Exact Einstein-Gauss-Bonnet spacetime in six dimensions

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A new exact solution to the field equations in the Einstein-Gauss-Bonnet modified theory of gravity, for a six-dimensional spherically symmetric static distribution of a perfect fluid source is presented. The pressure isotropy equation is integrated after a form for the temporal potential proportional to the radius is postulated to close the system of equations. For a specific choice of the coupling parameter it is demonstrated that the matching of the interior and exterior spacetimes is explicitly achievable. The general model has been tested to be physically acceptable in this framework using criteria extrapolated from the standard four dimensional theory and after locating a suitable parameter space through fine-tuning. A vanishing pressure hypersurface signifying a boundary exists and the speed of sound is subluminal throughout the interior of the matter distribution. Furthermore, all energy conditions are satisfied. Finally, the Chandrasekhar adiabatic stability bound is satisfied.

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I. INTRODUCTION

The case for considering higher dimensional gravitational fields was made during the pioneering work of Kaluza [1] and Klein [2] who introduced the idea of a 15 component metric tensor: four of which were connected to the Maxwell field, one to a scalar field called a dilaton, and the remaining ten to the usual four-dimensional spacetime. The fact that extra dimensions could not be accessed physically did not hamper investigations. The standard explanation is that the extra spatial dimensions are topologically curled as microscopic circles and are of negligible magnitude compared to the Planck scale; however, they are sufficient to influence the dynamical evolution of the system. It must be noted that the Large Hadron Collider experiment failed to confirm the existence of large scale extra dimensions but did not rule out small scale extra angular dimensions. Subsequent to Kaluza-Klein theory, several other theories delved into higher dimensions including brane-world cosmologies [3,4] and more recently Lovelock gravity [5,6]. The former has been motivated by developments in 10-dimensional string theory and its generalization *M* theory, which requires up to 11 dimensions. A notable five-dimensional advance at low energies is inherent in the Dvali-Gabadadze-Porrati models [7]. Another important development in the study of higher dimensional spacetimes was the construction of a background spacetime by Chamseddine [8,9] using Chern-Simons gauge theory, which generated nontrivial perturbations. The action constructed consisted of the Gauss-Bonnet, the Einstein and the cosmological constant term.

Our interest in this article lies not only in higher dimensional gravity but also higher curvature effects. In particular the Lovelock [5,6] theory has been shown to be the most general tensorial theory generating up to second order equations of motion and consequently being totally ghost free. Moreover, the standard requirements of the Bianchi identities or diffeomorphism invariance are satisfied. The Lovelock lagrangian is constructed from scalar invariants comprising quadratic forms of the Riemann and Ricci tensors as well as the square of the Ricci scalar. Amazingly all derivatives of order higher than 2 cancel off. Another important feature of Lovelock gravity is that it regains all known results of general relativity as higher curvature effects are only dynamic from dimension 5 onwards and reduce to general relativity when the dimension is 4 or lower. To zeroth order the cosmological constant term is regained. The second order Lovelock polynomial gives the Einstein-Gauss-Bonnet (EGB) invariant, which uncannily appears in the effective action principle of low energy heterotic string theory [10]. Herein lies a further motivation to specialize to the EGB Lovelock case in studying the impact of higher curvature terms on astrophysical phenomena. It generates a string theory inspired gravitational theory which should stand the test in cosmology, galaxy formation as well as in astrophysics. The last mentioned is what is of interest.

Following the work of Lovelock in the 1970s Boulware and Deser [11] established the exterior five-dimensional spacetime for static fluid spheres in 1985, and a year later Wiltshire [12] not only extended the vacuum result to

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include the effects of the electromagnetic field but obtained a general result for all spacetime dimensions. Both these solutions are unique up to branch cuts. The nonlinearity of the field equations prohibited the immediate location of exact solutions for interior spacetimes that could be used to model stellar structure. The first exact solutions for five-dimensional perfect fluid spheres with isotropic stresses were reported some 30 years after Wiltshire [12] by [13–15]. Just prior to this Dadhich and coworkers [16] proved that the Schwarzschild interior is universal in Lovelock gravity and is independent of the spacetime dimension. In other words the usual Schwarzschild metric of Einstein gravity generates the most general incompressible Lovelock star. A five-dimensional model proposed by Kang *et al.* [17] was incomplete in that a further integration was required to unlock the true dynamics of the star model.

For a variety of reasons six-dimensional spacetimes in Einstein-Gauss-Bonnet theory are of greater interest than five. Firstly it is well known that pure Lovelock gravity, where the lagrangian consists solely of the Nth order Lovelock terms and not the sum of terms up to N, that in the odd dimension bounded objects do not occur due to a generic failure of the pressure to vanish on a suitable hypersurface. Moreover for some spacetimes such as Gödel universes [18] and vacuum spacetimes [19], the odd dimensional pure Lovelock terms are nondynamical or kinematic. This is not necessarily true for general Lovelock lagrangians with terms of all orders, but does expose a potential defect in odd dimensional spaces. Of course, odd dimensional Lovelock theory (not pure Lovelock) does indeed admit compact objects [20,21]. Nevertheless the odd dimensional case eliminates the contribution of a number of terms in the field equations. For even dimensional pure Lovelock gravity no such restrictions apply. The total effect of all relevant higher curvature terms become evident in six-dimensional EGB theory. To date, besides the universal Schwarzschild interior solution, only one other exact solution for six-dimensional EGB has been found [22]. This underscores the difficulty of finding exact solutions given the presence of extra nonlinear terms in the stellar structure equations. Note that a large number of articles have focused on the seriously easier version of five dimensional anisotropic EGB stars—usually the isotropic system consists of three equations in four unknowns, but if anisotropy is allowed then there are now five unknowns making it trivial to write down exact "solutions" at will. The problem can be weakened even further by introducing charge in which case there are now six unknowns and up to three *a priori* prescriptions are allowed. The challenge in these weak mathematical problems is to test *post facto* if the elementary physical conditions are met.

Another motivation to analyze extensions to the Einstein theory is that the belief is that the observed accelerated expansion of the universe is not a natural result of general relativity. The Λ-CDM model in use relies on the existence of dark matter to drive the expansion. Till now there has been no experimental support for the existence of the pervasive dark matter. An alternative idea is to modify the standard theory geometrically. Some success has been achieved with the $f(R)$ theory of Starobinksy [23]; however, the caveat is that higher order derivatives arise and these generate ghosts of the theory. The Lovelock paradigm cannot claim any success in this problem yet, however, it does have such potential. We examine the EGB amendments to the Einstein field equations and we endeavor to check its prospects in providing explanations to phenomena such as stars and cold fluid planets. Not much work on exact solutions has been reported in this framework because of the complexity of the field equations.

Standard Einstein theory is known to comprise a complex system of partial differential equations and locating exact solutions which are physically relevant has proved to be difficult. Following Schwarzschild's two solutions in 1916, the next major advance was in 1939 with the Tolman [24] eight classes of solutions due to a cunning rearrangement of the equation of pressure isotropy. Indeed this approach may be attempted in EGB as well, although the equations are more complicated. Kuchowicz [25] provided further methods for solving the field equations using curvature coordinates and also by considering isotropic coordinates. This approach also offers a route to solving the EGB system and is currently being pursued. To date, exact solutions are still being discovered using different approaches available and presently some 127 exact solutions for static spherically symmetric isotropic fluids are known in Einstein gravity. The work has profited from substantial advances in computing software. Delgaty and Lake [26] show that only about 13% of these exact solutions satisfy all the elementary physical requirements. A comprehensive, albeit not exhaustive, list of exact solutions may be found in [27].

The traditional approach to solving Einstein's equations for a static spherically symmetric distribution has been to make an assumption for one of the gravitational potentials, since the system of field equations has three equations in four unknowns. This is an *ad hoc* technique that was invoked in four-dimensional spacetime by Finch and Skea [28] in correcting Duorah and Ray [29]. This work was recently extended by Chilambwe and Hansraj [30] for the higher dimensions. One would prefer adopting a physically reasonable equation of state; however, this approach leads to a dead end with an intractable differential equation. Although it should be checked, but it is highly unlikely this tactic will have any success with the more complicated EGB equations. In our approach we speculate on a number of possible potentials in the hope of finding an exact solution. The solution reported herein is generally well behaved physically; however, it does suffer the undesirable presence of a singularity at the stellar center. Of course, this singularity may be eliminated by supposing that our six-dimensional solution is enveloping another nonsingular solution such as the Schwarzschild interior. Matching across the inner core boundary and the interface with the vacuum will then have to be achieved. However, we concentrate on the behavior of the model elsewhere in the interior of the hyperstar.

Our work is arranged as follows: in Sec. II, we very briefly review rudiments of the Einstein-Gauss-Bonnet theory. In the following Sec. III, we present the six-dimensional line element and a transformation of the Einstein-Gauss-Bonnet field equations to an equivalent form by a coordinate redefinition. The exterior gravitational field applicable to our study is derived in Sec. IV. Section V details a new exact solution obtained by a prescription of the spatial gravitational potential in order to solve the six-dimensional EGB equations. Matching of the interior and exterior spacetimes is dealt with in Sec. VI. A qualitative physical analysis of the model is conducted in Sec. VII for a suitable parameter space. Finally in Sec. VIII, we conclude our investigation by summarizing and discussing our major results.

II. EINSTEIN-GAUSS-BONNET GRAVITY

We require an adapted action, different from the Einstein case, to generate the field equations in EGB gravity. The Gauss-Bonnet action in six dimensions can be written as

$$S = \int \sqrt{-g}\left[\frac{1}{2}(R - 2\Lambda + \alpha L_{GB})\right] d^6x + S_{\text{matter}}, \quad (1)$$

where the parameter α denotes the Gauss-Bonnet coupling constant. The value of the dimensionful coupling constant, which goes as $1/(l)^2$ with l being length in the Planck scale, has been the subject of debate with the expectation that it should be very small in comparison with the Planck length. On the other hand, in their analysis of constraints on Gauss-Bonnet gravity in dark energy cosmologies Amendola *et al.* [31] motivate a value for α of the order up to 10^{23}. Additionally, it is not even settled whether α should necessarily be positive. In the paper just mentioned, the authors acknowledge that the prospects of constraining α experimentally are encouraging although to date it has not been achieved yet. Note that the Lagrangian is quadratic in the geometric quantities: Ricci tensor, Ricci scalar, and the Riemann tensor. Observe that the equations of motion for this action are second order and quasilinear which are distinguishing features in EGB gravity. This is an advantage when compared with other modified theories of gravity. The Gauss-Bonnet term L_{GB} is dynamic for $n > 4$ but does not contribute to the gravitational field when $n \leq 4$.

The field equations in EGB gravity can be written as

$$G_{ab} + \alpha H_{ab} = T_{ab}, \quad (2)$$

where we have adopted the metric signature $(- + + + + +)$ in what is to follow. The tensor G_{ab} is the Einstein tensor in six dimensions. The Lanczos tensor H_{ab} can be expressed in the form

$$H_{ab} = 2(RR_{ab} - 2R_{ac}R^c_b - 2R^{cd}R_{acbd} + R_a^{cde}R_{bcde}) - \frac{1}{2}g_{ab}L_{\text{GB}}. \quad (3)$$

The Lovelock term of order 2 is defined by

$$L_{\text{GB}} = R^2 + R_{abcd}R^{abcd} - 4R_{cd}R^{cd}, \quad (4)$$

also known as the Gauss-Bonnet term that is present in the action of low energy string theory. The presence of the L_{GB} term substantially increases the complexity of the field equations in comparison to the already complicated Einstein equations.

III. FIELD EQUATIONS

The six-dimensional line element for static spherically symmetric spacetimes is taken as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2, \quad (5)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2 + \sin^2\theta \times \sin^2\phi \sin^2\psi d\eta^2$ and where $\nu(r)$ and $\lambda(r)$ are arbitrary functions representing the gravitational field with coordinates $(x^a) = (t, r, \theta, \phi, \psi, \eta)$. We use the timelike comoving fluid velocity $u^a = e^{-\nu}\delta^a_0$ with the property $u^a u_a = -1$. The matter field is defined by the energy momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \quad (6)$$

which is characteristic of perfect fluids and where ρ and p are the energy density and isotropic pressure, respectively.

Then the EGB field equations (2) when expanded amount to the system

$$\rho = \frac{1}{e^{4\lambda}r^4}[(4r^3e^{2\lambda} - 48\alpha r(1 - re^{2\lambda}))\lambda' - 6r^2e^{2\lambda}(1 - e^{2\lambda}) + 12\alpha(e^{2\lambda} - 1)^2], \quad (7)$$

$$p = \frac{1}{e^{4\lambda}r^4}[(1 - e^{2\lambda})(6r^2e^{2\lambda} - 48\alpha r\nu' + 12\alpha e^{2\lambda} - 12\alpha) + 4r^3e^{2\lambda}\nu'], \quad (8)$$

$$p = \frac{1}{e^{4\lambda}r^2}((12\alpha(e^{2\lambda} - 1) + r^2e^{2\lambda})(\nu'' + (\nu')^2 - \nu'\lambda') + 24\alpha\nu'\lambda') + \frac{1}{e^{4\lambda}r^3}((3r^2e^{2\lambda} + 12\alpha(e^{2\lambda} - 1)) \times (\nu' - \lambda') + 3re^{2\lambda}(1 - e^{2\lambda})), \quad (9)$$

in the canonical spherical coordinates (x^a). Eliminating α yields the stellar structure equations of Einstein gravity in six dimensions. Note that the system (7)–(9) consists of three field equations in four unknowns which is similar to the standard Einstein case for spherically symmetric perfect fluids. However, the nonlinearity in the system (7)–(9) has now greatly increased because of the presence of the EGB coupling parameter α. The presence of terms containing α makes the system more complex and difficult to solve in general.

We attempt to rewrite the system (7)–(9) in a simpler form by utilizing new coordinates due to Durgapal and Bannerji [32]. We make the coordinate change $x = Cr^2$, $e^{2\nu} = y^2(x)$ and $e^{-2\lambda} = Z(x)$. Equations (7)–(9) now assume the form

$$\frac{12\beta x(Z-1)\dot{Z} - 4x^2\dot{Z} - 6x(Z-1) + 3\beta(Z-1)^2}{x^2} = \frac{\rho}{C}, \tag{10}$$

$$\frac{(24\beta x(1-Z) + 8x^2)Z\dot{y} + (Z-1)(6x + 3\beta(1-Z))y}{x^2 y} = \frac{p}{C}, \tag{11}$$

$$\begin{aligned} &4x^2Z(x + 3\beta[1-Z])\ddot{y} \\ &\quad + 2x(x^2\dot{Z} + 3\beta[(1-3Z)\dot{Z}x - 2Z(1-Z)])\dot{y} \\ &\quad + 3(\beta(1-Z) + x)(\dot{Z}x - Z + 1)y = 0, \end{aligned} \tag{12}$$

where the condition (12) is the equation of pressure isotropy in six-dimensional EGB theory and we have redefined $\beta = 4\alpha C$. It has been written as a linear second order differential equation in y (if Z is a known quantity). This is the distinctive advantage of the coordinate transformation. An equivalent form of the condition of pressure of isotropy is

$$\begin{aligned} &[x^2(2x\dot{y} + 3y) + 3\beta x(2x\dot{y} + y - (6x\dot{y} + y)Z)]\dot{Z} \\ &\quad - 3\beta[4x^2\ddot{y} - 4x\dot{y} - y]Z^2 + [x(4x^2\ddot{y} - 3y) \\ &\quad + 6\beta(2x^2\ddot{y} - 2x\dot{y} - y)]Z + 3(x+\beta)y = 0, \end{aligned} \tag{13}$$

which may lend itself to finding exact solutions. Equation (13) is a nonlinear first order differential in Z (if y is a known quantity). Note that (13) is an Abel differential equation of the second kind and only few exact solutions are known for isolated cases. We want to find exact solutions to the generalized pressure isotropy conditions (12) and (13) in the presence of α. When $\alpha = 0$ we find that (12) reduces to the simpler form

$$4x^2Z\ddot{y} + 2x^2\dot{Z}\,\dot{y} + 3(x\dot{Z} - Z + 1)y = 0, \tag{14}$$

which is the pressure isotropy condition in six-dimensional Einstein gravity.

IV. EXTERIOR SOLUTION

The exterior vacuum solution is already known and for six dimensions can be deduced from the solution of Wiltshire [12]. However, we compute it in our coordinates in order to complete the matching later. It is well known that vacuum metrics have the general structure

$$ds^2 = -Fdt^2 + \frac{1}{F}dr^2 + r^2 d\Omega^2, \tag{15}$$

for some function F and which in our context translates to $y = \sqrt{Z}$ as a relationship between the metric potentials. Substituting into (13) gives the differential equation

$$\begin{aligned} &3\beta - 3Z(2\beta + 2\beta x^2\ddot{Z} - \beta x\dot{Z} + x) - 6\beta x^2\dot{Z}^2 + 6\beta x^2\ddot{Z} \\ &\quad + 2x^3\ddot{Z} + 3x(x-\beta)\dot{Z} + 3\beta Z^2 + 3x = 0 \end{aligned} \tag{16}$$

governing the spatial potential. Equation (16) has the solution

$$Z = 1 + \frac{x}{3\beta}\left(1 \pm \sqrt{c_2 + \frac{c_1}{x^{\frac{5}{2}}}}\right), \tag{17}$$

which is unique up to branch cuts. Wiltshire [12] expressed the exterior metric in the form

$$F = 1 + \frac{R^2}{8\kappa^2\tilde{\alpha}}\left(1 \pm \sqrt{1 + \frac{32\kappa^2\tilde{\alpha}GM}{R^5}}\right), \tag{18}$$

where $\kappa^2 = 4\pi G$ (G being the gravitational constant) and $\tilde{\alpha} = \alpha^*(d-3)(d-4)$ where d is the spacetime dimension and α^* is the Gauss-Bonnet coupling used by Wiltshire. The potential in (17) reduces to the Wiltshire solution upon setting the values of the integration constants as $c_2 = 1$ and $c_1 = \frac{4GM\sqrt{C}}{3\alpha}$ and harmonizing the coupling constants $\alpha = \frac{2\kappa^2\tilde{\alpha}}{3} = 4\kappa^2\alpha^*$. Note that the presence of two integration constants is only an artifact of the nonlinear system of equations—we have elected to use the second order pressure isotropy equation; hence, two independent constants arise. We could also have used the vanishing of the energy density (10) that is $\rho = 0$ to generate the solution as

$$Z = 1 + \frac{R^2}{12\alpha}\left(1 \pm \sqrt{1 + \frac{144\alpha^2 c_3}{\sqrt{C}R^5}}\right), \tag{19}$$

reverting to $\beta = 4\alpha C$, setting $r = R$ at the boundary interface and where c_3 is the sole integration constant.

In this case we set $c_3 = \frac{2\kappa^2\tilde{\alpha}GM\sqrt{C}}{9\alpha^2}$ to obtain the form of Wiltshire (18).

At this point it may be worthwhile clarifying what we mean by the radius of the hypersphere. In standard Einstein general relativity it is known that the matching of the second fundamental form across a common boundary surface is tantamount to the vanishing of the pressure at a finite radius. These are the well-known Israel-Darmois junction conditions. However, whether this holds true in EGB theory is unknown to date. Davis [33] generated the conditions to accomplish such a matching; however, the tangible resolution of these conditions to the spherically symmetric case has still not been realized. It is still unknown whether the vanishing of the pressure along a boundary hypersurface does indeed determine the radius of the sphere. Consequently, the matching we have achieved later is independent of this and instead relies on the matching of the metric potentials across a common hypersurface for a finite radial value $r = R$. However, we do make the assumption of a pressure-free hypersurface to locate a specific radial value that allows us to construct a specific stellar model to test the other remaining physical requirements.

The active gravitational mass m of a d-dimensional star is calculated with the help of the formula $m = \int \rho r^{d-2} dr$ where ρ is the energy density of the star. As will be seen later, it is generally not easy to evaluate this integral in light of the complexity of the density expressions in six-dimensional EGB gravity. However, the mass as measured by an observer situated at spatial infinity may be established with the aid of the Boulware-Deser or neutral Wiltshire exterior solution. In fact to complete a stellar model it will be necessary to express all constants of integration in terms of the mass M and radius R of the distribution of perfect fluid. This will indeed be achieved as will be demonstrated later for a specific case of the coupling parameter $\alpha = \beta/4C$.

V. NEW EXACT INTERIOR SOLUTION

In this section, we postulate a form for the metric potential form y to solve the Abel differential equation of the second kind (13). A number of exact solutions have been found with this approach in Einstein gravity. For example, following the discovery of the Schwarzschild exterior and interior metrics and the constant potential Einstein universe, the next significant advance in locating exact solutions to the nonlinear field equations came from Tolman [24], who displayed five new classes of solutions arising out of a cunning rearrangement of the pressure isotropy condition. Since then various *ad hoc* prescriptions of one of the potentials have resulted in some 120 exact solutions of the Einstein field equations for perfect fluid spheres have emerged [26]. While it would be desirable to seek solutions based on physical grounds, such as by imposing an equation of state, such an approach has failed even in the simpler Einstein gravity theory, so it is not expected to hold much promise in the current EGB context with several new higher curvature terms contributing to the nonlinearity in the equations. In light of this we explore choices of potentials that lead to the location of an exact solution.

Consider the metric potential stipulation $y = \sqrt{x}$ in (13). We obtain the real valued solution

$$Z(x) = \frac{1}{36}\left[\frac{h}{\sqrt[3]{2}Q} + \frac{81\sqrt[3]{2}Q}{h} + \frac{3x}{B} + 27\right], \tag{20}$$

and two complex valued solutions

$$Z(x) = \pm\left[\frac{9B + x}{12B} - \frac{(1 - i\sqrt{3})h}{72\sqrt[3]{2}Q} - \frac{9(1 + i\sqrt{3})Q}{4\sqrt[3]{4}h}\right], \tag{21}$$

where we have set

$$h = \sqrt[3]{Q^2x^3 - 1458Q^3 + j} \quad \text{and}$$
$$j = \sqrt{(Q^2x^3)^2 - 2916Q^5x^3}.$$

Note that A is an integration constant and we have put both $B = \frac{1}{4}\beta$ and $Q = AB^3$ for ease of reference. Inserting the solution (20) into (10)–(11), the energy density and the pressure are given by

$$\frac{\rho}{C} = [B^2 j(486h + 5\sqrt[3]{2}x^3)h - 90x^2h^2 + 52488\sqrt[3]{4}(A^2B^7)^2x^3 - 54\sqrt[3]{2}A^3B^{11}x^3(54h + \sqrt[3]{2}x^3) - 54\sqrt[3]{2}AB^5 j(81h + \sqrt[3]{2}x^3) + \sqrt[3]{2}(AB^4)^2(5x^6h + 39366\sqrt[3]{2}j)]/216Bx^2jh^2, \tag{22}$$

$$\frac{p}{C} = [(13122\sqrt[3]{4}Q^4 - 54\sqrt[3]{4}Qj - 5\sqrt[3]{2}jh + Q^2(162h - 5\sqrt[3]{2}x^3)h - 54\sqrt[3]{2}Q^3(27h + \sqrt[3]{2}x^3) + 90(AB^2)^2x^2h^2)]/216A^2B^5x^2h^2, \tag{23}$$

respectively, while the expressions

$$\frac{\rho-p}{C}=[(5-8748Qx)(AB^6)^2\sqrt[3]{2}hx^2+2(6561\sqrt[3]{4}Q-729\sqrt[3]{2}h)AB^9jh-52488\sqrt[3]{4}(A^2B^9)^2x^2 + (162B^2h-90x^2h+5\sqrt[3]{2}B^2x^3)B^4jh]/108B^5jh^2x^2, \tag{24}$$

$$\frac{\rho+p}{C}=[1944\sqrt[3]{4}A^4B^{13}x^3+486\sqrt[3]{4}A^2B^7j+6Bjh^2-\sqrt[3]{2}A^3B^{10}x^3(\sqrt[3]{2}x^3-108h)-\sqrt[3]{2}AB^4j(54h+\sqrt[3]{2}x^3)]/2x^2jh^2, \tag{25}$$

$$\frac{3\rho+5p}{C}=[157464\sqrt[3]{4}(AB^3)^6x^3-162\sqrt[3]{2}(AB^3)^5x^3(54h+\sqrt[3]{2}x^3)-270\sqrt[3]{4}AB^3j^2 + (AB^2)^2h(B^2j(2268h+5\sqrt[3]{2}(3x^3-5))+90h(5j-3)x^2)-25\sqrt[3]{2}hj^2 + 3\sqrt[3]{2}(AB^3)^4[-108\sqrt[3]{2}(AB^3)^3j(189h+4\sqrt[3]{2}x^3)]/216A^2B^5h^2jx^2 \tag{26}$$

will be helpful in analyzing the energy conditions later. Note that setting $\alpha=0$ in (10) with the potential choice $Z=1+kx$ yields constant density $\rho=-10kC$, where k is constant. This matches to the four-dimensional Schwarzschild interior metric. Dadhich *et al.* [16] proved that this result holds independent of the spacetime dimension. Our calculation corroborates that of Dadhich *et al.* in six-dimensional EGB gravity. Observe that this model does not allow us to express the isotropic pressure as a function of the energy density, that is, the equation of state may not be readily established for this model. Utilizing (10) and (11), the expression for sound speed is computed to be

$$\frac{dp}{d\rho}=[(2916Q-x^3)h^6(6jh^2+1944\sqrt[3]{4}Q^4x^3+486\sqrt[3]{4}Q^2j-\sqrt[3]{2}Qj(54h+\sqrt[3]{2}x^3) - \sqrt[3]{2}Q^3x^3(\sqrt[3]{2}x^3-108h))]/4A^4B^{12}[12050326889856\sqrt[3]{4}Q^7x^3-9x^9jh^2-4860Q^4\sqrt[3]{4}x^{12} - 1033121304\sqrt[3]{2}Q^6x^3(1134h+19\sqrt[3]{2}x^3)+Qx^6j(52488h^2-135\sqrt[3]{2}x^3h+\sqrt[3]{4}x^6) + 177147Q^5(7128\sqrt[3]{2}x^6h+629856x^3h^2+65\sqrt[3]{4}x^9+12754584\sqrt[3]{4}j) - 486Q^4(516560652\sqrt[3]{2}jh-81\sqrt[3]{2}x^9h+314928x^6h^2+21789081\sqrt[3]{4}x^3j) - 9Q^2x^3(17496\sqrt[3]{2}x^3jh+x^9h^2+9565938jh^2+378\sqrt[3]{4}x^6j)+Q^3(889632234\sqrt[3]{2}x^3jh-135\sqrt[3]{2}x^{12}h+65610x^9h^2 + 27894275208jh^2+\sqrt[3]{4}x^{15}+7617321\sqrt[3]{4}x^6j)]. \tag{27}$$

The adiabatic stability index is found, with the help of (23), (25), and (27), to be of the form

$$\kappa=[27jh^6(\sqrt[3]{2}Qj(54h+\sqrt[3]{2}x^3)-6jh^2-1944\sqrt[3]{4}Q^4x^3+\sqrt[3]{2}Q^3x^3(\sqrt[3]{2}x^3-108h) - 486\sqrt[3]{4}Q^2j)^2]/A^6[B^{18}(Q^2(5\sqrt[3]{2}x^3-162h)h+5\sqrt[3]{2}jh-90(AB^2)^2x^2h^2-13122\sqrt[3]{4}Q^4 + 54\sqrt[3]{2}Q^3(27h+\sqrt[3]{2}x^3)+54\sqrt[3]{4}Qj)(12050326889856\sqrt[3]{4}Q^7x^6-9x^{12}jh^2 - 1033121304\sqrt[3]{2}Q^6x^6(1134h+19\sqrt[3]{2}x^3)+Qx^9j(52488h^2+\sqrt[3]{4}x^6-135\sqrt[3]{2}x^3h) + 177147Q^5x^3(7128\sqrt[3]{2}x^6h+629856x^3h^2+65\sqrt[3]{4}x^9+12754584\sqrt[3]{4}j) - 486Q^4x^3(516560652\sqrt[3]{2}jh-81\sqrt[3]{2}x^9h+314928x^6h^2+10\sqrt[3]{4}x^{12}+21789081\sqrt[3]{4}x^3j) - 9Q^2x^6(17496\sqrt[3]{2}x^3jh+x^9h^2+9565938jh^2+378\sqrt[3]{4}x^6j)+Q^3x^3(65610x^9h^2 - 135\sqrt[3]{2}x^{12}h+889632234\sqrt[3]{2}x^3jh+27894275208jh^2+\sqrt[3]{4}x^{15}+7617321\sqrt[3]{4}x^6j))], \tag{28}$$

and we finally evaluate the surface redshift, which is then given by the simple expression

$$z=\frac{1}{\sqrt{x}}-1. \tag{29}$$

We have been successfully obtained necessary quantities for our model to be examined. In the following section we shall analyze our solution and test the model for physical meanings using the standard requirements.

VI. MATCHING

In order to complete the six-dimensional Einstein-Gauss-Bonnet stellar model, it is necessary to perform the appropriate matching with the exterior at the radial value $r = R$, R being the bounding radius, which will determine the integration constant as well as the parameters inserted. In other words we require A, B, and C in terms of the mass M and radius R of the hypersphere. Recall that B actually contains the coupling constant α and an extra constant C introduced in our coordinate transformation. The model we have constructed can be represented by the line element

$$ds^2 = -xdt^2 + 36\left[V + \frac{81}{V} + \frac{3x}{B} + 27\right]^{-1} dr^2 + r^2(d\Omega^2), \tag{30}$$

where we have put $V = \frac{h}{\sqrt[3]{2}Q}$ to simplify the expressions. A comparison of the components of this line element with the vacuum solution (19) allows us to determine the required constants and parameters. The matching of the g_{00} components gives the equation

$$X = 1 + \frac{X}{3\beta}\left(1 \pm \sqrt{1 + \frac{M}{X^{5/2}}}\right), \tag{31}$$

where we have replaced $X = CR^2$ to make the calculations transparent. After some simplifications the algebraic equation

$$3\beta(3\beta - 2)X^{5/2} + 6\beta(1 - 3\beta)X^{3/2} + 9\beta^2 X^{1/2} = M \tag{32}$$

results. Effectively, (32) is a quintic for which no general solution is known. However, it may be observed that on setting $\beta = \frac{2}{3}$, Eq. (32) reduces to a cubic equation and the real valued solution may be written in the form

$$X = \frac{1}{6}\left(W + \frac{4}{W} + 4\right), \tag{33}$$

where we have put $W = \frac{\sqrt[3]{27M^2 + 3\sqrt{3}\sqrt{27M^4 - 64M^2} - 32}}{2^{2/3}}$. Backtracking this allows us to establish the constant C in the form

$$C = \frac{(W + 2)^2}{WR^2} \tag{34}$$

in terms of the mass M and radius R of the sphere. We now consider matching the g_{11} components. Comparing the line element (30) with the exterior metric (19) requires a solution of the equation

$$36\left[V + \frac{81}{V} + \frac{12X}{\beta} + 27\right] = 1 + \frac{X}{3\beta}\left(1 \pm \sqrt{1 + \frac{M}{X^{5/2}}}\right) \tag{35}$$

at $r = R$ and recalling $B = \frac{1}{4}\beta = \alpha C$. The solution of the quadratic type Eq. (35) has the form

$$V = \frac{-2913\beta \pm \sqrt{\left(2913\beta + \sqrt{\frac{M}{X^{1/2}} + X^2} + 1295X\right)^2 - 3779136\beta^2} - \sqrt{\frac{M}{X^{1/2}} + X^2} - 1295X}{216\beta} \tag{36}$$

for all values of β. Note that the resolution of C in terms of M and R was only possible for $\beta = \frac{2}{3}$; hence, the coupling constant α has the value

$$\alpha = \frac{1}{6C} = \frac{WR^2}{6(W + 2)^2} \tag{37}$$

in terms of the stellar mass M and radius R in this model. By writing capital letters for the equivalent small letters to indicate that we are working at the boundary $r = R$, we have

$$V = \frac{H}{\sqrt[3]{2}Q}, \quad \text{where } H = \sqrt[3]{Q^2(CR^2)^3 - 1458Q^3 + \sqrt{(Q^2(CR^2)^3)^2 - 2916Q^5(CR^2)^3}} \tag{38}$$

from (23). Solving for Q in (38) gives the remarkably compact form

$$Q = \frac{C^3R^6V}{(729 + V)^2} \tag{39}$$

in terms of V and C and effectively in terms of M and R via (36). The matching is now complete with all parameters and integration constants expressed in terms of the total mass and radius of the hypersphere as measured by an observer at spatial infinity.

VII. QUALITATIVE PHYSICAL ANALYSIS

We require that matter distribution be well behaved and the metric potentials to be regular. Now observing that we have obtained a complicated form for our matter variable $y = \sqrt{x}$, we then make choice of parameters $d = -4.3$, $\alpha = 2004$, and $C = -1$. These particular values allow us to examine our results using graphical plots of the physical quantities to ensure a compatible model. Figure 1 displays the energy density that is a monotonically decreasing function everywhere within the matter distribution. The pressure in Fig. 2 displays a similar behavior as the energy density inside the boundary and decreasing monotonically. Most importantly we observe a pressure-free hypersurface occurring at approximately $x = 312$ radial units. Additionally, the criterion: $(0 < \frac{dp}{d\rho} < 1)$ for the speed of sound is satisfied as plotted in Fig. 3 and shows that the speed of sound is never superluminal within the distribution. It is crucial also to study how the energy conditions behave in the interior of the star. The applicable energy conditions are these: weak energy condition $(\rho + p \geq 0, \rho \geq 0)$, strong energy condition $(\rho + p \geq 0, (d-3)\rho + (d-1)p \geq 0)$, and the dominant energy condition $(\rho - |p| \geq 0, \rho \geq 0)$. That these conditions are all satisfied within the star is demonstrated by Fig. 4. The graphical representation of the decreasing surface redshift in the interior as radius increases is given by Fig. 5. For radial values greater than 0.1 the redshift is less than 2 units and is therefore consistent with redshift values associated

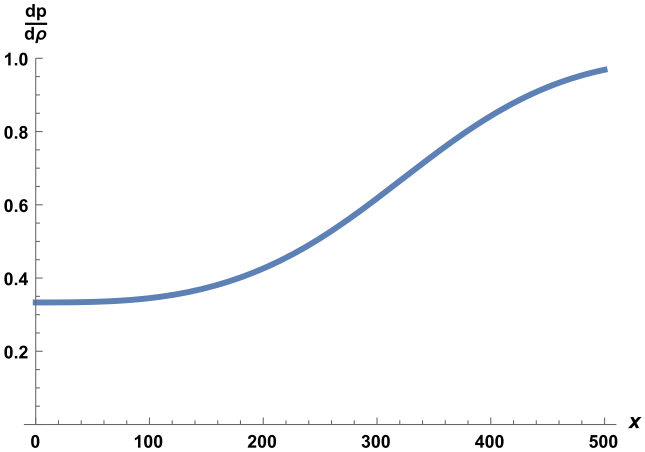

FIG. 3. Plot of sound speed $(\frac{dp}{d\rho})$ versus radial coordinate (x).

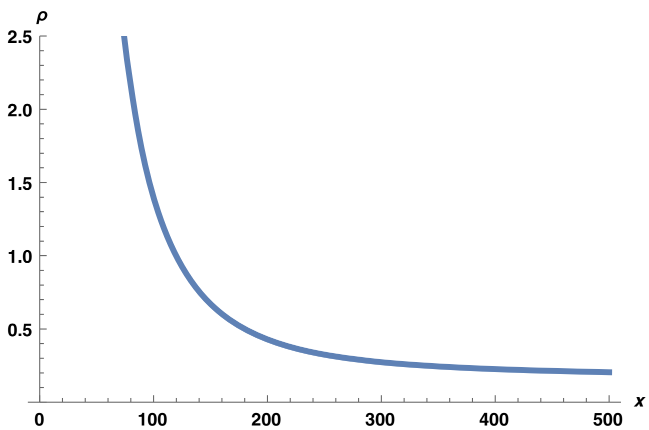

FIG. 1. Plot of energy density (ρ) versus radial coordinate (x).

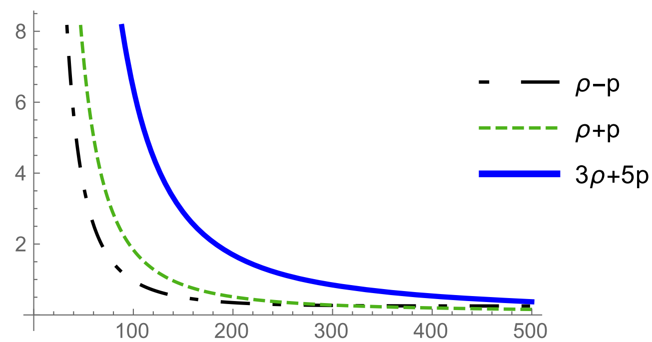

FIG. 4. Plot of energy conditions $(\rho - p$, $\rho + p$, and $3\rho + 5p)$ versus radial coordinate (x).

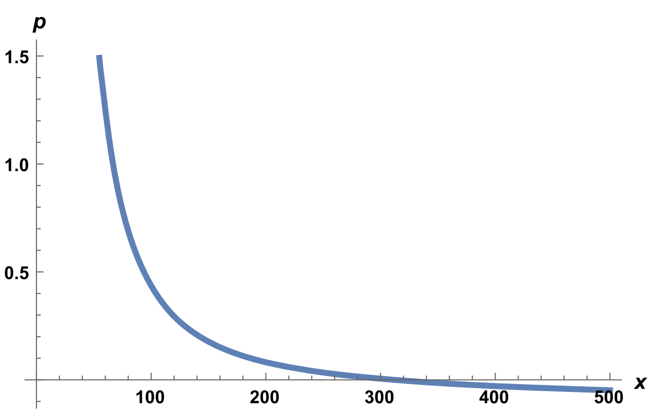

FIG. 2. Plot of pressure (p) versus radial coordinate (x).

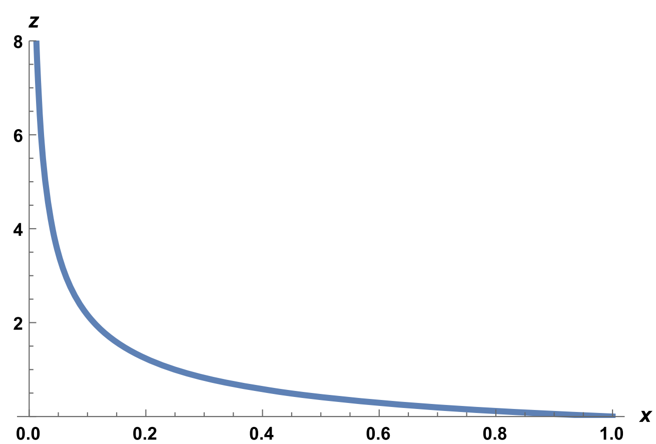

FIG. 5. Plot of surface redshift (z) versus radial coordinate (x).

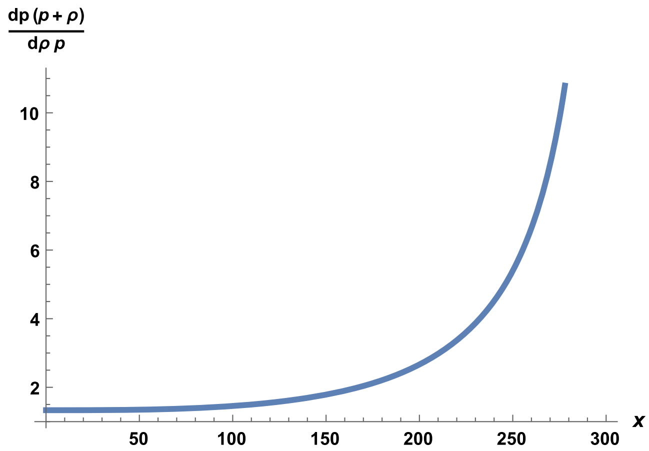

FIG. 6. Plot of adiabatic stability index ($\kappa = \frac{\rho+p}{p}\frac{dp}{d\rho}$) versus radial coordinate (x).

with neutron stars [34,35]. It is shown in Fig. 6 that the Chandrasekhar adiabatic stability index ratio, $\frac{\rho+p}{p}\frac{dp}{d\rho} > \frac{4}{3}$, is satisfied as demanded. One negative feature of this model is that a singularity at the stellar center is unavoidable. Nevertheless, given the rarity of exact solutions in this area, the positive features of the model cannot be discounted. In the standard Einstein theory it is not uncommon to remove central singularities by utilizing a core of well-behaved, perfect fluid surrounded by a spacetime such as is reported here. This of course introduces a new common hypersurface requiring further matching, which we do not pursue at this time, save to mention that the singularity may be adequately dealt with.

VIII. DISCUSSION

We have constructed an exact model for spherical distributions of perfect fluids in the Einstein-Gauss-Bonnet theory of gravitation by solving the associated equation of pressure isotropy. The prescribed simple form for the temporal metric potential $y = \sqrt{x}$ allowed for the solution of the second order ordinary differential equation in elementary functions. The exact solution enabled a study of the impact of the higher curvature terms on the evolution of the hypersphere of perfect fluid matter for an appropriate parameter space. This solution was subjected to a battery of tests for physical applicability and found to be reasonably well behaved. For the selected parameter space, adiabatic stability and causality were guaranteed. In addition all the energy conditions were satisfied. The use of plots assisted in verifying the compliance with the basic physical requirements. In general it is extremely difficult to find exact solutions for compact objects in higher curvature gravity due to the added complexity of the defining equations and it is pleasing that an explicit solution was located. The pleasing astrophysical behavior of the solution suggests that the higher curvature corrections to the standard theory may hold promise in resolving other problems, such as in cosmology, where general relativity appears to require modification.

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