be written explicitly in terms of the other as

$$\Gamma^\lambda{}_{\nu\mu} = e_a{}^\lambda (\partial^*_\mu e^a{}_\nu + A^a{}_{b\mu} e^b{}_\nu), \tag{40}$$

$$A^a{}_{b\mu} = e^a{}_\lambda (\partial^*_\mu e_b{}^\lambda + \Gamma^\lambda{}_{\nu\mu} e_b{}^\nu). \tag{41}$$

Using (32) and (39), one finds that

$$\nabla^*_\sigma g_{\mu\nu} = 0, \tag{42}$$

and so this derivative operator commutes with raising and lowering of coordinate indices. Equivalently, one may write this semimetricity condition as

$$\nabla_\sigma g_{\mu\nu} = -2B_\sigma g_{\mu\nu}, \tag{43}$$

which shows that the spacetime has, in general, a Weyl-Cartan Y_4 geometry. Hence, as discussed in Sec. II, the connection $\Gamma^\lambda_{\ \mu\nu}$ must satisfy the conditions (3)–(5).

Moreover, substituting (41) into the expressions (25)–(27) for the gauge field strengths $R^{ab}_{\ \ \mu\nu}$, $H_{\mu\nu}$, and $T^{*a}_{\ \ \mu\nu}$, one finds that

$$R^\rho_{\ \sigma\mu\nu} = 2(\partial_{[\mu}\Gamma^\rho_{\ |\sigma|\nu]} + \Gamma^\rho_{\ \lambda[\mu}\Gamma^\lambda_{\ |\sigma|\nu]}) - H_{\mu\nu}\delta^\rho_\sigma, \tag{44}$$

$$H_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}, \tag{45}$$

$$T^{*\lambda}_{\ \ \mu\nu} = 2\Gamma^\lambda_{\ [\nu\mu]}, \tag{46}$$

where $R^\rho_{\ \sigma\mu\nu} = e_a^{\ \rho}e^b_{\ \sigma}R^a_{\ b\mu\nu}$ and $T^{*\lambda}_{\ \ \mu\nu} = e_a^{\ \lambda}T^{*a}_{\ \ \mu\nu}$. Thus, although we recognize $T^{*\lambda}_{\ \ \mu\nu}$ as (minus) the torsion tensor of the Y_4 spacetime, we see that $R^\rho_{\ \sigma\mu\nu}$ is not simply its Riemann tensor. Rather, the Riemann tensor of the Y_4 spacetime is given by

$$\tilde{R}^\rho_{\ \sigma\mu\nu} \equiv R^\rho_{\ \sigma\mu\nu} + H_{\mu\nu}\delta^\rho_\sigma. \tag{47}$$

One should note that, although $\tilde{R}_{\rho\sigma\mu\nu}$ is antisymmetric in (μ, ν), it is no longer antisymmetric in (ρ, σ) (indeed $\tilde{R}_{(\rho\sigma)\mu\nu} = g_{\rho\sigma}H_{\mu\nu}$) and does not satisfy the familiar cyclic and Bianchi identities of the Riemann tensor in a Riemannian V_4 spacetime. One may also show that, with the given arrangements of indices, both $\tilde{R}^\rho_{\ \sigma\mu\nu}$ (or $R^\rho_{\ \sigma\mu\nu}$) and $T^{*\lambda}_{\ \ \mu\nu}$ transform covariantly with weight $w = 0$ under a local dilation. It is also worth noting that $\tilde{R}_{\mu\nu} \equiv \tilde{R}_{\mu\lambda\nu}^{\ \ \ \lambda} = R_{\mu\nu} - H_{\mu\nu}$ and $\tilde{R} \equiv \tilde{R}^\mu_{\ \mu} = R$. As one might expect, the quantities (44)–(47) arise naturally in the expression for the commutator of two derivative operators acting on a vector V^ρ (say) of Weyl weight w, which is given by

$$[\nabla^*_\mu, \nabla^*_\nu]V^\rho = \tilde{R}^\rho_{\ \sigma\mu\nu}V^\sigma + wH_{\mu\nu}V^\rho - T^{*\sigma}_{\ \ \mu\nu}\nabla^*_\sigma V^\rho. \tag{48}$$

A key point to note in the above geometric interpretation is that it leads to the identification of a covariant derivative $\nabla^*_\mu = \partial^*_\mu + \Gamma^\sigma_{\ \rho\mu}\mathsf{X}^\rho_{\ \sigma}$ (often termed the scale covariant or Weyl covariant derivative, although it was first introduced by Dirac [23], who called it the co-covariant derivative) that clearly differs from the conventional covariant derivative $\nabla_\mu = \partial_\mu + \Gamma^\sigma_{\ \rho\mu}\mathsf{X}^\rho_{\ \sigma}$ in Weyl-Cartan spacetimes, since $\nabla^*_\mu = \nabla_\mu + wB_\mu$. Indeed, this form immediately leads to the important property $\nabla^*_\sigma g_{\mu\nu} = 0$. A further important feature of the Weyl covariant derivative is that, when acting on an object of weight w, the resulting object also transforms covariantly with weight w; this is not the case for the conventional covariant derivative ∇_μ, which does not, in general, produce an object that transforms covariantly, unless $w = 0$.

It is also noteworthy that the Weyl covariant derivative cannot, in general, be written in the form $\nabla^*_\mu = \partial_\mu + {}^*\Gamma^\sigma_{\ \rho\mu}\mathsf{X}^\rho_{\ \sigma}$ for some alternative connection ${}^*\Gamma^\sigma_{\ \rho\mu}$, even if the latter is permitted to depend on w. Indeed, this is a manifestation of a larger issue. Whereas the geometric interpretation of PGT captures all of its gravitational interactions (at least for tensor fields) in terms of the metric and connection of an underlying Riemann-Cartan U_4 spacetime in which the matter fields reside, the geometric interpretation of WGT does not describe all of its gravitational interactions in an analogous manner. In particular, when acting on fields with nonzero Weyl weight w, the gravitational interactions mediated by the dilational gauge field B_μ cannot be fully "geometrized" in terms of the metric and connection of a Weyl-Cartan Y_4 spacetime, as is clear from (38) and (48), and one must augment the Y_4 spacetime interpretation by using the Weyl covariant derivative in such cases.

V. SECOND CLOCK EFFECT IN WGT

We now reconsider the second clock effect in the context of the above geometric interpretation of WGT, in particular making use of the Weyl covariant derivative that it identifies. Following our discussion in Sec. II, we will consider the SCE both in terms of the norms of parallel transported vectors and in terms of an appropriately defined proper time.

As discussed in the Introduction, in the geometric interpretation of WGT, for a test particle moving along some timelike worldline $\mathcal{C}$ given by $x^\mu = x^\mu(\lambda)$, the components of the tangent vector to this worldline as measured in the local Lorentz frame of an observer will be $u^a(\lambda) = e^a_{\ \mu}u^\mu(\lambda)$, which should be invariant under Weyl gauge transformations since they are physical observables in WGT. Since the vierbein $e^a_{\ \mu}$ has weight $w = 1$, the weight of the components $u^\mu(\lambda)$ in the coordinate basis is thus $w = -1$.

One may perform calculations in either the tetrad or coordinate basis, denoted by Latin and Greek indices, respectively. By virtue of the geometric interpretation of WGT described in Sec. IV, these two approaches yield consistent results, but we will work in terms of the coordinate basis to facilitate a more straightforward comparison with the calculations performed in Sec. II.

A. Parallel transported vectors

Defining parallel transport in terms of the Weyl covariant derivative, as in (38), and using the condition $\nabla^*_\sigma g_{\mu\nu} = 0$, one immediately finds that, in contrast to (6), the evolution of the inner product of any two vectors v^μ and w^μ parallel transported along some curve $\mathcal{C}$ is now given by

$$\frac{d}{d\lambda}[g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda)] = \frac{D^*}{D\lambda}[g_{\mu\nu}v^\mu(\lambda)w^\nu(\lambda)],$$
$$= (u^\sigma(\lambda)\nabla^*_\sigma g_{\mu\nu})v^\mu(\lambda)w^\nu(\lambda) = 0. \tag{49}$$

Hence, by setting $v^\mu = w^\mu$ and considering parallel transport around a closed curve $\mathcal{C}$, the length ℓ of a vector is unchanged on completing a loop, and so the original basis for suggesting the existence of an SCE is removed.

B. Proper time

As discussed in Sec. II, however, the intuitive argument above is not rigorous, and so we now reconsider how to define a physically sensible notion of proper time along (timelike) worldlines.

We begin by following an analogous procedure to that adopted in Sec. II D. In particular, we first seek to identify a parameter ξ (it will become clear shortly that this cannot be interpreted as proper time and so we do not denote this variable by τ here) that satisfies an analogous condition to (9), namely,

$$g_{\mu\nu} u^\mu(\xi) a^{*\nu}(\xi) = 0, \qquad (50)$$

where we define $a^{*\mu} = D^* u^\mu / D\xi$. It is straightforward to show that the condition (50) is consistent across the whole equivalence class defined by the gauge transformations (2), since

$$\bar{g}_{\mu\nu} \bar{u}^\mu(\xi) \bar{a}^{*\nu}(\xi) = e^{-\rho} g_{\mu\nu} u^\mu(\xi) a^{*\nu}(\xi). \qquad (51)$$

Following through the calculations in Sec. II D, but working instead in terms of the Weyl covariant derivative, one finds that (12) is replaced by

$$
\begin{aligned}
\frac{d}{d\lambda}[g_{\mu\nu} u^\mu(\lambda) u^\nu(\lambda)] &= \frac{D^*}{D\lambda}[g_{\mu\nu} u^\mu(\lambda) u^\nu(\lambda)], \\
&= (u^\sigma \nabla^*_\sigma g_{\mu\nu}) u^\mu u^\nu + 2 g_{\mu\nu} u^\mu a^{*\nu}, \\
&= 2 g_{\mu\nu} u^\mu a^{*\nu}, \qquad (52)
\end{aligned}
$$

where we used the condition (42), and in the last two lines (and the remainder of this section), we drop the explicit dependence of quantities on the arbitrary parameter λ for brevity. Thus, as might be expected, the condition (50) corresponds simply to finding a parametrization ξ for which the length of the tangent vector remains constant under parallel transport along its worldline, as in Riemann-Cartan U_4 spacetime. Consequently, (14) is replaced by

$$\Delta\xi(\lambda) = \left.\frac{d\xi/d\lambda}{\sqrt{g_{\mu\nu} u^\mu u^\nu}}\right|_{\lambda_0} \int_{\lambda_0}^\lambda \sqrt{g_{\mu\nu} u^\mu u^\nu}\, d\lambda', \qquad (53)$$

which now gives the parameter interval $\Delta\xi$ between two events corresponding to the parameter values λ_0 and λ along the worldline. As was the case in Sec. II D, if ξ is a solution then so too is $a\xi + b$, where a and b are constants. Thus, without loss of generality, one may choose ξ such that the length of the tangent vector $g_{\mu\nu} u^\mu(\xi) u^\nu(\xi)$ is unity along the entire worldline, so that $u^\mu(\xi)$ may be interpreted as the particle four-velocity, and hence identified with the timelike unit basis $\hat{\mathbf{e}}_0$ of a local Lorentz frame for an observer moving along the worldline.

As we discussed in Sec. II E, however, the differential $d/d\xi$ has Weyl weight $w = -1$ (indeed this holds for any arbitrary parametrization λ of the worldline). Thus, ξ is not invariant under Weyl gauge transformations, and so cannot be interpreted as the proper time of a particle (or observer) moving along the worldline, which is a physical observable and hence should be independent of any gauge transformations.

To address this issue, one must recognize that Einstein's original objection to Weyl's theory requires a massive Dirac field to represent atoms and observers, and also take seriously the physical mechanism by which such an observer might measure their proper time as they move along their worldline. One such method would be to carry with them some form of atomic clock, which provides a good physical approximation to an ideal clock, and is used to define the standard for the unit of time. The functioning of such a clock is based on the spacing of energy levels in atoms (this is, of course, also directly relevant to the consideration of spectral lines, the sharp nature of which is considered as the key observational evidence against the existence of the SCE). Although not particularly practical, one could in principle make use of the energy levels in the hydrogen atom, the spacings of which are characterized in the clock's local Lorentz frame by the Rydberg (ground state to free) energy $E_R = \frac{1}{2} m\alpha^2$ (in natural units), where m is the rest mass of the electron and α is the (dimensionless) fine structure constant.

As pointed out in Sec. III, however, to incorporate a Dirac field describing ordinary matter in WGT, one must also introduce a scalar compensator field ϕ and make the replacement $m\bar{\psi}\psi \to \mu\phi\bar{\psi}\psi$ in the Dirac action, where μ is a dimensionless parameter but $\mu\phi$ has the dimensions of mass in natural units. Thus, the Rydberg energy then becomes $E_R = \frac{1}{2}\mu\phi\alpha^2$ and so in general varies with spacetime position according to the value of ϕ. A photon emitted in a ground state to free electronic transition has energy E_R, defined as the projection of the photon 4-momentum onto the timelike basis vector $\hat{\mathbf{e}}_0$ of the atom's local Lorentz frame, such that $E_R = p_\mu dx^\mu/d\xi$. Therefore, in a small parameter interval $d\xi$, the number of cycles traversed in the photon wave train is $dN \propto E_R d\xi \propto \phi d\xi$, which is invariant under a Weyl gauge transformation, as it must be, since ϕ and $d\xi$ have weights $w = -1$ and $w = 1$, respectively. A proper time interval $d\tau$ in the atom's rest frame is, however, *defined* as the duration of a given number of cycles, and so $d\tau \propto \phi d\xi$, where one can take the constant of proportionality to equal unity, without loss of generality. Hence, the proper time interval measured by an atomic clock between two events corresponding to the parameter values ξ_0 and ξ along the worldline is given simply by

$$\Delta\tau(\xi) = \int_{\xi_0}^\xi \phi\, d\xi', \qquad (54)$$

which is invariant under Weyl gauge transformations, as required for a physically observable quantity. Indeed, the Rydberg energy E_R is then equal (in natural units) to the angular frequency of the photon as measured in terms of the proper time τ, and is itself also invariant under Weyl gauge transformations, as it should be.

Finally, applying this proper time definition to the two-clock thought experiment discussed in the Introduction, one sees immediately from (53) and (54) that the ratio of the elapsed proper time measured by each clock between their reunion and some subsequent event along their joint worldline is unity. Thus, the clock rates are the same after their reunion, and so WGTs do not predict an SCE.

VI. CONCLUSIONS

We have critically reconsidered the prevailing view in the literature that WGTs predict an SCE, which has previously been argued to rule out such theories as unphysical. Although WGTs are interpreted geometrically in terms of a Weyl-Cartan Y_4 spacetime, the gravitational interactions mediated by the dilational gauge field (or Weyl potential) B_μ cannot be fully geometrized in terms of the metric and connection of such a spacetime when acting on quantities with nonzero scaling dimensions (or Weyl weight) w.

Rather, the geometric interpretation of WGTs identifies a covariant derivative ∇_μ^* (often termed the Weyl covariant derivative) that differs from the conventional covariant derivative ∇_μ in Weyl-Cartan spacetime when acting on quantities of nonzero Weyl weight. The Weyl covariant derivative has the important properties that $\nabla_\sigma^* g_{\mu\nu} = 0$ and, when acting on quantities that transform covariantly with arbitrary weight w under Weyl gauge transformations, it produces objects that

also transform in this way; neither of these properties is shared by ∇_μ. If one defines parallel transport in terms of the Weyl covariant derivative, then the condition $\nabla_\sigma^* g_{\mu\nu} = 0$ immediately implies that the inner product of any two vectors is preserved as they are parallel transported along some curve, which removes the basis for Einstein's original concerns regarding the existence of an SCE.

Moreover, we show that more recent derivations of the SCE, which are based on defining proper time in Weyl-Cartan spacetime, rely on the assumption that the components u^μ of the tangent vector to an observer's worldline are invariant $(w = 0)$ under Weyl gauge transformations, whereas we show that, in fact, they have weight $w = -1$.

Furthermore, we point out that Einstein's original objection to Weyl's theory requires the presence of a massive Dirac matter field to represent atoms, observers, and clocks, so it is meaningless in this context to consider an empty Weyl-Cartan spacetime. The requirement of a Dirac field to represent such ordinary matter in turn necessitates the inclusion of a scalar compensator field in order that the total action obeys local Weyl invariance and the Dirac field may acquire a mass dynamically. We show that this scalar field is key to a physically meaningful definition of proper time.

When one makes use of the Weyl covariant derivative to define variation along a worldline, assigns the components u^μ of the tangent vector to an observer's worldline, the correct Weyl weight of $w = -1$, and includes the effect of the required compensator field in defining a physically sensible proper time variable that is invariant under Weyl gauge transformations, one immediately concludes that WGTs do not predict an SCE.

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