Fluid pulsation modes from strange stars in a higher-dimensional spacetime

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In this work, we make the first step to derive nonradial pulsation equations in extra dimensions and investigate how the f- and p_1-mode frequencies of strange quark stars, within the Cowling approximation, change with the number of dimensions. In this regard, the study is performed by solving numerically the nonradial pulsation equations, adjusted for a d-dimensional spacetime ($d \geq 4$). We connect the interior to a Schwarzschild-Tangherlini exterior metric and analyze the f- and p_1- mode frequencies. We found that the frequencies could become higher than those found in four-dimensional spacetime. The f-mode frequency is essentially constant and only for large gravitational radius values grows monotonically and fast with the gravitational radius. In a gravitational radius range, where f-mode frequencies are constant, they increase for spacetime dimensions $4 \leq d \leq 6$ and decrease for $d \geq 7$. Regarding p_1-mode frequencies they are always larger for higher dimensions and decay monotonically with the increase of the gravitational radius. In extra dimensions, as it happens for four-dimensional spacetime, we found p_1-mode frequencies are always larger than the f-modes ones. In the Newtonian gravity, for a homogeneous star in d dimensions, we observe that the f-mode eigenfrequencies are constant and given by the relation $\omega^2 = l M G_d / R^{d-1}$, where l represents the spherical harmonic index, $M G_d$ is the total star mass and R the stellar radius. For some gravitational radius interval, we show that a homogeneous star in Newtonian gravity is a good approximation to investigate the f-mode frequency of strange stars in the relativistic frame. In each dimension considered, we find that the f-mode frequencies of strange stars are essentially constant since they depend on the average star energy density that is almost constant as a function of the total star mass. Moreover, for a fixed energy density, we also find that the f-mode frequency changes with the volume of the unitary sphere in the $d - 1$ dimension, which attains its maximum value at $d = 6$. In neutron stars in four dimensions, where the average energy density of the star increases with the central energy density, the f-mode frequencies will increase with the star mass. Thus, the possibility to measure in gravitational wave detectors the f-mode oscillation frequency coming from compact stars with different pulsar masses and observe almost constant frequency values, for $d = 4$, in the range $f \sim 2$–3 [kHz] with $M \leq 1.8\ M_\odot$, it would be a good sign of the existence of strange quark stars that still lack an astronomical confirmation. Finally, if the f-mode frequencies are still constant and different from the range values of $d = 4$ for larger total masses, it would be evidence that quarks can propagate in extra spacetime dimensions and strange quark stars in d dimension could exist.

I. INTRODUCTION

The observation carried out by the LIGO and Virgo collaborations, the gravitational waves (GWs) from a binary black hole merger [1–3], turns out to be consistent with that foretold from general relativity theory. Shortly after this measurement, GWs coming from a merging

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neutron star pair were also detected [4]. The latter event, together with its electromagnetic counterpart [5], has paved a new route on fundamental physics to explore the astrophysical observation.

A few years after Einstein's theory of gravity was established and influenced by the idea of a generalized theory of gravitation—which, in principle, would unify gravitational and the electromagnetic force—Kaluza [6] and Klein [7] argued that extra dimensions could be the route toward a unified theory of gravitation. Many decades after this proposal and motivated by the idea that GWs

could assist to prove the existence of extra dimensions, there has been a great deal of interest in the physics community for studying various astrophysical phenomena in extra dimensions.

Within this gravity theory context, aiming to understand the extra-dimensional spacetime effect of some phenomena on compact objects research, several theoretical investigations have been carried out. For instance, the implications of spacetime dimension on static equilibrium configurations [8–10], radial stability [11], compactness [12–14] and gravitational collapse [15–18] of compact stars have been widely investigated.

In particular, Refs. [8,9] investigated the equilibrium configurations in higher-dimensional spacetime, by solving the stellar structure equations. Thus, considering a hypersphere composed by an incompressible fluid, it has been found that the extra dimensions increase the total mass of compact objects. In this regard, in Ref. [10] a comparative analysis of boson and Dirac stars in a d-dimensional spacetime has been investigated. There, the authors found that the solutions of gravitating matter systems depend on the number of dimensions.

Inspired in the aforementioned works, within the radial perturbation approach, Ref. [11] investigated the stability of compact fluid spheres in an extra-dimensional spacetime. Considering a hypersphere composed of a perfect fluid that follows the MIT bag model equation of state (EOS), it has been shown that extra dimension affects the radial stability of these objects. For a range of central energy densities and total masses, it is shown that the radial stability of strange stars increases with the dimension. Strange quark stars are a possible structure for compact stars but quite different from neutron stars since they are essentially self-bound stars, can have smaller radii, and for bare quark stars, they present a sharp surface where the energy density is not zero. In the MIT quark model, this energy density at the star surface is directly related to the bag constant (the deconfinement energy). These compact objects and their radial stability have been studied [19,20], and also anisotropic pressures and charge effects were investigated [21–23]. As mentioned in [11], the stability of these stars in a higher-dimensional spacetime could be associated with the fact that the deconfinement density energy is present. These all stable equilibrium solutions are far from reaching the Buchdahl limit [24] for a d-dimensional spacetime [12–14]. Thus, it is important to investigate the fluid pulsation modes of these d-dimensional strange stars, and see how their frequencies change when the spacetime dimensions increase.

The Buchdahl bound for the d-dimensional case states that the radius-mass ratio, i.e., the reason between R and $MG_d/(d-3)$, of a compact object follows the inequality $(d-3)R^{d-3}/MG_d \geq (d-1)^2/2(d-2)$ [12–14]. Certainly, a compact object that violates this bound would result in a gravitational collapse. The end outcome of a gravitational

collapse depends on the initial conditions imposed, its final stage could be either a naked singularity or a black hole [15–18]. It is worth mentioning that the influence of extra dimension on some black hole properties have been also investigated; see, e.g., [25–32].

Although the influence of the extra dimensions was addressed in the aforementioned contexts, it is of interest to see, within the Cowling approximation, how these could influence the fluid pulsation modes from stable strange stars in higher-dimensional spacetime (review [11]). In our knowledge, it is the first time that the nonradial pulsation equation in d spacetime dimensions is obtained, in the Cowling approximation, and solved numerically. For this purpose, we numerically integrate the hydrostatic equilibrium equations [33,34] and the nonradial pulsation equations [35,36], properly modified to include the extra dimensions.

We regard this paper as follows: The general relativistic formulation in higher-dimensional spacetime is shown in Sec. II; moreover, the steps to follow to obtain both the stellar structure and the nonradial oscillation equations and the equation of state are displayed. The influence of the spacetime dimension on the f- and p_1-mode frequencies (f and p_1 mode for short) are presented in Sec. III. Additionally, in this section, there is also presented the oscillation spectrum in the Newtonian limit. We conclude and make some final remarks in Sec. IV. The units in which $c = 1 = G_4$, with c and G_4 being respectively the speed of light and the four-dimensional gravitational constant, is adopted through the paper.

II. GENERAL RELATIVISTIC FORMULATION IN HIGH DIMENSIONS

A. Field equation

The nonradial oscillations of neutron stars in higher dimensions within the Cowling approximation are investigated in the framework of general relativity. For a framework in high spacetime dimensions, $d \geq 4$, the Einstein field equation can be represented through the equality [37]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{d-2}{d-3} S_{d-2} G_d T_{\mu\nu}. \qquad (1)$$

The left side of Eq. (1) contains the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the metric tensor $g_{\mu\nu}$. In turn, the right-hand side bears the universal constant G_d, in $d = 4$ it represents to the Newton's gravitational constant, and the area of unitary hypersphere $S_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$, with Γ being the usual gamma function and the factor $(d-2)S_{d-2}/(d-3)$ corresponds to the 8π term in four dimensions. Moreover, $T_{\mu\nu}$ depicts the perfect fluid energy-momentum tensor given by

$$T_{\mu\nu} = (\rho_d + p_d)U_\mu U_\nu + p_d g_{\mu\nu}, \qquad (2)$$

where ρ_d and p_d are respectively the energy density and the fluid pressure and U_μ is the velocity of the fluid in a d-dimensional spacetime, where $U_\mu U^\mu = -1$.

All aforementioned Greek indices previously defined μ, ν, etc., run from 0 to $d-1$, with 0 representing the time, and the $d-1$ the spacelike coordinates.

B. Static structure equations for a higher-dimensional spacetime

To describe the static perfect fluid hypersphere, the higher-dimensional spacetime is assumed to be of the form [11]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2 \sum_{i=1}^{d-2}\left(\prod_{j=1}^{i-1}\sin^2\theta_j\right)d\theta_i^2, \quad (3)$$

where the functions $\nu = \nu(r)$ and $\lambda = \lambda(r)$ depend on the radial coordinate r only.

To analyze the stellar equilibrium configurations of compact objects, the stellar structure equations in d dimensions must be resolved. These equations could be expressed through the equalities,

$$\frac{dm}{dr} = S_{d-2}\rho_d r^{d-2}, \quad (4)$$

$$\frac{dp_d}{dr} = -(p_d + \rho_d)G_d\left[\frac{S_{d-2}p_d r}{(d-3)} + \frac{m}{r^{d-2}}\right]e^{2\lambda}, \quad (5)$$

$$\frac{d\nu}{dr} = -\frac{1}{(p_d + \rho_d)}\frac{dp_d}{dr}, \quad (6)$$

where the metric function $e^{2\lambda}$ takes the form

$$e^{2\lambda} = \left(1 - \frac{2mG_d}{(d-3)r^{d-3}}\right)^{-1}. \quad (7)$$

Function m denotes the gravitational mass within the hypersphere of radius r. Equation (5) is the Tolman-Oppeheimer-Volkoff modified from its original form [33,34] to introduce the effects of the extra dimensions [11].

The higher-dimensional static structure equations are integrated along the radial coordinate, from the center to the surface of the object. In the center of the hypersphere, we adopt the conditions

$$m(0) = 0, \qquad \lambda(0) = 0, \qquad \nu(0) = \nu_c,$$
$$p_d(0)G_d = p_{cd}G_d \quad \text{and} \quad \rho_d(0)G_d = \rho_{cd}G_d. \quad (8)$$

The surface of the object $r = R$ is reached when

$$p_d(r = R)G_d = 0. \quad (9)$$

The parameters p_cG_d and ρ_cG_d stand for the central pressure and the central energy density, respectively.

It is important to mention that, at the object surface, the interior line element matches to the Schwarzschild-Tangherlini vacuum exterior solution [38,39]:

$$e^{2\nu(R)} = \frac{1}{e^{2\lambda(R)}} = 1 - \frac{2MG_d}{(d-3)r^{d-3}}, \quad (10)$$

where $MG_d/(d-3)$ represents the total mass of the object.

C. Nonradial oscillations in the Cowling approximation

The Cowling approximation in a four-dimensional spacetime [35,36] is employed frequently in literature on compact stars oscillations, e.g., Refs. [40,41]. This approximation helps to simplify equations and, by having a relatively small influence on the solutions, the results obtained are qualitatively correct. Indeed, the Cowling approximation shows a discrepancy of less than 20% and 10% to those obtained respectively by a relativistic numerical approach for f and p_1 modes [42]. This validates its employment to investigate, for instance, the implication of the rotation rate [43,44], crust elasticity [45] and the internal anisotropy [46] in the fluid pulsation modes from compact objects.

In this subsection, the f and p_1 modes from strange stars within the Cowling approximation in a d-dimensional spacetime are investigated. For such purpose, we keep the metric functions fixed, namely, we consider that the metric perturbation is zero $\delta g_{\mu\nu} = 0$ [41]. The fluid pulsation equations can be derived by considering a variation on the Bianchi identity, $\delta(\nabla_\mu T^{\mu\nu}) = 0$. Keeping the metric functions fixed, we get

$$\nabla_\mu(\delta T^{\mu\nu}) = 0, \quad (11)$$

with

$$\delta T^{\mu\nu} = (\delta p_d + \delta\rho_d)U^\mu U^\nu + (p_d + \rho_d)(\delta U^\mu)U^\nu$$
$$+ (p_d + \rho_d)U^\mu(\delta U^\nu) + \delta p_d g^{\mu\nu}. \quad (12)$$

To investigate what Eq. (11) means, it is necessary to project it orthogonal and along to the d-velocity U^μ.

To project Eq. (11) orthogonally to the d-velocity U^μ, it is multiplied by the orthogonal tensor $P_\nu^\sigma = \delta_\nu^\sigma + U^\sigma U_\nu$. After some algebra, we derive

$$(\delta p_d + \delta\rho_d)U^\mu\nabla_\mu U^\sigma + \nabla^\sigma(\delta p_d) + U^\sigma U^\mu\nabla_\mu(\delta p_d)$$
$$+ (p_d + \rho_d)U^\mu(\nabla_\mu(\delta U^\sigma) - \nabla^\sigma(\delta U_\mu)) = 0. \quad (13)$$

On the other hand, the projection of Eq. (11) along the d-velocity field U_ν yields

$$\nabla_\mu[(p_d + \rho_d)\delta U^\mu] + (p_d + \rho_d)U^\mu\nabla_\mu U_\nu(\delta U^\nu)$$
$$+ U^\mu\nabla_\mu(\delta\rho_d) = 0. \tag{14}$$

For the higher-dimensional spacetime, it is convenient to consider the Lagrangian fluid displacement vector components $\varsigma = (\varsigma^r, \varsigma^{\theta_1}, \ldots, \varsigma^{\theta_{d-2}})$ of the form

$$\varsigma^r = \frac{e^{-\lambda}}{r^{d-2}}\tilde{Q}Y_l^m, \tag{15}$$

$$\varsigma^{\theta_i} = -\frac{\tilde{Z}}{r^2}\left(\prod_{j=1}^{i-1}\frac{1}{\sin^2\theta_j}\right)\frac{\partial Y_l^m}{\partial\theta_i}, \tag{16}$$

where i goes from 1 to $d - 2$. In Eqs. (15) and (16), the functions $\tilde{Q} = \tilde{Q}(t,r)$ and $\tilde{Z} = \tilde{Z}(t,r)$ depend on both the temporal t and radial coordinate r, and $Y_l^m = Y_l^m(\theta_1, \ldots, \theta_{d-2})$ represents the spherical harmonics for a higher-dimensional spacetime. Then, the perturbations of the velocity of the fluid in a d-dimensional spacetime, $\delta U^\mu = (0, \delta U^r, \delta U^{\theta_1}, \ldots, \delta U^{\theta_{d-2}})$, can be placed as

$$\delta U^r = \frac{d\varsigma^r}{d\tau} = e^{-\nu}\frac{d\varsigma^r}{dt}, \tag{17}$$

$$\delta U^{\theta_i} = \frac{d\varsigma^{\theta_i}}{d\tau} = e^{-\nu}\frac{d\varsigma^{\theta_i}}{dt}. \tag{18}$$

It is important to say that Eqs. (15) and (16) are reduced to the form used in [41] considering $d = 4$.

Taking into account that $U^\sigma = (e^{-\nu}, 0, \ldots, 0)$, the explicit form of Eq. (13) for $\sigma = r, \theta_1$ is, respectively,

$$(p_d + \rho_d)e^{2(\lambda-\nu)}\frac{d^2\varsigma^r}{dt^2} + (\delta\rho_d + \delta p_d)\frac{d\nu}{dr} = -\frac{\partial\delta p_d}{\partial r}, \tag{19}$$

$$(p_d + \rho_d)e^{-2\nu}r^2\frac{d^2\varsigma^{\theta_1}}{dt^2} = -\frac{\partial\delta p_d}{\partial\theta_1}. \tag{20}$$

From Eqs. (14) and (A7), after some algebra, it is found that $\delta\rho_d$ is given by the expression

$$\frac{\delta\rho_d}{p_d + \rho_d} = -\frac{\tilde{Q}}{p_d + \rho_d}\frac{d\rho_d}{dr}\frac{e^{-\lambda}}{r^{d-2}}Y_l^m$$
$$-\left(\frac{e^{-\lambda}}{r^{d-2}}\frac{\partial\tilde{Q}}{\partial r}Y_l^m + \frac{\tilde{Z}}{r^2}l(l+d-3)Y_l^m\right). \tag{21}$$

Considering that the fluid pressure depends on the energy density $p_d = p_d(\rho_d)$, we have

$$\frac{\delta p_d}{p_d + \rho_d} = -\frac{\tilde{Q}}{p_d + \rho_d}\frac{dp_d}{dr}\frac{e^{-\lambda}}{r^{d-2}}Y_l^m$$
$$-\frac{dp_d}{d\rho_d}\left(\frac{e^{-\lambda}}{r^{d-2}}\frac{\partial\tilde{Q}}{\partial r}Y_l^m + \frac{\tilde{Z}}{r^2}l(l+d-3)Y_l^m\right). \tag{22}$$

Since within the Cowling approximation the metric perturbation is neglected, the density perturbation sets to zero $\delta\rho_d = 0$, however, the pressure perturbation δp_d is not set to zero. Qualitative correct results can be derived with this approach, see, e.g., [35]. Using the equalities (21) and (22) with $\delta\rho_d = 0$, Eqs. (19) and (20) can be written in the form

$$\frac{(p_d + \rho_d)}{r^{d-2}}e^{\lambda-2\nu}\frac{\partial^2\tilde{Q}}{\partial t^2} - \left(\frac{dp_d}{dr} + \frac{d\rho_d}{dr}\right)\frac{d\nu}{dr}\frac{\tilde{Q}e^{-\lambda}}{r^{d-2}}$$

$$-\frac{\partial}{\partial r}\left[p_d\Gamma_1\left(\frac{e^{-\lambda}}{r^{d-2}}\frac{\partial\tilde{Q}}{\partial r} + \frac{\tilde{Z}}{r^2}l(l+d-3)\right) + \frac{\tilde{Q}e^{-\lambda}}{r^{d-2}}\frac{dp_d}{dr}\right]$$

$$+\frac{dp_d}{dr}\left(1 + \frac{dp_d}{d\rho_d}\right)\left(\frac{e^{-\lambda}}{r^{d-2}}\frac{\partial\tilde{Q}}{\partial r} + \frac{\tilde{Z}}{r^2}l(l+d-3)\right) = 0, \tag{23}$$

$$(p_d + \rho_d)e^{-2\nu}\frac{\partial^2\tilde{Z}}{\partial t^2} + p_d\Gamma_1\left[\frac{e^{-\lambda}}{r^{d-2}}\frac{\partial\tilde{Q}}{\partial r} + \frac{\tilde{Z}}{r^2}l(l+d-3)\right]$$

$$+\frac{e^{-\lambda}\tilde{Q}}{r^{d-2}}\frac{dp_d}{dr} = 0, \tag{24}$$

with $\Gamma_1 = \frac{p_d + \rho_d}{p_d}\frac{dp_d}{d\rho_d}$ being the adiabatic index.

Assuming a harmonic dependence on time of the perturbative variables of the form $\tilde{Q}(t,r) = Q(r)e^{i\omega t}$ and $\tilde{Z}(t,r) = Z(r)e^{i\omega t}$, with ω being the eigenfrequency, Eqs. (23) and (24) can be placed in a form more appropriate for the numerical calculation. Considering $d[\text{Eq.}(24)]/dr$-[Eq. (23)] in Eq. (24), we get

$$\frac{dZ}{dr} = 2Z\frac{d\nu}{dr} - \frac{e^\lambda Q}{r^{d-2}}. \tag{25}$$

On the other hand, from Eqs. (24) and (25), we obtain

$$\frac{dQ}{dr} = \frac{d\rho_d}{dp_d}\left[\omega^2 r^{d-2}e^{\lambda-2\nu}Z + \frac{d\nu}{dr}Q\right] - l(l+d-3)e^\lambda Z r^{d-4}. \tag{26}$$

It is the first time that these equations are obtained, and to determine the oscillation spectrum of compact stars in a spacetime in high dimensions in the Cowling approximation, it is necessary to solve Eqs. (25) and (26). These two differential equations are reduced to those derived in [41] for a four-dimensional spacetime.

To integrate Eqs. (25) and (26) from the center ($r = 0$) to the surface of the hypersphere ($r = R$), the boundary conditions have to be defined. In order to find a regular solution in the center, in a similar way as realized in [41,46], we consider that the functions Q and Z take the form

$$Q = Cr^{l+d-3}, \qquad Z = -C\frac{r^l}{l}, \tag{27}$$

where C represents a dimensionless constant. In turn, at the surface of the object is found

$$\left[\omega^2 e^{\lambda-2\nu}Z r^{d-2} + \frac{d\nu}{dr}Q\right]_{r=R} = 0. \tag{28}$$

D. Equation of state

For the fluid contained in the compact object, we consider that both the d-dimensional energy density ρ_d and pressure p_d are related by a linear equation of state of form [11]

$$p_d = \frac{(\rho_d - d\mathcal{B}_d)}{(d-1)}, \qquad (29)$$

where $\mathcal{B}_d$ is a constant. In four-dimensional spacetime, Eq. (29) displays the MIT bag model EOS. It is well known that this EOS represents a fluid composed of up, down, and strange quarks. In [47], Witten has proposed that the strange quark matter might be the true fundamental state of strongly interacting matter. This conjecture is corroborated by Farhi and Jaffe [48] taking into account massless and noninteracting quarks.

Due to the volume, the constant $\mathcal{B}_d$ and the functions ρ_d and p_d are dimension dependent units. With the aim to have these units independent of the spacetime dimension, those are used of the following form $\mathcal{B}_d G_d$, $\rho_d G_d$, and $p_d G_d$. In such a form, the units of the aforementioned variables are $[\mathrm{MeV/fm^3}]$.

In this article, following [11], we consider $d\mathcal{B}_d G_d = 240\ [\mathrm{MeV/fm^3}]$. It means that, for $d = 4$, the bag constant is $\mathcal{B}_4 = 60\ [\mathrm{MeV/fm^3}]$. It is important to say that the bag constant considered is within the hadronic mass spectroscopy interval [49], $60 \leq \mathcal{B}_4 \leq 90\ [\mathrm{MeV/fm^3}]$.

III. RESULTS

A. Numerical method

To analyze the extra dimensions' influence in the oscillation spectrum of compact stars, the stellar structure equations, and the nonradial oscillation equations are numerically solved from the center toward the surface of the compact object.

Once defined the EOS, Eqs. (4)–(6) are solved for different dimensions d and central energy densities $\rho_{cd} G_d$. The numerical solutions begin by integrating Eqs. (4) and (5) from the center to the surface of the compact object using the fourth order Runge-Kutta method, for a value of d and $\rho_{cd} G_d$.

After finding the parameters $p_d G_d$, $\rho_d G_d$, mG_d and λ, we solve Eq. (6) and then the nonradial pulsation equations (25) and (26) are integrated. The whole system of equations is solved by using the shooting method as described:

(i) In the case of Eq. (6), the method starts taking into account a test value of ν_c. If after the integration the condition (10) is not attained, the process is repeated until finding a ν_c that satisfies that condition.

(ii) The nonradial pulsation equations (25) and (26) are numerically integrated. It begins considering the correct coefficient of ν_c found, in the solution of the stellar structure equations, for a certain value of
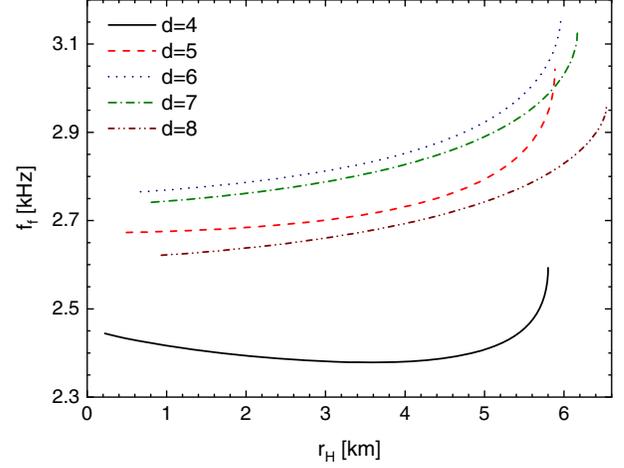

FIG. 1. The f-mode frequency versus the gravitational radius, for some spacetime dimensions.

$\rho_{cd} G_d$ and d, $l = 2$ and a trial value of ω^2. If, at the final of the numerical solution, the condition (28) is not fulfilled, ω^2 is corrected until satisfied in the next integration.

The numerical method implemented in this work reproduces the results of Ref. [46], for the study of the f and p_1 modes of a neutron star in the perfect fluid case.

B. Oscillation spectrum of relativistic strange stars in a d-dimensional spacetime

In Fig. 1 is shown the behavior of the f-mode frequencies as a function of the gravitational radius (see, e.g., [29]),

$$r_H = \left(\frac{2MG_d}{d-3}\right)^{\frac{1}{d-3}}\ [\mathrm{km}] \qquad (30)$$

for some different spacetime dimensions. In all cases presented, we only consider stable compact objects against radial perturbations, see [11]. Note in the curves that the f mode are almost constant [50,51] and only show a fast increase for higher values of the gravitational radius. For $d = 4$, we can see that the frequency shows a tiny small decrease with the gravitational radius until it reaches its minimum value. From this point, the frequency increases with r_H. In turn, for $d > 4$, the f mode is essentially constant and for large gravitational radius values grows monotonically with r_H.

In Fig. 1 is also shown the influence of extra dimensions in the f mode. For a gravitational radius range, we see that the f mode increases in the dimensions $4 \leq d \leq 6$ and decreases in spacetime dimensions $d \geq 7$ (see also top panel of Fig. 3). Moreover, it is important to say that, for all spacetime dimensions considered, we note that the f-mode frequencies are in the 2.38–3.18 [kHz] range. These limit values of f-mode frequencies are related respectively to the
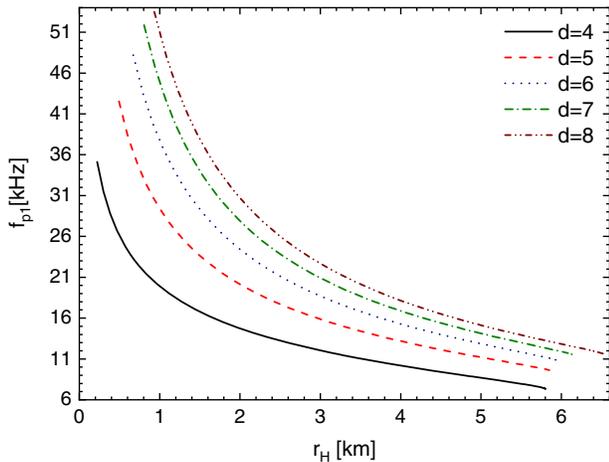
FIG. 2. The p_1-mode frequency as a function the gravitational radius, for four spacetime dimensions.

lowest and highest value found in the dimensions $d = 4$ and $d = 6$.

The dependence of the p_1-mode frequencies with the gravitational radius for four different spacetime dimensions is plotted in Fig. 2. Such as it happens in four-dimensional spacetime, in higher dimensions we found that the p_1-mode frequencies are larger than the f modes (check Ref. [52]). Further note that the p_1 mode decays monotonically with r_H, thus attaining the lowest frequencies of oscillation in the maximum gravitational radius points. On the other hand, for a gravitational radius interval, the p_1-mode frequencies are also affected by the increment of the spacetime dimension. For higher dimensions larger p_1 modes are derived.

The change of the f- and p_1-mode frequencies with the spacetime dimensions have respectively schemed on the top and bottom of Fig. 3. In the figure, there are considered five different gravitational radii, as a function of spacetime dimensions. The results obtained, marked by circles in the figure (review also Table I), are connected by a third and second-order polynomial fitting curvature correlating f- and p_1-mode frequencies with the spacetime dimensions, respectively.

In Fig. 3, in the upper panel, for the range of r_H chosen, the fitting curves that correlate the f mode with d are close to each other. This proximity in the curves is related to the interval chosen for the gravitational radii analyzed, where, in each dimension, the f-mode frequency values are almost constant and close to each other (see Fig. 1). For the mentioned gravitational radius range, we note that there is a universality among these curves. These can be represented by a "universal" curve that connects all points (review Table II). As can be noted in the figure, the f-mode frequency grows with the spacetime dimensions until attaining its maximum value is close to $d = 6$, after this point, the f mode decreases with the dimension. In the bottom panel, in all total gravitational radii considered, we
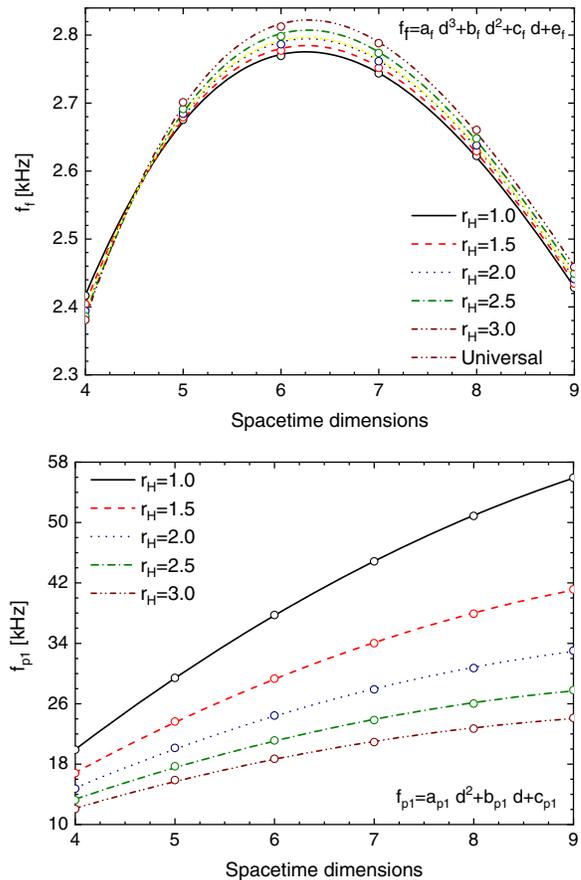

FIG. 3. The f- and p_1-mode frequencies as a function of the spacetime dimensions are plotted on the top and bottom panel, respectively. Circles represent the frequency values calculated with the shooting method. The curves and lines are resulting from the polynomials fitting of third and second order. In both cases are considered five different gravitational radii.

obtain that the curves derived for the p_1-mode frequency against the number of dimensions can be reproduced quite well by a polynomial of second order in d. In this mode, as found in the f mode, we also observe a universality among all curves $f_{p_1} = f_{p_1}(d)$ (check, again, Table II).

In Table I, we can see the values of frequencies for f and p_1 modes shown in Fig. 3, for each spacetime dimension d and five gravitational radii.

In Table II, the constants of the fitting curves that correlate f- and p_1-mode frequencies with the spacetime dimension in Fig. 3, for each r_H, are presented.

C. Oscillation spectrum of Newtonian homogeneous stars in d spacetime dimensions

In this subsection, we analyze the oscillation spectrum of homogeneous stars in Newtonian gravity. We consider that the energy density $\rho_d = d\mathcal{B}_d$ is constant along the whole star. In this limit, the integration of the static equilibrium structure equations, the fluid pressure, and the mass of the

TABLE I. Values of the f- and p_1-mode frequencies for four different r_H. The units of the frequencies and gravitational radii are, respectively, [kHz] and [km].

| d | $r_H = 1.0$ | | $r_H = 1.5$ | | $r_H = 2.0$ | | $r_H = 2.5$ | | $r_H = 3.0$ | |
	f	p_1	f	p_1	f	p_1	f	p_1	f	p_1
4	2.4165	19.910	2.4039	16.789	2.3939	14.745	2.3862	13.237	2.3808	12.042
5	2.6784	29.411	2.6791	23.642	2.6843	20.141	2.6913	17.705	2.7010	15.867
6	2.7694	37.747	2.7771	29.327	2.7865	24.427	2.7982	21.126	2.8124	18.703
7	2.7438	44.856	2.7518	34.022	2.7617	37.844	2.7736	23.835	2.7880	20.913
8	2.6222	50.863	2.6291	37.905	2.6377	30.701	2.6480	26.018	2.6604	22.681
9	2.4282	55.932	2.4342	41.135	2.4409	33.022	2.4490	27.804	2.4586	24.121

TABLE II. Values of polynomial fitting parameters of the f- and p_1-mode frequencies as a function of dimensional parameter d, for four different r_H. We use third and second order polynomial fitting for f and p_1 mode, respectively. The units of all fitting parameters are [kHz].

| $f$mode | | | | | p_1 mode | | |
r_H	a_f ($\times 10^{-3}$)	b_f ($\times 10^{-1}$)	c_f	e_f	a_{p1} ($\times 10^{-1}$)	b_{p1} ($\times 10$)	c_{p1} ($\times 10$)
1.0	4.9380	−1.5213	1.3232	−0.7575	−5.6208	1.4495	−2.9042
1.5	5.5806	−1.6656	1.4291	−1.0034	−4.5216	1.0713	−1.8741
2.0	6.1148	−1.7890	1.5217	−1.2207	−3.7948	0.8548	−1.3270
2.5	6.6250	−1.9080	1.6118	−1.4307	−3.2789	0.7133	−0.9937
3.0	7.1370	−2.0285	1.7033	−1.6420	−2.8926	0.6133	−0.7746
Universal	6.0791	−1.7825	1.5178	−1.2108			

object within a radius r in d spacetime dimensions are, respectively, described by

$$p_d = \frac{S_{d-2}G_d}{2(d-1)}(d\mathcal{B}_d)^2(R^2 - r^2),\qquad(31)$$

$$m = \frac{S_{d-2}}{d-1}(d\mathcal{B}_d)r^{d-1}.\qquad(32)$$

It is important to say that, within the Newtonian gravity, the nonradial oscillations in Cowling approximation, Eqs. (23) and (24), follow the relations

$$\frac{dZ}{dr} = -\frac{Q}{r^{d-2}},\qquad(33)$$

$$\frac{dQ}{dr} = -Zr^{d-4}l(l+d-3).\qquad(34)$$

From these equations, to obtain regular solutions in the center, in a similar way as considered in the relativistic case, we regard

$$Q = C_2 r^{l+d-3}, \qquad Z = -C_2\frac{r^l}{l},\qquad(35)$$

with C_2 being a dimensionless constant. At the object's surface is found

$$\left[\omega^2 Z r^{d-2} - \frac{dp_d}{dr}\frac{Q}{d\mathcal{B}_d}\right]_{r=R} = 0.\qquad(36)$$

Considering Eqs. (31) and (32), this last equation yields

$$\omega^2 = \frac{lMG_d}{R^{d-1}}.\qquad(37)$$

We must point out that when $l = 2$ for a four-dimensional spacetime $d = 4$, Eq. (37) is reduced to that obtained by Emden, see, e.g., Refs. [53,54].

In addition, considering the relation (32) for $r = R$, Eq. (37) can be reduced to the form

$$\omega^2 = ldG_d\mathcal{B}_d\frac{S_{d-2}}{d-1}.\qquad(38)$$

Through Eqs. (37) and (38), the dependence of the eigenfrequency of oscillation ω (or the f-mode frequency of oscillation, since $f_f = \omega/2\pi$) with the spacetime dimension can be analyzed. In that way, these two equalities are employed to investigate the dependence of the fundamental mode frequency with the squared root of the average density,

$$\bar{\rho} = \frac{MG_d}{R^{d-1}}\,[\text{km}]^{-2},\qquad(39)$$

and different spacetime dimensions. In Newtonian gravity, we can note that the f-mode frequency is directly proportional to $\bar{\rho}^{1/2}$ with the proportionality constant $\sqrt{l}/2\pi$; see, e.g., [52,55], for $d = 4$.

TABLE III. Eigenfrequency of f-mode oscillation of homogeneous star in Newtonian gravity for different spacetime dimensions, $l = 2$ and constant energy density 240 [MeV/fm^3].

d	f[kHz]
4	2.4612
5	2.6714
6	2.7590
7	2.7337
8	2.6139

In Table III, the f mode of oscillations of homogeneous stars in Newtonian approximation considering $l = 2$, the energy density 240 [MeV/fm^3], and few spacetime dimensions are presented. As can be seen in Table III, the higher f mode is determined in $d = 6$. This is easily explained since it is well known that the volume of the unitary sphere ($S_{d-2}/(d-1)$) has a maximum for $d = 6$ [56] and, as mentioned above, the eigenfrequency of f-mode oscillation [Eq. (38)] depends on it. If we use a larger bag constant value, consistent with hadron spectroscopy, such as 90 [MeV/fm^3] [49] for $d = 4$, which corresponds to a constant energy density of 360 [MeV/fm^3], the f-mode frequencies will increase by a factor of $\sqrt{360/240} = 1.2$ because they are proportional to the squared root of the average star density that goes with $\mathcal{B}_4^{1/2}$ as we can see in Eq. (38). Thus, the f-mode oscillation frequencies for homogeneous stars in d dimensions are still in a narrow range, $f \sim 2.8$–3.3 [kHz].

From Eq. (38), we see that for a constant f-mode frequency, the product $G_d d \mathcal{B}_d S_{d-2}/(d-1)$ is equivalent to $4G_4 \mathcal{B}_4 S_{d-2}/(d-1)$ (since $dG_d \mathcal{B}_d = 4G_4 \mathcal{B}_4$). From this relation, we note that the value of the constant f-mode frequency for each dimension of the homogeneous star depends on the square root of the product $\mathcal{B}_4 S_{d-2}/(d-1)$. However, for any dimension, we can separate the dimensional effect from the bag constant if we divide the constant f-mode frequency by the squared root of $S_{d-2}/(d-1)$. In this case, the normalized f-mode frequency will depend essentially only in the square root of $\mathcal{B}_4$. Thus, if we change the value of the bag constant in $d = 4$, the new normalized f-mode frequency will scale according to the square root of the new value of the bag constant in $d = 4$. The dimensional effects can be separated if we calculate the ratio between the f-mode frequency in one dimension with the other in another dimension, which will depend only on the square root of the ratio between their $S_{d-2}/(d-1)$ volume factors.

D. f-mode frequencies of relativistic strange stars and of Newtonian homogeneous stars

The main difference between neutron stars and quark stars is that for these self-bound compact stars, the average density is almost constant with the variation of the total star
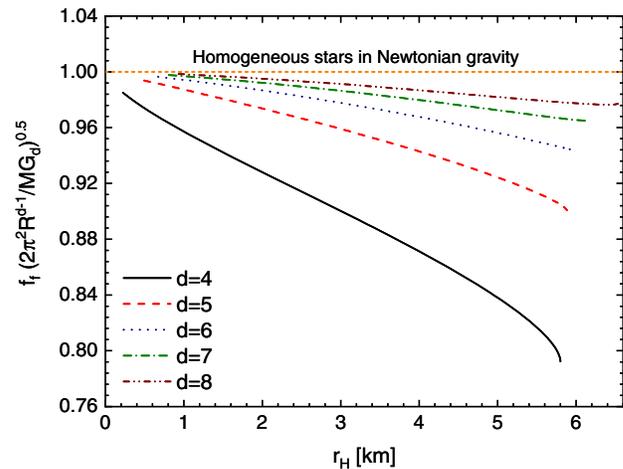

FIG. 4. The f mode for strange quark stars in the relativistic formalism, normalized by the square root of the average density $\bar{\rho}$ divided by $2\pi^2$, i.e., $(MG_d/2\pi^2 R^{d-1})^{0.5}$, as a function of the gravitational radius.

mass for several relativistic strange quarks stars, see Fig. 4 for any dimensions, and the homogeneous density star is a good approximation. Since the star mass grows with the homogeneous energy density times the volume $S_{d-2}R^{d-1}/(d-1)$ [see Eq. (32)], the f-mode frequencies are almost constant for any strange quark star (independent of their mass); unlike neutron stars, where in four-dimensional spacetime the f-mode frequencies scale and increase with average star density and in particular with the stellar mass. Moreover, for quark stars, with fixed $dG_d \mathcal{B}_d$, f-mode depends only on the volume of the unitary sphere $S_{d-2}/(d-1)$. For instance, in Table III, for $dG_d \mathcal{B}_d = 240$ [MeV/fm^3], we find that the larger value of the f mode is attained in $d = 6$ (five spatial dimensions).

The f-mode frequencies, shown in Fig. 1 as a function of the gravitational radius, are almost constant and begin to increase only for a large gravitational radius. In four dimensions, we note that the f-mode frequency is almost constant up to $r_H \sim 5.0$ [km] (in $\sim 1.7 M_\odot$). In larger spacetime dimensions, similar behavior of the f mode with the gravitational radius is found, the f mode remains approximately constant until $r_H \sim 5.6$ [km]. In addition, in each spacetime dimension, at the almost constant f-mode frequency interval, we note that the f-mode frequencies are very similar to those derived for homogeneous stars in Newtonian gravity, see Table III and Fig. 4. From this, we can understand that the quark energy density is also almost constant inside the star for any number of dimensions. Additionally, for $d > 6$, in the almost constant f-mode frequency interval, we see that the f mode decreases with the spacetime dimension. This is exactly the same dimensional dependence of f-mode frequencies of a homogeneous star in the Newtonian gravity (see, again, Table III). This implies that the total quark star mass goes with the volume in $d - 1$ dimensions and $\bar{\rho}$ is constant for

TABLE IV. Physical parameters of detected strange star candidates derived using the MIT EOS with $\mathcal{B}_4 = 60$ [MeV/fm^3].

SS candidate	Observed mass $M/M_\odot$	Predicted radius [km]	M/R	f[kHz]
Vela X-1	1.77 ± 0.08 [57]	$11.17^{+0.01}_{-0.04}$	0.235	2.4234
4U 1608 − 52	1.74 ± 0.14 [58]	$11.16^{+0.01}_{-0.12}$	0.231	2.4164
PSR J1903 + 327	1.667 ± 0.021 [59]	$11.11^{+0.02}_{-0.02}$	0.222	2.4035
4U 1820 − 30	1.58 ± 0.06 [60]	$11.02^{+0.07}_{-0.07}$	0.212	2.3930
Cen X-3	1.49 ± 0.08 [57]	$10.91^{+0.02}_{-0.12}$	0.202	2.3863
EXO 1745 − 248	1.3 ± 0.2 [61]	$10.58^{+0.34}_{-0.35}$	0.182	2.3794
LMC X-4	1.29 ± 0.05 [57]	$10.57^{+0.12}_{-0.11}$	0.181	2.3791
SMC X-1	1.04 ± 0.09 [57]	$9.98^{+0.23}_{-0.25}$	0.154	2.3804
SAX J1808.4 − 3658	0.9 ± 0.3 [62]	$9.73^{+0.65}_{-0.85}$	0.137	2.3839
4U 1538 − 52	0.87 ± 0.07 [57]	$9.49^{+0.22}_{-0.12}$	0.136	2.3852
HER X-1	0.85 ± 0.15 [63]	$9.42^{+0.48}_{-0.32}$	0.134	2.3860

any d dimensions (almost constant for a self-bound quark star in $d = 4$). Due to the fact that the f-mode frequency is proportional to the volume of the unitary sphere, we note that it diminishes with the increase of the spacetime dimension (for $d > 6$). Thus, for a large dimension, the f-mode frequency tends to zero, this being the minimum value that the frequency can attain. This result seems very reasonable since the star pressure goes with $1/(d-1)$ and, due to it decreases with the spacetime dimension, the star is more prone to the gravitational collapse.

In Fig. 4, the f mode of stable relativistic quark stars—normalized by the square root of the average density $\bar{\rho}$ divided by $2\pi^2$, i.e., $(MG_d/2\pi^2 R^{d-1})^{0.5}$—as a function of the gravitational radius. For each dimension considered, for a range of r_H, we see that the normalized f mode follows the constant Newtonian frequencies. At this range of r_H, we note that Eq. (38) is a good approximation of the eigenfrequency squared for the relativistic case. Furthermore, it is even more constant for higher r_H and approaches the homogeneous star limit when the dimension increases; only for large r_H a fast decrease of the f mode with the gravitational radius is seen. These results can be understood by analyzing the equation of state, $p_d = (\rho_d - d\mathcal{B}_d)/(d-1)$, where the fluid pressure gets much smaller than the energy density since it decreases when the spacetime dimension increases, due to the factor $1/(d-1)$. Thus, for large spacetime dimensions, the homogeneous star becomes a good approximation since $p_d \ll \rho_d$ and to the fact that the relations $MG_d/(d-3)R^{d-3}$ [11] and $S_{d-2}p_d r^{d-1}/(d-3)m$ become smaller with the growth of d. This indicates that, for a sufficiently large dimension, the relativistic results are similar to those ones derived from the homogeneous stars in Newtonian gravity. From this result, as similarly happens on homogeneous stars in Newtonian gravity, the f-mode frequency of oscillation in relativistic quark stars could have much lower values for $d \gg 6$.

The results of our model for $d = 4$ can be compared with the parameters of strange star candidates. In Table IV, we present the mass of strange star candidates and the predicted total radius, compactness, and f-mode frequencies using the MIT EOS with $\mathcal{B}_4 = 60$ [MeV/fm^3]. Very different values of star masses in the range $0.85 \leq M/M_\odot \leq 1.77$ have f-mode oscillation frequencies essentially constant with values in a very small range $f \sim 2.38$–2.42 [kHz]. From these results, we can understand that if f-mode frequencies measured by gravitational wave astronomy are almost constant, for very different pulsar masses, it is a good signature to identify strange star candidates.

IV. CONCLUSIONS AND FINAL REMARKS

In this work, we investigated how the oscillation spectrum changes with the spacetime dimensions in the context of general relativity. Thus, we derived for the first time the nonradial perturbation equations for a d-dimensional spacetime within the Cowling approximation. For the fluid, we consider that pressure and energy density are related by a linear EOS, the MIT bag model equation of state extended for d dimensions. We also assumed that the interior solutions are connected smoothly with the vacuum Schwarzschild-Tangherlini exterior solution. We analyzed the oscillation spectrum only for stable compact objects against radial perturbations [11] for different gravitational radii, r_H, and spacetime dimensions, d.

By analyzing the properties of the oscillation spectrum in strange quark stars, for a gravitational radius range and some spacetime dimension values, it is observed that f-mode frequencies are almost constant and only show a fast increase for higher values of the gravitational radius, in contrast with p_1-mode frequencies that change significantly with r_H and d. From one side, for a range of gravitational radii, we found that the minimum and maximum f-mode frequencies were determined on the

dimensions 4 and 6, respectively. Moreover, for a range of gravitational radii, the f-mode frequency dependence with the number of dimensions d follows a universal curve as a polynomial of third order in d. On the other hand, at the same gravitational radii interval, we found that the p_1-mode frequencies grow significantly with d, and the frequency dependence with the number of dimensions can be reproduced quite well with a polynomial of second order in d. As happens in $d = 4$, for $d > 4$, we found that the f-mode frequencies are lower than those obtained for p_1 mode.

Within the Cowling approximation, the nonradial oscillation equations in the Newtonian gravity were also analyzed. For this framework, we investigated the oscillation spectrum of a compact object made of homogeneous energy density, for such a case, it is considered $\rho_d = d\mathcal{B}_d = $ constant along the whole star. Such as it is found in the four-dimensional case, for a d-dimensional spacetime, the eigenfrequency of oscillation of a compact object also depends on its total mass MG_d and radius R. Furthermore, since quark star mass goes with the volume in d dimensions, such as it happens in $d = 4$, the eigenfrequency squared of the f mode in the Newtonian gravity is constant and depends only on the volume of the unitary sphere $[S_{d-2}/(d-1)]$, which has a maximum for $d = 6$.

We prove relativistic f-mode frequencies for $d \geq 4$, which we have calculated here for the first time, are almost constant for any strange quark star independent of their masses if $M \leq 1.8\ M_\odot$. From this, for the gravitational radius range where the frequencies are essentially constant, we can also note that quark energy density is also almost constant inside the star for any number of dimensions. This implies that for relativistic strange stars, for any d dimensions, the total quark star mass still goes approximately with the volume in $d - 1$ dimensions and the average star density $\bar{\rho}$ that is almost constant as a function of the total mass.

It is important to stress that for neutron stars in $d = 4$ dimensions, the star mass increases when the stellar radius decreases and, as a consequence the average density increases with the central density and is not constant, which implies that f-mode frequencies increase with the neutron star mass [40]. Thereby, the possibility of measuring in gravitational wave detectors the f-mode oscillation frequency emitted by compact stars [64–66], for different pulsar masses, and obtaining almost constant frequency values, for $d = 4$, in the interval $f \sim 2$–3 [kHz] with $M \leq 1.7\ M_\odot$, it would be a good indication of the existence of strange quark stars which still lack an astronomical confirmation. Finally, if the f-mode frequencies are still constant and larger than those values found in $d = 4$ for a range of larger total masses, it would be a sign that quarks can propagate in extra spacetime dimensions and strange quark stars in d dimension could exist.

It is worth noting that strange quark stars with crust are also investigated in the literature [67]. The crust on the top of such stars could be supported by strong electric fields at the surface. On that work, it is concluded that the existence of a crust results in large radii for small stellar masses $\sim 0.01\ M_\odot$, but it does not considerably affect the radii of quark stars with larger masses. Therefore, the existence of a crust does not seem to have an important effect on the quadrupole properties of the star [67,68] for quark star masses larger than one solar mass. From these findings, our conclusion of f-mode oscillation frequencies being almost constant for very different quark star masses should be still valid for strange quark stars with the crust. This could be a good signature to confirm the existence of strange quark stars, which still lack an astronomical confirmation, and might be found in gravitational-wave astronomy [66].

APPENDIX: DIVERGENCE OF THE VELOCITY

Within the Cowling approximation, the divergence of the velocity $\nabla_\nu(\delta U^\nu)$ is derived through the equality

$$\nabla_\nu(\delta U^\nu) = \frac{1}{\sqrt{-||g||}} \partial_\nu(\sqrt{-||g||}\delta U^\nu), \qquad (A1)$$

with $\sqrt{-||g||}$ being of the form

$$\sqrt{-||g||} = e^{\lambda+\nu} r^{d-2} \prod_{i=1}^{d-3}(\sin\theta_i)^{d-2-i}. \qquad (A2)$$

The divergence of the velocity is represented by the equality

$$\nabla_\nu(\delta U^\nu) = \left(\frac{d\lambda}{dr} + \frac{d\nu}{dr} + \frac{d-2}{r}\right)\delta U^r + \partial_r(\delta U^r)$$
$$+ \sum_{i=1}^{d-2}\partial_{\theta_i}(\delta U^{\theta_i}) + \sum_{i=1}^{d-3}(d-2-i)\frac{\delta U^{\theta_i}}{\tan\theta_i}. \qquad (A3)$$

In a similar way as defined in four dimensions, for a higher-dimensional spacetime, Eq. (3), the spherical harmonic follows the identity [69]

$$\Delta_{S^{d-2}}Y_l^m = -l(l + d - 3)Y_l^m, \qquad (A4)$$

where, following Ref. [70], $\Delta_{S^{d-2}}$ is given by the expression

$$\Delta_{S^{d-2}} = \sum_{i=1}^{d-2}\prod_{j=1}^{i-1}\frac{\partial_{\theta_i}^2}{\sin^2\theta_j} + \sum_{i=1}^{d-3}\prod_{j=1}^{i-1}(d-2-i)\frac{\cot\theta_i\partial_{\theta_i}}{\sin^2\theta_j}. \qquad (A5)$$

Taking into account the fluid velocity component, Eq. (18), after some algebra Eq. (A4) is reduced to the form

$$\sum_{i=1}^{d-2} \partial_{\theta_i}(\delta U^{\theta_i}) + \sum_{i=1}^{d-3} (d-2-i) \frac{\delta U^{\theta_i}}{\tan \theta_i} = \frac{l(l+d-3)}{r^2 e^\nu} \frac{\partial \tilde{Z}}{\partial t} Y_l^m.$$

(A6)

Replacing Eqs. (17) and (A6) in Eq. (A3), it becomes

$$\nabla_\nu(\delta U^\nu) = \frac{e^{-(\lambda+\nu)}}{r^{d-2}} \frac{\partial^2 \tilde{Q}}{\partial t \partial r} Y_l^m + \frac{l(l+d-3)}{r^2 e^\nu} \frac{\partial \tilde{Z}}{\partial t} Y_l^m.$$

(A7)

[1] B. P. Abbott *et al.*, Observation of Gravitational Waves from Binary Black Hole Merger, Phys. Rev. Lett. **116,** 061102 (2016).

[2] B. P. Abbott *et al.*, GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116,** 241103 (2016).

[3] B. P. Abbott *et al.*, GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118,** 221101 (2017).

[4] B. P. Abbott *et al.*, GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. **119,** 161101 (2017).

[5] B. P. Abbott *et al.*, Multi-messenger observational of a binary neutron star merger, Astrophys. J. Lett. **848,** L12 (2017).

[6] T. Kaluza, On the unification problem in physics, Int. J. Mod. Phys. D **27,** 1870001 (2018).

[7] O. Klein, Quantum theory and five-dimensional theory of relativity, Z. Phys. **37,** 895 (1926).

[8] J. Ponce de Leon and N. Cruz, Hydrostatic equilibrium of a perfect fluid sphere with exterior higher-dimensional Schwarzschild space-time, Gen. Relativ. Gravit. **32,** 1207 (2000).

[9] T. Harko and M. Mak, Anisotropic charged fluid spheres in D space-time dimensions, J. Math. Phys. (N.Y.) **41,** 4752 (2000).

[10] J. L. Blázquez-Salcedo, C. Knoll, and E. Radu, Boson and Dirac stars in $D \geq 4$ dimensions, Phys. Lett. B **793,** 161 (2019).

[11] J. D. V. Arbañil, G. A. Carvalho, R. V. Lobato, R. M. Marinho, Jr., and M. Malheiro, Extra dimensions' influence on the equilibrium and radial stability of strange quark stars, Phys. Rev. D **100,** 024035 (2019).

[12] N. Dadhich and S. Chakraborty, Buchdahl compactness limit for a pure Lovelock static fluid star, Phys. Rev. D **95,** 064059 (2017).

[13] S. Chakraborty and S. Sengupta, Packing extra mass in compact stellar structures: An interplay between Kalb-Ramond field and extra dimensions, J. Cosmol. Astropart. Phys. 05 (2018) 032.

[14] B. C. Paul, On the mass of a uniform density star in higher dimensions, Classical Quantum Gravity **18,** 2637 (2001).

[15] S. G. Ghosh and A. Beesham, Higher dimensional inhomogeneous dust collapse and cosmic censorship, Phys. Rev. D **64,** 124005 (2001).

[16] S. G. Ghosh and A. Beesham, Naked singularities in higher dimensional inhomogeneous dust collapse, Classical Quantum Gravity **17,** 4959 (2000).

[17] S. G. Ghosh and N. Dadhich, Naked singularities in higher dimensional Vaidya space-times, Phys. Rev. D **64,** 047501 (2001).

[18] U. Debnath and S. Chakraborty, Gravitational collapse in higher dimensional space-time, Gen. Relativ. Gravit. **36,** 1243 (2004).

[19] M. Malheiro, M. Fiolhais, and A. R. Taurines, Metastable strange matter and compact quark stars, J. Phys. G **29,** 1045 (2003).

[20] F. Weber, M. Meixner, R. P. Negreiros, and M. Malheiro, Ultradense neutron star matter strange quarks stars, and the nuclear equation of state, Int. J. Mod. Phys. E **16,** 1165 (2007).

[21] J. D. V. Arbañil and M. Malheiro, Radial stability of anisotropic strange quark stars, J. Cosmol. Astropart. Phys. 11 (2016) 012.

[22] R. P. Negreiros, F. Weber, M. Malheiro, and V. Usov, Electrically charged strange quark stars, Phys. Rev. D **80,** 083006 (2009).

[23] J. D. V. Arbañil and M. Malheiro, Equilibrium and stability of charged strange quark stars, Phys. Rev. D **92,** 084009 (2015).

[24] H. A. Buchdahl, General relativistic fluid spheres, Phys. Rev. **116,** 1027 (1959).

[25] A. Ishibashi and H. Kodama, Stability of higher-dimensional Schwarzschild black holes, Prog. Theor. Phys. **110,** 901 (2003).

[26] V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Gravitational radiation in *D*-dimensional space-times, Phys. Rev. D **67,** 064026 (2003).

[27] V. Cardoso, J. P. S. Lemos, and S. Yoshida, Quasinormal modes of Schwarzschild black holes in four and higher dimensions, Phys. Rev. D **69,** 044004 (2004).

[28] R. C. Myers and M. J. Perry, Black holes in higher dimensional space-times, Ann. Phys. (N.Y.) **172,** 304 (1986).

[29] R. Emparan and H. S. Reall, Black holes in higher dimensions, Living Rev. Relativity **11,** 6 (2008).

[30] G. W. Gibbons, D. Ida, and T. Shiromizu, Uniqueness and Nonuniqueness of Static Black Holes in Higher Dimensions, Phys. Rev. Lett. **89,** 041101 (2002).

[31] J. L. Blázquez-Salcedo and C. Knoll, Slowly damped quasinormal modes of the massive Dirac field in d-dimensional Tangherlini space-time, Phys. Rev. D **97,** 044020 (2018).

[32] R. André, J. P. S. Lemos, and G. M. Quinta, Thermodynamics and entropy of self-gravitating matter shells and black holes in d dimensions, Phys. Rev. D **99,** 125013 (2019).

[33] R. C. Tolman, Static solution of Einstein's field equation for spheres of fluid, Phys. Rev. D **55,** 364 (1939).

[34] J. R. Oppenheimer and G. Volkoff, On massive neutron cores, Phys. Rev. D **55,** 374 (1939).

[35] P. N. McDermott, H. M. Van Horn, and J. F. Scholl, Nonradial g-mode oscillations of warm neutron stars, Astrophys. J. **268,** 837 (1983).

[36] L. Lindblom and R. J. Splinter, The accuracy of the relativistic Cowling approximation, Astrophys. J. **348,** 198 (1990).

[37] J. P. S. Lemos and V. T. Zanchin, Bonnor stars in d space-time dimensions, Phys. Rev. D **77,** 064003 (2008).

[38] F. L. Tangherlini, Schwarzschild field in n dimensions and the dimensionality of space problem, Nuovo Cimento **27,** 636 (1963).

[39] J. P. S. Lemos and V. T. Zanchin, Electrically charged fluids with pressure in Newtonian gravitation and general relativity in d space-time dimensions: Theorems and results for Weyl type systems, Phys. Rev. D **80,** 024010 (2009).

[40] C. Vásquez Flores and G. Lugones, Discriminating hadronic and quark stars through gravitational waves of fluid pulsation modes, Classical Quantum Gravity **31,** 155002 (2014).

[41] H. Sotani, N. Yasutake, T. Maruyama, and T. Tatsumi, Signatures of hadron-quark mixed phase in gravitational waves, Phys. Rev. D **83,** 024014 (2011).

[42] S. Yoshida and Y. Kojima, Accuracy of the relativistic Cowling approximation in slowly rotating stars, Mon. Not. R. Astron. Soc. **289,** 117 (1997).

[43] A. Stavridis, A. Passamonti, and K. Kokkotas, Nonradial oscillations of slowly and differentially rotating compact stars, Phys. Rev. D **75,** 064019 (2007).

[44] S. Boutloukos and H. P. Nollert, Eigenmode frequency distribution of rapidly rotating neutron stars, Phys. Rev. D **75,** 043007 (2007).

[45] L. Samuelsson and N. Andersson, Neutron star asteroseismology. Axial crust oscillations in the Cowling approximation, Mon. Not. R. Astron. Soc. **374,** 256 (2007).

[46] D. D. Doneva and S. S. Yazadjiev, Nonradial oscillations of anisotropic neutron stars in the Cowling approximation, Phys. Rev. D **85,** 124023 (2012).

[47] E. Witten, Cosmic separation of phases, Phys. Rev. D **30,** 272 (1984).

[48] E. Farhi and R. L. Jaffe, Strange matter, Phys. Rev. D **30,** 2379 (1984).

[49] G. H. Bordbar, H. Bahri, and F. Kayanikhoo, Calculation of the structural properties of a strange quark star in the presence of a strong magnetic field using a density dependent bag constant, Res. Astron. Astrophys. **12,** 9 (2012).

[50] H. Sotani and T. Harada, Nonradial oscillations of quark stars, Phys. Rev. D **68,** 024019 (2003).

[51] Y. Kojima and K.-i. Sakata, Discrimination of quark stars from neutron stars in quadrupole oscillations, Prog. Theor. Phys. **108,** 801 (2002).

[52] C. Chirenti, P. R. Silveira, and O. D. Aguiar, Nonradial oscillations of neutron stars and the detection of gravitational waves, Int. J. Mod. Phys. **18,** 48 (2012).

[53] T. G. Cowling, The nonradial oscillations of polytropic stars, Mon. Not. R. Astron. Soc. **101,** 367 (1941).

[54] P. Gregorian, Nonradial neutron star oscillations, Master's thesis, Utrecht University, Institute for Theoretical Physics, Netherlands, 2014.

[55] K. D. Kokkotas and B. G. Schmidt, Quasinormal modes of stars and black holes, Living Rev. Relativity **2,** 2 (1999).

[56] G. Huber, Gamma function derivation of n-sphere volumes, Am. Math. Mon. **89,** 301 (1982).

[57] M. L. Rawls, J. A. Orosz, J. E. McClintock, M. A. P. Torres, C. D. Bailyn, and M. M. Buxton, Refined neutron star mass determinations for six eclipsing x-ray pulsar binaries, Astrophys. J. **730,** 25 (2011).

[58] T. Güver, F. Özel, A. Cabrera-Lavers, and P. Wroblewski, The distance, mass, and radius of the neutron star in 4U 1608-52, Astrophys. J. **712,** 964 (2010).

[59] P. C. C. Freire *et al.*, On the nature and evolution of the unique binary pulsar J1903 + 0327, Mon. Not. R. Astron. Soc. **412,** 2763 (2011).

[60] T. Güver, P. Wroblewski, L. Camarota, and Feryal Özel, The mass and radius of the neutron star in 4U 1820-30, Astrophys. J. **719,** 1807 (2010).

[61] F. Özel, T. Güver, and D. Psaltis, The mass and radius of the neutron star in EXO 1745-248, Astrophys. J. **693,** 1775 (2009).

[62] P. Elebert *et al.*, Optical spectroscopy and photometry of SAX J1808.4-3658 in outburst, Mon. Not. R. Astron. Soc. **395,** 884 (2009).

[63] M. K. Abubekerov, E. A. Antokhina, A. M. Cherepashchuk, and V. V. Shimanskii, The mass of the compact object in the low-mass x-ray binary 2S 0921-630, Astronomy Reports **52,** 379 (2008).

[64] M. G. Alford, S. Han, and K. Schwenzer, Signatures for quark matter from multimessenger observations, J. Phys. G **46,** 114001 (2019).

[65] K. Glampedakis and L. Gualtieri, Gravitational waves from single neutron stars: An advanced detector era survey, Astrophysics and Space Science Library **457,** 673 (2018).

[66] W. C. G. Ho, D. I. Jones, N. Andersson, and C. M. Espinoza, Gravitational waves from transient neutron stars f-mode oscillations, Phys. Rev. D **101,** 103009 (2020).

[67] S. Postnikov, M. Prakash, and J. M. Lattimer, Tidal Love numbers of neutron and self-bound quark stars, Phys. Rev. D **82,** 024016 (2010).

[68] C. W. Yip, M.-C. Chu, and P. T. Leung, The quadrupole oscillations of strange-quark stars, Astrophys. J. **513,** 849 (1999).

[69] C. R. Frye and C. J. Efthimiou, Spherical harmonics in p dimensions, arXiv:1205.3548.

[70] Y. Umemura and N. Kono, Infinite dimensional Laplacian and spherical harmonics, Publ. RIMS **1,** 163 (1965).