Boson stars in $f(T)$ extended theory of gravity

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Spherically symmetric configurations of the noninteracting massive complex scalar field, representing nonrotating boson stars, are considered within the framework of the modified torsion based $f(T)$ gravity, with $f(T) = T + \alpha T^2/2$. We find that with sufficiently large negative value of α the mass of the boson stars can be made arbitrarily large. This is in contrast to general relativity where an upper bound, $M_{\rm max} \sim M_{\rm Planck}^2/m$, to the mass of the boson stars built from the noninteracting scalar field exists and where the masses of boson stars in the astrophysical regime can be obtained only with the introduction of the scalar field self-interaction. With sufficiently large negative α we also find negative gravitational binding energy for all masses, which can be seen as an indication of the stability of such configurations. In its positive regime, α can not be made arbitrarily large as a phase transition in the stress-energy components of the $f(T)$-fluid develops. This phenomenon has already been reported to occur in polytropic stars constructed within the $f(T)$ gravity theory.

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I. INTRODUCTION

Recent cosmological observations [1,2], as well as the ever open quest for the complete quantum theory of gravity [3,4], motivate the research in the field of modified theories of gravity. One of the common ways to modify general relativity (GR) is to allow for a nonlinear function f of the scalar curvature R in the Einstein-Hilbert action. The resulting modified theories of gravity based on curvature are known as $f(R)$ gravity theories [5–7]. They have been successfully applied from cosmological settings [8–10] to high curvature gravity regimes [11–13]. However, it is well known that a theory of gravity equivalent to general relativity (GR) can be formulated in terms of torsion instead of curvature [14]. This theory replaces the scalar curvature of the Einstein-Hilbert action by the torsion scalar T; it uses the tetrad field as the dynamical degree of freedom and is known as the teleparallel equivalent of GR (TEGR). In the same spirit as the $f(R)$ gravity theories modify the curvature-based GR, a nonlinear function f can be used to modify the action of TEGR. The resulting theories of gravity, known as $f(T)$ gravity theories, have recently gained considerable attention. While most of applications of the $f(T)$ gravity are in the field of cosmology [15–26], the applications dealing with the static spherical symmetry are somewhat less in number [27–34]. An important problem found in the early formulations of $f(T)$ gravity was the lack of Lorentz invariance in the sense that the equations of motion were not invariant with respect to the particular choice of the tetrad fields, regardless of the latter satisfying the expected metric compatibility condition. This problem was pointed out and investigated by many authors [35–42] and is still not fully understood [43,44]. In this work we will rely on the covariant formulation of $f(T)$ gravity as proposed by Krššák and Saridakis [40]. As the particular form of f we will use $f(T) = T + \alpha T^2/2$, since this form of f guarantees the correct GR-limit when the parameter $\alpha \to 0$.

Static spherically symmetric vacuum solutions in $f(T)$ gravity have been considered in [45–47], while the solutions involving the polytropic fluid and Yang-Mills field have been considered in [48,49]. In this work we will construct static spherically symmetric self-gravitating configurations of the noninteracting complex scalar field. Our motivation to study this matter model comes in one part from its relative simplicity, while in the other part it comes from the fact that scalar field is the key component of the standard Λ cold dark matter model of cosmology. It is therefore interesting to explore objects that could be constructed of self-gravitating scalar fields in the primordial or in any other cosmological epoch. The self-gravitating configurations of the scalar field are in GR commonly referred to as boson stars [50,51]. The maximal mass of a boson star formed by the noninteracting scalar field is in GR estimated to be $M_{\rm max} \sim M_{\rm Planck}^2/m$, where m is the scalar field mass, the estimate being based on the assumption that the scalar field is confined within a radius comparable to the Compton wavelength and that it is bound by the uncertainty principle and gravity [51,52]. Such configurations are sometimes called miniboson stars [53]. Boson stars that have masses that are in the astrophysical regime can in GR be obtained only with the introduction of the scalar field self-interaction [54].

In this work we will numerically construct boson stars in $f(T) = T + \alpha T^2/2$ gravity. We will first inspect the global parameters of the solutions, such as the gravitational mass and the particle number, and compare these to the corresponding values in GR. We will also inspect the radial profiles of the energy density and the principal pressures, as these may reveal features that are specific to the $f(T)$ gravity theory. The paper is organized as follows: in Sec. II we introduce the $f(T)$ gravity action and derive the field equations specific to the scalar field in the static spherical symmetry. In Sec. III we discuss the boundary conditions and our numerical procedure. In Sec. IV we compute the gravitational mass and the particle number for boson stars in $f(T)$ gravity with positive and negative values of the parameter α, and we compare these to the GR-case. In Sec. V we investigate a specific feature that develops in the radial profiles of the energy density and the principal pressures as α reaches a critical value in its positive regime. We conclude the paper in Sec. VI. We use natural units $c = 1 = \hbar$ throughout the paper so that $G = 1/M^2_{\mathrm{Planck}}$, and we use the metric signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

II. FIELD EQUATIONS IN STATIC SPHERICAL SYMMETRY IN $f(T)$

The $f(T)$ gravity theory action can be written as

$$S = \int \left(\frac{f(T)}{16\pi G} + \mathcal{L}_{\mathrm{matter}} \right) h \, \mathrm{d}^4 x, \tag{1}$$

where f is in general a nonlinear function of the torsion scalar T, h is the determinant of the tetrad $h^a{}_\mu$, and $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian density due to matter fields. We use latin symbols for the tetrad indices, and greek symbols for the spacetime indices. The tetrad satisfies the metric compatibility condition $h^a{}_\mu h^b{}_\nu g^{\mu\nu} = \eta^{ab}$, where $g^{\mu\nu}$ is the spacetime metric tensor and η^{ab} is the metric of Minkowski. If $f(T) = T$, the variation of the action (1) with respect to the tetrad gives field equations that are equivalent to those of general relativity (GR), and the resulting theory of gravity is known as the teleparallel equivalent of GR (TEGR). If f is nonlinear in T, the field equation resulting from the variation of (1) with respect to the tetrad can be written as

$$h^{-1} h^a{}_\mu \partial_\sigma \left(h \frac{\mathrm{d}f(T)}{\mathrm{d}T} S_a{}^{\nu\sigma} \right) - \frac{\mathrm{d}f(T)}{\mathrm{d}T} T_{\alpha\beta\mu} S^{\alpha\beta\nu} + \frac{1}{2} f(T) \delta_\mu{}^\nu + \frac{\mathrm{d}f(T)}{\mathrm{d}T} S_a{}^{\alpha\nu} h^b{}_\mu \omega^a{}_{b\alpha} = 8\pi G \mathcal{T}_\mu{}^\nu, \tag{2}$$

where

$$\mathcal{T}_a{}^\mu = -\frac{1}{h} \frac{\delta(h\mathcal{L}_{\mathrm{matter}})}{\delta h^a{}_\mu} = -\frac{\partial \mathcal{L}_{\mathrm{matter}}}{\partial h^a{}_\mu} - h_a{}^\mu \mathcal{L}_{\mathrm{matter}} \tag{3}$$

is the standard stress–energy tensor and

$$T^\alpha{}_{\beta\gamma} = h_a{}^\alpha (\partial_\beta h^a{}_\gamma - \partial_\gamma h^a{}_\beta) + h_a{}^\alpha \omega^a{}_{b\beta} h^b{}_\gamma - h_a{}^\alpha \omega^a{}_{b\gamma} h^b{}_\beta \tag{4}$$

is the torsion tensor. The quantity $\omega^a{}_{b\alpha}$ is the inertial spin connection which is, in the covariant formulation of $f(T)$ gravity proposed by Kršš́ák and Saridakis [40], determined from the requirement that the torsion tensor vanishes in the flat space limit of the metric. The tensors,

$$K_{\alpha\beta\gamma} = \frac{1}{2} (T_{\alpha\gamma\beta} + T_{\beta\alpha\gamma} + T_{\gamma\alpha\beta}) \tag{5}$$

and

$$S_{\alpha\beta\gamma} = K_{\beta\gamma\alpha} + g_{\alpha\beta} T_{\sigma\gamma}{}^\sigma - g_{\alpha\gamma} T_{\sigma\beta}{}^\sigma \tag{6}$$

are known as the contorsion tensor and the modified torsion tensor respectively. The torsion scalar,

$$T = T^{\alpha\beta\gamma} S_{\alpha\beta\gamma}, \tag{7}$$

is then defined as the contraction of the torsion tensor with the modified torsion tensor.

In order to study the most simple form of $f(T)$ we chose to work with

$$f(T) = T + \frac{\alpha}{2} T^2, \tag{8}$$

where the parameter α is allowed to have both positive and negative values. In the limit $\alpha \to 0$ the resulting $f(T)$ gravity theory reduces to TEGR, implying the equivalence of the resulting field equations with those of GR. For nonzero α, terms proportional to α appear in the field equation (2), and for sufficiently small values of α, the solutions to the field equations are expected to differ continuously from their GR counterparts. The departure of the solutions from the well-known GR solutions is expected to reveal features of $f(T)$ gravity theory which we aim to study in this paper. The form (8) can also be seen as the lowest two terms in the power expansion of a more general nonlinear $f(T)$ having the correct GR-limit as $\alpha \to 0$.

In order to facilitate the comparison of the solutions obtained within $f(T)$ gravity to those of GR, it is convenient to write the field equations in the form that allows for the "GR-picture interpretation." The field equations can be written as

$$G_\mu{}^\nu = 8\pi G \mathcal{T}_{\mathrm{eff}\,\mu}{}^\nu = 8\pi G (\mathcal{T}_\mu{}^\nu + \tilde{\mathcal{T}}_\mu{}^\nu), \tag{9}$$

where $G_\mu{}^\nu$ on the lhs is the Einstein's tensor of GR, while on the rhs we introduce the effective stress-energy tensor $\mathcal{T}_{\mathrm{eff}\,\mu}{}^\nu$ as the sum of the matter stress-energy tensor (3) and $\tilde{\mathcal{T}}_\mu{}^\nu$ (denoted with the tilde) which consists of terms proportional to α. The tensor $\tilde{\mathcal{T}}_\mu{}^\nu$ can therefore be interpreted as the stress-energy introduced by the nonlinearity of $f(T)$ or the stress-energy of the "$f(T)$-fluid."

As we intend to work in static spherical symmetry, we use spherical coordinates $x^\mu = (t, r, \vartheta, \varphi)$ and write the tetrad field as

$$h^a{}_\mu = \mathrm{diag}(\mathrm{e}^{\Phi(r)}, \mathrm{e}^{\Lambda(r)}, r, r\sin\theta), \tag{10}$$

which through the metric compatibility condition implies the static spherically symmetric metric,

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab} = \mathrm{diag}(\mathrm{e}^{2\Phi(r)}, -\mathrm{e}^{2\Lambda(r)}, -r^2, -r^2\sin^2\vartheta). \tag{11}$$

The flat space limit of the above metric is obtained by letting the metric profile functions $\Phi(r) \to 0$ and $\Lambda(r) \to 0$. The condition that the components of the torsion tensor vanish in the flat space limit allows one to construct the spin connection $\omega^a{}_{b\alpha}$. If a local Lorentz transformation $\Lambda(x)$ which satisfies $\eta_{ab} = \eta_{cd}\Lambda^c{}_a\Lambda^d{}_b$, then the tetrad and the spin connection transform as [43]

$$h'^a{}_\mu = \Lambda^a{}_b h^b{}_\mu, \qquad \omega'^a{}_{b\mu} = \Lambda^a{}_c\omega^c{}_{d\mu}\Lambda_b{}^d + \Lambda^a{}_c\partial_\mu\Lambda_b{}^c. \tag{12}$$

In order to get the flat space spin connection in the corresponding tetrad, one must satisfy the condition,

$$\omega^a{}_{\mu b} = \Lambda^a{}_c\partial_\mu\Lambda_b{}^c. \tag{13}$$

For our tetrad its nonzero components are found to be $\omega^{\hat{r}}{}_{\hat{\vartheta}\vartheta} = -\omega^{\hat{\vartheta}}{}_{\hat{r}\vartheta} = -1$, $\omega^{\hat{r}}{}_{\hat{\varphi}\varphi} = -\omega^{\hat{\varphi}}{}_{\hat{r}\varphi} = -\sin\vartheta$, and $\omega^{\hat{\vartheta}}{}_{\hat{\varphi}\varphi} = -\omega^{\hat{\varphi}}{}_{\hat{\vartheta}\varphi} = -\cos\vartheta$ (coordinate labels are used as indices and the orthonormal ones are denoted with the hat symbol) [55]. The resulting torsion scalar is

$$T = \frac{2\mathrm{e}^{-2\Lambda(r)}(\mathrm{e}^{\Lambda(r)} - 1)(\mathrm{e}^{\Lambda(r)} - 2r\Phi'(r) - 1)}{r^2}, \tag{14}$$

where prime denotes differentiation with respect to r.

The matter Lagrangian for the noninteracting (free) complex scalar field ϕ is

$$\mathcal{L}_{\mathrm{matter}} = \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi^*\partial_\nu\phi + \partial_\mu\phi\partial_\nu\phi^*) - m^2\phi^*\phi, \tag{15}$$

where $\partial_\mu = \partial/\partial x^\mu$ and m is the field mass. According to (3), the stress-energy tensor of the scalar field is

$$\mathcal{T}_\nu{}^\mu = \partial_\nu\phi^*\partial^\mu\phi + \partial_\nu\phi\partial^\mu\phi^* - \delta^\mu_\nu(\partial^\sigma\phi^*\partial_\sigma\phi - m^2\phi^*\phi). \tag{16}$$

For the scalar field we use the standard time-stationary harmonic ansatz compatible with the assumed static spherical symmetry,

$$\phi(r, t) = \phi(r)\mathrm{e}^{-\mathrm{i}\omega t}, \tag{17}$$

where from here on $\phi(r)$ denotes a real profile function depending on the radial coordinate only. The constant ω can be interpreted as the energy of the quantum of the scalar field. The above field ansatz avoids the instability problems given by the Derric's Theorem [56], while for the other methods see e.g., [57,58].

In order to write down the field equations of $f(T)$ gravity theory in the GR-picture (9), we start with the well-known components of the Einstein tensor,

$$G_t{}^t = r^{-2}(1 - \mathrm{e}^{-2\Lambda}(1 - 2r\Lambda')), \tag{18}$$

$$G_r{}^r = r^{-2}(1 - \mathrm{e}^{-2\Lambda}(1 + 2r\Phi')), \tag{19}$$

$$G_\vartheta{}^\vartheta = G_\varphi{}^\varphi = r^{-1}\mathrm{e}^{-2\Lambda}((\Lambda' - \Phi')(1 + r\Phi') - r\Phi''), \tag{20}$$

and proceed to the components of the stress-energy tensor $\tilde{\mathcal{T}}_\mu{}^\nu$ of the $f(T)$-fluid which can be given by

$$\begin{aligned}8\pi G\tilde{\mathcal{T}}_t{}^t = &-\alpha r^{-4}\mathrm{e}^{-4\Lambda}(\mathrm{e}^\Lambda - 1)\\ &\times((\mathrm{e}^\Lambda - 1)((\mathrm{e}^\Lambda - 5)(\mathrm{e}^\Lambda - 1) - 4r^2(2\Phi'' + \Phi'2))\\ &+ 4r\Lambda'(3(\mathrm{e}^\Lambda - 1) + 2(\mathrm{e}^\Lambda - 3)r\Phi'))\end{aligned} \tag{21}$$

$$\begin{aligned}8\pi G\tilde{\mathcal{T}}_r{}^r = &-\alpha r^{-4}\mathrm{e}^{-4\Lambda}(\mathrm{e}^\Lambda - 1)(\mathrm{e}^\Lambda - 2r\Phi' - 1)\\ &\times((\mathrm{e}^\Lambda - 1)(\mathrm{e}^\Lambda + 3) + 2(\mathrm{e}^\Lambda - 3)r\Phi')\end{aligned} \tag{22}$$

$$\begin{aligned}8\pi G\tilde{\mathcal{T}}_\vartheta{}^\vartheta = 8\pi G\tilde{\mathcal{T}}_\varphi{}^\varphi = &\alpha r^{-4}\mathrm{e}^{-4\Lambda}((\mathrm{e}^\Lambda - 1)((\mathrm{e}^\Lambda + 3)(\mathrm{e}^\Lambda - 1)^2\\ &+ 2r(\Phi'(\mathrm{e}^\Lambda - 2\mathrm{e}^{2\Lambda} - 4r^2\Phi'' + 1) - 2r^2\Phi'^3\\ &+ 3(\mathrm{e}^\Lambda - 1)r\Phi'' + 3(\mathrm{e}^\Lambda - 1)r\Phi'^2))\\ &+ 2r\Lambda'(r\Phi'(2(2\mathrm{e}^\Lambda - 3)r\Phi'\\ &-3(\mathrm{e}^\Lambda - 3)(\mathrm{e}^\Lambda - 1)) - 3(\mathrm{e}^\Lambda - 1)^2)).\end{aligned} \tag{23}$$

Finally, the stress-energy components due to the scalar field are given by

$$\mathcal{T}_t{}^t = \mathrm{e}^{-2\Lambda}\phi'^2 + \phi^2(m^2 + \mathrm{e}^{-2\Phi}\omega^2) = \rho, \tag{24}$$

$$\mathcal{T}_r{}^r = -\mathrm{e}^{-2\Lambda}\phi'^2 + \phi^2(m^2 - \mathrm{e}^{-2\Phi}\omega^2) = -p, \tag{25}$$

$$\mathcal{T}_\vartheta{}^\vartheta = \mathcal{T}_\varphi{}^\varphi = \mathrm{e}^{-2\Lambda}\phi'^2 + \phi^2(m^2 - \mathrm{e}^{-2\Phi}\omega^2) = -q, \tag{26}$$

and can be interpreted in terms of the energy density ρ, the radial pressure p and the transverse pressure q of the boson fluid.

The variation of the action (1) with respect to the scalar field gives the field equation $\nabla^\mu \nabla_\mu \phi - m^2\phi = 0$, where ∇^μ is the partial derivative involving the Levi-Civita connection, and which can be written out as

$$r\phi(\omega^2 \mathrm{e}^{-2\Phi} - m^2) + \mathrm{e}^{-2\Lambda}(\phi'(-r\Lambda' + r\Phi' + 2) + r\phi'') = 0. \tag{27}$$

The above differential equation, which has the same structure as in GR, together with the three independent differential equations that follow from (9) and whose parts are given by (18)–(26), complete the set of equations to be satisfied by the tetrad profile functions $\Phi(r)$ and $\Lambda(r)$ and the scalar field profile function $\phi(r)$.

III. BOUNDARY CONDITIONS AND THE NUMERICAL PROCEDURE

In order for the solutions to the field equations derived in the preceding section to represent nonrotating boson stars, they must be global and must satisfy certain boundary conditions. As $r \to \infty$, one expects the energy density and the pressures of the boson fluid to vanish and the spacetime metric to approach that of the static spherically symmetric vacuum of the gravity theory that is being considered. As can be seen from (24)–(26), vanishing of the stress-energy components of the boson fluid implies $\phi \to 0$ and $\phi' \to 0$. The static spherically symmetric vacuum in $f(T)$ gravity with f given by (8) is asymptotically flat [47] so the boundary conditions on the metric profile functions are $\Phi \to 0$ and $\Lambda \to 0$. At $r = 0$, in GR one requires orthonormal components of the Riemann tensor to be finite, which leads to the boundary conditions $\Lambda(0) = 0$, $\Phi'(0) = 0$, and $\Lambda'(0) = 0$. These conditions apply also in the case of $f(T)$ gravity with f given by (8), which can be shown by expanding the field equations (9) in powers of r near $r = 0$. One finds the conditions,

$$\alpha \mathrm{e}^{-4\Lambda(0)}(\mathrm{e}^{\Lambda(0)} - 1)^3(\mathrm{e}^{\Lambda(0)} - 5)r^{-4} + \mathcal{O}(r^{-3}) = 0, \tag{28}$$

$$\alpha \mathrm{e}^{-4\Lambda(0)}(\mathrm{e}^{\Lambda(0)} - 1)^3(\mathrm{e}^{\Lambda(0)} + 3)r^{-4} + \mathcal{O}(r^{-3}) = 0, \tag{29}$$

where the leading terms are due to the nonlinearity of f. The above conditions are satisfied at $r = 0$ if $\Lambda(0) = 0$, which is exactly one of the condition known form GR. Plugging $\Lambda(0) = 0$ into higher-order terms of the same power expansion, as well as into the power expansion of the field equation (27), one finds the conditions,

$$4\Lambda'(0)(1 + 2\alpha\Lambda'(0)(\Lambda'(0) - 2\Phi'(0)))r^{-1} + \mathcal{O}(r^0) = 0, \tag{30}$$

$$(\Lambda'(0) - \Phi'(0))(1 + 2\alpha\Lambda'(0)(\Lambda'(0) - 2\Phi'(0)))r^{-1} + \mathcal{O}(r^0) = 0, \tag{31}$$

$$\phi'(0)r^{-1} + \mathcal{O}(r^0) = 0, \tag{32}$$

that are satisfied if $\Lambda'(0) = 0$, $\Phi'(0) = 0$ and $\phi'(0) = 0$, which are equivalent to the boundary conditions known from GR. Interestingly, another solution to the above conditions exists as a consequence of introducing the T^2 term in (8) and which appears to be $\Phi'(0) = \Lambda'(0)/2 + 1/(4\alpha\Lambda'(0))$. However, we will consider only the first result as it corresponds to the GR case. The problem with the second case is that in the limit $\alpha \to 0$, i.e., in the GR-limit, Φ diverges at $r \to 0$. This case could be interesting in the regime where α is large, but this theory would considerably deviate from GR and would not pass the standard solar system tests. Summarizing the derived boundary conditions, at $r = 0$ we have $\Lambda = 0$ and $\Phi' = \Lambda' = \phi' = 0$ for the derivatives (regularity at the origin), while as $r \to \infty$ we have $\Phi \to 0$, $\Lambda \to 0$, and $\phi \to 0$ (asymptotic flatness).

A quantity of interest in any asymptotically flat solution in GR is its gravitational or ADM mass, M, which can, in a static spherically symmetric spacetime, be interpreted as the mass of the central body (star). In the modified theory, using the GR-picture it can be expressed as

$$M = 4\pi \int_0^\infty r^2 \rho_{\mathrm{eff}} \mathrm{d}r, = 4\pi \int_0^\infty r^2(\tilde{\rho} + \rho)\mathrm{d}r, \tag{33}$$

where the effective energy density ρ_{eff} is the sum of the energy density $\tilde{\rho} = \tilde{\mathcal{T}}_t{}^t$ of the $f(T)$-fluid given by (21) and the energy density ρ of the boson fluid given by (24). Another quantity relevant to boson stars is the particle number, N, which is given by

$$N = \int j^0 h\, \mathrm{d}^3x = 4\pi \int_0^\infty j^0 \mathrm{e}^{\Lambda+\Phi} r^2 \mathrm{d}r, \tag{34}$$

where $h = \det[h^a{}_\mu] = \sqrt{-\det[g_{\mu\nu}]}$, and j^0 is the component of the conserved Noether current implied by the $U(1)$ symmetry, which is given by

$$j^\mu = \mathrm{i}(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) = 2\mathrm{e}^{-2\Phi}\omega\phi^2\delta^\mu_0. \tag{35}$$

The mass of the boson star and the particle number are both expected to be finite.

Due to the complexity of the field equations we proceed to construct the solutions numerically. We introduce dimensionless radial coordinate and variables,

$$\tilde{r} = mr, \qquad \Omega = \frac{\omega}{m}, \qquad \tilde{\alpha} = \alpha m^2,$$
$$\sigma = \sqrt{4\pi G}\phi = \sqrt{4\pi}\phi/M_{\mathrm{Planck}}, \tag{36}$$

and also the rescaled radial coordinate,

$$x = \frac{\tilde{r}}{\tilde{r}+1}, \tag{37}$$

mapping $0 \leq \tilde{r} < \infty$ onto $0 \leq x < 1$, which is more appropriate for the numerical treatment. Out of the four field equations presented in the preceding section involving profile functions Φ, Λ, and ϕ as unknowns, only three must be independent one of another. It was pointed out in [59] that if the antysimmetric part of field equations in $f(T)$ gravity are satisfied (as it is the case in the covariant formulation) then the contracted Bianchi identities are satisfied as well. Therefore, the usual treatment as in GR can be exploited in establishing the number of independent equations needed to solve the system. As the three independent equations that are required to compute three unknown functions, Λ, Φ and ϕ, we chose to work with the ${}^t_t$ and the ${}^\vartheta_\vartheta$-component of (9) and with the field equation (27), as this choice allows simplest extraction of the highest order derivatives Φ'', Λ', and σ''. The ${}^r_r$-component of (9) is used to verify the solutions. The value σ_0 of the profile function σ at $x = 0$ and the value of the parameter $\tilde{\alpha}$ are used to parametrize the solutions, while the *a priori* unknown value of Ω (the rescaled scalar field frequency) acquires the role of the eigenvalue of the boundary value problem. Technically, for the chosen values of σ_0 and $\tilde{\alpha}$ and a trial eigenvalue Ω we use power expansions of the rescaled field equations at $x = 0$ to derive initial data at a point close to $x = 0$ and evolve the equations in x starting from that point. Trial value of Ω is then fine-tuned until boundary conditions as $x \to 1$ are satisfied. This procedure allowed us to construct solutions over a wide range of the parameter space. Pairs of parameters σ_0 and $\tilde{\alpha}$ exist for which we could obtain solutions having different eigenvalues Ω with zero or more nodes in σ. Solutions with one or more nodes in σ have been found in GR long ago [52] and are usually referred to as excited boson stars. They are generally considered to be unstable [60], so in this work we are considering only solutions with no nodes in ϕ.

IV. MASS AND PARTICLE NUMBER

In order to obtain insight into the effects that the T^2 term in (8) has on the structure of boson stars we first generate a family of solutions that has the fixed central value of the rescaled field profile function, $\sigma(0) = \sigma_0 = \sqrt{4\pi} \times 0.1$, while the value of $\tilde{\alpha}$ ranges over positive and negative values. For each solution we obtain the eigenvalue $\Omega = \omega/m$ and compute the corresponding gravitational mass M and the particle number N of the boson star. The results of this computation are shown in Fig. 1. With $\tilde{\alpha} = 0$ we reproduce the well-known GR solution. With $\tilde{\alpha}$ going into the negative regime the mass M and the particle number N are increasing, while the eigenvalue Ω is decreasing. We did not find any indication that for even larger negative values of $\tilde{\alpha}$ than the ones shown in the figure the solutions

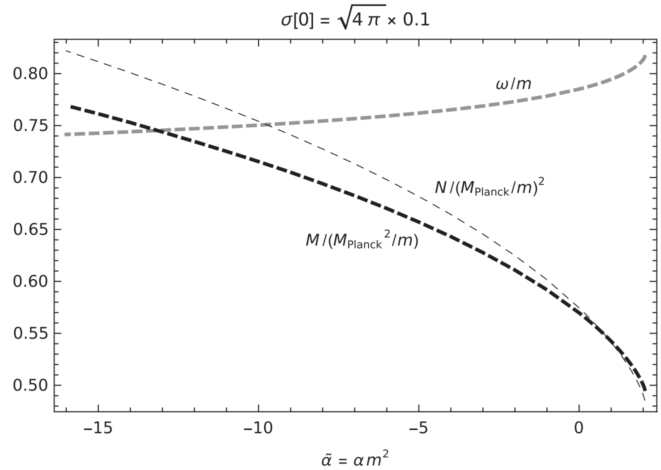

FIG. 1. Gravitational mass M (thick black dashed line, in units of $M^2_{\rm Planck} m^{-1}$), particle number N (thin black dashed line, in units of $M^2_{\rm Planck} m^{-2}$), and energy of scalar field quantum ω (thick gray dashed line, in units of m) in boson stars with field mass m, no field self interaction, central field amplitude $\sigma_0 = \sqrt{4\pi} \times 0.1$, and a range of values of $\tilde{\alpha} = \alpha m^2$.

would cease to exist or that an upper bound on the mass would be reached. With positive values of $\tilde{\alpha}$ less than a critical value, M and N kept decreasing, while ω is increasing. For the fixed value $\sigma_0 = \sqrt{4\pi} \times 0.1$ that we used, the critical value of $\tilde{\alpha}$ is somewhat greater than 2, but since the stability of the numerical solutions becomes doubtful as one approaches the critical $\tilde{\alpha}$, its precise value could not be obtained.

The maximal mass of a stable boson star built from noninteracting (free) scalar field in GR is known to be small, $M_{\rm max} \sim M^2_{\rm Planck}/m$, and such boson stars are sometimes referred to as mini boson stars [53]. However, according to another well-known result [54], the introduction of field self-interaction allows for solutions with masses in the astrophysical range, $M \sim \lambda^{1/2} M_{\rm Ch} = \lambda^{1/2} M^3_{\rm Planck}/m^2_H$, where m_H is the mass of the hydrogen atom. Our result shows that with negative $\tilde{\alpha}$, even without the field self-interaction, the mass and particle number of a boson star can become greater than that of a mini boson star in GR.

Our second family of solutions uses the fixed value $\tilde{\alpha} = -5$ and varies the central value of the rescaled field profile function σ_0. Mass M, particle number N, and the eigenvalue $\Omega = \omega/m$, are shown as functions of σ_0 in Fig. 2 with dashed lines, while the reference GR results ($\tilde{\alpha} = 0$) are shown with solid lines. In GR, boson star mass M and particle number N increase with σ_0 up to the respective maxima, followed by "damped oscillations." With $\tilde{\alpha} = -5$ no maximum in M or N was found which can be considered as qualitatively different behavior. This result is in line with our earlier finding that with constant σ_0 and increasing negative value of $\tilde{\alpha}$ no bound of M or N was found.

Another important feature one can observe in Fig. 2 is that in the case of GR, at a critical value of σ_0 somewhat

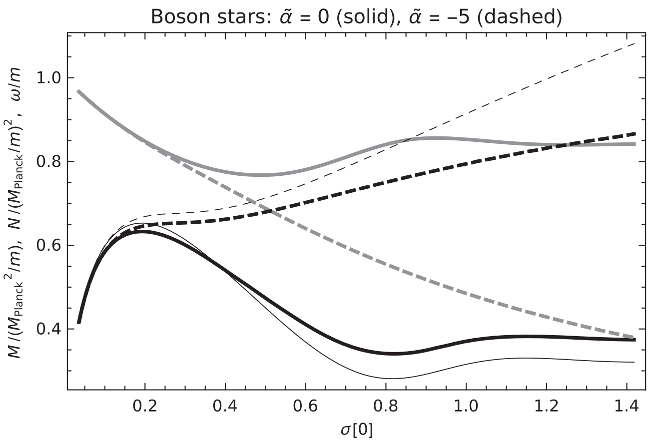

FIG. 2. Gravitational mass M (thick black lines), particle number N (thin black lines), and energy of scalar field quantum ω (thick gray lines) of boson stars with field mass m, no field self interaction, a range of values of central field amplitude $\sigma_0 = \sqrt{4\pi}\phi_0$, and $\tilde{\alpha} = 0$ (solid lines) and $\tilde{\alpha} = -5$ (dashed lines).

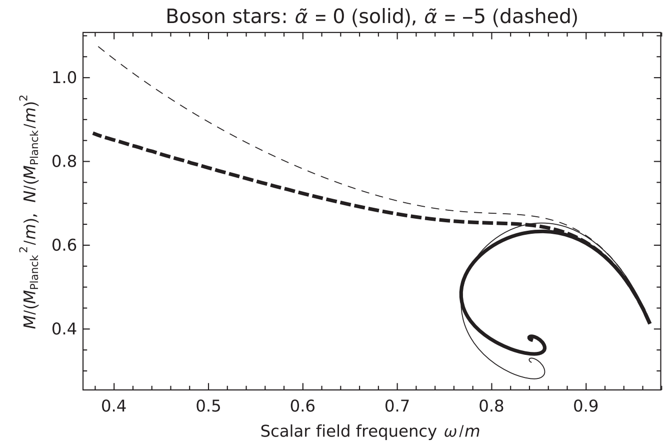

FIG. 3. Gravitational mass M (thick lines) and particle number N (thin lines) shown as functions of energy of scalar field quantum ω in boson stars with field mass m, no field self interaction, and $\tilde{\alpha} = 0$ (solid lines) and $\tilde{\alpha} = -5$ (dashed lines).

greater than the one corresponding to the maxima of M and N, a crossing of the M-curve and the N-curve takes place. This implies that at the critical configuration we have $M = Nm$, m being the mass parameter of the scalar field. At values of σ_0 below the critical value, we have $M < Nm$. If the gravitational mass is understood as a sum of the mass of the particles and the gravitational binding energy, this would imply that gravitational energy is negative which could further be understood as an indication of the stability of the configurations. At values of σ_0 greater than critical we have $M > Nm$, which would imply positive gravitational energy, i.e., that positive work was done in order to bring the infinitely dispersed particles into the given configuration. Such configurations can hardly be imagined as stable. Interestingly, in case of $\tilde{\alpha} = -5$ we have $M < Nm$ for all values of $\tilde{\alpha}$ that we have tested, which can be seen as an indication of stability. We provide further discussion of the issue of stability of static spherically symmetric solutions in $f(T)$ gravity in Sec. VI.

Figure 3 shows the dependence of the mass M and the particle number N of boson stars vs the eigenvalue Ω, which corresponds to the rescaled energy quanta of the scalar field. In GR case the solutions are always bounded within some range of ω where the minimal $\omega_{\min}$ is present (solid lines). With $\tilde{\alpha} = -5$ the energy quanta are becoming arbitrary small, even though the stellar mass is increasing. In the case of positive $\tilde{\alpha}$ this feature is not observed and the theory behaves like GR with the increased gravitational constant.

V. ENERGY DENSITY AND PRESSURE PROFILES

The first family of solutions whose mass M, particle number N, and the eigenvalue $\Omega = \omega/m$, are shown in Fig. 1, revealed that in the positive regime of the parameter $\tilde{\alpha}$ there is a critical value beyond which the solutions could not be obtained. As one approaches the critical value of $\tilde{\alpha}$, M and N approach zero, Ω approaches unity from below (i.e., ω is approaching the scalar field mass parameter m), and the solutions are becoming increasingly more difficult to construct numerically. In order to trace down the cause of this phenomenon we looked into the radial profiles of the components of the stress-energy tensor. In the upper plot of Fig. 4 the radial profile of the effective energy density, as well as the separate contributions due to the boson fluid and due to the $f(T)$-fluid, are shown for a solution with a close-to-critical value of $\tilde{\alpha}$. The energy density of the boson fluid has a smooth outwardly decreasing profile that is similar to the behavior of this quantity in the GR solutions. The energy density due to the $f(T)$-fluid is vanishing at the center, it is outwardly increasing up to a radially thin layer within which it steeply drops to zero, becomes negative, and asymptotically approaches zero (vacuum value) from the negative regime. We will refer to this abrupt feature involving the change of sign of the $f(T)$-fluid energy density as the "phase transition." The effective energy density [the sum of the energy densities of the boson and the $f(T)$-fluid] is everywhere positive and outwardly decreasing, but within the thin layer within which the phase transition of the $f(T)$-fluid takes place it has a an abrupt step.

The lower plot of Fig. 4 shows the radial and the transverse effective pressures as well as the separate contributions to these quantities due to the boson fluid and the $f(T)$-fluid. The radial and the transverse pressure profiles of the $f(T)$-fluid vanish at the center and are outwardly increasing up to the phase transition layer, where the radial pressure profile smoothly becomes outwardly decreasing, while the transverse pressure abruptly changes sign. In the most part of the interior of the boson star the effective pressure appears to be isotropic (up to numerical

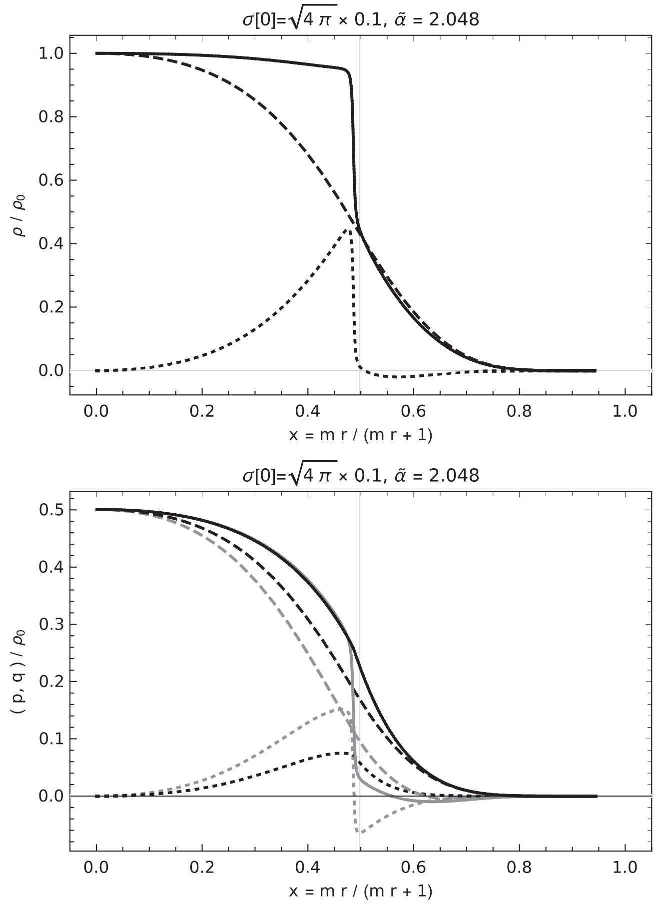

FIG. 4. Radial profiles of stress-energy tensor components for $\sigma_0 = \sqrt{4\pi} \times 0.1$ and $\tilde{\alpha} = 2.048$ (close-to-critical configuration). Upper plot: effective energy density (solid line), energy density due to boson fluid (long dashed line), and energy density due $f(T)$-fluid (short dashed line). Lower plot: effective radial and transverse pressures (black and gray solid lines, respectively), radial and transverse pressures due to boson fluid (black and gray long dashed lines), and radial and transverse pressures due to nonlinear terms in $f(T)$ (black and gray short dashed lines). All quantities are shown relative to value of effective energy density at $r = 0$.

noise), regardless of both fluids having manifestly anisotropic pressures. Outside of the phase transition layer effective pressure is anisotropic, as is the static spherically symmetric vacuum solution in $f(T)$ gravity theory [47].

Attempts to increase the value of $\tilde{\alpha}$ beyond the value used in Fig. 4 made the steep portions of the radial profiles of the energy density and the transverse pressure of the $f(T)$-fluid even steeper. It is therefore reasonable to expect that the critical value of $\tilde{\alpha}$ corresponds to a discontinuity in these profiles. It is also obvious that such solutions can not be obtained by the numerical procedure we use. We should also note that qualitatively similar phase transition in the $f(T)$-fluid was found in [48], where compact stars composed of the polytropic fluid were considered. It is therefore likely that with positive close-to-critical values of $\tilde{\alpha}$ the phase transition described above is a genuine feature of static spherically symmetric solutions in the $f(T)$ gravity theory with f given by (8).

Another interesting observation is that if M is the mass of the close-to-critical (in the sense of $\tilde{\alpha}$ being as large as technically possible) configuration of the boson star in $f(T)$, the approximate value of the radial coordinate at which the phase transition takes place coincides with the value at which in GR the horizon of the Schwarzschild black hole of mass M would form. However, at this point we have no analytical arguments supporting that above assertion.

VI. CONCLUSION

Recently, many modified theories of gravity have arisen from the theoretical and experimental evidence that GR may be incomplete. In the desire to provide the fundamental description of the nature of spacetime, which is the goal of any theory of gravity, new views have been opened, in particular the concept of torsion. Thus, the modified theories based on torsion deserve attention as their curvature based counterparts do. However, a modified theory of gravity that is to be taken seriously must explain the complete spectrum of physical phenomena, ranging from cosmological singularities, CMB, inflation, to high energy physics, gravitational waves, black holes, compact objects, etc. In this work we have explored the static spherically symmetric self-gravitating configurations of the simplest bosonic matter—the noninteracting complex scalar field—within the framework of $f(T) = T + \alpha T^2/2$ torsion based modified theory of gravity. Within GR, such solutions are known as boson stars. In particular, when the noninteracting scalar field is used, the mass of the star is bounded from above by $M_{\mathrm{max}} \sim M_{\mathrm{Planck}}^2/m$, and such stars are referred to as the mini boson stars, while only with the introduction of the scalar field self-interaction higher stellar masses can be obtained.

We have derived the field equations and the boundary conditions relevant for the boson stars in $f(T)$ gravity, and we have numerically constructed the solutions over a wide range of values of the parameters α and the central field amplitude ϕ_0. We have found that if α is negative and sufficiently large, the mass of the boson is increasing with the central field amplitude ϕ_0, i.e., that it is no longer bounded as in GR. This implies that within $f(T)$ with negative α boson stars formed of noninteracting scalar field are not necessarily mini boson stars, but may acquire masses in the astrophysical range. The finding that the mass is no longer bounded might also be relevant for the discussion of the dynamical stability of boson stars in $f(T)$, since in GR the solutions with the central field amplitude greater than the value corresponding to the maximal mass were in some works found to be unstable [60–63]. Another finding that might be relevant for the discussion of the stability of boson stars in $f(T)$ gravity is that with sufficiently large negative α the binding energy of the boson star, defined as the difference between its gravitational mass M and the particle number N multiplied by the field mass m, is negative for all values of ϕ_0 we have tested. This implies that (positive) work must be applied in order to disperse the boson fluid into individual particles with negligible gravitational interaction, which indicates stability. In GR, the binding energy of boson stars is negative only for ϕ_0 less than a critical value, which is interpreted as onset of dynamical instability [63]. An analysis that could provide conclusive answers to the problem of stability of boson stars within $f(T)$ gravity would require perturbation of the time-dependent field equations around the static solutions we have computed. However, this endeavor lies outside of the scope of the present work, and we leave it for a future project. With α in the positive regime, we have found that the solutions can be obtained only up to a critical value of α. In an attempt to trace down the cause of this phenomenon we have adopted the "GR-picture" where the features of the $f(T)$ gravity theory are viewed as the presence of the stress energy tensor of a quantity that we refer to as the "$f(T)$-fluid," which together with the stress energy tensor of the scalar field constitutes effective stress energy on the rhs of the Einstein equation. As we approached the critical positive value of α, we could observe the development of abrupt sign changes in the energy density and the transverse pressure of the $f(T)$-fluid, eventually becoming steplike.

The most important findings of this work—existence of stable boson stars in $f(T) = T + \alpha T^2/2$ gravity with a negative α with masses greater than those achievable in GR, and the occurrence of the phase transition in the pressure of the $f(T)$-fluid with a positive α—agree with the findings of our earlier work [48] where compact objects were modeled using the perfect fluid governed by the polytropic equation of state. The most important difference between the compact objects considered there and the boson stars considered here is that compact objects have a sharply defined surface radius R beyond which the energy density and the pressure of the polytropic fluid vanishes, while in the case of boson stars the boson fluid takes up all space and the surface radius is not a meaningful quantity. At the surface of the compact objects, the standard conditions for joining the exterior vacuum metric are satisfied, but as the spherically symmetric vacuum solution for $f(T)$ gravity is at present not available in closed form, we could not reliably access the gravitational mass M of the compact object. Therefore, rather than computing the binding energy to access the stability properties like we did in this work with boson stars, in [48] we inspected the relation between the particle number and the surface radius, which mimics the well-known mass-to-radius method used in GR. We find it remarkable, however, that the two different matter models and two different approaches to stability yielded the same conclusion, namely, that negative α in $f(T) = T + \alpha T^2/2$ has a stabilizing effect on the static spherically symmetric self-gravitating structures (stars).

We can conclude that the structure of boson stars formed of noninteracting complex scalar field minimally coupled to $f(T) = T + \alpha T^2/2$ gravity is vastly different from its GR counterpart. We believe that further research that could include perturbative analysis of spherically symmetric time-dependent field equations (stellar pulsations) or axially symmetric solutions (rotating boson stars) could reveal further interesting features specific to torsion based $f(T)$ gravity.

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