Rigidly rotating perfect fluid stars in 2+1 dimensions

Carsten Gundlach and Patrick Bourg
Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

(Received 23 July 2020; accepted 21 September 2020; published 9 October 2020)

Cataldo has found all rigidly rotating self-gravitating perfect fluid solutions in $2+1$ dimensions with a negative cosmological constant Λ, for a density that is specified *a priori* as a function of a certain radial coordinate. We rewrite these solutions in standard polar-radial coordinates, for an arbitrary barotropic equation of state $p(\rho)$. For any given equation of state, we find the two-parameter family of solutions with a regular center and finite total mass M and angular momentum J (rigidly rotating stars). For analytic equations of state, the solution is analytic except at the surface, but including at the center. Defining the dimensionless spin $\tilde{J} \coloneqq \sqrt{-\Lambda}J$, there is precisely one solution for each $(\tilde{J}, M)$ in the region $|\tilde{J}| - 1 < M < |\tilde{J}|$, which consists of parts of the point-particle region $M < -|\tilde{J}|$ and overspinning regions $|\tilde{J}| > |M|$. In an adjacent compact part of the black-hole region $|\tilde{J}| < M$ (whose extent depends on the equation of state), there are precisely two solutions for each $(\tilde{J}, M)$. Hence, exterior solutions exist in all three classes of Bañados, Teitelboim, and Zanelli solution (black hole, point particle, and overspinning), but not all possible values of $(\tilde{J}, M)$ can be realized as stars. Regardless of the values of $\tilde{J}$ and M, the causal structure of all stars for all equations of state is that of anti–de Sitter space, without horizons or closed timelike curves.

DOI: 10.1103/PhysRevD.102.084023

I. INTRODUCTION

Classical Einstein gravity in $2+1$ spacetime dimensions may appear to be dynamically trivial because in $2+1$ dimensions the Weyl tensor is identically zero. This means that the full Riemann tensor is determined by the Ricci tensor and so by the stress-energy tensor of the matter. Hence, there are no gravitational waves, and the vacuum solution is locally unique: Minkowski in the absence of a cosmological constant Λ, de Sitter for $\Lambda > 0$, and anti–de Sitter for $\Lambda < 0$.

However, in 1992, Bañados, Teitelboim, and Zanelli [1] (from now on, BTZ) noticed that $2+1$-dimensional vacuum Einstein gravity with $\Lambda < 0$ admits rotating black-hole solutions that are in close analogy with the family of Kerr solutions in $3+1$ dimensions. They can be found easily by solving an axistationary ansatz for the metric, but their existence was unexpected because the metric has to be locally that of the $2+1$-dimensional anti–de Sitter solution (from now on, AdS3). In fact, these metrics can be derived as highly nontrivial identifications of AdS3 under an isometry [2].

We define the cosmological length scale

$$\ell \coloneqq (-\Lambda)^{-\frac{1}{2}} \tag{1}$$

and the dimensionless spin

$$\tilde{J} \coloneqq \frac{J}{\ell}. \tag{2}$$

The gravitational mass M is already dimensionless in $2+1$ dimensions. A key difference to axistationary vacuum solutions in $3+1$ dimensions is the existence of a mass gap: while AdS3 is given by the BTZ solution with parameters $M = -1$ and $\tilde{J} = 0$, only the BTZ solutions with $M > 0$ and $|\tilde{J}| < M$ represent black holes. Solutions with $-1 < M < 0$ and $|\tilde{J}| < -M$ represent point particles, similar to those for $\Lambda = 0$ described in [3]. The status of those with $|\tilde{J}| > |M|$, which we call "overspinning," remains unclear.

The relevance of the BTZ solutions goes beyond vacuum because, roughly speaking, the vacuum exterior of any rotating isolated object must be a BTZ solution, even if the object itself is neither stationary nor axisymmetric.

More precisely, consider a region of spacetime with a timelike world tube removed. We can make this region simply connected by making a cut from the world tube to the outer boundary of the region. In the resulting simply connected region, the spacetime must be AdS3. However, when we make the region multiply connected again by identifying the two sides of the cut, this identification is parametrized by an isometry of AdS3. The isometry group of AdS3 is six-dimensional, but it was shown in [2] that the gauge-invariant part of the identification is characterized by only two parameters $(\tilde{J}, M)$, parametrizing precisely the BTZ solutions. A region of spacetime with several world tubes removed requires one identification around each world tube, and so is described by a pair $(\tilde{J}_i, M_i)$ for each world tube representing a compact object.

By contrast, in $3 + 1$ dimensions, the exterior of a rotating object is not in general the Kerr solution, even if the object is axisymmetric and stationary. The argument we have just given does not apply because in more than $2 + 1$ dimensions a vacuum spacetime need not be Minkowski even locally. Put more physically, compact objects in $3 + 1$ dimensions can make not only their mass and spin, but also their internal structure felt in their vacuum exteriors through tidal forces and gravitational waves.

Perhaps the simplest example of axistationary matter solutions are rotating perfect fluid stars. In this paper, we examine if rigidly rotating perfect fluid stars exist in $2 + 1$ dimensions for reasonable equations of state. Here we define a star to be a perfect fluid solution with a regular center and finite mass and spin. We allow both for stars which have a surface at finite radius and are surrounded by vacuum, and stars which fill all of space but whose density falls off sufficiently rapidly. Given the existence of three different classes of BTZ solutions, we ask if point-particle, black-hole, and overspinning BTZ solutions can all be realized as exterior or asymptotic spacetimes of rigidly rotating perfect fluid stars.

Hence, in this paper, we solve the Einstein-fluid equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{3}$$

with $\Lambda \leq 0$ and the perfect-fluid stress-energy tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{4}$$

making an ansatz of stationarity and axisymmetry. The vector field u^a is tangential to the fluid worldlines, with $u^a u_a = -1$, and p and ρ are the pressure and total energy density measured in the fluid rest frame. We formally assume a barotropic equation of state $p = p(\rho)$ given *a priori*. However, as we consider only axistationary solutions, where all variables depend only on the radial coordinate r, any solution with a given barotropic equation of state could also *a posteriori* be a interpreted as a solution of a two-parameter equation of state $p = p(\rho, s)$ (where s is, e.g., the specific entropy), together with a given stratification $s = s(\rho)$. We set $c = G = 1$ throughout.

Cruz and Zanelli [4] have shown that static perfect fluid solution require a nonpositive cosmological constant $\Lambda \leq 0$ and also studied in more detail the case of constant energy density. In [5], the special cases of a polytropic equation of state with and without cosmological constant were also studied in [5,6]. In [7], García and Campuzano derived all static circularly symmetric spacetimes with $\Lambda \leq 0$. Rigidly rotating configurations were also studied [8,9]. Cataldo [10] has found all axistationary rigidly rotating perfect fluid solutions in $2 + 1$ spacetime dimensions with $\Lambda < 0$. The total energy density ρ is specified *a priori* as a function of the radial coordinate $\rho(\bar{r})$. The metric and $p(\bar{r})$ are then given explicitly in terms of $\rho(\bar{r})$ and four parameters C, D, E, and ω_0. The equation of state $p(\rho)$ is implied only *a posteriori* by comparing $p(\bar{r})$ and $\rho(\bar{r})$. We summarize these results in Sec. II A below, followed by a list of questions that remained open: how does one find the general solution if not $\rho(\bar{r})$ but the equation of state $p(\rho)$ is given *a priori*? Which solutions have a regular center? Which solutions have a vacuum exterior solution, and what is its form? What are the BTZ mass and angular momentum of such starlike solutions?

To answer these questions, we translate Cataldo's solution into the standard $2 + 1$ form in terms of a lapse, shift and two-metric, introduce an area radius coordinate, identify Cataldo's radial coordinate $\bar{r}$ with a certain integral over the equation of state, and identify the subset of solutions with a regular center, which as expected have only two free parameters (not four). We give expressions for M and $\tilde{J}$ in terms of these two parameters and certain integrals involving only the equation of state.

Our solutions for a general equation of state are in implicit form. They can be made explicit by evaluating an integral, inverting the resulting function, and evaluating another integral. As already obtained by Cataldo, this can be done for the linear equation of state $p = \kappa\rho$ and the "polytropic" equation of state $p = K\rho^k$. As a further example, we also consider the equation of state $p = \kappa(\rho - \rho_s)$ for $\rho_s > 0$.

II. GENERAL EQUATION OF STATE

A. Rigidly rotating axistationary perfect fluid solutions

Cataldo [10] found axisymmetric, stationary, rigidly rotating perfect fluid solutions of the Einstein equations in comoving coordinates, defined by $u^a \propto (\partial_t)^a$, for a certain choice of radial coordinate, in the form

$$ds^2 = -(\bar{r}d\bar{t} + \omega d\theta)^2 + h^{-1}d\bar{r}^2 + h d\theta^2, \tag{5}$$

where

$$\omega(\bar{r}) := \frac{\omega_0}{\bar{r}} + E\bar{r}, \tag{6}$$

$$h(\bar{r}) := C - \Lambda\bar{r}^2 + D\bar{r} + \frac{\omega_0^2}{\bar{r}^2} + 16\pi\bar{f}(\bar{r}), \tag{7}$$

$$\bar{f}(\bar{r}) := \int_{\bar{r}_0}^{\bar{r}} \bar{r}'\rho(\bar{r}')d\bar{r}' - \bar{r}\int_{\bar{r}_0}^{\bar{r}} \rho(\bar{r}')d\bar{r}', \tag{8}$$

$$p(\bar{r}) := \frac{D}{16\pi\bar{r}} - \frac{1}{\bar{r}}\int_{\bar{r}_0}^{\bar{r}} \rho(\bar{r}')d\bar{r}'. \tag{9}$$

Here $\bar{r}_0$ is an arbitrary integration limit. (In solutions with a regular center, we will later choose it to correspond to the center.) These solutions are parametrized by the function $\rho(\bar{r})$ and the constants ω_0, E, C, and D. (We denote the time and radial coordinates of [10] by $\bar{t}$ and $\bar{r}$ to distinguish them from rescaled coordinates t and r that we introduce below, and the area radius, which we will denote by R.)

At this point, it appears that the density ρ has to be specified as a function of the radial coordinate $\bar{r}$, which only afterward implies an equation of state $p(\rho)$ through the expression (9) for $p(\bar{r})$. This issue was partly addressed in [10] by deriving explicit solutions for two simple barotropic equations of state, but it remained unclear if and how solutions can be obtained for an arbitrary equation of state $p(\rho)$ given *a priori*.

It also remained unclear which solutions have a regular center. This issue was partly addressed in [10] by giving explicit solutions with a regular center for the above-mentioned equations of state. There was, however, no systematic construction of all solutions with a regular center for an arbitrary given equation of state in terms of precisely two free parameters that control the mass and spin of the star. Also lacking was a criterion on the equation of state for a solution with a regular center to either have a vacuum exterior, or to be asymptotically AdS3 with finite BTZ mass M and spin J.

In the remainder of this paper, we resolve all these questions.

B. The equation of state

We first clarify the role of the equation of state. Differentiating (9), we obtain

$$\bar{r}\frac{dp}{d\bar{r}} + p + \rho = 0. \quad (10)$$

Solving this separable ordinary differential equation by integration, we find

$$\ln\frac{\bar{r}}{\bar{r}_0} = -\int_{p_0}^{p(\bar{r})}\frac{dp}{p+\rho(p)} = -\int_{\rho_0}^{\rho(\bar{r})}\frac{p'(\rho)d\rho}{p(\rho)+\rho}, \quad (11)$$

where $\rho_0 := \rho(\bar{r}_0)$ is the density at $\bar{r}_0$ and $p_0 := p(\rho_0)$ the corresponding pressure, given by the equation of state $p(\rho)$. For stars, we will later choose $\bar{r}_0$ as the value of $\bar{r}$ at the regular center, so that p_0 is the central pressure.

Unless stated otherwise, we assume throughout that the equation of state $p(\rho)$ is at least continuous and piecewise continuously differentiable, with $0 \le p'(\rho) < 1$, and where $p'(\rho) = 0$ is allowed only at $p = 0$. As a consequence, the sound speed $\sqrt{p'(\rho)}$ is real and less than the speed of light, and the inverse equation of state $\rho(p)$ also exists as a continuous function that is piecewise once differentiable for $p > 0$. We allow for the possibility that $p(\rho_s) = 0$ for some $\rho_s \ge 0$.

In obtaining (10) by differentiating (9), we have lost the constant D. To find its value, we evaluate (9) at $\bar{r}_0$, obtaining

$$D = 16\pi\bar{r}_0 p_0. \quad (12)$$

C. Standard form of the metric

For further analysis, we rearrange the metric in the usual 2 + 1 form, and with the two-metric expressed in terms of an area radius R, that is, as

$$ds^2 = -\bar{\alpha}^2 d\bar{t}^2 + a^2\left(\frac{dR}{d\bar{r}}\right)^2 d\bar{r}^2 + R^2(d\theta + \bar{\beta}d\bar{t})^2, \quad (13)$$

where a, $\bar{\alpha}$, $\bar{\beta}$, and R are all functions of $\bar{r}$. Hence $\bar{\alpha}$ is the lapse, $\bar{\beta}$ the shift in the angular direction, both with respect to the time coordinate $\bar{t}$, $g_{\theta\theta} = R^2$ defines the area radius R as the length of the Killing vector ∂_θ (and hence R is a scalar), and $g_{RR} = a^2$ if we use R as the radial coordinate. We read off

$$R^2 = h - \omega^2, \quad (14)$$

$$\bar{\beta} = -\frac{\bar{r}\omega}{R^2}, \quad (15)$$

$$\bar{\alpha}^2 = \bar{r}^2 + R^2\bar{\beta}^2, \quad (16)$$

$$a^2 = \frac{1}{(\frac{dR}{d\bar{r}})^2 h} = \frac{4R^2}{(\frac{dR^2}{d\bar{r}})^2 h} \quad (17)$$

as functions of $\bar{r}$. We see that $\bar{t}$ and $\bar{r}$ have nonstandard dimensions, namely, length^{-1} and length^2, respectively. We use $\bar{r}_0$ to define a length scale

$$s := \sqrt{\bar{r}_0} \quad (18)$$

and then define

$$t := s^2\bar{t}, \qquad r := \frac{\bar{r}}{s}, \quad (19)$$

which have the usual dimension length. We correspondingly rescale the lapse and shift as

$$\alpha := \frac{\bar{\alpha}}{s^2}, \qquad \beta := \frac{\bar{\beta}}{s^2}. \quad (20)$$

The metric now takes the form

$$ds^2 = -\alpha^2 dt^2 + a^2\left(\frac{dR}{dr}\right)^2 dr^2 + R^2(d\theta + \beta dt)^2. \quad (21)$$

We introduce the dimensionless cosmological constant and spin parameters

$$\lambda := s\sqrt{-\Lambda} \ge 0, \quad (22)$$

$$\Omega := \frac{\omega_0}{s^3}, \quad (23)$$

and their combination

$$\mu := \lambda^2 - \Omega^2. \tag{24}$$

Note that $\lambda \ll 1$ corresponds to the length scale s being small compared to the cosmological length scale ℓ, but also, equivalently, to the cumulative effects of the cosmological constant being small over length scales of size s. We will in general consider $\lambda > 0$, but at one point also $\lambda = 0$, interpreted as $\Lambda = 0$. Otherwise, we always express λ in terms of the two independent parameters μ and Ω.

To write all our equations in fully nondimensional form, we introduce the dimensionless radial coordinate y and dimensionless area radius x defined by

$$y := \frac{r}{s}, \qquad x := \frac{R}{s}. \tag{25}$$

For a given equation of state $p(\rho)$ and reference density ρ_0, the relation between the density ρ and the dimensionless radial coordinate y is

$$y(\rho_0;\rho) = \exp\left(-\int_{\rho_0}^{\rho} \frac{p'(\tilde\rho)d\tilde\rho}{p(\tilde\rho)+\tilde\rho}\right), \tag{26}$$

or equivalently

$$y(p_0;p) = \exp\left(-\int_{p_0}^{p} \frac{d\tilde p}{\tilde p+\rho(\tilde p)}\right), \tag{27}$$

where ρ_0 and $p_0 = p(\rho_0)$ are the density and pressure at $y = 1$, $p'(\rho) := dp/d\rho$, and $\rho(p)$ is the inverse equation of state; compare also Eq. (50) of [9]. We define the dimensionless function $f(y) := s^{-2}\bar f(\bar r)$, that is,

$$f(y) = s^2\left(\int_1^y \rho(\tilde y)\tilde y d\tilde y - y\int_1^y \rho(\tilde y)d\tilde y\right). \tag{28}$$

We primarily use s rather than ℓ to adimensionalize all other variables and parameters in order to keep the limit $\Lambda = 0$ regular. However, when we want to compare different solutions with the same $\Lambda < 0$, it is more natural to express the dimensionful quantities R, ρ, and p in terms of ℓ, using

$$s = \lambda\ell = \sqrt{\mu+\Omega^2}\,\ell. \tag{29}$$

In particular, we have

$$R = \ell\sqrt{\mu+\Omega^2}\,x \tag{30}$$

and

$$s^2\rho = (\mu+\Omega^2)\ell^2\rho. \tag{31}$$

D. Local mass and angular momentum

For an arbitrary time-dependent axisymmetric spacetime in $2+1$ spacetime dimensions, regardless of matter content, there exist two conserved currents $\nabla_a j^a_{(J)} = 0$ and $\nabla_a j^a_{(M)} = 0$: the conserved current due to the angular Killing vector, and a second, more mysterious, one that generalizes the Misner-Sharp mass that exists for spherical symmetry in any dimension, to a conserved mass that exists for axisymmetry in $2+1$ dimensions only. In terms of the metric (21), the corresponding conserved quantities are given by

$$J = \frac{R^3\frac{\partial\beta}{\partial r}}{\frac{dR}{dr}a\alpha}, \tag{32}$$

$$M = \frac{R^2}{\ell^2} + \frac{J^2}{4R^2} - \frac{1}{a^2}. \tag{33}$$

Note that these expressions hold in the axsymmetric but time-dependent case. In the axistationary case that we consider here, $\partial\beta/\partial r$ simply becomes $d\beta/dr$. In any vacuum region, M and J are constant with values equal to the BTZ parameters of the same name, that is, the Einstein equations give $M_{,r} = M_{,t} = J_{,r} = J_{,t} = 0$. In particular, for constant (J, M), the polar-radial metric (21) takes the form

$$c_0^2\alpha^2 = -M + \frac{R^2}{\ell^2} + \frac{J^2}{4R^2}, \tag{34}$$

$$a^2 = \frac{1}{c_0^2\alpha^2}, \tag{35}$$

$$c_0\beta = -\frac{J}{2R^2} + \beta_0. \tag{36}$$

We can further set $c_0 = 1$ by rescaling t by the constant factor c_0, and $\beta_0 = 0$ by a rigid rotation of the coordinate system that corresponds to shifting θ by $\beta_0 t$. The result is the standard form of the BTZ metric first given in [1].

E. Solutions with a regular center

We now demand that the solution has a regular center at some value of the radial coordinate $\bar r$. Without loss of generality, we choose the center to be at the reference radius $\bar r_0$, so that $R(\bar r_0) = 0$ and $\bar f(\bar r_0) = (d\bar f/d\bar r)(\bar r_0) = 0$. With these conditions, (7) can be solved for the parameter C, which is now replaced as a free parameter by $\bar r_0$.

We also demand that there is no conical singularity at the center, $a(\bar r_0) = 1$. However, a necessary condition for this limit to be finite, given that $R(\bar r_0) = 0$ (by definition) and $(dR^2/d\bar r)(\bar r_0) \neq 0$ (by observation) is that $h(\bar r_0) = 0$, and hence that $\omega(\bar r_0) = 0$. This last condition can be solved for the parameter E. Applying l'Hôpital's rule, we then have

$$\lim_{\bar r\to\bar r_0} a = \lim_{\bar r\to\bar r_0} \frac{4}{\frac{dR^2}{d\bar r}\frac{dh}{d\bar r}} = \frac{4}{\frac{dh}{d\bar r}(\bar r_0)^2}, \tag{37}$$

and so we need $(dh/d\bar r)(\bar r_0) = 2$, which can be solved for D. The result, expressed for brevity in terms of our dimensionless parameters μ and Ω and reference scale s, is

$$E = -\frac{\Omega}{s}, \tag{38}$$

$$C = s^2(\mu - 2(1+\Omega^2)), \tag{39}$$

$$D = 2(1-\mu). \tag{40}$$

For a given barotropic equation of state, the general solution with a regular center now has two dimensionless free parameters μ, Ω, which govern, roughly speaking, the mass and spin of the star. This is the number of free physical parameters one would expect after imposing regularity at the center. Note that, for fixed Λ, s is given in terms of μ and Ω by (29), and from (12) and (40), the central pressure is given in terms of μ by

$$p_0 = \frac{1-\mu}{8\pi s^2}, \tag{41}$$

or equivalently

$$p_0 = \frac{1-\mu}{8\pi(\mu+\Omega^2)\ell^2}. \tag{42}$$

The expression for the metric coefficients, for an arbitrary equation of state, can be written concisely as

$$x^2 = \mu(y-1)^2 + 2(y-1) + 16\pi f, \tag{43}$$

$$\alpha^2 = y^2 + \frac{\Omega^2(y^2-1)^2}{x^2}, \tag{44}$$

$$a^2 = \frac{4y^2}{(\frac{dx^2}{dy})^2\alpha^2}, \tag{45}$$

$$\beta = \frac{\Omega(y^2-1)}{sx^2}, \tag{46}$$

where x, a, α, and β are all functions of y. Note that $y \geq 1$ with $y=1$ at the regular center. Recall that $f(y)$ was defined in Eq. (28), where $\rho(y)$ is given implicitly by inverting the integral (26), with the integration limit $\rho_0 = \rho(p_0)$ defined in terms of our free parameters μ and Ω by Eq. (42).

Equations (26), (28), and (42)–(46) together fully specify our solutions and can be taken as the starting point for the analysis that follows.

For an analytic equation of state, $f(y)$ is analytic with $f(y) = O(y-1)^2$ near the center, and hence

$$x^2 = 2(y-1) + O(y-1)^2, \tag{47}$$

$$\begin{aligned}\beta &= \frac{\Omega}{s} + O(y-1)\\ &= \frac{\Omega}{\sqrt{\mu+\Omega^2}\ell} + O(y-1)\end{aligned} \tag{48}$$

near the center. We note for later use that, while β is proportional to Ω for small Ω, it remains finite everywhere as $|\Omega| \to \infty$.

We obtain a fully explicit solution in the radial coordinate y if and only if the integral (26) can be evaluated for $y(\rho_0;\rho)$, this can then be inverted to give $\rho(\rho_0;y)$, and if the integral (28) can then also be evaluated. Furthermore, we obtain a fully explicit solution in terms of the area radius R if and only if Eq. (43) can also be inverted to give $y(x)$.

However, we do not need explicit solutions to establish analyticity of the solution in the area radius R. In an open interval of ρ where the equation of state $p(\rho)$ is analytic and $p+\rho > 0$, Eq. (26) defines y as a monotonically decreasing analytic function of ρ in this interval of ρ, and so $\rho(y)$ exists and is analytic in the corresponding interval of y. It follows that f is an analytic function of y in this interval. Hence, a, α, and β are all analytic functions of y at least for $y > 1$. A closer look shows that they are analytic also at $y = 1$, which corresponds to $x = 0$. Moreover, x^2 is an analytic function of y for $y \geq 1$, and so implicitly ρ, p, a, α, β are all analytic functions of x^2. In other words, they are even analytic functions of R for $R \geq 0$. For typical equations of state, analyticity breaks down at the surface of the star where $p = 0$.

By a standard argument, analyticity in R^2 implies that if we rewrite the metric in terms of Cartesian coordinates $X \coloneqq R\cos\theta$, $Y \coloneqq R\sin\theta$, all coefficients of the metric in the coordinates (t, X, Y) are analytic functions of X and Y (and independent of t), including at the center $X = Y = 0$.

The expressions for the local mass and angular momentum as functions of y are

$$\begin{aligned}M = {}&(\mu+2\Omega^2)x^2 - \frac{1}{4}\left(\frac{dx^2}{dy}\right)^2\\ &-\frac{\Omega^2(y^2-1)\frac{dx^2}{dy}}{y},\end{aligned} \tag{49}$$

$$J = s\Omega\left(2x^2 - \frac{(y^2-1)\frac{dx^2}{dy}}{y}\right), \tag{50}$$

or equivalently

$$\tilde{J} = \sqrt{\mu + \Omega^2}\Omega\left(2x^2 - \frac{(y^2-1)\frac{dx^2}{dy}}{y}\right). \tag{51}$$

These are also even analytic functions of R.

F. The AdS3 and test fluid cases

For $\mu = 1$, the central pressure is zero, and so this must correspond to the AdS3 solution. Indeed, with $\mu = 1$, the metric takes the form

$$x^2 = y^2 - 1, \tag{52}$$

$$\alpha^2 = (1+\Omega^2)y^2 - \Omega^2, \tag{53}$$

$$a^2 = \alpha^{-2}, \tag{54}$$

$$\beta = s^{-1}\Omega =: \beta_0, \tag{55}$$

and we have $M = -1$ and $J = 0$. Hence, this is the AdS3 solution in a rigidly rotating coordinate system, with constant angular velocity β_0. In the vacuum solution, β_0 has no physical significance and can be set to zero.

Expanding in $\mu - 1$, to leading order, we obtain the test fluid limit, in which a stationary, rigidly rotating, fluid configuration is held together only by the cosmological constant (as well as being pulled apart by rotation), but in which its self-gravity can be ignored. The metric is that of AdS3, but in a coordinate system that rotates with the fluid. As in the self-gravitating case, the equation of state and the central density ρ_0 implicitly determine a function $\rho = \rho(\rho_0; y)$ through Eq. (26). In the test fluid case, from (52), y is given in terms of the area radius R, the cosmological constant Λ, and the constant angular velocity β_0 as

$$y^2 = 1 + x^2 = 1 + R^2(-\Lambda - \beta_0^2), \tag{56}$$

where we have used (22), (24) with $\mu = 1$ and (55) to eliminate s. Hence, we have an implicit expression $\rho(R)$ for any rigidly rotating test fluid solution, for arbitrary central density ρ_0 and arbitrary constant angular velocity β_0 (with respect to the Killing vector ∂_t), given a cosmological constant $\Lambda < 0$ and equation of state.

G. Starlike solutions

We now look for solutions in which either $p = 0$ occurs at finite radius or $p \to 0$ and $\rho \to 0$ sufficiently rapidly as $R \to \infty$ so that the solution has finite M and J. We shall call such solutions "stars." Without any attempt at rigor, we classify the possibilities by assuming that the fluid is polytropic at low pressure, that is,

$$p \sim \rho^k \quad \text{as} \quad p \to 0 \tag{57}$$

for some $k \geq 1$. We note that for $k < 1$, the sound speed $\sqrt{p'(\rho)}$ diverges as $\rho \to 0$. We therefore disregard this range as unphysical.

From (42), we require $\mu \leq 1$ for the central pressure to be non-negative, and from (43) we further require $\mu \geq 0$ for $x(y)$ to be a monotonically increasing function for all y, in particular at large y. Stars therefore exist only with $\Lambda < 0$ and for $0 \leq \mu \leq 1$. Physically, from (24), $\mu > 0$ means that the Hubble acceleration is centripetal ($\Lambda < 0$) and larger than the centrifugal acceleration due to the rigid rotation ($\lambda^2 > \Omega^2$). Both the Hubble and the centrifugal acceleration depend on radius in the same way, and so this is true either for all y or for none.

1. Stars with a surface

From (27), we see that the solution has a surface $p(y_*) = 0$ at some finite coordinate radius y_* and finite area radius x_* if and only if the integral

$$y_*(p_0) := y(p_0; 0) = \exp\int_0^{p_0} \frac{dp}{p + \rho(p)} \tag{58}$$

converges. Note that in this case $y_*(0) = 1$. In the approximation (57), this is the case for $k > 1$. The limiting case $k = \infty$ can be interpreted as a fluid where $\rho = \rho_s > 0$ is finite at $p = 0$. (One may think of such a perfect fluid as a liquid, rather than a gas).

In the exterior $y > y_*$, the solution must be equal to a BTZ solution with constant M and J. To verify this, we note that in the exterior, (28) reduces to

$$16\pi f = m - 2(1-\mu)y, \tag{59}$$

where we have defined the integrated fluid mass

$$m := 16\pi s^2 \int_1^{y_*} \rho y \, dy. \tag{60}$$

We have identified the coefficient of y in (59) as $-D$ by demanding that (9) holds in the vacuum region $p = 0$ and have then used (40) to eliminate D.

As $y \geq 1$ in the integral in (60), we have

$$m \geq 16\pi s^2 \int_1^{y_*} \rho \, dy = 2(1-\mu), \tag{61}$$

where to obtain the last equality we have evaluated (9) in the vacuum region $p = 0$ and used (40).

To clarify what free parameters determine m, we use (27) to eliminate y and (41) to eliminate s in favor of the central pressure p_0 and then (42) to in turn express p_0 in terms our free parameters μ and Ω. We obtain

$$m = 2(1-\mu)I\left(\frac{1-\mu}{8\pi(\mu+\Omega^2)\ell^2}\right), \tag{62}$$

where

$$I(p_0) := \int_0^{p_0} \exp\left(-2\int_{p_0}^{p} \frac{d\tilde{p}}{\tilde{p}+\rho(\tilde{p})}\right)\frac{\rho(p)}{p_0}\frac{dp}{p+\rho(p)}. \tag{63}$$

So, in general, m depends on μ and Ω^2, as well as of course on the equation of state. Note that from (61), we have $I(p_0) \geq 1$.

To simplify the expressions that follow, we define the auxiliary quantity

$$A(\mu,\Omega) := m(\mu,\Omega) + \mu - 2. \tag{64}$$

By definition, $A(1,\Omega) = -1$ in the vacuum or test fluid case, where $m = 0$. From (61), we have

$$A + \mu \geq 0. \tag{65}$$

With f given by (59), the metric coefficients in the vacuum exterior are given by (43)–(46) as

$$x^2 = \mu y^2 + A, \tag{66}$$

$$\alpha^2 = y^2 + \frac{\Omega^2(y^2-1)^2}{\mu y^2 + A}, \tag{67}$$

$$a^2 = \frac{1}{\mu^2\alpha^2}, \tag{68}$$

$$\beta = \frac{\Omega(y^2-1)}{s(\mu y^2 + A)}. \tag{69}$$

Substituting (66) into the expressions (49) and (50) for M and J, we obtain the constant values

$$M = M_{\mathrm{tot}} := A\mu + 2(A+\mu)\Omega^2, \tag{70}$$

$$\tilde{J} = \tilde{J}_{\mathrm{tot}} := 2\sqrt{\mu+\Omega^2}(A+\mu)\Omega, \tag{71}$$

or equivalently

$$J_{\mathrm{tot}} = 2s(A+\mu)\Omega. \tag{72}$$

It is then easy to verify that the exterior metric (66)–(69), is (34)–(36), generally with $c_0 \neq 1$ and $\beta_0 \neq 0$.

2. Stars without a surface

If the integral (58) diverges but the integral (60) with $y_* = \infty$ converges to a finite value of m, the star has no surface but finite mass.

Taking the limit of $M(y)$ and $J(y)$ as $y \to \infty$, we again obtain the finite total values given by (70) and (72). The metric is now asymptotic (rather than strictly equal) to the BTZ metric (21) and (34)–(36).

In these stars without a sharp surface, we can nevertheless roughly identify a central region where self-gravity of the star is important and M and $|J|$ still increase, and an outer region, or stellar atmosphere, where M and J are essentially constant and the fluid is essentially a test fluid on the BTZ spacetime with parameters M_{tot} and J_{tot}.

In our approximation (57), this happens in the marginal case $k = 1$; we need to also specify the constant of proportionality, as the dimensionless parameter κ in

$$p \simeq \kappa\rho \quad \text{as} \quad p \to 0 \tag{73}$$

for some $0 < \kappa < 1$. The pressure and density fall off as $\rho \sim p \sim y^{-1-\frac{1}{\kappa}}$, and so once again m is finite, but there is now no surface at finite radius, and the metric is only asymptotically BTZ, with $y_* = \infty$. The sound speed is also less than the speed of light for $0 < \kappa < 1$.

3. Nonstars

When not only y_* but m diverges, $f(y)$ grows faster than y as $y \to \infty$. In the approximation (57), this is the case for $1/2 \leq k < 1$, when $\rho \sim y^{-\frac{1}{k}}$ and $f \sim y^{2-(1/k)}$ as $y \to \infty$. However, we have already ruled out $k < 1$ on the grounds that the sound speed $\sqrt{p'(\rho)}$ diverges at the surface. The expressions for $M(y)$ and $J(y)$ also diverge, and so the spacetime is not asymptotically BTZ. Such solutions do not describe stars. Recall again that we have already ruled out $k < 1$ on the grounds of diverging sound speed.

H. The manifold of solutions

In contrast to 3 + 1 and higher dimensions, the vacuum exterior metric, or the asymptotic metric at infinity, of a rotating star is given by a BTZ metric. It is therefore of interest what region in the $(\tilde{J}, M)$ plane is covered by possible stellar exterior solutions. Recall that for stars the parameters μ and Ω can take any values in the strip

$$0 < \mu \leq 1, \quad -\infty < \Omega < \infty. \tag{74}$$

In the following, we suppress the suffix "tot" for brevity, and for the rest of this section, M and $\tilde{J}$ always denote the total mass and spin of the spacetime, measured at infinity.

The manifold of solutions is uniquely parametrized by (Ω, μ). However, if we are interested more in the values of $(\tilde{J}, M)$, we can present the solution manifold as a hypersurface in $(\tilde{J}, M, \mu)$ space. The case of the linear equation of state $p = \kappa\rho$ is nongeneric in that A is a function of μ only, but it, and in particular the value $\kappa = 1/2$, can serve as a concrete illustration of the general considerations presented below. The solution manifold parametrized by (Ω, μ) for the equation of state $p = \rho/2$ is shown in Fig. 1. The same solution manifold embedded in $(\tilde{J}, M, \mu)$ space is shown in Fig. 2 and the projection of this embedding down into the $(\tilde{J}, M)$ plane in Fig. 3. We stress that the following

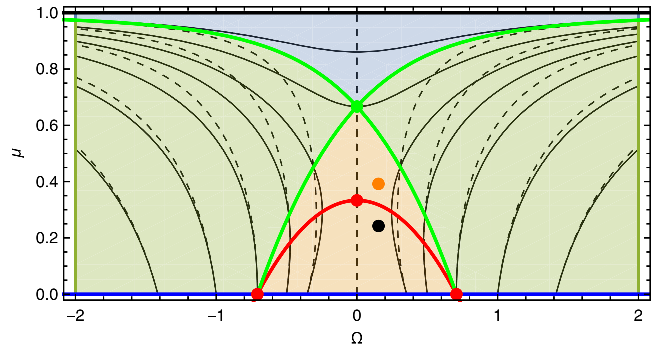

FIG. 1. The nature of the asymptotic metric for starlike solutions with the linear equation of state $p = \rho/2$. All solutions lie in the strip $0 < \mu < 1$, $-\infty < \Omega < \infty$. The asymptotic metric is of black-hole type in the orange (bottom) region, of point particle type in the blue (top) region, and of overspinning type in the green (left and right) regions. The parameter values of the two solutions shown in Fig. 4, and which have the same $(\tilde{J}, M)$, are indicated by an orange dot and a black dot. The contours of $M = -1, -1/2, 0, 1/2, 1, 2, 4, 8, 16$ (from top to bottom, solid) and $\tilde{J} = 0, \pm 1/2, \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$ (outward from the center, dashed) are also shown. The crossing green lines indicate $\tilde{J} = \pm M$. The bottom region is split into two regions by the red line, each of which covers the same region in the $(\tilde{J}, M)$ plane. Solutions in the bottom half, such as the one indicated by the black dot, are conjectured to be unstable. The green dot is at $(0, \mu_c)$, and the three red dots are at $(\pm\Omega_0, 0)$ and $(0, \mu_r)$.

arguments hold for all equations of state that admit starlike solutions, and so these figures apply qualitatively to all equations of state.

1. Boundary $\mu = 1$ of solution space

We have already seen that $\mu = 1$ at finite Ω (the thick black line in Fig. 1) corresponds to a rotating test fluid on the AdS3 spacetime with $M = -1$ and $\tilde{J} = 0$. However, taking the simultaneous limit $\mu \to 1_-$, $\Omega \to \pm\infty$ of (70) and (71) such that

$$\mu = 1 - \frac{\tilde{q}}{\Omega^2} \tag{75}$$

for some fixed constant $\tilde{q} > 0$, we have $s \to \infty$ and $p_0 \to 0$ and so, for finite $I(0)$, we obtain

$$m \simeq 2(1-\mu) I(0), \tag{76}$$

giving

$$A + \mu \simeq \frac{q}{\Omega^2}, \qquad q \coloneqq 2[I(0) - 1]\tilde{q}, \tag{77}$$

and hence two 1-parameter families of solutions with

$$M = -1 + q, \qquad \tilde{J} = \pm q. \tag{78}$$

From (65), we have that $q \geq 0$. See the blue region in Fig. 1 as $\Omega \to \pm\infty$ and the thick dashed black line in Fig. 2. In this limit, the fluid is infinitely dilute but infinitely extended. Note that even though $\Omega \to \infty$, the angular velocity β is finite everywhere. The integrated fluid rest mass m vanishes, but $M > -1$. Intuitively, this nontrivial gravitational mass comes from rotational energy.

We now show, assuming an analytic equation of state for small $p > 0$, that $I(0) = 1$ if the star has a surface at finite radius. To see this, we write

$$I(p_0) = \int_0^{p_0} y^2(p_0; p) \frac{\rho}{p_0} \frac{dp}{p + \rho} \geq 0. \tag{79}$$

We can bound $1 \leq y^2 \leq y_*^2$ in the integrand, and so

$$\frac{1}{p_0} \int_0^{p_0} \frac{\rho}{p + \rho} dp \leq I(p_0) \leq \frac{y_\star^2}{p_0} \int_0^{p_0} \frac{\rho}{p + \rho} dp. \tag{80}$$

From $y_*(0) = 1$ (as noted above) and the squeeze theorem, we then have

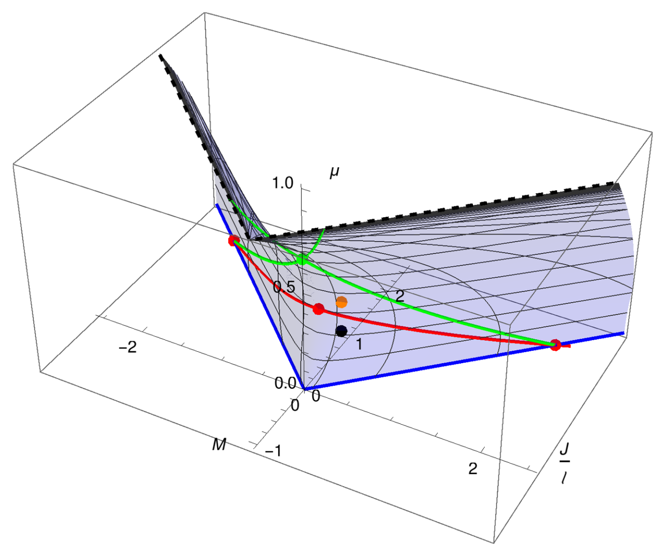

FIG. 2. Parametric plot of $(\tilde{J}, M)$ as a function of (μ, Ω), embedded in three dimensions as $(\tilde{J}, M, \mu)$. All dots and thick curves correspond to those of the same color in Fig. 1. Contours of Ω and μ are shown as thin lines. The thick red line denotes the locus of $|\partial(\tilde{J}, M)/\partial(\mu, \Omega)| = 0$, where the embedded surface is vertical. The intersecting thick green lines denote the loci of $\tilde{J} = \pm M$ at nontrivial values of μ. The bottom edge of the plot, $\mu = 0$, is at $\tilde{J} = \pm M$, for $M > 0$. The top edge of the plot (dashed black line) $\mu = 1$ is at $\tilde{J} = \pm(M + 1)$, for $M \geq -1$, with $M = -1$ only at $\mu = 1$. The single point $M = -1$, $\tilde{J} = 0$ in this plot corresponds to a two-parameter family of test fluid solutions. Solutions in the area below the red line are conjectured to be unstable. The orange dot and the black dot represent two solutions with the same M and $\tilde{J}$ that are presumed stable and unstable, respectively. The green dot is at $(0, M_c, \mu_c)$, and the three red dots are at $(\pm M_0, M_0, 0)$ and $(0, 0, \mu_r)$.

$$I(0) = \lim_{p_0 \to 0} \frac{1}{p_0} \int_0^{p_0} \frac{dp}{1 + \frac{p}{\rho}}. \tag{81}$$

From causality, p/ρ must remain bounded as $p \to 0$. If in fact $p/\rho \to 0$ as $p \to 0$, we have $I(0) = 1$.

In the other case, where $p/\rho \to \kappa$ remains finite as $p \to 0$, the surface of the star is at infinity and so we cannot rely on (81). However, one can see by explicit calculation that $I(0) = 1/(1-\kappa)$ for this case, which is finite, see also (109) below.

2. Boundary $\mu = 0$ of solution space

If $A(0,\Omega)$ is finite, the boundary $\mu = 0$ of solution space corresponds to a family of solutions with

$$M = 2A(0,\Omega)\Omega^2, \qquad \tilde{J} = 2A(0,\Omega)|\Omega|\Omega. \tag{82}$$

Note that $A(0,\Omega) \geq 0$ from (65), and so these solutions obey $M \geq 0$ with $|\tilde{J}| = M$. See the thick blue line in Figs. 1 and 2.

3. Second family of critically spinning solutions

There is a second family of solutions with $|\tilde{J}| = |M|$, over a finite range of M including both positive and negative values of M, namely,

$$\Omega = \pm\Omega_c(\mu), \tag{83}$$

where $\Omega_c(\mu)$ is defined by solving

$$A^2 = 4(A+\mu)\Omega^2 \tag{84}$$

for Ω^2, given μ. Along these curves, parametrized by μ, we have

$$|M| = |\tilde{J}| = A\left(\mu + \frac{A}{2}\right). \tag{85}$$

The range $1 > \mu > 0$ corresponds to the range $-1/2 < M < M_0$. Here

$$M_0 \coloneqq 8\Omega_0^2, \tag{86}$$

where Ω_0 is the positive solution of

$$A(0,\Omega_0) = 4\Omega_0^2. \tag{87}$$

[Note that therefore $\Omega_0 = \Omega_c(0)$.] See the thick green lines in Figs. 1 and 2. The two curves intersect at $M = \tilde{J} = 0$, which corresponds to $\mu = \mu_c$ defined by

$$A(\mu_c, 0) = 0. \tag{88}$$

This always has a solution in the range $0 \leq \mu_c < 1$ because $A(\mu,\Omega)$ is continuous with $A(0,\Omega) \geq 0$ and $A(1,\Omega) = -1$. [We assume without proof that there is only one solution.]

At their upper ends, the two curves are asymptotic to $\mu = 1$ as $\Omega \to \pm\infty$ in the (Ω,μ) strip, but in the $(\tilde{J}, M)$ plane they end at the finite points $M = -1/2$, $\tilde{J} = \pm 1/2$. At their lower ends, they intersect $\mu = 0$ at finite $|\Omega| = \Omega_0$, corresponding to $|\tilde{J}| = M = M_0 > 0$.

4. Double cover of a region in the $(\tilde{J}, M)$ plane

As there are two solutions for $M = |\tilde{J}|$ for $0 \leq M < M_0$, by continuity there must be a region of the $(\tilde{J}, M)$ plane that is doubly covered by the manifold of solutions. As the solutions $M = |\tilde{J}|$ corresponding to $\mu = 0$ lie on one boundary of the solution manifold, they also form one boundary of the doubly covered region [in (Ω,μ) and $(\tilde{J}, M)$, respectively]. The other boundary of the doubly covered region in the $(\tilde{J}, M)$ plane occurs where the solution manifold of Fig. 2 folds over. This occurs where

$$\left|\frac{\partial(\tilde{J}, M)}{\partial(\Omega,\mu)}\right| = 0, \tag{89}$$

which is equivalent to

$$2(A+\mu)(\mu A_{,\mu} + A - 4\Omega^2) + (A - 4\Omega^2 - 3\mu)\Omega A_{,\Omega} = 0. \tag{90}$$

This implicitly defines a curve

$$\Omega = \pm\Omega_r(\mu), \quad 0 < \mu < \mu_r, \tag{91}$$

where μ_r is defined by $\Omega_r(\mu_r) = 0$, giving

$$\mu_r A_{,\mu}(\mu_r, 0) + A(\mu_r, 0) = 0. \tag{92}$$

[Note that $\Omega_r(0) = \Omega_0$. We assume without proof that there is only one such curve, that is, the solution manifold is not folded over more than double.]

In fluid parameter space (Ω,μ), the doubly covered region lies between the curves (83) for $0 < \mu < \mu_c$ (the lower part of the two green curves in Fig. 1) and the curve $\mu = 0$ for $-\Omega_0 < \Omega < \Omega_0$ (part of the blue line). It is divided into two halves by (91) (the red curve). All three curves intersect at the two points $\mu = 0$, $\Omega = \pm\Omega_0$. Pairs of points from those two halves of the doubly covered region have the same values of M and $\tilde{J}$.

In BTZ parameter space $(\tilde{J}, M)$, the doubly covered region lies between $|\tilde{J}| = M$ for $0 < M < M_0$ (corresponding to both the blue and green curves in Fig. 2), and the red curve,

$$\Omega = \pm\tilde{\Omega}_r(M), \qquad 0 < M < M_0, \tag{93}$$

which is given implicitly by (70) and (71) with (91). The double cover becomes clearer by comparing Fig. 2 with its top view, Fig. 3. The corner points at $\mu = 0$, $\Omega = \pm\Omega_0$ have

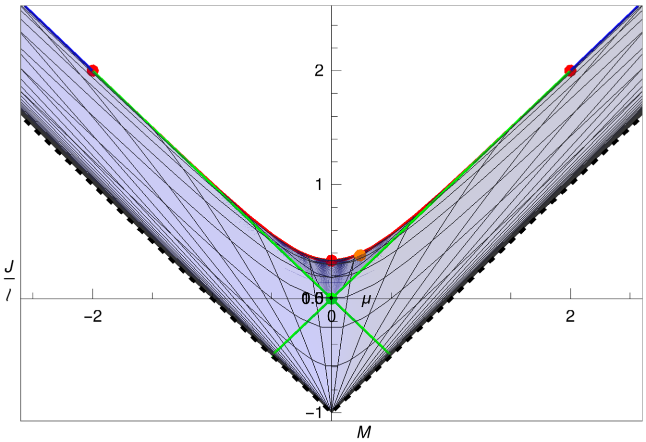

FIG. 3. A top view, suppressing the dimension μ, of the plot in Fig. 2. All dots and curves are as described in Fig. 2. Note that the orange dot lies on top of, and so hides, the black one.

$M = |\tilde{J}| = M_0$. Hence, the maximum possible M for given $|\tilde{J}| < M_0$ is obtained on the red curve. In particular, the maximum possible mass without rotation is given by $\Omega = 0$ and $\mu = \mu_r$, and is

$$M_r \coloneqq M(\mu_r, 0) = A(\mu_r, 0)\mu_r. \tag{94}$$

The red curve (91) corresponds to a curve of solutions that have a zero mode, a static linear perturbation that corresponds to an infinitesimal change of (μ, Ω) that leaves $(\tilde{J}, M)$ invariant to linear order. This signals that a linear perturbation mode changes from stable to unstable across the red curve. This is familiar from nonrotating stars in $3+1$ dimensions, where an extremum of the mass as a function of central density signals a separation between stable and unstable stars, with the less dense stars stable and the more dense ones unstable. We conjecture that the solutions in the doubly covered region with smaller μ (and hence larger central density) are unstable, corresponding to region below the red curve in Fig. 1. As their asymptotic metrics are of black-hole type, it is possible that these unstable solutions collapse to a black hole when perturbed in a suitable way.

We have obtained some evidence for this conjecture by time evolving the two solutions with the equation of state $p = \rho/2$ represented by orange and black dots in Fig. 1. Adding a small perturbation of the density with either sign to the less dense (orange) solution sets up propagating perturbations that remain small. Adding a small density perturbation to the denser (black) solution results in a highly nonlinear oscillation for one sign of the perturbation, where the central density repeatedly decreases below that of the orange solution, while perturbing the initial density with the opposite sign triggers prompt collapse to a black hole.

5. Summary of $\Lambda < 0$

In summary, the manifold of solutions contains a unique solution with given $(\tilde{J}, M)$ in the chevron-shaped region,

$$|\tilde{J}| - 1 < M < |\tilde{J}|, \tag{95}$$

that is bounded by the curves (78) and (82), while in a contiguous compact region bounded by $|\tilde{J}| = M$ for $0 < M < M_0$ and the curve (91) there are two solutions with the same given $(\tilde{J}, M)$. There are no solutions with $(\tilde{J}, M)$ outside these two regions.

6. The case $\Lambda = 0$

We now consider the limit where the length scale s remains finite but $\Lambda \to 0$. Then $\lambda^2 = \mu + \Omega^2 = 0$, so in this limit $\mu = \Omega = 0$. Therefore, no rigidly rotating stars can exist. Intuitively, only the cosmological contraction due to $\Lambda < 0$ can balance the centrifugal acceleration of rigid rotation, while the curvature generated by stress energy cannot. Setting $\Omega = 0$, replacing $(\mu + \Omega^2)\ell^2$ with s^2, and then setting $\mu = 0$, we obtain

$$m = 2I\left(\frac{1}{8\pi s^2}\right). \tag{96}$$

Equations (26) and (28) still hold, and so do (43)–(46) and (49), reduced to

$$x^2 = 2(y-1) + 16\pi f, \tag{97}$$

$$\alpha^2 = y^2, \tag{98}$$

$$M = -\frac{1}{a^2} = -\frac{1}{4}\left(\frac{dx^2}{dy}\right)^2, \tag{99}$$

with $\beta = 0$ and $J = 0$. They define an analytic interior solution for analytic equation of state, with in particular a regular center. However, in the vacuum exterior to this interior solution, (97) with (59) gives $x^2 = m - 2$, which is constant, so from (99) $M = 0$. This means that a diverges at the surface, but the metric expressed in terms of y remains regular, and in the exterior, it is

$$ds^2 = -y^2 dt^2 + s^2\left(\frac{dy^2}{m-2} + (m-2)d\theta^2\right) \tag{100}$$

for $y_* < y < \infty$. The spatial geometry is a cylinder; see also Eq. (79) of [9]. If $y_\star$ is finite, we do not consider such a solution as a star.

I. Causal structure

If we apply the standard compactification of AdS3, namely,

$$R = \ell \tan \frac{\psi}{\ell} \tag{101}$$

to the BTZ metric in its standard form (21) and (34)–(36) with $c_0 = 1$ and $\beta_0 = 0$, we obtain

$$ds^2 = \frac{1}{\cos^2 \frac{\psi}{\ell}} \left[-F\, dt^2 + G^{-1} d\psi^2 + \ell^2 \sin^2 \frac{\psi}{\ell} (d\theta + H\, dt)^2 \right], \tag{102}$$

where $F = G$ and

$$G = 1 - (M+1)\cos^2 \frac{\psi}{\ell} + \frac{J^2 \cos^4 \frac{\psi}{\ell}}{4 \sin^2 \frac{\psi}{\ell}}, \tag{103}$$

$$H = \frac{J \cos^2 \frac{\psi}{\ell}}{2 \sin^2 \frac{\psi}{\ell}}. \tag{104}$$

This is conformal to a metric (the one in the large square brackets) that is regular everywhere, or in the black-hole case everywhere outside the event horizon, but always including at $\psi/\ell = \pi/2$, which is therefore revealed as a timelike conformal boundary. In our starlike solutions, $F \neq G$ and H are different functions from those given above, but they are finite and nonzero for $0 \leq \psi/\ell \leq \pi/2$.

For the BTZ metrics corresponding to black holes, the familiar Penrose diagram [2] is a different one, being a square that is compact in the time as well as the radial direction. At first sight, this seems to contradict the above conformal picture for a star, in which the conformal metric has an infinite range of t. The apparent contradiction is resolved by noticing that the black-hole conformal diagram contains at its top and right corner a point representing timelike infinity where the curve representing the future branch of the event horizon meets the curve representing the timelike conformal boundary. If we now cover up the black-hole region with a star, the timelike curve representing the surface of the star and the timelike conformal boundary meet at the same point in the conformal diagram. Both have infinite proper length and are tangential to the stationary Killing vector. Moreover, a radial light ray reflected at both curves travels between them an infinity number of times before reaching the point in the conformal diagram where they meet. Hence, there must be a conformal transformation where these two curves remain parallel and have infinite coordinate length in the resulting Penrose diagram, as derived above.

A second question about the causal structure is if the spacetime admits closed timelike curves. It is obvious that closed timelike curves exist if there is a region where the metric coefficient $g_{\theta\theta} = R^2$ is negative. Conversely, Bañados *et al.* [2] have proved that the BTZ metrics do not contain closed timelike curves if there is no region with $R^2 < 0$, or if such regions are excluded. The proof only relies on the signature of the metric coefficients, not their form, and so generalizes to metrics of the form (13), as long as a^2 and α^2 remain positive. Hence, as a^2, α^2, and R^2 are manifestly non-negative in our starlike solutions, they do not contain closed timelike curves. (The examples of solutions with closed timelike curves given by Cataldo [10] can therefore not be starlike, that is, have both a regular center and be asymptotically BTZ.)

III. SIMPLE EQUATIONS OF STATE

A. Ultrarelativistic linear equation of state $p = \kappa\rho$

In the following, we concentrate on solutions with the ultrarelativistic (linear) equation of state $p = \kappa\rho$, assuming the physical range $0 < \kappa < 1$ of the equation of state parameter, which gives a real speed of sound smaller than the speed of light. (With the value $\kappa = 1/2$ in particular, this equation of state can be interpreted as a gas of massless particles without internal degrees of freedom.) We have already seen above that starlike solutions with this equation of state have no surface at finite radius but are asymptotically BTZ. From (26), we have

$$\rho(\rho_0; y) = \rho_0 y^{-\frac{1+\kappa}{\kappa}}, \tag{105}$$

and hence from (28)

$$8\pi f(y) = (1-\mu)\left((1-y) + \frac{\kappa}{1-\kappa}\left(1 - y^{-\frac{1-\kappa}{\kappa}}\right)\right). \tag{106}$$

Of the metric coefficients, we here write out only

$$x^2 = \mu(y^2 - 1) + \frac{2\kappa(1-\mu)}{1-\kappa}\left(1 - y^{-\frac{1-\kappa}{\kappa}}\right). \tag{107}$$

The other metric coefficients are given by (44)–(46).

In the test fluid case $\mu = 1$, we have $x^2 = y^2 - 1$, and so the density in terms of the area radius takes the simple form

$$\rho = \rho_0 [1 + R^2(-\Lambda - \beta_0^2)]^{-\frac{1+\kappa}{2\kappa}}, \tag{108}$$

where the central density ρ_0 is arbitrary (but assumed so small that self-gravity can be neglected) and β_0 is the constant angular velocity.

Integrating (105), we have

$$m = 2\frac{1-\mu}{1-\kappa} \Leftrightarrow I(p_0) = \frac{1}{1-\kappa} \tag{109}$$

and so

$$A = \frac{2\kappa - (1+\kappa)\mu}{1-\kappa}. \tag{110}$$

For this particular equation of state, $I(p_0)$ is constant, and so m and A depend on μ only but (untypically) not on Ω. The total mass and spin at infinity are

$$M_{\rm tot} = \frac{-(1+\kappa)\mu^2 + 2\kappa\mu(1-2\Omega^2) + 4\kappa\Omega^2}{1-\kappa}, \tag{111}$$

$$\tilde J_{\rm tot} = \frac{4\kappa(1-\mu)\Omega\sqrt{\mu+\Omega^2}}{1-\kappa}. \tag{112}$$

The loci of $\tilde J_{\rm tot} = \pm M_{\rm tot}$ are the two intersecting critical curves $\Omega = \pm\Omega_c(\mu)$ with

$$\Omega_c(\mu) = \frac{2\kappa - (1+\kappa)\mu}{\sqrt{8\kappa(1-\kappa)(1-\mu)}}. \tag{113}$$

They cross at

$$\mu_c = \frac{2\kappa}{1+\kappa}, \tag{114}$$

which is inside the strip for all $0 < \kappa < 1$, and they intersect the edge $\mu = 0$ of the strip at

$$\Omega_0 = \sqrt{\frac{\kappa}{2(1-\kappa)}}. \tag{115}$$

Hence, for all physical values of κ, the strip contains regions corresponding to point-particle, black-hole, and overspinning values of the pair $(\tilde J, M)$, as we have already shown in general.

The parameter space $0 < \mu < 1$, $-\infty < \Omega < \infty$ of solutions is shown in Fig. 1 for $\kappa = 1/2$, together with contour lines of M and $\tilde J$, the lines $|\tilde J| = |M|$, color coding of the asymptotic metric as black hole, point particle, or overspinning, and the curve that divides the black-hole region of parameter space into two halves that cover the corresponding region of $(\tilde J, M)$ space twice. This second curve is given by

$$\Omega_r^2(\mu) = \frac{1+\kappa}{2(1-\kappa)}(\mu_r - \mu), \qquad \mu_r := \frac{\kappa}{1+\kappa} \tag{116}$$

for $0 < \mu < \mu_r$. We can deparametrize this curve to obtain J^2 as a function of M involving only square roots, but the result is messy.

Solutions of black-hole type exist only for $M < M_0$ with

$$M_0 = 8\Omega_0^4 = \frac{2\kappa^2}{(1-\kappa)^2}. \tag{117}$$

The maximum possible mass without rotation is

$$M_r = A(\mu_r)\mu_r = \frac{\kappa^2}{1-\kappa^2}. \tag{118}$$

The manifold of solution is shown embedded in $(\tilde J, M, \mu)$ space in Fig. 2 to show the double cover more clearly, using the same color coding. A top view, suppressing the μ direction and thus hiding the double cover, is given in Fig. 3.

In all these figures, we have marked a specific pair of solutions with black-hole class asymptotic metrics, both of which have the same total mass $M = 0.38$ and angular momentum $\tilde J = 0.24$, but which have different parameter values $(\Omega, \mu) \simeq (0.154, 0.242)$ and $(0.153, 0.392)$. These solutions themselves are illustrated in Fig. 4 by plotting M, $\tilde J$ and $\ell^2\rho$ as functions of R/ℓ.

B. Modified linear equation of state $p = \kappa(\rho - \rho_s)$

A simple equation of state that admits solutions with a surface at finite radius is the inhomogeneous linear one,

$$p = \kappa(\rho - \rho_s) \tag{119}$$

for $0 < \kappa < 1$ and $\rho_s \geq 0$. Obviously, this reduces to the previous example for $\rho_s = 0$. Proceeding as before, we find

$$\rho = \rho_0 y^{-\frac{1+\kappa}{\kappa}} + \frac{\kappa\rho_s}{1+\kappa}\left(1 - y^{-\frac{1+\kappa}{\kappa}}\right). \tag{120}$$

We then obtain

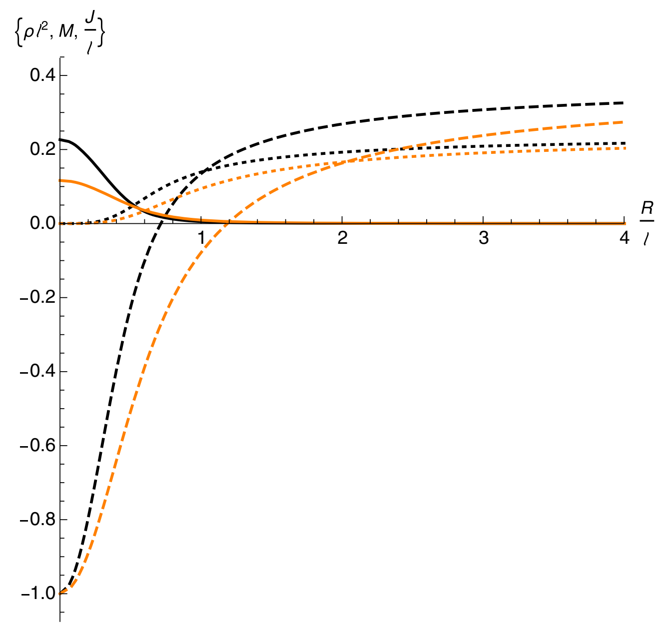

FIG. 4. An example of two starlike solutions with the equation of state $p = \rho/2$. Both have $M = 0.38$ and $\tilde J = 0.24$, but different central densities. We plot $\ell^2\rho$ (solid), M (dashed), and $\tilde J$ (dotted) against R/ℓ. The less compact solution, with $(\Omega, \mu) \simeq (0.154, 0.242)$ and lower central density (stable in nonlinear numerical time evolutions) is plotted in orange, and the more compact one with $(\Omega, \mu) \simeq (0.153, 0.392)$ and higher density (numerically found to be unstable) in blue.

$$x^2 = \tilde{\mu}(y^2 - 1) + \frac{2\kappa(1-\tilde{\mu})}{1-\kappa}\left(1 - y^{-\frac{1-\kappa}{\kappa}}\right), \quad (121)$$

which is just (107) again, only with μ replaced by

$$\tilde{\mu} := \mu - \sigma, \quad (122)$$

where

$$\sigma := \frac{\kappa}{1+\kappa} 8\pi s^2 \rho_s = \frac{\kappa}{1+\kappa}(\mu + \Omega^2) 8\pi \ell^2 \rho_s. \quad (123)$$

The other metric components follow, and we do not give them here. The stellar surface is now at finite radius,

$$y_*(\rho_0) = \left(\frac{(1+\kappa)\rho_0}{\rho_s} - \kappa\right)^{\frac{\kappa}{1+\kappa}}. \quad (124)$$

Note that $y_*(\rho_s) = 1$ as expected. We have

$$m = 2\frac{1-\mu}{1-\kappa} + \frac{1+\kappa}{1-\kappa}\sigma\left(1 - \left(\frac{1-\mu+\sigma}{\sigma}\right)^{\frac{2\kappa}{1+\kappa}}\right), \quad (125)$$

which now depends also on Ω through $\sigma(\mu, \Omega)$. We do not write down further expressions, which are complicated and do not add new insight.

C. Polytropic equation of state $p = K\rho^k$

For

$$p = K\rho^k, \quad (126)$$

the star has a surface at finite radius,

$$y_*(\rho_0) = (1 + K\rho_0^{k-1})^{\frac{k}{k-1}}, \quad (127)$$

if and only if $k > 1$, consistent with the analysis in Sec. II G. We find

$$\rho(\rho_0; y) = K^{-\frac{1}{k-1}}\left(\left(\frac{y}{y_*(\rho_0)}\right)^{-\frac{k-1}{k}} - 1\right)^{\frac{1}{k-1}}. \quad (128)$$

The functions $f(y)$ and hence $x^2(y)$ can be expressed in closed form in terms of hypergeometric functions, as already noticed in [10]. The same is true for m and hence M_{tot} and J_{tot}. We do not write down these expressions as they do not give further insight.

IV. CONCLUSIONS

We have constructed rotating perfect fluid starlike solutions in $2+1$-dimensional general relativity with a negative cosmological constant $\Lambda < 0$. We defined these to have a regular center, and finite mass M and spin J at infinity. (We again suppress the suffix "tot" in this section.) We have found these solutions in standard polar-radial coordinates (t, R, θ), in terms of two free parameters μ and Ω that control their mass and spin, and we have given expressions for the total mass M and spin J in terms of the two free parameters. We have thus established that starlike solutions in $2+1$ dimensions exist for generic equations of state.

Furthermore, we have shown that these solutions are analytic in suitable coordinates, including at the center, for analytic equations of state (except at the surface, if there is a sharp surface). We have also shown that their causal structure is that of the AdS3 cylinder, without closed timelike curves.

For any equation of state with $0 < p'(\rho) < 1$ and where either $p \sim \rho^k$ with $k > 1$ as $\rho \to 0$, or $p = 0$ occurs at finite ρ, we have shown that rotating and nonrotating stars with a sharp surface exist. The spacetime in the vacuum exterior is then the BTZ solution. In the limiting case where the equation of state is linear at low density, $p \simeq \kappa\rho$ with $0 < \kappa < 1$ as $\rho \to 0$, the density goes to zero only asymptotically, but sufficiently fast so that the spacetime is asymptotically BTZ with finite M and J.

We stress that the necessary and sufficient criterion for the existence of stars with a surface at finite radius and finite M and J is simply that the integral (58) converges at $p = 0$. We have not assumed further constraints on the equation of state except the causality constraint $0 < p'(\rho) < 1$ for all $p > 0$.

We have shown that for a generic equation of state the (Ω, μ) parameter space contains exterior/asymptotic metrics of all three BTZ types: black hole, point particle, and overspinning, but not for all values $(\tilde{J}, M)$. More precisely, solutions for generic equations of state cover all of the infinite region (95) of the $(\tilde{J}, M)$ plane and a finite region bounded by (93). In this second region, there are two solutions for the same values of M and $\tilde{J}$, with the more compact one conjectured to be unstable.

For an arbitrary barotropic equation of state $p = p(\rho)$, our solutions are in implicit form, involving two integrals and one function inversion. The integrals can be solved in closed form for the linear equation of state $p = \kappa\rho$, explicitly constructing the space of solutions, and we have shown that this is possible also for two other simple equations of state in which stars have sharp surfaces.

In spite of the local triviality of gravity, two compact self-gravitating objects in $2+1$ dimensions can interact gravitationally through global effects [3] and, for $\Lambda < 0$, even merge to form a black hole; see [11] for an explicit construction of a spacetime representing the formation of a spinning black hole from two massless point particles colliding with impact parameter. However, because there are no tidal forces or gravitational waves, unless and until the two objects actually touch they do not affect each other's local dynamics. In particular, if they start in an axistationary state, then they remain so unless and until they touch. This makes axistationary matter solutions even more relevant for representing interacting compact objects than they are in $3+1$ dimensions.

ACKNOWLEDGMENTS

The early stage of this work was partly funded by a 2019 EPSRC Vacation Bursary to the University of Southampton. C. G. gratefully acknowledges conversations with Jorma Louko about $2+1$-dimensional gravity.

[1] M. Bañados, C. Teitelboim, and J. Zanelli, Black Hole in Three-Dimensional Spacetime, Phys. Rev. Lett. **69**, 1849 (1992).

[2] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the $2+1$ black hole, Phys. Rev. D **48**, 1506 (1993).

[3] S. Deser, R. Jackiw, and G. T'Hooft, Three-dimensional Einstein gravity: Dynamics of flat space, Ann. Phys. (N.Y.) **152**, 220 (1984).

[4] N. Cruz and J. Zanelli, Stellar equilibrium in $2+1$ dimensions, Classical Quantum Gravity **12**, 975 (1995).

[5] P. M. Sá, Polytropic stars in three-dimensional spacetime, Phys. Lett. B **467**, 40 (1999).

[6] N. Cornish and N. Frankel, Gravitation in $2+1$ dimensions, Phys. Rev. D **43**, 2555 (1991).

[7] A. García and C. Campuzano, All static circularly symmetric perfect fluid solutions of gravity, Phys. Rev. D **67**, 064014 (2003).

[8] N. Cornish and N. Frankel, Gravitation vs rotation in $2+1$ dimensions, Classical Quantum Gravity **11**, 723 (1994).

[9] M. Lubo, M. Rooman, and Ph. Spindel, $(2+1)$-dimensional stars, Phys. Rev. D **59**, 044012 (1999).

[10] M. Cataldo, Rotating perfect fluids in $(2+1)$-dimensional Einstein gravity, Phys. Rev. D **69**, 064015 (2004).

[11] S. Holst and H.-J. Matschull, The anti-de Sitter Gott universe: A rotating BTZ wormhole, Classical Quantum Gravity **16**, 3095 (1999).