Quantum tunneling rate of dilute axion stars close to the maximum mass

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We compute the quantum tunneling rate of dilute axion stars close to the maximum mass [P.H. Chavanis, Phys. Rev. D **84**, 043531 (2011)] using the theory of instantons. We confirm that the lifetime of metastable states is extremely long, scaling as $t_{\rm life} \sim e^N t_D$ (except close to the critical point), where N is the number of axions in the system and t_D is the dynamical time ($N \sim 10^{57}$ and $t_D \sim 10$ hrs for typical QCD axion stars; $N \sim 10^{96}$ and $t_D \sim 100$ Myrs for the quantum core of a dark matter halo made of ultralight axions). Therefore, metastable equilibrium states can be considered as stable equilibrium states in practice. We develop a finite size scaling theory close to the maximum mass and predict that the collapse time at criticality scales as $t_{\rm coll} \sim N^{1/5} t_D$ instead of being infinite when fluctuations are neglected. The collapse time at criticality is smaller than the age of the universe for QCD axion stars and larger than the age of the universe for dark matter cores made of ultralight axions. We also consider the thermal tunneling rate and reach the same conclusions. We compare our results with similar results obtained for Bose-Einstein condensates in laboratory, globular clusters in astrophysics, and quantum field theory in the early Universe.

I. INTRODUCTION

The nature of dark matter (DM) is still unknown and constitutes one of the greatest mysteries of modern cosmology. The cold dark matter (CDM) model in which DM is assumed to be made of weakly interacting massive particles (WIMPs) of mass $m \sim {\rm GeV}/c^2$ works remarkably well at large (cosmological) scales [1] but encounters problems at small (galactic) scales. These problems are known as the cusp-core problem [2], the missing satellite problem [3], and the too big to fail problem [4]. In addition, there is no current evidence for any CDM particle such as the WIMP. In order to solve this "CDM crisis" [5], it has been proposed to take the quantum nature of the particles into account. For example, it has been suggested that DM may be made of bosons in the form of Bose-Einstein condensates (BECs) at absolute zero temperature [6–117] (see the Introduction of [37] and Ref. [118] for an early history of this model and Refs. [119–124] for recent reviews). In this model, DM halos are interpreted as gigantic boson stars described by a scalar field (SF) that represents the wave function ψ of the BEC. The mass of the DM boson has to be very small (see below) for quantum mechanics to manifest itself at galactic scales. By contrast, quantum mechanics is completely negligible at astrophysical scales for "heavy" particles of mass $m \sim {\rm GeV}/c^2$ such as WIMPs.

One possible DM particle candidate is the axion [125]. Axions are hypothetical pseudo-Nambu-Goldstone bosons of the Peccei-Quinn [126] phase transition associated with a $U(1)$ symmetry that solves the strong charge parity (CP)

problem of quantum chromodynamics (QCD). The QCD axion is a spin-0 particle with a very small mass $m = 10^{-4}$ eV/c^2 and an extremely weak self-interaction $a_s = -5.8 \times 10^{-53}$ m arising from nonperturbative effects in QCD (a_s is the scattering length of the axion) [127,128]. Their role in cosmology has been first investigated in [129–132]. Axions have huge occupation numbers so they can be described by a classical relativistic quantum field theory with a real scalar field $\varphi(\mathbf{r}, t)$ whose evolution is governed by the Klein-Gordon-Einstein (KGE) equations. In the relativistic regime, the particle number is not conserved. In the nonrelativistic limit, axions can be described by an effective field theory with a complex scalar field $\psi(\mathbf{r}, t)$ whose evolution is governed by the Gross-Pitaevskii-Poisson (GPP) equations (see the Appendix A). In the nonrelativistic regime, the particle number is conserved. One particularity of the QCD axion is to have a negative scattering length ($a_s < 0$) corresponding to an attractive self-interaction.

The formation of structures in an axion-dominated Universe was first investigated by Hogan and Rees [133] and Kolb and Tkachev [134]. In the very early Universe, the axions are relativistic but self-gravity can be neglected with respect to their attractive self-interaction. These authors found that the attractive self-interaction of the axions generates very dense structures corresponding to pseudo-soliton configurations that they called "axion miniclusters" [133] or "axitons" [134] (these nongravitational solitons are also called "oscillons"). These axitons have a mass $M_{\rm axiton} \sim 10^{-12} M_\odot$ and a radius $R_{\rm axiton} \sim 10^9$ m. At later

times, self-gravity must be taken into account. Kolb and Tkachev [134] mentioned the possibility to form boson stars[1] by Jeans instability. This possibility was originally proposed by Tkachev [139,140] who introduced the names "gravitationally bound axion condensates" [139] and "axionic Bose stars" [140], becoming later "axion stars." Tkachev [139,140] and Kolb and Tkachev [134] discussed the maximum mass of these axion stars due to general relativity but, surprisingly, they considered the case of a repulsive self-interaction ($a_s > 0$). Since axions have an attractive self-interaction ($a_s < 0$), their result does not apply to axion stars.

The case of boson stars with an attractive self-interaction ($a_s < 0$), possibly representing axion stars, has been considered only recently [37,38,44,75,86,93,110,112,141–170] (see a review in [171]). The Jeans instability of a Newtonian self-gravitating BEC with an attractive $|\psi|^4$ self-interaction was studied by Chavanis [37,44] and Guth et al. [143]. An infinite homogeneous BEC of axions is unstable to the formation of localized denser clumps of axions. The clumps can be axitons bound by axion self-interaction or axion stars bound by self-gravity. In the case of axion stars, gravitational cooling [172–174] provides an efficient mechanism for relaxation to a stable configuration. The existence of a maximum mass for axion stars was envisioned by Barranco and Bernal [141] but they did not determine this critical mass.[2] The maximum mass of Newtonian self-gravitating BECs with an attractive $|\psi|^4$ self-interaction, and the corresponding radius, were first calculated by Chavanis and Delfini [37,38] who obtained the explicit expressions[3]

$$M_{\text{max}}^{\text{exact}} = 1.012 \frac{\hbar}{\sqrt{Gm|a_s|}} \tag{1}$$

and

$$(R_{99}^*)^{\text{exact}} = 5.5 \left(\frac{|a_s|\hbar^2}{Gm^3} \right)^{1/2}. \tag{2}$$

For $M > M_{\text{max}}$ there is no equilibrium state. For $M < M_{\text{max}}$ there are two possible equilibrium states for the same mass M. The solution with $R > R_{99}^*$ is stable (minimum of energy) while the solution $R < R_{99}^*$ is unstable (maximum of energy). For $R \gg R_{99}^*$ we are in the noninteracting limit and for $R \ll R_{99}^*$ we are in the nongravitational limit. Starting from the KGE equations with the axion potential, Braaten et al. [145,146] showed that the results of Chavanis and Delfini [37,38] apply to dilute axion stars because, for these objects, it is possible to make the Newtonian approximation and to expand the axion potential to order φ^4, leading to the GPP equations with an attractive $|\psi|^4$ self-interaction (see also Eby et al. [142], Davidson and Schwetz [147], and Appendix A). Thus, dilute axion stars can exist only below the maximum mass M_{max} and above the minimum radius R_{99}^* given by Eqs. (1) and (2). We stress that the maximum mass of dilute axion stars [37,38] has a nonrelativistic origin unlike the maximum mass of boson stars [135–138].

For QCD axions, the maximum mass $M_{\text{max}}^{\text{exact}} = 6.46 \times 10^{-14} M_\odot = 1.29 \times 10^{17}$ kg $= 2.16 \times 10^{-8} M_\oplus$ and the corresponding radius $(R_{99}^*)^{\text{exact}} = 3.26 \times 10^{-4} R_\odot = 227$ km $= 3.56 \times 10^{-2} R_\oplus$ are very small, much smaller than galactic sizes. Therefore, QCD axions are expected to form mini axion stars of the size of asteroids ("axteroids"). They could be the constituents of DM in the form of massive compact halo objects (MACHOs) [75,86]. However, mini axion stars behave essentially as CDM and do not solve the small-scale crisis of CDM.

On the other hand, string theory [175] predicts the existence of axions with a very small mass leading to the notion of string axiverse [176]. This new class of axions is called ultralight axions (ULA) [122]. For an ULA with a mass $m = 2.19 \times 10^{-22}$ eV/c^2 and a very small attractive self-interaction $a_s = -1.11 \times 10^{-62}$ fm, one finds that the maximum mass and the minimum radius of axionic DM halos are $M_{\text{max}} = 10^8 M_\odot$ and $R_{99}^* = 1$ kpc. For smaller (absolute) values of the scattering length, the maximum mass is larger. Therefore, ULAs can form giant BECs with the dimensions of DM halos. These objects may correspond either to ultracompact DM halos like dwarf spheroidal galaxies (dSphs) or to the quantum core (soliton) of larger DM halos. In that second case, the quantum core is surrounded by a halo of scalar radiation (arising from quantum interferences) resulting from a process of violent relaxation [177] and gravitational cooling [172–174]. This "core-halo" structure has been evidenced in direct

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[1]Boson stars, that are the solutions of the KGE equations, were introduced by Kaup [135] and Ruffini and Bonazzola [136] in the case where the bosons have no self-interaction. Boson stars in which the bosons have a repulsive self-interaction ($a_s > 0$) were considered later by Colpi et al. [137] using field theory and by Chavanis and Harko [138] using a hydrodynamic treatment valid in the Thomas-Fermi (TF) limit. These authors showed that boson stars can exist only below a maximum mass, $M_{\text{max}} = 0.633 \hbar c/Gm$ for noninteracting bosons and $M_{\text{max}} = 0.307 (a_s \hbar^2 c^4/G^3 m^3)^{1/2}$ for bosons with a repulsive self-interaction in the TF limit, due to general relativistic effects.

[2]Barranco and Bernal [141] developed a general relativistic formalism based on the KGE equations which is well-suited to the limit where $|a_s|$ is very small (or the axion decay constant f is close to the Planck energy $M_P c^2$). However, their scaling relations were not adapted to study dilute axion stars for which $f \ll M_P c^2$ and they could not explore this type of stars thoroughly, nor determine their maximum mass. The solutions that they found ($M \sim 10^{14}$ kg and $R \sim 10$ m for QCD axions) have a mass much lower than the maximum mass $M_{\text{max}} = 1.29 \times 10^{17}$ kg and correspond to the unstable branch $R < R_{99}^* = 227$ km of dilute axion stars (see below).

[3]Equivalent expressions, written in terms of different parameters (e.g., the dimensionless self-interaction constant λ or the axion decay constant f), are given in [86].

numerical simulations of noninteracting BECDM [55,56,81,84,109,115] and it is expected to persist for self-interacting bosons. In the case of ULAs, the quantum core (ground state of the GPP equations) stems from the equilibrium between the quantum pressure (Heisenberg's uncertainty principle), the attractive self-interaction of the axions and the gravitational attraction. On the other hand, the "atmosphere" has an approximately isothermal [177] or Navarro-Frenk-White (NFW) profile [178] as obtained in classical numerical simulations of collisionless matter (see, e.g., [85] for the Schrödinger-Vlasov correspondance). It is the atmosphere that determines the mass and the size of large DM halos and explains why the halo radius r_h increases with the halo mass M_h while the core radius R_c decreases with the core mass M_c (see Appendix L of [99] for a more detailed discussion). The core mass–halo mass relation $M_c(M_h)$ of BECDM halos with an attractive self-interaction has been determined in [110,112]. It is found that the core mass M_c increases with the halo mass M_h up to the maximum mass $(M_c)_{max}$. Of course, these core-halo configurations are stable only if the mass of their core is smaller than the maximum mass $(M_c < (M_c)_{max})$. In sufficiently large DM halos (see [110,112] for a precise account), the core mass passes above the maximum mass, becomes unstable, and undergoes gravitational collapse.

The collapse of dilute axion stars above M_{max} was first discussed by Chavanis [75] using a Gaussian ansatz and assuming that the self-interaction is purely attractive and that the system remains spherically symmetric and non-relativistic. In that case, the system is expected to collapse toward a mathematical singularity (Dirac peak).[4] Less idealized scenarios were considered in later works from numerical simulations. For example, Cotner [151] showed that the system may break into several stable pieces (axion "drops" [147]) of mass $M' < M_{max}$, thereby avoiding its catastrophic collapse toward a singularity. This type of fragmentation has been observed experimentally in the case

of nongravitational BECs with an attractive self-interaction in a magnetic trap [182]. On the other hand, when the system becomes dense enough, the $|\psi|^4$ approximation is not valid anymore and one has to take into account higher order terms in the expansion of the SF potential (or, better, consider the exact axionic self-interaction potential). These higher order terms, which can be repulsive (unlike the φ^4 term for axions), can account for strong collisions between axions. These collisions may have important consequences on the collapse dynamics. Three possibilities have been considered in the literature:

(i) The first possibility, proposed by Braaten *et al.* [145], is to form a *dense axion star* in which the gravitational attraction and the attractive φ^4 self-interaction are balanced by the repulsive φ^6 (or higher order) self-interaction. They used a non-relativistic approximation and determined the mass-radius relation of axion stars numerically. They recovered the stable branch of dilute axion stars and the unstable branch of nongravitational axion stars found by Chavanis and Delfini [37,38] and evidenced, in addition, a new stable branch of dense axion stars. On this branch, self-gravity is negligible (except for very large masses). The mass-radius relation of axion stars presents therefore a maximum mass M_{max}^{dilute} and a minimum mass M_{min}^{dense}. Following Ref. [75], Eby *et al.* [148,152,153] studied the collapse of dilute axion stars to dense axion stars with the Gaussian ansatz[5] and argued that collapsing axion stars evaporate a large fraction of their mass through the rapid emission of relativistic axions.

(ii) The second possibility is a *bosenova* phenomenon in which the collapse of the axion star may be accompanied by a burst of outgoing relativistic axions (radiation) produced by inelastic reactions when the density reaches high values. In that case, the collapse (implosion) is followed by an explosion. This phenomenon was shown experimentally by Donley *et al.* [183] for nongravitational relativistic BECs with an attractive self-interaction and has been demonstrated by Levkov *et al.* [154] for relativistic axion stars from direct numerical simulations of the KGE equations in the Newtonian limit with the exact axionic potential taking collisions into account. These equations predict multiple cycles of collapses and explosions with a self-similar scaling regime and a series of singularities at finite times. These multiple cycles can lead either to a dilute axion star with a mass $M' < M_{max}$ or *no remnant* at all because of complete disappearance of the axion star into scalar waves.

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[4]In Ref. [75] this mathematical singularity was abusively referred to as a "black hole." This terminology is clearly not correct since a nonrelativistic approach is used in [75]. What we meant by "black hole" was actually a Newtonian "Dirac peak" in the sense of, e.g., [179]. On the other hand, the Gaussian ansatz used in [75] provides an inaccurate description of the late stage of the collapse dynamics. Indeed, in the late stage of the collapse, the system is dominated by the attractive self-interaction and the BEC is described by the nongravitational GP equation with an attractive self-interaction. In that case, it is well-known [180,181] that the collapse is self-similar and leads to a finite time singularity. The central density becomes infinite in a finite time t_{coll} at which a singular density profile $\rho \propto r^{-2}$ is formed. The Dirac peak may be formed in the post-collapse regime $t > t_{coll}$ as in [179]. This complex late dynamics cannot be studied with the Gaussian ansatz. However, the Gaussian ansatz is relevant to determine the collapse time of the system which is dominated by the early evolution of the system. It is found in Ref. [75] that $t_{coll} \propto (M - M_{max})^{-1/4}$ when $M \to M_{max}^+$.

[5]As noted in Appendix B of [86], replacing a mathematical singularity (Dirac peak) by a dense axion star with a small radius does not change the estimate of the collapse time obtained in [75].

(iii) The third possibility, when the mass of the axion star is sufficiently large or when the self-interaction is sufficiently weak, is the formation of a *black hole* [155,158]. In that case, general relativity must be taken into account. Helfer *et al.* [155] and Michel and Moss [158] produced a phase diagram displaying a tricritical point joining phase boundaries between dilute axion stars, relativistic bosenova (no remnant), and black holes.[6]

The importance of relativistic effects during the collapse of axion stars has been stressed by Visinelli *et al.* [157]. In particular, they argued that special relativistic effects are crucial on the dense branch[7] while self-gravity can generally be neglected. As a result, dense axion stars correspond to pseudobreathers or oscillons which are described by the sine-Gordon equation. These objects are known to be unstable and to decay via emission of relativistic axions (more precisely, they are dynamically stable but they decay rapidly because of relativistic effects). They have a very short lifetime much shorter than any cosmological timescale. Eby *et al.* [159,168,169] confirmed the claim of Visinelli *et al.* [157] that dense axion stars are relativistic and short-lived.[8] It is important to stress that these authors considered axions (like QCD axions) described by a *real* scalar field for which the particle number is not conserved in the relativistic regime. This is the reason for their fast decay. Alternatively, if we consider ULAs described by a *complex* scalar field (like, e.g., in Refs. [63,64]) for which the particle number is conserved, the dense axion stars should be long-lived. This is suggested by the recent work of Guerra *et al.* [170] on "axion boson stars."

Phase transitions between nonrelativistic dilute and dense axion stars have been studied in [86] using the Gaussian ansatz. This allowed us to recover analytically the mass-radius relation of axion stars obtained numerically in [145].[9] There exists a transition mass M_t such that dilute axion stars are fully stable (global minima of energy) for $M < M_t$ and metastable (local minima of energy) for $M_t < M < M_{max}^{dilute}$. Inversely, dense axion stars are metastable for $M_{min}^{dense} < M < M_t$ and fully stable for $M > M_t$. If a dilute axion star gains mass, for instance by merger and accretion, it can overcome the maximum mass M_{max}^{dilute}, collapse and form a dense axion star (it may also emit a relativistic radiation—bosenova—and disappear into scalar waves as discussed above). Inversely, if a dense axion star loses mass, decaying by emitting axion radiation because of relativistic effects, it can pass below the minimum mass M_{min}^{dense} and disperse outwards (explosion) due to the repulsive kinetic pressure (quantum potential). This mechanism determines the lifetime of dense axion stars in the nonrelativistic regime. As noted by Braaten and Zhang [171] their lifetime may be too short to be astrophysically relevant. However, dense axion stars may have an important cosmological effect by transforming nonrelativistic axions into relativistic axions. The above mentioned phase transitions, involving collapses and explosions, are similar to those studied in [191,192] for self-gravitating fermions at finite temperature enclosed within a "box." They also share similarities with the phase transitions of compact objects (white dwarfs, neutron stars and black holes) as discussed in Sec. XI.C of Ref. [86]. This analogy has been recently confirmed by Guerra *et al.* [170] who numerically solved the KGE equations for a complex scalar field. Their mass-radius relations display the Newtonian maximum mass of dilute axion stars M_{max}^{dilute} derived in [37,38] and the general relativistic maximum mass of dense axion stars $M_{max,GR}^{dense}$ predicted qualitatively in [86] (see Appendix D for a complementary discussion).

Close to the maximum mass M_{max}^{dilute}, the dilute axion stars are metastable (local but not global minima of energy). They are rendered unstable by the quantum mechanical process of barrier-penetration (tunnel effect). We can determine the tunneling rate of axion stars, and their lifetime, by using the theory of path integrals and instantons that was originally elaborated in the context of quantum field theory [193,194]. The instanton theory was applied to the Gross-Pitaevskii (GP) equation by Stoof [195] in order to determine the lifetime of a (nongravitational) metastable BEC with an attractive self-interaction in a confining

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[6]Their phase diagram is consistent with the maximum mass of nonrelativistic dilute axion stars with quartic attractive self-interaction obtained in [37,38] (see the solid line in Fig. 1 of [155] and the solid line in Fig. 3 of [158]).

[7]Braaten and Zhang [171] argue that their evidence is not completely convincing except close to the minimum mass M_{min}^{dense}. The accuracy of the nonrelativistic approximation may improve as M increases along the dense branch.

[8]By contrast, dilute axion stars are long-lived with respect to decay in photons with a lifetime far longer than the age of the Universe [150,160,161,184]. However, photons can be emitted during collisions between dilute axion stars and neutron stars. In particular, it has been proposed that fast radio bursts (FRBs), whose origin is one of the major mysteries of high energy astrophysics, could be caused by axion stars that can engender bursts when undergoing conversion into photons during their collision with the magnetosphere of neutron stars (magnetars), during their collision with the magnetized accretion disk of a black hole, or during their collapse above the maximum mass. We refer to [185–189] for the suggestion of this scenario and to [190] for an interesting critical discussion.

[9]In Ref. [86] we have argued that, at very large masses where general relativistic effects are important, the mass-radius relation of dense axion stars should form a spiral. This implies the existence of another maximum mass $M_{max,GR}^{dense}$, of general relativistic origin, above which the dense axion stars collapse toward a black hole. We have estimated this maximum mass qualitatively in [86]. In this manner, we could recover analytically [86] the phase diagram and the tricritical point obtained numerically in Refs. [155,158].

harmonic potential.[10] Using a Gaussian ansatz, he showed that this problem can be reduced to the simpler problem of the quantum tunneling rate of a fictive particle in a one dimensional potential. This basically leads to the WKB formula [217]. The approach of Stoof [195] was further developed by Ueda and Leggett [218] and Huepe et al. [219] who studied the behavior of the tunneling rate close to the critical point. The correctness of Stoof's analytical approach was studied by Freire and Arovas [220] who used a more rigorous instanton theory based on field theory and showed that the results of Stoof provide a relevant approximation of the exact solution. We will assume that the collective coordinate approach (Gaussian ansatz) remains valid in the case of self-gravitating BECs with an attractive self-interaction and we will use this analytical approach in line with our previous works on the subject [37,38,75,86,93].[11] A similar investigation was recently made by Eby et al. [160]. Here, we explicitly derive the analytical expression of the quantum tunneling rate close to the maximum mass emphasizing the scaling $(1 - M/M_{\max})^{5/4}$ of the reduction factor. Despite this reduction factor, we show that the lifetime of metastable axion stars is considerable, scaling as $e^N t_D$ (where t_D is the dynamical time) with $N \sim 10^{57}$ and $t_D \sim 10$ hrs for QCD axions, and $N \sim 10^{96}$ and $t_D \sim 100$ Myrs for ULAs.[12] Therefore, in practice, metastable states can be considered as stable equilibrium states, except for masses extraordinarily close

to the maximum mass $M_{\max}$. We develop a finite size scaling theory close to the maximum mass and predict that the collapse time at criticality scales as $t_{\text{coll}} \sim N^{1/5} t_D$ instead of being infinite as in Ref. [75] where fluctuations are neglected. The collapse time at criticality is smaller than the age of the universe for QCD axion stars and larger than the age of the universe for the quantum cores of DM halos made of ULAs. On the other hand, our detailed calculation of the quantum tunneling rate may be useful if one is able in the future to perform direct N-body simulations or laboratory experiments of self-gravitating BECs with an attractive self-interaction mimicking dilute axion stars. In that case, the number of bosons N will not be very large and metastability effects should be observed, especially close to the maximum mass.

This paper is organized as follows. In Sec. II we recall the basic equations describing dilute axion stars. In Sec. III we use a Gaussian ansatz to transform these equations into the simpler mechanical problem of a fictive particle in a one dimensional potential. In Sec. IV we determine the quantum tunneling rate of the BEC from the theory of instantons. We give its general expression and its approximate expression close to the maximum mass. In Sec. V we briefly consider the thermal tunneling (or thermal activation) rate of the BEC by using the analogy with Brownian motion. In Sec. VI we consider corrections to the maximum mass due to quantum and thermal fluctuations and show that they are generally negligible. We emphasize the very long lifetime of dilute axion stars. Finally, in Sec. VII we determine the correction to the collapse time at criticality due to quantum and thermal fluctuations. We finally conclude by discussing analogies and differences with other systems of physical interest.

II. DILUTE AXION STARS

In this section, we recall the basic equations describing dilute axion stars in the nonrelativistic limit.

A. GPP equations

Dilute axion stars can be interpreted as Newtonian self-gravitating BECs with an attractive φ^4 self-interaction [86]. They are described by the GPP equations[13]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + \frac{4\pi a_s \hbar^2}{m^2} |\psi|^2 \psi, \quad (3)$$

$$\Delta \Phi = 4\pi G |\psi|^2, \quad (4)$$

where $\psi(\mathbf{r}, t)$ is the wave function of the condensate, $\Phi(\mathbf{r}, t)$ is the gravitational potential, and a_s is the scattering

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[10]Experimental evidence of Bose-Einstein condensation was reported by several groups in 1995 [196–198]. Some laboratory BECs like ^{7}Li are made of atoms that have a negative scattering length ($a_s < 0$), hence an attractive self-interaction [197]. When they are confined by a harmonic potential, they are stable (actually metastable) only below a maximum particle number $N_{\max}$. This maximum particle number was obtained by Ruprecht et al. [199] and Kagan et al. [200] by solving the GP equation numerically and by Baym and Pethick [201] and Stoof [195] by solving the GP equation analytically using a Gaussian ansatz. The approximate analytical approach of Baym and Pethick [201] and Stoof [195]—called the method of collective coordinates or the Ritz optimization procedure—was further developed by [202–204] and finds its origin in the works of [205–210] in the context of nonlinear optics. The existence of a maximum particle number was confirmed experimentally in Ref. [211]. Near the stability limit, quantum tunneling or thermal fluctuations cause the condensate to collapse. During the collapse, the density rises until collisions cause atoms to be ejected from the condensate in an energetic explosion similar to supernova [212]. After the explosion, the condensate regrows fed by collisions between thermal atoms in the gas. This leads to a series of sawtoothlike cycles of growth (explosion) and collapse [204,213–216] until the gas reaches thermal equilibrium.

[11]We refer to, e.g., [220–222] for more general methods, including the formalism of coherent state path integration [223], that could be used to go beyond the Gaussian ansatz.

[12]The scaling $e^N t_D$ was anticipated in Ref. [86] by analogy with similar results obtained for other systems with long-range interactions, such as globular clusters [224], where the destabilization of the metastable state is due to thermal (or energetic) fluctuations instead of quantum fluctuations.

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[13]See Appendix A for the derivation of the GPP equations from the more general KGE equations describing relativistic axion stars.

length of the bosons ($a_s < 0$ for axions with an attractive self-interaction). The GP equation (3) involves a cubic nonlinearity associated with a quartic effective potential (see Eq. (A31) of Appendix A).

B. Hydrodynamic equations

Making the Madelung [225] transformation

$$\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}, \quad \rho = |\psi|^2, \quad \mathbf{u} = \frac{\nabla S}{m}, \quad (5)$$

where $\rho(\mathbf{r}, t)$ is the mass density, $S(\mathbf{r}, t)$ is the action and $\mathbf{u}(\mathbf{r}, t)$ is the velocity field, it can be shown (see, e.g., [79]) that the GPP equations (3) and (4) are equivalent to hydrodynamic equations of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (6)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + m\Phi + \frac{4\pi a_s \hbar^2}{m^2}\rho + Q = 0, \quad (7)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi - \frac{1}{m}\nabla Q, \quad (8)$$

$$\Delta\Phi = 4\pi G\rho, \quad (9)$$

where

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m}\left[\frac{\Delta\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right] \quad (10)$$

is the quantum potential taking into account the Heisenberg uncertainty principle and $P(\rho)$ is the pressure arising from the self-interaction of the bosons. For a cubic nonlinearity (i.e., a $|\psi|^4$ effective potential), the equation of state is quadratic

$$P = \frac{2\pi a_s \hbar^2}{m^3}\rho^2. \quad (11)$$

This is a polytropic equation of state of index $n = 1$. For an attractive self-interaction between the bosons ($a_s < 0$), the pressure is negative. Equations (6)–(9) are called the quantum Euler-Poisson equations [37]. They are equivalent to the GPP equations (3) and (4). In the following, we will exclusively use the hydrodynamic formalism. In that case, the normalization condition of the wave function is equivalent to the conservation of mass $M = \int \rho d\mathbf{r}$. We refer to [79] for the expression of the following results in terms of the wave function.

C. Equilibrium state

In the hydrodynamic representation, an equilibrium state of the quantum Euler-Poisson equations (6)–(9), obtained by taking $\partial_t = 0$ and $\mathbf{u} = \mathbf{0}$, satisfies the equation

$$\nabla P + \rho\nabla\Phi + \frac{\rho}{m}\nabla Q = \mathbf{0}. \quad (12)$$

This equation can be interpreted as a condition of quantum hydrostatic equilibrium. It describes the balance between the pressure force due to the self-interaction of the bosons, the gravitational force, and the quantum force arising from the Heisenberg uncertainty principle. Combining Eq. (12) with the Poisson equation (9), we obtain the fundamental differential equation of quantum hydrostatic equilibrium [79]

$$-\nabla \cdot \left(\frac{\nabla P}{\rho}\right) + \frac{\hbar^2}{2m^2}\Delta\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 4\pi G\rho. \quad (13)$$

For the quadratic equation of state (11), this differential equation has been solved numerically in Ref. [38] in the general case of attractive or repulsive self-interaction.

D. Total energy

The total energy associated with the quantum Euler-Poisson equations is given by

$$E_{\text{tot}} = \Theta_c + \Theta_Q + U + W, \quad (14)$$

where Θ_c is the classical kinetic energy, Θ_Q is the quantum kinetic energy, U is the internal energy, and W is the gravitational energy. It can be explicitly written as

$$E_{\text{tot}} = \int \rho\frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{1}{m}\int \rho Q\, d\mathbf{r}$$
$$+ \frac{2\pi a_s \hbar^2}{m^3}\int \rho^2\, d\mathbf{r} + \frac{1}{2}\int \rho\Phi\, d\mathbf{r}. \quad (15)$$

We can easily show [79] that the quantum Euler-Poisson equations (6)–(9) conserve the total energy ($\dot{E}_{\text{tot}} = 0$).

E. Variational principle

It can be shown that the minimization problem

$$\min_{\rho, \mathbf{u}}\{E_{\text{tot}}[\rho, \mathbf{u}] \quad | \quad M \text{ fixed}\} \quad (16)$$

determines an equilibrium state of the quantum Euler-Poisson equations that is dynamically stable. This is a criterion of nonlinear dynamical stability resulting from the fact that E_{tot} and M are conserved by the quantum Euler-Poisson equations. It provides a necessary and sufficient condition of dynamical stability since it takes into account all the invariants of the quantum Euler-Poisson equations.

The variational principle for the first variations (extremization) can be written as

$$\delta E_{\text{tot}} - \frac{\mu}{m}\delta M = 0, \quad (17)$$

where μ is a Lagrange multiplier (global chemical potential) taking into account the mass constraint. This variational problem gives $\mathbf{u} = \mathbf{0}$ (the equilibrium state is static) and the Gibbs condition

$$m\Phi + \frac{4\pi a_s \hbar^2}{m^2}\rho + Q = \mu \qquad (18)$$

expressing the fact that the gravitational potential + the enthalpy (equal to the local chemical potential) + the quantum potential is constant. Taking the gradient of Eq. (18) and using Eq. (11), we recover the condition of quantum hydrostatic equilibrium (12). Therefore, an extremum of total energy at fixed mass is a steady state of the quantum Euler-Poisson equations. Furthermore, it can be shown that the star is linearly stable with respect to the quantum Euler-Poisson equations if, and only if, it is a local minimum of energy at fixed mass (a maximum or a saddle point is linearly unstable).

The series of equilibria of dilute axion stars with an attractive φ^4 self-interaction presents a maximum mass $M_{\max}$ [37]. Using the Poincaré-Katz turning point criterion [226,227], the Wheeler $M(R)$ theorem [228], or the catastrophe (or bifurcation) theory [229],[14] one can show that the series of equilibria is dynamically stable before the turning point of mass M or energy E_{tot} [they coincide since $\delta E_{\text{tot}} = 0 \Leftrightarrow \delta M = 0$ according to Eq. (17)] and becomes dynamically unstable afterwards [37,38].[15] Furthermore, the curve $E_{\text{tot}}(M)$ displays cusps at its extremal points [38].

F. Quantum virial theorem

The time-dependent scalar virial theorem associated with the quantum Euler-Poisson equations can be written as (see Appendix G of [79])

$$\frac{1}{2}\ddot{I} = 2(\Theta_c + \Theta_Q) + 3\int P d\mathbf{r} + W, \qquad (19)$$

where $I = \int \rho r^2 \, d\mathbf{r}$ is the moment of inertia. At equilibrium, we obtain the quantum virial theorem

$$2\Theta_Q + 3\int P d\mathbf{r} + W = 0. \qquad (20)$$

[14]The Poincaré-Katz turning point criterion [226,227] states that a mode of stability is lost at an extremum of mass if the curve $\mu(M)$ rotates anticlockwise and gained if it rotates clockwise (see Refs. [86,191,227] for a precise account and for illustrations). The mass-radius theorem of Wheeler [228] introduced in the physics of compact objects like white dwarfs and neutron stars states that a mode of stability is lost at an extremum of mass if the curve $M(R)$ rotates anticlockwise and gained if it rotates clockwise. To be complete, we also quote the necessary (but not sufficient) Vakhitov-Kolokolov condition of stability $dM/d\rho_0 > 0$ [181,230].

[15]Note that in more general situations (e.g., in the case of dense axion stars with an additional repulsive φ^6 potential [86]) the stability can be regained after a second turning point of mass (minimum mass $M_{\min}$).

III. GAUSSIAN ANSATZ

A. Total energy

We can obtain an approximate analytical solution of the GPP equations (3) and (4) by developing a mechanical analogy. Making a Gaussian ansatz for the wave function (see, e.g., Sec. 8.2 of Ref. [79] for details):

$$\psi(\mathbf{r}, t) = \left[\frac{M}{\pi^{3/2}R(t)^3}\right]^{1/2} e^{-r^2/2R(t)^2} e^{imH(t)r^2/2\hbar}, \qquad (21)$$

where $R(t)$ is the typical radius of the BEC and $H = \dot{R}/R$, we find that the energy functional (14) can be written as a function of R and $\dot{R}$ (for a fixed mass M) as[16]

$$E_{\text{tot}} = \frac{1}{2}\alpha M \left(\frac{dR}{dt}\right)^2 + V(R) \qquad (22)$$

with the effective potential

$$V(R) = \sigma \frac{\hbar^2 M}{m^2 R^2} - \zeta \frac{2\pi|a_s|\hbar^2 M^2}{m^3 R^3} - \nu \frac{GM^2}{R}. \qquad (23)$$

The coefficients are

$$\alpha = \frac{3}{2}, \quad \sigma = \frac{3}{4}, \quad \zeta = \frac{1}{(2\pi)^{3/2}}, \quad \nu = \frac{1}{\sqrt{2\pi}}. \qquad (24)$$

The first term in Eq. (22) is the classical kinetic energy while the effective potential (23) comprises the quantum kinetic energy, the internal energy and the gravitational energy. Using the conservation of total energy, $\dot{E}_{\text{tot}} = 0$, we get

$$\alpha M \frac{d^2 R}{dt^2} = -\frac{dV}{dR}. \qquad (25)$$

This equation is similar to the equation of motion of a particle of mass αM and position R moving in a one-dimensional potential $V(R)$. This equation can also be obtained from the quantum virial theorem (19) (see Sec. 8.4 of Ref. [79] for details). Instead of starting from the total energy, the same results can be obtained from the Lagrangian of the GPP equations (see Appendix B of Ref. [75] for details). Finally, we can draw some analogies between the equation of motion (25) for the radius of a BEC and the Friedmann equations in cosmology governing the evolution of the scale factor of the Universe where $H = \dot{R}/R$ plays the role of the Hubble constant (see Sec. 8.8 of Ref. [79] for details).

[16]For a Gaussian density profile, the relation between the radius R and the radius R_{99} containing 99% of the mass is $R_{99} = 2.38167R$ [37].
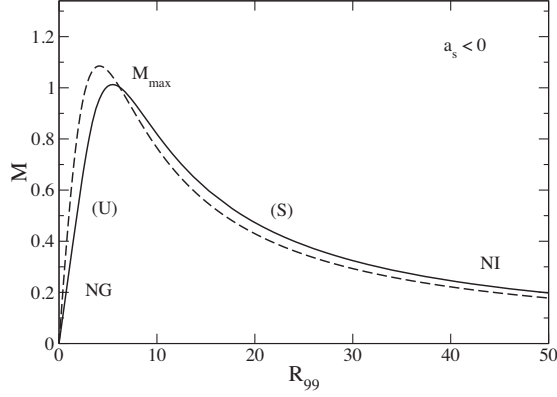
FIG. 1. Mass-radius relation of dilute axion stars interpreted as self-gravitating BECs with an attractive self-interaction ($a_s < 0$). We have chosen a normalization such that $\hbar = G = m = |a_s| = 1$. The solid line is the exact mass-radius relation obtained by solving the GPP equations numerically [38]. The dotted line corresponds to the approximate analytical mass-radius relation (26) obtained from the Gaussian ansatz [37].

B. Mass-radius relation

We have seen that an extremum of total energy E_{tot} given by Eq. (14) at fixed mass M is an equilibrium state of the GPP equations (3) and (4). On the other hand, a (local) minimum of total energy is (meta)stable while a maximum or a saddle point is unstable. Within the Gaussian ansatz, we have to minimize the total energy E_{tot} given by Eq. (22) at fixed mass M. An extremum corresponds to $dR/dt = 0$ and $V'(R) = 0$. The second condition leads to the mass-radius relation [37]

$$M = \frac{2\sigma \frac{\hbar^2}{m^2 R^3}}{\frac{\nu G}{R^2} + 6\pi\zeta \frac{|a_s|\hbar^2}{m^3 R^4}}. \tag{26}$$

This equation can also be obtained from the equilibrium quantum virial theorem (20) [79]. The mass-radius relation is plotted in Fig. 1 in the case of an attractive self-interaction ($a_s < 0$). It displays a maximum mass [37]

$$M_{\text{max}} = \left(\frac{\sigma^2}{6\pi\zeta\nu}\right)^{1/2} \frac{\hbar}{\sqrt{Gm|a_s|}} \tag{27}$$

at

$$R_* = \left(\frac{6\pi\zeta}{\nu}\right)^{1/2} \left(\frac{|a_s|\hbar^2}{Gm^3}\right)^{1/2}. \tag{28}$$

The prefactors are 1.085 and 1.73. We have the identity

$$M_{\text{max}} = \frac{\sigma}{\nu} \frac{\hbar^2}{Gm^2 R_*}. \tag{29}$$

There is no equilibrium state with $M > M_{\text{max}}$. When $M < M_{\text{max}}$, two equilibrium states exist with the same

mass. By computing the second derivative of $V(R)$ or by using the identity

$$\frac{dM}{dR} = -\alpha M \frac{m^2 R^3}{2\sigma\hbar^2} \omega^2, \tag{30}$$

where $\omega^2 = V''(R)/\alpha M$ is the square radial pulsation of the BEC (see Sec. 8 of Ref. [79] for details and generalizations) one can analytically show [37] that the equilibrium states with $R > R_*$ are stable while the equilibrium states with $R < R_*$ are unstable. Therefore, R_* is the minimum radius of stable equilibrium states. This result can also be obtained from the mass-radius relation $M(R)$ by using the Poincaré-Katz turning point criterion [226,227] or the Wheeler theorem [228] (see footnote 14) stating that the change of stability occurs at the turning point of mass (this result is valid beyond the Gaussian ansatz as discussed in Sec. II E).

In the noninteracting limit, corresponding to $R \gg R_*$, the mass-radius relation reduces to

$$M \sim \frac{2\sigma}{\nu} \frac{\hbar^2}{Gm^2 R}. \tag{31}$$

These equilibrium states are stable. In the nongravitational limit, corresponding to $R \ll R_*$, we get

$$M \sim \frac{\sigma}{3\pi\zeta} \frac{mR}{|a_s|}. \tag{32}$$

However, these equilibrium states are unstable.

Remark: The above results apply to dilute axion stars. Stable dilute axion stars exist only below a maximum mass M_{max} and above a minimum radius R_* given by Eqs. (27) and (28) within the Gaussian ansatz [37].[17] The exact values of the maximum mass $M_{\text{max}}^{\text{exact}}$ and of the corresponding radius $(R_{99}^*)^{\text{exact}}$ [see Eqs. (1) and (2)] have been obtained in Ref. [38] by computing the steady states of the GPP equations (3) and (4) numerically. If we take into account a φ^6 repulsion in the self-interaction potential (or consider the exact potential of axions), an additional stable branch appears in the mass-radius relation at small radii corresponding to dense axion stars [86,145].

C. Collapse, gravitational cooling, or explosion

When $M < M_{\text{max}}$ there are two possible equilibrium states for the same mass with radius $R_S > R_*$ and $R_U < R_*$. The equilibrium state R_S is stable (S) and the equilibrium state R_U is unstable (U). The evolution of the unstable state depends on the sign of its energy $E_{\text{tot}}^{(U)}$. In [75] we have identified a critical mass

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[17] The expressions of the maximum mass in the presence of a central black hole or in the presence of dark energy (cosmological constant) are given in [93,231].
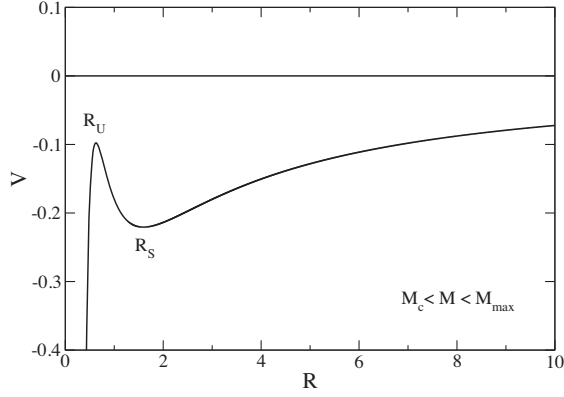
FIG. 2. Effective potential $V(R)$ as a function of the radius R for $M_c < M < M_{max}$. In that case $V_{max} < 0$.

FIG. 3. Effective potential $V(R)$ as a function of the radius R for $M < M_c$. In that case $V_{max} > 0$.

$$M_c = \frac{\sqrt{3}}{2} M_{max} \qquad (33)$$

at which $E_{tot}^{(U)} = 0$.

When $M_c < M < M_{max}$, the energy $E_{tot}^{(U)}$ of the unstable state is negative (see Fig. 2). If slightly perturbed, the unstable star can either collapse toward a Dirac peak ($R \to 0$) or migrate toward a stable dilute axion star by gravitational cooling ($R \to R_S$) [172–174]. This is a dissipative process similar to violent relaxation [177] during which the star undergoes damped oscillations and emits a scalar field radiation. Through this process, it loses energy (and mass) and settles on a stable equilibrium state (S) with a larger radius and a lower energy than the initial configuration (U).

When $M < M_c$, the energy $E_{tot}^{(U)}$ of the unstable state is positive (see Fig. 3). If slightly perturbed, the unstable star can either collapse toward a Dirac peak ($R \to 0$), migrate toward a stable dilute axion star by gravitational cooling ($R \to R_S$), or explode and disperse away ($R \to +\infty$).

Remark: In the following, we shall assume that the mass of the dilute axion star (S) is relatively close to M_{max}. As a result, if it can reach the unstable state (U) by quantum or thermal tunneling, thereby reducing its radius, it then generically collapses toward a Dirac peak.

D. Normal form of the potential close to the maximum mass

Expanding the effective potential from Eq. (23) to third order close to the maximum mass M_{max}, we obtain [75]

$$\frac{V(R)}{V_0} = \frac{1}{3R_*^3} (R - R_*)^3 - \frac{2}{R_*} \left(1 - \frac{M}{M_{max}}\right)(R - R_*)$$
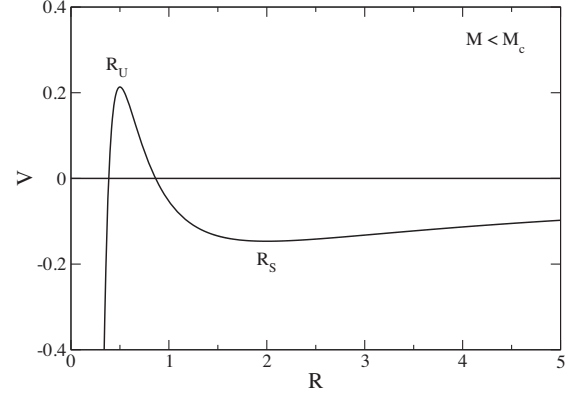$$- \frac{1}{3} + \frac{5}{3}\left(1 - \frac{M}{M_{max}}\right), \qquad (34)$$

where

$$V_0 = \nu \frac{GM_{max}^2}{R_*} = \frac{\sigma^2 \nu^{1/2}}{(6\pi\zeta)^{3/2}} \frac{\hbar m^{1/2} G^{1/2}}{|a_s|^{3/2}}. \qquad (35)$$

Equation (34) is the normal form of a potential $V(R)$ close to a saddle-center bifurcation (see Fig. 4). With this approximation, the equation of motion (25) of the fictive particle becomes

$$\alpha M \frac{d^2 R}{dt^2} = -\frac{V_0}{R_*^3}(R - R_*)^2 + \frac{2V_0}{R_*}\left(1 - \frac{M}{M_{max}}\right). \qquad (36)$$

The mass-radius relation close to M_{max}, corresponding to $V'(R) = 0$, is given by

$$R - R_* = \pm\sqrt{2}R_*\left(1 - \frac{M}{M_{max}}\right)^{1/2}. \qquad (37)$$

The upper sign corresponds to the branch $R > R_*$ and the lower sign corresponds to the branch $R < R_*$. On the other hand, the square radial pulsation $\omega^2 = V''(R)/\alpha M$ of the BEC is given by
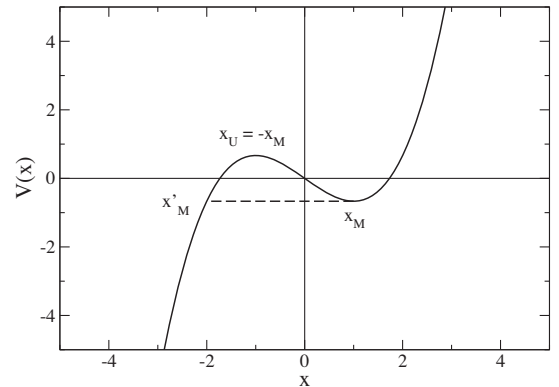

FIG. 4. Normal form of the potential close to a saddle-center bifurcation.

$$\omega^2 = \pm \frac{2\sqrt{2}}{t_D^2}\left(1 - \frac{M}{M_{\max}}\right)^{1/2}, \qquad (38)$$

where we have introduced the dynamical time

$$t_D = \left(\frac{\alpha}{\nu}\right)^{1/2}\frac{1}{\sqrt{G\rho_0}} = \frac{6\pi\zeta}{\nu}\left(\frac{\alpha}{\sigma}\right)^{1/2}\frac{|a_s|\hbar}{Gm^2} \qquad (39)$$

constructed with the density

$$\rho_0 = \frac{M_{\max}}{R_*^3} = \frac{\sigma\nu}{(6\pi\zeta)^2}\frac{Gm^4}{a_s^2\hbar^2}. \qquad (40)$$

The expression from Eq. (38) confirms that the branch $R > R_*$ is stable ($\omega^2 > 0$) while the branch $R < R_*$ is unstable ($\omega^2 < 0$). From Eqs. (37) and (38) we obtain the relation

$$\frac{dM}{dR} = -\frac{\omega^2}{2}\frac{M_{\max}t_D^2}{R_*} \qquad (41)$$

that links the stability of the system (through the sign of the square pulsation ω^2) to the slope of the mass-radius relation. This is a particular case of the Poincaré-Katz turning point criterion close to the maximum mass (see Sec. 8.7 of Ref. [79] for details). Another manner to investigate the stability of an equilibrium state is to compute its energy. The energies of the equilibrium states close to the maximum mass are

$$\frac{E_{\text{tot}}}{V_0} = \frac{V}{V_0} = -\frac{1}{3} + \frac{5}{3}\left(1 - \frac{M}{M_{\max}}\right)$$
$$\mp \frac{4}{3}\sqrt{2}\left(1 - \frac{M}{M_{\max}}\right)^{3/2}. \qquad (42)$$

The energy of the stable state ($R > R_*$, upper sign) is lower than the energy of the unstable state ($R < R_*$, lower sign) for the same mass M. This is expected since an equilibrium state is a minimum energy state.

When $M > M_{\max}$, there is no equilibrium state. In that case, the dilute axion star is expected to collapse. Within our approximations (nonrelativistic treatment + purely attractive self-interaction + spherical collapse), it should form a classical singularity (Dirac peak). The collapse time has been investigated in [75] using a Gaussian ansatz. It is found that, close to the maximum mass, the collapse time is given by

$$\frac{t_{\text{coll}}}{t_D} \sim 2.90178\ldots\left(\frac{M}{M_{\max}} - 1\right)^{-1/4} \quad (M \to M_{\max}^+). \qquad (43)$$

If we consider that the dilute axion star collapses toward a dense axion star of finite radius $R_{\text{dense}} > 0$ [86,148] instead of forming a singularity at $R = 0$ (Dirac peak), the results

of [75] remain valid because R_{dense} is generically very small (this point is specifically addressed in Appendix B of [86]).

IV. QUANTUM TUNNELING RATE OF THE BEC

When $M < M_{\max}$, the potential $V(R)$ has two equilibrium states (see Fig. 4): a stable equilibrium state at $R_M > R_*$ (local minimum) and an unstable equilibrium state at $R_U < R_*$ (local maximum). Since the potential $V(R)$ has no global minimum (it tends to $-\infty$ when $R \to 0$), the stable equilibrium state at R_M is actually metastable. This metastable equilibrium state represents a dilute axion star. In principle, because of quantum fluctuations, the metastable BEC can decay toward a more stable state—a dense axion star if we take into account the repulsive φ^6 term in the self-interaction potential—or collapse. In this section, we compute the tunneling rate of the BEC and the lifetime of the metastable state by using the instanton theory (a pedagogical exposition of this theory is presented in [232]). This path integral formulation lends itself naturally to the study of the semiclassical limit $\hbar \to 0$ via a steepest-descent approach. As explained in the Introduction, we use a Gaussian ansatz and reduce the problem to the tunneling rate of a particle in a one dimensional potential following the approach of Stoof [195].

A. General expression

The equation of motion of the fictive particle representing the BEC is given by Eq. (25). Classically ($\hbar = 0$), the fictive particle can be in equilibrium in the local minimum R_M of the potential $V(R)$. If slightly displaced from its equilibrium position, it will oscillate with a pulsation $\omega_M^2 = V''(R_M)/\alpha M$. However, because of quantum fluctuations ($\hbar \neq 0$), this equilibrium state is metastable and the particle can cross the potential barrier and escape. In the present formalism, quantum fluctuations are incorporated in the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\alpha M}\frac{d^2\psi}{dR^2} + V(R)\psi \qquad (44)$$

for the fictive particle. In the semiclassical limit $\hbar \to 0$, the quantum tunneling rate of the BEC is given by

$$\Gamma \sim Ae^{-B/\hbar}, \qquad (45)$$

where the prefactor A is specified below and the exponent B is equal to

$$B = S[R_b(t)] - S[R_M], \qquad (46)$$

where

$$S[R(t)] = \int\left[\frac{1}{2}\alpha M\left(\frac{dR}{dt}\right)^2 + V(R)\right]dt \qquad (47)$$

is the Euclidean action of the fictive particle representing the BEC. It is obtained from the classical action by replacing $V(R)$ by $-V(R)$ (this is achieved by making the Wick rotation $t \to -it$ in the Feynman path integral [232]). The trajectory $R_b(t)$ occuring in B is the one that makes the Euclidean action (47) extremal. This is the so-called instanton (or bounce) solution. Therefore, the bounce exponent B is equal to the value of the Euclidean (imaginary-time) action evaluated along the bounce trajectory (instanton). The condition $\delta S = 0$ leads to the equation

$$\alpha M \frac{d^2 R_b}{dt^2} = V'(R_b), \qquad (48)$$

which is analogous to the classical equation of motion of a fictive particle in the reversed potential $-V(R)$. If we consider the classical equation of motion (25) of the particle in the potential $V(R)$, the only solution consistent with the initial condition $\dot{R} = 0$ at $R = R_M$ is $R(t) = R_M$ (see Fig. 4). It corresponds to the stable equilibrium state of the original problem. When we make the Wick rotation, we are led to the equation of motion (48) for the particle in the reversed potential $-V(R)$. There are now two solutions consistent with the initial condition $\dot{R} = 0$ at $R = R_M$ (see Fig. 5). The first solution is the trivial solution $R(t) = R_M$ mentioned previously. The second solution is a nontrivial topological solution which extends far from R_M. This is the standard example of an instanton. It starts at $t \to -\infty$ from the top of the hill R_M with zero initial velocity, rolls down the hill, bounces off the wall at the turning point R'_M such that $V(R'_M) = V(R_M)$ at some time t_c (this defines the center of the instanton) and returns to the top of the hill R_M with zero velocity at $t \to +\infty$. Using the classical analogy, the so-called "bounce" solution $R_b(t)$ has the property that the particle spends a very long time around R_M but in a relatively short time oscillates once in the potential minimum of $-V(R)$. The first integral of motion of Eq. (48) is
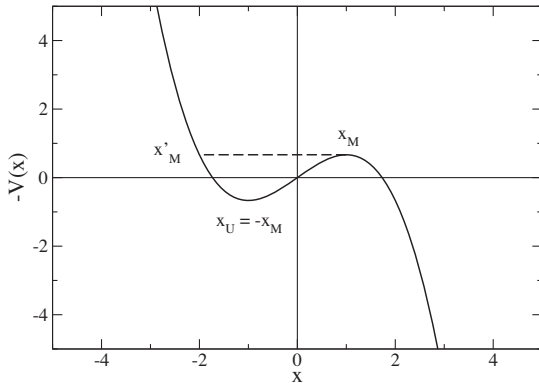

FIG. 5. Inverted potential occurring in the instanton theory (the dashed line locates the bounce).

$$E = \frac{1}{2}\alpha M \dot{R}_b^2 - V(R_b), \qquad (49)$$

where E is a constant that can be called the energy of the instanton. It is determined by the initial condition $\dot{R}_b = 0$ at $R_b = R_M$ giving $E = -V(R_M)$. As a result, the equation of the instanton is

$$\frac{1}{2}\alpha M \dot{R}_b^2 = V(R_b) - V(R_M), \qquad (50)$$

or, equivalently,

$$\dot{R}_b = \mp \sqrt{\frac{2}{\alpha M}[V(R_b) - V(R_M)]}, \qquad (51)$$

where we should use the sign $-$ before the bounce at R'_M and the sign $+$ after the bounce. The instanton profile is given by an integral of the form

$$\int_{R'_M}^{R_b(t)} \frac{dR}{\sqrt{[V(R) - V(R_M)]}} = \mp \sqrt{\frac{2}{\alpha M}}(t - t_c). \qquad (52)$$

An arbitrary parameter t_c indicates its center (defined by $\dot{R}_b(t_c) = 0$).

It is now easy to obtain a closed expression for the Euclidean action of the instanton in the limit $t \to +\infty$. Using Eq. (50) the bounce exponent

$$B = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\alpha M \left(\frac{dR_b}{dt}\right)^2 + V(R_b) - V(R_M)\right] dt \qquad (53)$$

can be written under the equivalent forms

$$B = \int_{-\infty}^{+\infty} 2[V(R_b) - V(R_M)] dt, \qquad (54)$$

or

$$B = \int_{-\infty}^{+\infty} \alpha M \dot{R}_b^2 dt. \qquad (55)$$

The last integral can be rewritten as

$$B = 2\int_{-\infty}^{t_c} \alpha M \dot{R}_b^2 dt = 2\int_{R_M}^{R'_M} \alpha M \dot{R}_b dR_b, \qquad (56)$$

leading to [see Eq. (51)]

$$B = 2\int_{R'_M}^{R_M} \sqrt{2\alpha M[V(R) - V(R_M)]} dR, \qquad (57)$$

where we recall that R'_M is the turning point (bounce) defined by the condition $V(R'_M) = V(R_M)$. If we use Eq. (54) or Eq. (55) to compute B, we have to explicitly

determine the trajectory of the instanton (bounce). If we use Eq. (57), this is not necessary. We just need to know the expression of the potential $V(R)$. This leads to the following expression of the quantum tunneling rate

$$\Gamma \sim A e^{-\frac{2}{\hbar} \int_{R'_M}^{R_M} \sqrt{2\alpha M[V(R)-V(R_M)]} dR}. \quad (58)$$

This expression, which is valid in the semiclassical approximation $\hbar \to 0$, can also be obtained by using the WKB method to find the transmission amplitude across the potential barrier [217]. Therefore, it is oftentimes called the WKB transmittivity formula.

In many applications, the exponential behavior of the tunneling rate is sufficient. The calculation of the prefactor A is more involved. It requires the determination of a fluctuation determinant which was obtained by Duru *et al.* [233] using the method of Gel'fand and Yaglom [234]. This leads to the following expression of the prefactor

$$A = \sqrt{\frac{\alpha M \omega_M v_M^2}{\pi \hbar}}, \quad (59)$$

where v_M, which depends on the details of the potential, is determined by the asymptotic behavior of the instanton solution via the formula

$$R_b(t) \simeq R_M - \frac{v_M}{\omega_M} e^{-\omega_M |t|} \quad (t \to \pm\infty). \quad (60)$$

The complete expression of the tunneling rate including the prefactor is therefore

$$\Gamma \sim \sqrt{\frac{\alpha M \omega_M v_M^2}{\pi \hbar}} e^{-\frac{2}{\hbar} \int_{R'_M}^{R_M} \sqrt{2\alpha M[V(R)-V(R_M)]} dR}. \quad (61)$$

Finally, the typical lifetime of the metastable state can be estimated by

$$t_{\text{life}} \sim \Gamma^{-1} \sim \sqrt{\frac{\pi \hbar}{\alpha M \omega_M v_M^2}} e^{\frac{2}{\hbar} \int_{R'_M}^{R_M} \sqrt{2\alpha M[V(R)-V(R_M)]} dR}. \quad (62)$$

B. Expression valid close to the maximum mass

In this section, we determine the quantum tunneling rate of the BEC close to the maximum mass M_{max} by using the normal form of the potential close to a saddle-center bifurcation given by Eq. (34). It is convenient to set $x = R - R_*$. In that case, the potential can be rewritten as

$$V(x) = \frac{1}{3} a x^3 - b x, \quad (63)$$

where a and b are two positive constants given by

$$a = \frac{V_0}{R_*^3} \quad \text{and} \quad b = \frac{2V_0}{R_*} \left(1 - \frac{M}{M_{\text{max}}}\right). \quad (64)$$

For simplicity, we have taken the additional constant in the potential equal to zero (this is possible without restriction of generality since only differences of potential occur in our problem). The potential (63) presents a local minimum and a local maximum (see Fig. 4). The local minimum of $V(x)$ is located at $x_M = \sqrt{b/a}$ and the value of the potential at that point is $V(x_M) = -(2/3)ax_M^3$. The maximum of $V(x)$ is located at $x_U = -x_M$ and the value of the potential at that point is $V(x_U) = (2/3)ax_M^3$. The bouncing (or escape) point x'_M where $V(x'_M) = V(x_M)$ is given by $x'_M = -2x_M$. Finally, we note that $V(x) = 0$ for $x = 0$ and for $x = \pm\sqrt{3b/a}$. With these notations, the potential (63) can be rewritten as

$$V(x) - V(x_M) = a \left(\frac{1}{3} x^3 - x_M^2 x + \frac{2}{3} x_M^3\right). \quad (65)$$

The roots of the third degree equation defined by the term in parenthesis in Eq. (65) are x_M (double root) and x'_M (single root). We then find that the potential (63) can be written as

$$V(x) - V(x_M) = \frac{a}{3} (x - x_M)^2 (x - x'_M). \quad (66)$$

For future use, we note that the barrier of potential $\Delta V = V(x_U) - V(x_M)$ is

$$\Delta V = \frac{4}{3} a x_M^3. \quad (67)$$

On the other hand, the square pulsations ($\omega^2 = V''(x)/\alpha M$) of the fictive particle at the metastable and unstable positions are

$$\omega_M^2 = \frac{2}{\alpha M} \sqrt{ab} \quad \text{and} \quad \omega_U^2 = -\frac{2}{\alpha M} \sqrt{ab}. \quad (68)$$

When $\hbar \to 0$, the quantum tunneling rate of the BEC is given by Eq. (61). We propose two methods to compute the bounce exponent B in the exponential factor using respectively the WKB formula and the instanton solution. We also compute the prefactor A of the tunneling rate.

1. The WKB formula

The expression of B can be obtained from the WKB formula (57). When the potential is given by Eq. (66), the integral appearing in Eq. (57) takes the explicit form

$$B = 2\sqrt{\frac{2\alpha M a}{3}} \int_{-2x_M}^{x_M} (x_M - x)\sqrt{x + 2x_M} \, dx. \quad (69)$$

With the change of variables

$$X = \sqrt{\frac{x}{3x_M} + \frac{2}{3}}, \tag{70}$$

it can be rewritten as

$$B = 36\sqrt{2\alpha M}ax_M^{5/2}\int_0^1 (1 - X^2)X^2\, dX. \tag{71}$$

Using the identity

$$\int_0^1 (1 - X^2)X^2\, dX = \frac{2}{15}, \tag{72}$$

we obtain

$$B = \frac{24}{5}\sqrt{2\alpha M}ax_M^{5/2}. \tag{73}$$

2. The instanton solution

The expression of B can also be obtained from Eqs. (54) and (55) by explicitly calculating the instanton solution. The instanton (bounce) is determined by Eq. (52). When the potential is given by Eq. (66), this equation becomes

$$\int^{x_b(t)} \frac{dx}{(x_M - x)\sqrt{x + 2x_M}} = -\sqrt{\frac{2a}{3\alpha M}}t. \tag{74}$$

With the change of variables

$$X = \sqrt{\frac{x}{3x_M} + \frac{2}{3}}, \tag{75}$$

it can be rewritten as

$$\int^{\sqrt{\frac{x_b(t)}{3x_M}+\frac{2}{3}}} \frac{dX}{1 - X^2} = -\sqrt{\frac{ax_M}{2\alpha M}}t. \tag{76}$$

Using the identity

$$\int \frac{dX}{1 - X^2} = \tanh^{-1}(X) \qquad (-1 < X < 1), \tag{77}$$

we obtain

$$x_b(t) = x_M\left[3\tanh^2\left(\sqrt{\frac{ax_M}{2\alpha M}}t\right) - 2\right]$$

$$= x_M\left[1 - \frac{3}{\cosh^2(\sqrt{\frac{ax_M}{2\alpha M}}t)}\right]. \tag{78}$$

This is the instanton (bounce) solution (see Fig. 6). We have chosen the origin of time so that the instanton center is at $x'_M = -2x_M$ (bouncing point) at $t = 0$. For $t \to \pm\infty$, we
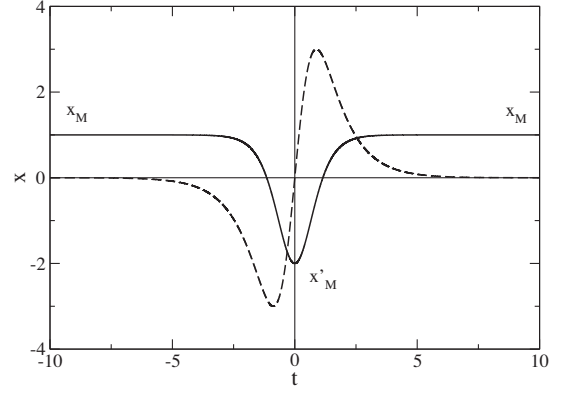

FIG. 6. Quantum instanton (bounce) $x_b(t)$ close to the maximum mass (the dashed line corresponds to the velocity $\dot{x}_b(t)$).

have $x_b(t) \to x_M$. It is precisely this type of solutions, which approach a static limit in the distant past and future, that are referred to as "instantons." The velocity of the fictive particle associated with the instanton solution is

$$\dot{x}_b(t) = 6x_M\sqrt{\frac{ax_M}{2\alpha M}}\frac{\sinh\left(\sqrt{\frac{ax_M}{2\alpha M}}t\right)}{\cosh^3\left(\sqrt{\frac{ax_M}{2\alpha M}}t\right)}. \tag{79}$$

Substituting this expression into Eq. (55) we obtain

$$B = 18\sqrt{2\alpha M}ax_M^{5/2}\int_{-\infty}^{+\infty} \frac{\sinh^2(x)}{\cosh^6(x)}\,dx. \tag{80}$$

Using the identity

$$\int_{-\infty}^{+\infty} \frac{\sinh^2(x)}{\cosh^6(x)}\,dx = \frac{4}{15}, \tag{81}$$

we recover Eq. (73). The same expression can also be obtained from Eq. (54).

3. The prefactor

To obtain the prefactor of the tunneling rate given by Eq. (59) we first note that $\omega_M^2 = V''(x_M)/\alpha M = 2ax_M/\alpha M$. On the other hand, from Eq. (78), we have

$$x_b(t) \simeq x_M - 12x_M e^{-\sqrt{\frac{2ax_M}{\alpha M}}|t|} \qquad (t \to \pm\infty). \tag{82}$$

Comparing this asymptotic behavior with the expression from Eq. (60), we obtain

$$v_M = 12\left(\frac{2a}{\alpha M}\right)^{1/2}x_M^{3/2}. \tag{83}$$

Therefore, the prefactor of the tunneling rate is

$$A = 12 \left(\frac{8a^3}{\pi^2 \alpha M \hbar^2} \right)^{1/4} x_M^{7/4}. \tag{84}$$

Combining Eqs. (45), (73) and (84), we find that the complete expression of the quantum tunneling rate of the BEC close to the maximum mass is given by

$$\Gamma \sim 12 \left(\frac{8a^3}{\pi^2 \alpha M \hbar^2} \right)^{1/4} x_M^{7/4} e^{-\frac{24}{5\hbar} \sqrt{2\alpha M a} x_M^{5/2}}. \tag{85}$$

Returning to the original variables, we get

$$\Gamma \sim 12 \left(\frac{8}{\pi^2} \right)^{1/4} \left[2 \left(1 - \frac{M}{M_{\max}} \right) \right]^{7/8} (\alpha \sigma)^{1/4} \sqrt{N}$$
$$\times e^{-\frac{24}{5} \sqrt{2} [2(1 - \frac{M}{M_{\max}})]^{5/4} \sqrt{\alpha \sigma} N} t_D^{-1}, \tag{86}$$

where we have introduced the particle number $N = M/m$ and we recall that the above expression is valid for $M \to M_{\max}$. We note that the bounce exponent scales as $B \propto (1 - M/M_{\max})^{5/4}$ and the prefactor as $A \propto (1 - M/M_{\max})^{7/8}$. These are the same scalings as those obtained in Refs. [218,219] for nongravitational BECs. These scalings are universal since they just depend on the normal form of the potential close to a saddle-center bifurcation.

V. THERMAL TUNNELING RATE OF THE BEC

In addition to quantum fluctuations, the BEC may also experience thermal fluctuations that can destabilize the metastable equilibrium state. Indeed, because of thermal fluctuations the system can overcome the energy barrier between the metastable state and the unstable state and collapse. We provide here a very heuristic treatment of thermal fluctuations in a BEC, using an analogy with the Kramers [235] problem in Brownian theory (a similar approach has been used in [195,219] for nongravitational BECs and in [224] for globular clusters).

In Sec. III, making a Gaussian ansatz, we have reduced the original problem (solving the GPP equations (3) and (4)) to the simpler mechanical problem of a particle with mass αM in a potential $V(R)$ governed by the deterministic equation (25). Within this framework, we have taken into account quantum fluctuations in Sec. IV by replacing the deterministic equation (25) by the Schrödinger equation (44). Similarly, we can take thermal fluctuations into account by replacing the determinsitic equation (25) by a stochastic Langevin equation of the form

$$\alpha M \frac{d^2 R}{dt^2} + \xi \alpha M \frac{dR}{dt} = -\frac{dV}{dR} + \sqrt{2\xi \alpha M k_B T} \eta(t), \tag{87}$$

where $\eta(t)$ is a Gaussian white noise with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$. This equation involves a friction force characterized by a friction coefficient ξ and a random force whose strength is measured by the temperature T. These two effects arise simultaneously on account of the fluctuation-dissipation theorem encapsulated in the Einstein relation $D = \xi k_B T / \alpha M$, where D is the diffusion coefficient. The thermal tunneling (or thermal activation) rate is of the general form

$$\Gamma \sim A e^{-\Delta V / k_B T}, \tag{88}$$

where $\Delta V = V(R_U) - V(R_M)$ is the potential barrier between the metastable state and the unstable state and A is a prefactor discussed below. The expression (88) is valid when $k_B T \ll \Delta V$. The exponential term in Eq. (88) was obtained long ago by Arrhenius [236] from an empirical analysis of chemical reaction rates and is called the Arrhenius law. It was later justified by Kramers [235] from the detailed study of the stochastic motion of a Brownian particle past a potential barrier. The prefactor has different expressions depending on the considered regime. Kramers [235] obtained the general formula

$$\Gamma \sim \frac{\omega_M}{2\pi |\omega_U|} \left[\sqrt{\frac{\xi^2}{4} + |\omega_U|^2} - \frac{\xi}{2} \right] e^{-\Delta V / k_B T}, \tag{89}$$

where we recall that $\omega_M^2 = V''(R_M)/\alpha M > 0$ and $\omega_U^2 = V''(R_U)/\alpha M < 0$. This formula was derived from a Fokker-Planck equation in phase space (Kramers equation). In the strong friction limit $\xi \to +\infty$, Eq. (89) reduces to

$$\Gamma \sim \frac{\omega_M |\omega_U|}{2\pi \xi} e^{-\Delta V / k_B T}. \tag{90}$$

This asymptotic result can be directly obtained from a Fokker-Planck equation in position space (Smoluchowski equation). In the weak friction limit $\xi \to 0$, Eq. (89) reduces to

$$\Gamma \sim \frac{\omega_M}{2\pi} e^{-\Delta V / k_B T}, \tag{91}$$

which corresponds to the result of the transition state theory. However, using a more careful treatment, Kramers [235] showed that this asymptotic formula is not perfectly correct and that it must be replaced by

$$\Gamma \sim \frac{\xi I_U}{k_B T} \frac{\omega_M}{2\pi} e^{-\Delta V / k_B T}, \tag{92}$$

where $I_U \sim 2\pi \Delta V / \omega_M$ is the action of the path at the barrier peak. This more accurate expression shows that, when $\xi \to 0$, the thermal tunneling rate Γ vanishes proportionally to ξ instead of tending to a constant. For sufficiently large values of ξ, the expressions (89) and (90) become valid.

Close to the maximum mass, using the normal form of the potential (66), we find that the thermal tunneling rate of the BEC based on the Arrhenius law (88) is given by

$$\Gamma \propto e^{-\frac{4}{3}[2(1-\frac{M}{M_{\max}})]^{3/2}\nu\eta N}, \qquad (93)$$

where we have introduced the particle number $N = M/m$ and the dimensionless inverse temperature

$$\eta = \frac{GM_{\max}m}{R_* k_B T}. \qquad (94)$$

We recall that Eq. (93) is valid for $M \to M_{\max}$. The complete expression of the thermal tunneling rate based on the Kramers formula (89) is

$$\Gamma \sim \frac{1}{2\pi} \left[\sqrt{\frac{\xi^2}{4} + \frac{2}{t_D^2} \left[2\left(1 - \frac{M}{M_{\max}}\right)\right]^{1/2}} - \frac{\xi}{2} \right]$$
$$\times e^{-\frac{4}{3}[2(1-\frac{M}{M_{\max}})]^{3/2}\nu\eta N}. \qquad (95)$$

In the strong friction limit $\xi \to +\infty$, we get

$$\Gamma \sim \frac{1}{\pi\xi t_D^2} \left[2\left(1 - \frac{M}{M_{\max}}\right)\right]^{1/2} e^{-\frac{4}{3}[2(1-\frac{M}{M_{\max}})]^{3/2}\nu\eta N}. \qquad (96)$$

In the weak friction limit $\xi \to 0$, we obtain

$$\Gamma \sim \frac{1}{\sqrt{2}\pi t_D} \left[2\left(1 - \frac{M}{M_{\max}}\right)\right]^{1/4} e^{-\frac{4}{3}[2(1-\frac{M}{M_{\max}})]^{3/2}\nu\eta N}, \qquad (97)$$

although this formula is not fully correct as mentioned above. We note that the potential barrier scales as $\Delta V \propto (1 - M/M_{\max})^{3/2}$. This is the same scaling as the one obtained in [219] for nongravitational BECs and in [224] for globular clusters. This scaling is universal since it just depends on the normal form of the potential close to a saddle-center bifurcation.

Remark: The thermal tunneling rate of the BEC can also be obtained by applying the instanton theory to the generalized stochastic GPP and quantum Smoluchowski-Poisson equations [232] (see also Appendix C for the related stochastic Ginzburg-Landau-Poisson equations).

VI. CORRECTION OF THE CRITICAL MASS DUE TO QUANTUM AND THERMAL FLUCTUATIONS

Let us summarize the preceding results. A self-gravitating BEC with an attractive self-interaction can exist only below a maximum mass $M_{\max}$ [37,38]. For $M < M_{\max}$ and $R > R_*$ the BEC is in a metastable state (local but not global minimum of energy) corresponding to a dilute axion star. Because of quantum fluctuations it can decay into a more stable state (dense axion star) if we account for the repulsive self-interaction between the bosons, or collapse if there is no repulsive self-interaction. The lifetime of the metastable state due to quantum fluctuations can be estimated by $t_{\text{life}}^Q \sim 1/\Gamma_Q$ where Γ_Q is the quantum tunneling rate of the BEC. According to Eq. (86), we have

$$t_{\text{life}}^Q \sim \frac{1}{12} \left(\frac{\pi^2}{8}\right)^{1/4} \left[2\left(1 - \frac{M}{M_{\max}}\right)\right]^{-7/8} \frac{1}{(\alpha\sigma)^{1/4}} \frac{1}{\sqrt{N}}$$
$$\times e^{\frac{24}{5}\sqrt{2}[2(1-\frac{M}{M_{\max}})]^{5/4}\sqrt{\alpha\sigma}N} t_D. \qquad (98)$$

The quantum lifetime of dilute axion stars scales as

$$t_{\text{life}}^Q \sim e^N t_D, \qquad (99)$$

except close to the critical point. Since N is very large ($N = 7.21 \times 10^{56}$ for QCD axions and $N = 5.09 \times 10^{95}$ for ULAs), the lifetime of a metastable state is considerable [86]. As a matter of fact, metastable states can be considered as stable states. Only extraordinarily close to the maximum mass will their lifetime decrease. In principle, the BEC will collapse at a mass M_{crit} smaller than $M_{\max}$. The mass at which the BEC collapses because of quantum tunneling can be estimated by writing that the exponent of the exponential term in Eq. (98) is of order unity. This gives

$$M_{\text{crit}}^Q \sim M_{\max}(1 - 0.103 N^{-4/5}). \qquad (100)$$

It displays the scaling $N^{-4/5}$. For large values of N, which is the case for axion stars, this correction is extremely small and can be neglected. Therefore, the value of the maximum mass of axion stars [37,38] is essentially unaffected by quantum tunneling. However, corrections due to quantum tunneling could be observed in laboratory experiments and numerical simulations attempting to mimic "axion stars" because, in that case, the number of particles will be necessarily reduced as compared to reality.

Similar results are obtained if we account for thermal tunneling. The lifetime of the metastable state due to thermal fluctuations can be estimated by $t_{\text{life}}^T \sim 1/\Gamma_T$ where Γ_T is the thermal tunneling rate of the BEC. According to Eq. (93), we have

$$t_{\text{life}}^T \propto e^{\frac{4}{3}[2(1-\frac{M}{M_{\max}})]^{3/2}\nu\eta N}, \qquad (101)$$

where the value of the prefactor is discussed in Sec. V. The thermal lifetime of dilute axion stars scales as

$$t_{\text{life}}^T \propto e^N, \qquad (102)$$

except close to the critical point. The mass at which the BEC collapses because of thermal tunneling can be estimated by

$$M_c^T \sim M_{\max}(1 - 0.762 N^{-2/3}), \qquad (103)$$

where we have taken $\eta \sim 1$ for simplicity. It displays the scaling $N^{-2/3}$. The same scaling was found in [224,237] in the case of globular clusters. For axion stars, this correction is extremely small and can be neglected. Therefore, the

value of the maximum mass of axion stars [37,38] is not altered by thermal effects.

VII. FINITE SIZE SCALING CLOSE TO THE MAXIMUM MASS

In Ref. [75] we have studied the collapse time of dilute axion stars when $M > M_{\max}$. For $M \to M_{\max}^+$, we have found that

$$\frac{t_{\text{coll}}}{t_D} \sim \frac{2.90}{(M/M_{\max} - 1)^{1/4}}. \qquad (104)$$

This study neglects quantum and thermal fluctuations. As a result, it predicts that the collapse time is infinite when $M = M_{\max}$. Actually, because of quantum and thermal fluctuations, the collapse time should be large but finite at $M = M_{\max}$. We develop below an argument to estimate the finite size scaling of the collapse time close to the maximum mass and its finite value at $M = M_{\max}$.

For $M < M_{\max}$, we have found that the quantum lifetime of dilute axion stars is given by Eq. (98). Finite size effects enter in the expression of the metastable state lifetime in the combination $N(1 - M/M_{\max})^{5/4}$. If we assume that a similar combination enters in the expression of the collapse time for $M > M_{\max}$ we expect a scaling of the form

$$\frac{t_{\text{coll}}}{t_D} \propto (M/M_{\max} - 1)^{-1/4} F[N(M/M_{\max} - 1)^{5/4}] \qquad (105)$$

with $F(x) \to 1$ for $x \to +\infty$ in order to recover Eq. (104) when $N \to +\infty$. At $M = M_{\max}$, the singular factor $(M/M_{\max} - 1)^{-1/4}$ must cancel out implying that $F(x) \sim x^{1/5}$ for $x \to 0$. Therefore, at $M = M_{\max}$, the collapse time taking into account quantum fluctuations scales as

$$t_{\text{coll}}^{\text{Q}} \propto N^{1/5} t_D, \qquad (M = M_{\max}). \qquad (106)$$

We can make similar calculations to account for thermal fluctuations. In that case, finite size effects enter in the expression of the metastable state lifetime in the combination $N(1 - M/M_{\max})^{3/2}$ [see Eq. (101)]. We therefore expect a scaling of the form

$$\frac{t_{\text{coll}}}{t_D} \propto (M/M_{\max} - 1)^{-1/4} F[N(M/M_{\max} - 1)^{3/2}] \qquad (107)$$

with $F(x) \to 1$ for $x \to +\infty$ and $F(x) \to x^{1/6}$ for $x \to 0$. Therefore, at $M = M_{\max}$, the collapse time taking into account thermal fluctuations scales as

$$t_{\text{coll}}^{\text{T}} \propto N^{1/6} t_D, \qquad (M = M_{\max}). \qquad (108)$$

For QCD axions with mass $m = 10^{-4}$ eV/c^2 and self-interaction $a_s = -5.8 \times 10^{-53}$ m, the maximum mass is

$M_{\max}^{\text{exact}} = 6.46 \times 10^{-14} \, M_\odot$ and the minimum radius is $(R_{99}^*)^{\text{exact}} = 227$ km. As a result, the typical number of axions is $N \sim 10^{57}$ and the typical dynamical time is $t_D \sim 10$ hrs. Then, we get $t_{\text{coll}}^Q \sim 10^8$ yrs and $t_{\text{coll}}^T \sim 10^6$ yrs. The collapse time of QCD axion stars at criticality is smaller than the age of the Universe ($\sim 14 \times 10^9$ yrs).

For ULAs with mass $m = 2.19 \times 10^{-22}$ eV/c^2 and self-interaction $a_s = -1.11 \times 10^{-62}$ fm, the maximum mass is $M_{\max}^{\text{exact}} = 10^8 \, M_\odot$ and the minimum radius is $(R_{99}^*)^{\text{exact}} = 1$ kpc. As a result, the typical number of axions is $N \sim 10^{96}$ and the typical dynamical time is $t_D \sim 10^8$ yrs. Then, we get $t_{\text{coll}}^Q \sim 10^{27}$ yrs and $t_{\text{coll}}^T \sim 10^{24}$ yrs. The collapse time of axion stars (or of the quantum core of DM halos) made of ULAs at criticality is much larger than the age of the Universe ($\sim 14 \times 10^9$ yrs).

Remark: We note that the scalings from Eqs. (106) and (108) for the collapse time at criticality are also valid for nongravitational BECs with an attractive self-interaction in a confining trap (see footnote 10). They do not seem to have been reported previously in that context.

VIII. CONCLUSION

In this paper, we have computed the quantum and thermal tunneling rates of dilute axion stars close to the maximum mass $M_{\max}$ [37,38]. In the quantum case, we have shown that the bounce exponent vanishes as $(1 - M/M_{\max})^{5/4}$ and the amplitude as $(1 - M/M_{\max})^{7/8}$. In the thermal case, we have shown that the energy barrier vanishes as $(1 - M/M_{\max})^{3/2}$. The same scalings were previously obtained in the case of nongravitational BECs with attractive self-interaction in a harmonic trap close to the maximum particle number [218,219]. The scaling for the bounce exponent of the quantum tunneling rate was also obtained long ago by [228] in the case of neutron stars close to the Oppenheimer-Volkoff maximum mass and the scaling for the thermal tunneling rate was also obtained by [224] in the case of globular clusters close to the point of gravitational collapse. These scalings reflect the universal form of the potential close to a saddle-center bifurcation. However, despite these attenuation factors, the lifetime of dilute axion stars generically scales as $e^N t_D$ as anticipated in [86]. In the case of axion stars, the number of bosons is very large ($N \sim 10^{50}$–10^{100}) implying that the lifetime of dilute axion stars is considerable. As a matter of fact, these metastable states can be considered as stable states [86,160] except extremely close to the critical point. Barrier penetration is a notoriously slow process. Indeed, similar results regarding the very long lifetime of metastable states have been previously obtained in the case of systems with long-range interactions [238–241], neutron stars [228],

quantum field theory in the early universe [242–252],[18] laboratory BECs [195,218,219], globular clusters [224,237],[19] and Q-balls [222]. In all these examples, quantum and thermal tunneling are very rare processes.

More precisely, we can draw the following conclusions depending on the number of particles in the system:

(i) In laboratory BECs, the number of bosons in a quantum gas is moderate ($N \sim 1000$) so that quantum and thermal tunnelings, even if they are small, can be observed and measured [218,219].

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[18]In the context of quantum field theory, it was believed in the 1970-1980s that the early Universe, by cooling below some critical temperature T_0, had undergone a first order phase transition from a metastable symmetic state $\varphi = 0$ (false vacuum) to a stable symmetry-breaking Higgs state $\varphi = \sigma$ (true vacuum) giving mass to the particles [193,194,253–256]. The tunneling was expected to proceed through the formation of bubbles like in the liquid-gas phase transition [257]. However, it was soon realized that the nucleation of bubbles, thermal or quantum, was a very rare event and that the tunneling probability was extremely small. As a result, the lifetime of the metastable vacuum state $\varphi = 0$ in gauge theories is usually extremely large [242–252], much larger than the age of the Universe. In practice, in these scenarios, the Universe remains in the supercooled symmetric vacuum state $\varphi = 0$, leading to a phase of inflation during which the scale factor increases exponentially with time [246–251,258–263]. Therefore, the transition from the symmetric state $\varphi = 0$ to the asymmetric state $\varphi = \sigma$ does not occur at T_0, where the asymmetric state becomes energetically favored, but at a much lower temperature $\sim T_c$ at which the symmetric state $\varphi = 0$ becomes unstable. Now that the Higgs mass has been measured and found to be much larger than the value required in the previous scenarios, it is rather believed that the early Universe experienced a second order phase transition from a symmetric phase $\varphi = 0$ at high temperatures ($T > T_c$) to a symmetry-breaking phase $\varphi = \sigma$ at low temperatures ($T < T_c$).

[19]The very long lifetime of metastable states justifies the notion of statistical equilibrium for self-gravitating systems [224]. It is well known since the works of Antonov [264] and Lynden-Bell and Wood [265] that no equilibrium state for self-gravitating systems exists in a strict sense, even if they are confined within a box in order to prevent their evaporation or if we use the King model to take into account tidal effects. They can always increase their entropy at fixed energy and mass by forming a "core-halo" structure made of a binary star (containing a very negative potential energy) surrounded by a hot halo (containing a very positive kinetic energy). In this sense, there is no global maximum of entropy at fixed energy and mass. The system is ultimately expected to collapse (gravothermal catastrophe). However, there exist metastable equilibrium states (local but not global maxima of entropy at fixed energy and mass) if the energy is not too low. If the system is initially in a metastable state (which is the most natural situation), it must cross a huge barrier of entropy to collapse. This is achieved by forming a condensed structure (or a binary star) similar to a "critical droplet" in the physics of phase transitions and nucleation problems. This requires nontrivial three-body or higher correlations. This is a very rare event whose probability scales as e^{-N}. Therefore the lifetime of the metastable state scales as e^N. This lifetime is generally larger or comparable to the age of the Universe making metastable states fully relevant. Therefore, in practice, metastable states can be considered as stable equilibrium states.

(ii) In globular clusters, the number of stars is of the order of $N \sim 10^6$. This number is large but not gigantic. In particular, finite N effects can advance the onset of gravitational collapse (gravothermal catastrophe) [224,237,266]. The critical density contrast taking into account the finite number of particles is $\mathcal{R}_c = 709 \times \exp(-3.30 N^{-1/3})$ giving $\mathcal{R}_c = 686$ for $N = 10^6$ instead of $\mathcal{R}_c = 709$. This may explain why observations reveal that a greater number of globular clusters than is normally believed may already be in an advanced stage of core collapse.

(iii) In axion stars, the number of bosons is of the order of $N \sim 10^{50}$–10^{100}. This number is gigantic so that quantum and thermal tunnelings are completely negligible and the onset of gravitational collapse (or the value of the maximum mass) is not altered [86]. Only extraordinarily close to the maximum mass does their lifetime decrease.

Our results show that the tunneling rate of dilute axion stars is usually negligible. This strengthens the validity of our former results [37,75,86] where this effect was neglected from the start. This is because, in axion stars, the number of bosons N is huge. However, we can imagine that, in a close future, it will be possible to make laboratory experiments and numerical simulations of "axion stars" or self-gravitating BECs with an attractive self-interaction. In such experiments, and in the first generation of numerical simulations, the number of bosons N will be relatively small and tunneling effects may be measurable. Our results will be useful to interpret such experiments and numerical simulations. The general methods presented in our paper may also find applications in other situations of physical interest, beyond axion stars, where the tunneling rate is larger.

On the other hand, the present study allowed us to take into account the effect of fluctuations during the collapse of axion stars which were ignored in our previous work [75]. In particular, at the maximum mass $M = M_{\text{max}}$, we found that the collapse time scales as $t_{\text{coll}}^{\text{Q}} \propto N^{1/5} t_D$ in the quantum case and as $t_{\text{coll}}^{\text{T}} \propto N^{1/6} t_D$ in the thermal case instead of being infinite as implied by Eq. (104) which does not take into account quantum and thermal fluctuations [75]. We then found that the collapse time is smaller than age of universe for QCD axion stars but larger for axion stars (or for the quantum core of DM halos) made of ULAs.

APPENDIX A: FROM THE KGE EQUATIONS TO THE GPP EQUATIONS

In this Appendix, we show that the GPP equations (3) and (4) can be derived from the KGE equations in the nonrelativistic limit $c \to +\infty$. For the sake of generality, we take into account the expansion of the Universe (the static case is recovered for $a = 1$). As an example, we consider the case of dilute axion stars and relate the scattering length a_s appearing in the GP equation (3) to

the axion decay constant f. This Appendix follows Secs. II and III of [86].

Let us consider the relativistic quantum field theory of a real SF $\varphi(\mathbf{r}, t)$ with an action

$$S = \int d^4x \sqrt{-g} \mathcal{L} \qquad (A1)$$

associated with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2 c^2}{2\hbar^2} \varphi^2 - V(\varphi) + \frac{c^4}{16\pi G} R, \quad (A2)$$

where $g_{\mu\nu}$ is the metric tensor, g is its determinant, R is the Ricci scalar, and $V(\varphi)$ is the potential of the SF. The least action principle $\delta S = 0$ leads to the KGE equations (see, e.g., [63])

$$\Box \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{dV}{d\varphi} = 0, \qquad (A3)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (A4)$$

where $\Box = D_\mu(g^{\mu\nu}\partial_\nu) = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu)$ is the d'Alembertian in a curved spacetime and

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi - \frac{m^2 c^2}{2\hbar^2} \varphi^2 - V(\varphi) \right] \qquad (A5)$$

is the energy-momentum tensor of the SF with $T_0^0 = \epsilon$ the energy density. The SF may represent the axion. The instanton potential of the axion is [126,267,268]

$$V(\varphi) = \frac{m^2 c f^2}{\hbar^3} \left[1 - \cos\left(\frac{\hbar^{1/2} c^{1/2} \varphi}{f} \right) \right] - \frac{m^2 c^2}{2\hbar^2} \varphi^2, \quad (A6)$$

where m is the mass of the axion and f is the axion decay constant. For this potential, the KG equation (A3) takes the form

$$\Box \varphi + \frac{m^2 c^{3/2} f}{\hbar^{5/2}} \sin\left(\frac{\hbar^{1/2} c^{1/2} \varphi}{f} \right) = 0. \qquad (A7)$$

This is the general relativistic sine-Gordon equation. Considering the dilute regime $\varphi \ll f/\sqrt{\hbar c}$ (which is valid in particular in the nonrelativistic limit $c \to +\infty$ considered below)[20] and expanding the cosine term in Eq. (A6) in Taylor series, we obtain at leading order the φ^4 potential

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[20]According to Eq. (A40), the axion decay constant f scales as $c^{3/2}$.

$$V(\varphi) = -\frac{m^2 c^3}{24 f^2 \hbar} \varphi^4. \qquad (A8)$$

In that case, the KG equation (A3) takes the form

$$\Box \varphi + \frac{m^2 c^2}{\hbar^2} \varphi - \frac{m^2 c^3}{6 f^2 \hbar} \varphi^3 = 0. \qquad (A9)$$

In general, a quartic potential is written as

$$V(\varphi) = \frac{\lambda}{4\hbar c} \varphi^4, \qquad (A10)$$

where λ is the dimensionless self-interaction constant. Comparing Eqs. (A8) and (A10), we find that

$$\lambda = -\frac{m^2 c^4}{6 f^2}. \qquad (A11)$$

We note that $\lambda < 0$, so that the φ^4 self-interaction term for axions is attractive. This attraction is responsible for the collapse of dilute axion stars above a maximum mass [37,75]. The next order φ^6 term has been considered in [86] and turns out to be repulsive. This repulsion, that occurs at high densities, may stop the collapse of dilute axion stars and lead to the formation of dense axion stars [145].

In the weak-field gravity limit of general relativity $\Phi/c^2 \ll 1$, using the simplest form of the Newtonian gauge, the Friedmann-Lemaître-Robertson-Walker (FLRW) line element is given by

$$ds^2 = c^2 \left(1 + 2\frac{\Phi}{c^2} \right) dt^2 - a(t)^2 \left(1 - 2\frac{\Phi}{c^2} \right) \delta_{ij} dx^i dx^j, \qquad (A12)$$

where $\Phi(\mathbf{r}, t)$ is the Newtonian potential and $a(t)$ is the scale factor. In the Newtonian limit $\Phi/c^2 \to 0$, the KGE equations (A3) and (A4) for the inhomogeneous SF reduce to

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + \frac{3H}{c^2} \frac{\partial \varphi}{\partial t} - \frac{1}{a^2} \Delta \varphi + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2\Phi}{c^2} \right) \varphi + \frac{dV}{d\varphi} = 0, \qquad (A13)$$

$$\frac{\Delta \Phi}{4\pi G a^2} = \frac{\epsilon}{c^2} - \frac{3H^2}{8\pi G}, \qquad (A14)$$

where $H = \dot{a}/a$ is the Hubble parameter and the energy density is given by

$$\epsilon = \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2a^2}(\nabla\varphi)^2 + \frac{m^2c^2}{2\hbar^2}\varphi^2 + V(\varphi). \quad \text{(A15)}$$

For the φ^4 potential (A8), we get

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{3H}{c^2}\frac{\partial\varphi}{\partial t} - \frac{1}{a^2}\Delta\varphi + \frac{m^2c^2}{\hbar^2}\left(1 + \frac{2\Phi}{c^2}\right)\varphi - \frac{m^2c^3}{6f^2\hbar}\varphi^3 = 0 \tag{A16}$$

and

$$\epsilon = \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2a^2}(\nabla\varphi)^2 + \frac{m^2c^2}{2\hbar^2}\varphi^2 - \frac{m^2c^3}{24f^2\hbar}\varphi^4. \tag{A17}$$

In Eqs. (A13)–(A15), we have neglected general relativity (i.e., we have treated gravity in the Newtonian framework)[21] but we have kept special relativity effects (see Eqs. (2) and (3) of [86] for more general equations valid at the order $O(\Phi/c^2)$ in the post-Newtonian approximation). Considering now the nonrelativistic limit $c \to +\infty$ where the SF displays rapid oscillations, these equations can be simplified. To that purpose, we write

$$\varphi = \frac{1}{\sqrt{2}}\frac{\hbar}{m}[\psi(\mathbf{r},t)e^{-imc^2t/\hbar} + \psi^*(\mathbf{r},t)e^{imc^2t/\hbar}], \quad \text{(A18)}$$

where the complex wave function $\psi(\mathbf{r},t)$ is a slowly varying function of time (the fast oscillations $e^{imc^2t/\hbar}$ of the SF have been factored out). This transformation allows us to separate the fast oscillations of the SF with pulsation $\omega = mc^2/\hbar$ caused by its rest mass from the slow evolution of $\psi(\mathbf{r},t)$. From Eq. (A18), we get

$$\dot\varphi = \frac{1}{\sqrt{2}}\frac{\hbar}{m}\left[\dot\psi e^{-imc^2t/\hbar} - \frac{imc^2}{\hbar}\psi e^{-imc^2t/\hbar} + \text{c.c.}\right], \quad \text{(A19)}$$

$$\nabla\varphi = \frac{1}{\sqrt{2}}\frac{\hbar}{m}(\nabla\psi e^{-imc^2t/\hbar} + \text{c.c.}), \tag{A20}$$

$$\ddot\varphi = \frac{1}{\sqrt{2}}\frac{\hbar}{m}\left[\ddot\psi e^{-imc^2t/\hbar} - \frac{2imc^2}{\hbar}\dot\psi e^{-imc^2t/\hbar} - \frac{m^2c^4}{\hbar^2}\psi e^{-imc^2t/\hbar} + \text{c.c.}\right], \tag{A21}$$

$$\Delta\varphi = \frac{1}{\sqrt{2}}\frac{\hbar}{m}(\Delta\psi e^{-imc^2t/\hbar} + \text{c.c.}), \tag{A22}$$

where c.c. denotes complex conjugation. These equations are exact.[22] On the other hand, if we compute $\dot\varphi^2$, $(\nabla\varphi)^2$, φ^2, φ^3 and φ^4 from Eqs. (A18)–(A20) and neglect terms with a rapidly oscillating phase factor $e^{inmc^2t/\hbar}$ with $n \geq 2$, we get[23]

$$\dot\varphi^2 = \frac{\hbar^2}{m^2}\left|\frac{\partial\psi}{\partial t}\right|^2 + c^4|\psi|^2 - 2\frac{\hbar c^2}{m}\text{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right), \quad \text{(A24)}$$

$$(\nabla\varphi)^2 = \frac{\hbar^2}{m^2}|\nabla\psi|^2, \tag{A25}$$

$$\varphi^2 = \frac{\hbar^2}{m^2}|\psi|^2, \tag{A26}$$

$$\varphi^3 \simeq \frac{1}{2\sqrt{2}}\frac{\hbar^3}{m^3}(3\psi^2\psi^*e^{-imc^2t/\hbar} + \text{c.c.}), \tag{A27}$$

$$\varphi^4 = \frac{3\hbar^4}{2m^4}|\psi|^4. \tag{A28}$$

Substituting these relations into the KGE equations (A14), (A16) and (A17), and neglecting oscillatory terms, we obtain the relativistic GPP equations (see Eqs. (7) and (8) of [86] for more general equations valid at the order $O(\Phi/c^2)$ in the post-Newtonian approximation):

$$i\hbar\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2mc^2}\frac{\partial^2\psi}{\partial t^2} - \frac{3}{2}H\frac{\hbar^2}{mc^2}\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2ma^2}\Delta\psi$$
$$- m\Phi\psi - m\frac{dV_{\text{eff}}}{d|\psi|^2}\psi + \frac{3}{2}i\hbar H\psi = 0, \tag{A29}$$

$$\frac{\Delta\Phi}{4\pi Ga^2} = |\psi|^2 + \frac{\hbar^2}{2m^2c^4}\left|\frac{\partial\psi}{\partial t}\right|^2 + \frac{\hbar^2}{2a^2m^2c^2}|\nabla\psi|^2$$
$$+ \frac{1}{c^2}V_{\text{eff}}(|\psi|^2) - \frac{\hbar}{mc^2}\text{Im}\left(\frac{\partial\psi}{\partial t}\psi^*\right) - \frac{3H^2}{8\pi G}, \tag{A30}$$

with the effective potential

$$V_{\text{eff}}(|\psi|^2) = -\frac{\hbar^3 c^3}{16 f^2 m^2} |\psi|^4. \qquad \text{(A31)}$$

We note that $V_{\text{eff}}(|\psi|^2) \equiv \overline{V(\varphi)}$ is different from the expression that one would have obtained by directly substituting Eq. (A26) into Eq. (A8).[24] They differ by a factor $2/3$. On the other hand, assuming $(\hbar/mc^2)|\ddot{\psi}| \ll |\dot{\psi}|$, $(\hbar/mc^2)|\dot{\psi}| \ll |\psi|$ and $mc^2/\hbar \gg H$, Eqs. (A29) and (A30) reduce to

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i\hbar H\psi = -\frac{\hbar^2}{2ma^2} \Delta\psi + m\Phi\psi + m\frac{dV_{\text{eff}}}{d|\psi|^2}\psi, \qquad \text{(A32)}$$

$$\frac{\Delta\Phi}{4\pi Ga^2} = |\psi|^2 + \frac{\hbar^2}{2a^2m^2c^2}|\nabla\psi|^2 + \frac{1}{c^2}V_{\text{eff}}(|\psi|^2) - \frac{3H^2}{8\pi G}. \qquad \text{(A33)}$$

Finally, taking the nonrelativistic limit $c \to +\infty$, we obtain the GPP equations

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{3}{2} i\hbar H\psi = -\frac{\hbar^2}{2ma^2} \Delta\psi + m\Phi\psi + m\frac{dV_{\text{eff}}}{d|\psi|^2}\psi, \qquad \text{(A34)}$$

$$\frac{\Delta\Phi}{4\pi Ga^2} = |\psi|^2 - \frac{3H^2}{8\pi G}. \qquad \text{(A35)}$$

In the last expression, the energy density is given by

$$\frac{\epsilon}{c^2} \simeq |\psi|^2. \qquad \text{(A36)}$$

Since ϵ/c^2 represents, in the nonrelativistic limit $c \to +\infty$, the rest-mass density ρ, we conclude that $\rho = |\psi|^2$. Therefore, the field equation (A14) reduces to the Poisson equation

$$\Delta\Phi = 4\pi Ga^2\left(\rho - \frac{3H^2}{8\pi G}\right). \qquad \text{(A37)}$$

Equation (A31) is valid for the φ^4 potential (A8). More generally, the effective potential associated with the axion potential (A6) is

$$V_{\text{eff}}(|\psi|^2) = \frac{m^2 c f^2}{\hbar^3}\left[1 - \frac{\hbar^3 c}{2f^2m^2}|\psi|^2 - J_0\left(\sqrt{\frac{2\hbar^3 c|\psi|^2}{f^2m^2}}\right)\right], \qquad \text{(A38)}$$

where J_0 is the Bessel function of zeroth order (see [86,148] for a detailed derivation). In the dilute regime $|\psi|^2 \ll f^2m^2/\hbar^3 c$ (which is valid in particular in the nonrelativistic limit $c \to +\infty$, see footnote 20) the effective potential $V_{\text{eff}}(|\psi|^2)$ is dominated by the $|\psi|^4$ term. If we expand Eq. (A38) in powers of $|\psi|$, we recover Eq. (A31) at leading order. A $|\psi|^4$ effective potential is usually written as

$$V_{\text{eff}}(|\psi|^2) = \frac{2\pi a_s \hbar^2}{m^3}|\psi|^4, \qquad \text{(A39)}$$

where a_s is the s-scattering length of the bosons [269].[25] Comparing Eqs. (A31) and (A39), we find that

$$a_s = -\frac{\hbar c^3 m}{32\pi f^2}. \qquad \text{(A40)}$$

We note that the scattering length is negative ($a_s < 0$) corresponding to an attractive self-interaction. On the other hand, comparing Eqs. (A11) and (A40) yields[26]

$$\frac{\lambda}{8\pi} = \frac{2a_s mc}{3\hbar}. \qquad \text{(A41)}$$

The nonrelativistic limit $c \to +\infty$ can also be performed directly in the action of the SF. Let us consider the nongravitational case for brevity of presentation (we also assume a static background). In that case, the action of the SF is

$$S = \int d^4x \mathcal{L} \qquad \text{(A42)}$$

with the Lagrangian density

$$\mathcal{L} = \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{m^2c^2}{2\hbar^2}\varphi^2 - V(\varphi). \qquad \text{(A43)}$$

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[24]This is because φ is a real SF. Therefore, substituting φ (exact) from Eq. (A18) into $V(\varphi)$ *then* averaging over the oscillations is different from substituting φ (already averaged over the oscillations) from Eq. (A26) into $V(\varphi)$. In other words, $\overline{V_1(\varphi^2)} \neq V_1(\overline{\varphi^2})$ where we have set $V(\varphi) = V_1(\varphi^2)$. The case of a complex SF φ has been considered in [63,64]. In that case, the transformation of the equations from φ to ψ is exact (there is no need to average over the oscillations) and the potential is unchanged ($V_{\text{eff}} = V$).

[25]The ordinary GP equation with a cubic nonlinearity [270–273] is usually derived from the mean field Schrödinger equation [274] with a pair contact potential [275,276] (see, e.g., Sec. II.A. of [37]). The present approach shows that the GP equation with a cubic nonlinearity may also be derived from the KG equation with a quartic self-interaction potential. More generally, the GP equation with a nonlinearity $V_{\text{eff}}(|\psi|^2)$ may be derived from the KG equation with a self-interaction potential $V(\varphi)$ (see Refs. [63,64,86] for a more detailed discussion in the case of a complex or a real SF respectively).

[26]We note that the relation between λ and a_s is different for a real SF and for a complex SF (see Appendix A of [75] for a complex SF). They differ by a factor $2/3$ for the reason indicated in footnote 24.

The least action principle $\delta S = 0$ leads to the Euler-Lagrange equation

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad (A44)$$

yielding the KG equation

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi + \frac{m^2 c^2}{\hbar^2}\varphi + \frac{dV}{d\varphi} = 0. \quad (A45)$$

On the other hand, substituting Eqs. (A24)–(A28) into the Lagrangian (A43) and neglecting oscillatory terms, we obtain the action

$$S = \int \mathcal{L} d\mathbf{r} \quad (A46)$$

with the Lagrangian

$$\mathcal{L} = \frac{\hbar^2}{2m^2 c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 - \frac{\hbar}{m} \text{Im}\left(\frac{\partial \psi}{\partial t} \psi^* \right)$$
$$- \frac{\hbar^2}{2m^2} |\nabla\psi|^2 - V_{\text{eff}}(|\psi|^2). \quad (A47)$$

In the nonrelativistic limit $c \to +\infty$, it reduces to

$$\mathcal{L} = \frac{\hbar}{2m} i \left(\frac{\partial \psi}{\partial t} \psi^* - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m^2} |\nabla\psi|^2 - V_{\text{eff}}(|\psi|^2), \quad (A48)$$

where we have used

$$2i\text{Im}\left(\frac{\partial \psi}{\partial t} \psi^* \right) = \frac{\partial \psi}{\partial t} \psi^* - \psi \frac{\partial \psi^*}{\partial t}. \quad (A49)$$

We recover the Lagrangian of a nonrelativistic BEC (see, e.g., Appendix B of [75]). The Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla\psi} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (A50)$$

yields the GP equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\frac{dV_{\text{eff}}}{d|\psi|^2}\psi. \quad (A51)$$

Remark: Basically, axions are described by a relativistic quantum field theory with a real scalar field φ that obeys the KGE equations. In that case, the particle number is not conserved. However, axions whose kinetic energies are much smaller than mc^2 (like in dilute axion stars) can be described by a nonrelativistic effective field theory with a complex SF ψ that obeys the GPP equations. In that case,

they are just spinless particles whose number $N = \frac{1}{m}\int |\psi|^2 d\mathbf{r}$ is conserved. Physically, the particle number is conserved because, by removing the fast oscillating terms, we have eliminated the particle number violating processes that are energetically forbidden for nonrelativistic particles.

APPENDIX B: GINZBURG-LANDAU-POISSON, CAHN-HILLIARD-POISSON, AND SMOLUCHOWSKI-POISSON EQUATIONS

In this Appendix, we introduce Ginzburg-Landau-Poisson (GLP), Cahn-Hilliard-Poisson (CHP) and Smoluchowski-Poisson (SP) equations that can serve as numerical algorithms to compute the stable equilibrium states of the GPP equations.[27]

1. Equations for ψ

The GP equation writes [79]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m[V'(|\psi|^2) + \Phi + \Phi_{\text{ext}}]\psi, \quad (B1)$$

where, for the sake of generality, we have considered an arbitrary potential of self-interaction $V(|\psi|^2)$ and we have added an external potential $\Phi_{\text{ext}}(\mathbf{r})$. We also recall that $\Phi(\mathbf{r}, t)$ is the gravitational potential determined by the Poisson equation (4). More generally, it can represent the mean field potential $\Phi(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|)\rho(\mathbf{r}', t)d\mathbf{r}'$ produced by particles interacting via a long-range binary potential $u(|\mathbf{r} - \mathbf{r}'|)$. The energy functional associated with the GPP equations is [79]

$$E_{\text{tot}} = \frac{\hbar^2}{2m^2} \int |\nabla\psi|^2 d\mathbf{r} + \int V(|\psi|^2)d\mathbf{r}$$
$$+ \frac{1}{2}\int |\psi|^2\Phi d\mathbf{r} + \int |\psi|^2\Phi_{\text{ext}}d\mathbf{r}. \quad (B2)$$

The GP equation (B1) can then be rewritten as

$$i\hbar \frac{\partial \psi}{\partial t} = m\frac{\delta E_{\text{tot}}}{\delta \psi^*}. \quad (B3)$$

The GPP equations conserve the mass $M = \int |\psi|^2 d\mathbf{r}$ and the energy E_{tot} [79]. A stationary solution is obtained by extremizing E_{tot} at fixed M, writing $\delta E_{\text{tot}} - \frac{\mu}{m}\delta M = 0$, where μ (global chemical potential) is a Lagrange multiplier accounting for the conservation of mass [79]. Using

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[27]Note that similar numerical algorithms, having the form of generalized Fokker-Planck equations, have been introduced in [277–280] in order to compute the stable equilibrium states of the Vlasov-Poisson and Euler-Poisson equations.

$$\frac{\delta E_{\text{tot}}}{\delta \psi^*} = -\frac{\hbar^2}{2m^2} \Delta \psi + [V'(|\psi|^2) + \Phi + \Phi_{\text{ext}}]\psi, \quad \text{(B4)}$$

we get

$$-\frac{\hbar^2}{2m} \Delta \psi + m[V'(|\psi|^2) + \Phi + \Phi_{\text{ext}}]\psi = \mu \psi. \quad \text{(B5)}$$

The same equation can be obtained by substituting the ansatz $\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar}$ into Eq. (B1). This establishes that the eigenenergy E is equal to the global chemical potential μ. It can be shown furthermore that an equilibrium state is stable if, and only if, it is a minimum of E_{tot} at fixed M [79].

We stress that the GPP equations do *not* relax toward the stationary state that minimizes the energy E_{tot} at fixed mass M (ground state)—except if the system is in this state initially—because they do not satisfy an H-theorem. Indeed, the GPP equations are reversible and the energy E_{tot} is conserved. In other words, the GPP equations are *not* relaxation equations in the usual sense. They can, however, display a complicated process of violent relaxation [177] and gravitational cooling [172–174] toward a quasistationary state with a core-halo structure. The characterization of this core-halo state is considered in [281]. Independently from the characterization of this core-halo state it is an important mathematical problem in itself to be able to construct the stable stationary solutions of the GPP equations. However, it is difficult in practice to numerically solve the nonlinear eigenvalue problem defined by Eq. (B5) and make sure that the solution is stable.

In order to compute the stable steady states of the GP equation, Huepe *et al.* [219] proposed to solve the GL equation[28]

$$-\hbar \frac{\partial \psi}{\partial t} = m \frac{\delta F}{\delta \psi^*}, \quad \text{(B6)}$$

where $F = E_{\text{tot}} - \frac{\mu}{m}M$ is a grand potential. This equation can be written explicitly as

$$-\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m[V'(|\psi|^2) + \Phi + \Phi_{\text{ext}}]\psi - \mu \psi. \quad \text{(B7)}$$

The GLP equations satisfy an H-theorem for the grand potential: $\dot{F} = -(2m/\hbar) \int |\delta F/\delta \psi|^2 d\mathbf{r} \le 0$. As a result, they relax toward a steady state of the form of Eq. (B5) which minimizes F at fixed μ.[29] This is therefore a stable steady state of the GPP equations with this value of μ. The

GL equation does not conserve the mass M contrary to the GP equation. This is because it is associated with a grand canonical description where the chemical potential μ is fixed instead of the mass M. In order to obtain, at equilibrium, the correct value of μ corresponding to a prescribed mass M, Huepe *et al.* [219] proposed to solve Eq. (B7) with a chemical potential $\mu(t)$ that evolves in time so as to conserve M.[30] This amounts to introducing formally a canonical description where the mass is fixed. In conclusion, the GL equation provides a useful numerical algorithm to construct the stable stationary solutions of the GP equation. By construction, the equilibrium solution reached by the GL equation (which truly is a relaxation equation) is guaranteed to be a stable stationary solution of the GP equation.

2. Equations for ρ

In the hydrodynamic representation, the energy functional associated with the GPP equations is [79]

$$E_{\text{tot}} = \frac{1}{m} \int \rho Q \, d\mathbf{r} + \int V(\rho) \, d\mathbf{r}$$
$$+ \frac{1}{2} \int \rho \Phi \, d\mathbf{r} + \int \rho \Phi_{\text{ext}} \, d\mathbf{r}, \quad \text{(B8)}$$

where we have not written the classical kinetic term $\Theta_c = (1/2) \int \rho \mathbf{u}^2 d\mathbf{r}$ since we are interested by equilibrium states only. The quantum Euler-Poisson equations (6)–(9), which are equivalent to the GPP equations (3) and (4), can be written in terms of functional derivatives of E_{tot} (see Sec. 3.6 of [79]). The quantum Euler-Poisson equations conserve the mass $M = \int \rho d\mathbf{r}$ and the energy E_{tot} (including Θ_c) [79]. A steady state is obtained by extremizing E_{tot} at fixed M, writing $\delta E_{\text{tot}} - \frac{\mu}{m} \delta M = 0$ [79]. Using

$$\frac{\delta E_{\text{tot}}}{\delta \rho} = \frac{Q}{m} + V'(\rho) + \Phi + \Phi_{\text{ext}}, \quad \text{(B9)}$$

we get

$$Q + m(V'(\rho) + \Phi + \Phi_{\text{ext}}) = \mu. \quad \text{(B10)}$$

The same equation can be obtained from the condition of quantum hydrostatic equilibrium (12) by using $V''(\rho) = h'(\rho) = P'(\rho)/\rho$ or $P = \rho V'(\rho) - V(\rho)$, where h is the enthalpy (equal to the local chemical potential) [79]. Equation (B10) is also equivalent to Eq. (B5). An equilibrium state is stable if, and only if, it is a minimum of E_{tot} at fixed M [79]. In order to compute the stable steady states of the quantum Euler equation, one can solve the GL equation

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[28]This amounts to integrating the GP equation in imaginary time [282,283].

[29]They can only relax toward minima of F, not toward maxima or saddle points that are unstable.

[30]This is similar to the time-dependent inverse temperature $\beta(t)$ enforcing the conservation of energy in [277].

$$\xi \frac{\partial \rho}{\partial t} = -\frac{\delta F}{\delta \rho}, \tag{B11}$$

where $F = E_{\text{tot}} - \frac{\mu}{m} M$ is a grand potential. This equation can be written explicitly as

$$-m\xi \frac{\partial \rho}{\partial t} = Q + m(V'(\rho) + \Phi + \Phi_{\text{ext}}) - \mu. \tag{B12}$$

The GLP equations satisfy an H-theorem for the grand potential: $\dot{F} = -(1/\xi) \int (\delta F/\delta \rho)^2 d\mathbf{r} \leq 0$. We can then make the same comments as those following Eq. (B7).

Remark: An alternative manner to compute the stable equilibrium states of the quantum Euler equation is to solve the CH equation

$$\xi \frac{\partial \rho}{\partial t} = \Delta \frac{\delta E_{\text{tot}}}{\delta \rho}, \tag{B13}$$

or, explicitly,

$$\xi \frac{\partial \rho}{\partial t} = \Delta \left(\frac{Q}{m} + V'(\rho) + \Phi + \Phi_{\text{ext}} \right), \tag{B14}$$

which conserves the mass M and satisfies an H-theorem for the energy: $\dot{E}_{\text{tot}} = -(1/\xi) \int [\nabla(\delta E_{\text{tot}}/\delta \rho)]^2 d\mathbf{r} \leq 0$. Following [79], we may also consider the generalized CH equation

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \frac{\delta E_{\text{tot}}}{\delta \rho} \right), \tag{B15}$$

which conserves the mass M and satisfies an H-theorem for the energy: $\dot{E}_{\text{tot}} = -(1/\xi) \int \rho [\nabla(\delta E_{\text{tot}}/\delta \rho)]^2 d\mathbf{r} \leq 0$. Explicitly, this equation has the form of a quantum Smoluchowski equation [79]

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q \right). \tag{B16}$$

It corresponds to the strong friction limit $\xi \to +\infty$ of the damped GP equation introduced in [79]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m[V'(|\psi|^2) + \Phi + \Phi_{\text{ext}}]\psi$$
$$- i\frac{\hbar}{2}\xi \left[\ln\left(\frac{\psi}{\psi^*}\right) - \left\langle \ln\left(\frac{\psi}{\psi^*}\right) \right\rangle \right]\psi. \tag{B17}$$

This dissipative equation may thus serve as a numerical algorithm to compute the stable equilibrium states of the GPP equations. The damped quantum Euler equations, which are equivalent to Eq. (B17), could be considered as well [79]. In these different examples, the mass is automatically conserved so it is not necessary to enforce its conservation with a Lagrange multiplier $\mu(t)$ as for the GL equation.

3. Equations for ϕ

Let us set $\rho = \phi^2$ where ϕ is real. In that case, the energy functional (B8) can be rewritten as

$$E_{\text{tot}} = \frac{\hbar^2}{2m^2} \int (\nabla\phi)^2 d\mathbf{r} + \int V(\phi^2) d\mathbf{r}$$
$$+ \frac{1}{2} \int \phi^2 \Phi d\mathbf{r} + \int \phi^2 \Phi_{\text{ext}} d\mathbf{r}, \tag{B18}$$

where, as before, we have not written the classical kinetic term Θ_c. The GPP equations conserve the mass $M = \int \phi^2 d\mathbf{r}$ and the energy E_{tot} (including Θ_c). A steady state is obtained by extremizing E_{tot} at fixed M, writing $\delta E_{\text{tot}} - \frac{\mu}{m} \delta M = 0$. Using

$$\frac{\delta E_{\text{tot}}}{\delta \phi} = -\frac{\hbar^2}{m^2} \Delta\phi + 2V'(\phi^2)\phi + 2\phi\Phi + 2\phi\Phi_{\text{ext}}, \tag{B19}$$

we get

$$-\frac{\hbar^2}{2m} \Delta\phi + m[V'(\phi^2) + \Phi + \Phi_{\text{ext}}]\phi = \mu\phi. \tag{B20}$$

An equilibrium state is stable if, and only if, it is a minimum of E_{tot} at fixed M.

In order to compute the stable steady states of the GP equation, one can solve the GL equation

$$\xi \frac{\partial \phi}{\partial t} = -\frac{\delta F}{\delta \phi}, \tag{B21}$$

where $F = E_{\text{tot}} - \frac{\mu}{m} M$ is a grand potential. This equation can be written explicitly as

$$-\frac{\xi m}{2} \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta\phi + m[V'(\phi^2) + \Phi + \Phi_{\text{ext}}]\phi - \mu\phi. \tag{B22}$$

The GLP equations satisfy an H-theorem for the grand potential: $\dot{F} = (-1/\xi) \int (\delta F/\delta \phi)^2 d\mathbf{r} \leq 0$. We can then make the same comments as those following Eq. (B7).

APPENDIX C: THERMAL TUNNELING IN THE STOCHASTIC GINZBURG-LANDAU EQUATION

In this Appendix, we take thermal fluctuations into account in the framework of the stochastic GL equation [284].[31] We compute the thermal tunneling rate of a field $\rho(\mathbf{r}, t)$ across a barrier of free energy by using the instanton theory [285]. Our approach provides, in this context, a justification of the Kramers formula giving the typical

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[31]This equation has been studied numerically recently by Verma *et al.* [108] in relation to self-gravitating BECs.

lifetime of a metastable state. Similar results can be obtained for the stochastic CH and generalized CH (or Smoluchowski) equations [286–288].

The stochastic GL equation writes

$$\xi \frac{\partial \rho}{\partial t} = -\frac{\delta F}{\delta \rho} + \sqrt{2\xi k_B T}\zeta(\mathbf{r}, t), \qquad (C1)$$

where $\zeta(\mathbf{r}, t)$ is a Gaussian white noise. The free energy $F[\rho]$ can be an arbitrary functional of ρ, but it is usually written under the form

$$F[\rho] = \int \left[\frac{1}{2}(\nabla \rho)^2 + V(\rho)\right] d\mathbf{r}. \qquad (C2)$$

The potential $V(\rho)$ is often approximated by its normal form close to a critical point according to the Landau theory of phase transitions but we shall treat it here as an arbitrary function of ρ. For a functional of the form of Eq. (C2), the stochastic GL equation (C1) can be written explicitly as

$$\xi \frac{\partial \rho}{\partial t} = \Delta \rho - V'(\rho) + \sqrt{2\xi k_B T}\zeta(\mathbf{r}, t). \qquad (C3)$$

In the absence of noise ($T = 0$), the deterministic GL equation writes

$$\xi \frac{\partial \rho}{\partial t} = -\frac{\delta F}{\delta \rho} = \Delta \rho - V'(\rho). \qquad (C4)$$

Its equilibrium states are extrema of F:

$$\frac{\partial \rho}{\partial t} = 0 \quad \Leftrightarrow \quad \frac{\delta F}{\delta \rho} = 0 \quad \Leftrightarrow \quad -\Delta \rho + V'(\rho) = 0. \quad (C5)$$

On the other hand, it satisfies an H-theorem

$$\dot{F} = \int \frac{\delta F}{\delta \rho} \frac{\partial \rho}{\partial t} d\mathbf{r} = -\frac{1}{\xi} \int \left(\frac{\delta F}{\delta \rho}\right)^2 d\mathbf{r} \le 0. \quad (C6)$$

As a result, the deterministic GL equation generically relaxes toward a stable equilibrium state which minimizes $F[\rho]$ (maxima or saddle points are linearly unstable). In the presence of noise ($T \neq 0$), the stochastic GL equation (C1) can be interpreted as a Langevin equation. The probability density $P[\rho, t]$ of the density field $\rho(\mathbf{r}, t)$ at time t is governed by the functional FP equation

$$\xi \frac{\partial P}{\partial t}[\rho, t] = \int d\mathbf{r} \frac{\delta}{\delta \rho(\mathbf{r})} \left\{ \left[k_B T \frac{\delta}{\delta \rho(\mathbf{r})} + \frac{\delta F}{\delta \rho(\mathbf{r})}\right] P[\rho, t] \right\}. \qquad (C7)$$

It relaxes toward the equilibrium Boltzmann distribution

$$P_{\mathrm{eq}}[\rho] = \frac{1}{Z(\beta)} e^{-\beta F[\rho]}. \qquad (C8)$$

Let us assume that the free energy functional $F[\rho]$ has a local minimum $\rho_M(\mathbf{r})$ (metastable state) and a global minimum $\rho_S(\mathbf{r})$ (stable state) separated by a maximum or a saddle point $\rho_U(\mathbf{r})$ (unstable state). In the absence of noise, the evolution of the system is deterministic and the density $\rho(\mathbf{r}, t)$ relaxes toward one of the minima of the potential as implied by the H-theorem (C6). In the presence of noise, the density switches back and forth between the two minima (attractors). When the noise is weak ($T \to 0$),[32] the transition between the two minima is a rare event. One important problem is to determine the rate Γ for the density profile, initially located in the metastable state $\rho_M(\mathbf{r})$, to cross the barrier of free energy and reach the stable state $\rho_S(\mathbf{r})$.

Since the distribution of the Gaussian white noise $\zeta(\mathbf{r}, t)$ is

$$P[\zeta(\mathbf{r}, t)] \propto e^{-\int_{-\infty}^{+\infty} dt \int d\mathbf{r} \zeta^2(\mathbf{r}, t)/2}, \qquad (C9)$$

the probability of the path $\rho(\mathbf{r}, t)$ is

$$P[\rho(\mathbf{r}, t)] \propto e^{-S[\rho(\mathbf{r}, t)]/k_B T}, \qquad (C10)$$

where S is the generalized Onsager-Machlup (OM) functional [289]

$$S[\rho(\mathbf{r}, t)] = \frac{1}{4\chi} \int dt \int d\mathbf{r} \left(\frac{\partial \rho}{\partial t} + \chi \frac{\delta F}{\delta \rho}\right)^2. \qquad (C11)$$

The functional S may be called an action by analogy with the path-integral formulation of quantum mechanics (the temperature T plays the role of the Planck constant $\hbar$ in quantum mechanics) [290]. It can be written as $S = \int L dt$ where L is the corresponding Lagrangian. The probability density to observe the system with the profile $\rho_2(\mathbf{r})$ at time t_2 given that it had the profile $\rho_1(\mathbf{r})$ at time t_1 is

$$P[\rho_2(\mathbf{r}), t_2 | \rho_1(\mathbf{r}), t_1] = \int \mathcal{D}\rho e^{-S[\rho]/k_B T}, \qquad (C12)$$

where the integral runs over all paths satisfying $\rho(\mathbf{r}, t_1) = \rho_1(\mathbf{r})$ and $\rho(\mathbf{r}, t_2) = \rho_2(\mathbf{r})$. For a given initial condition $\rho_0(\mathbf{r})$ at $t = t_0$, the probability density $P[\rho(\mathbf{r}), t] \equiv P[\rho(\mathbf{r}), t | \rho_0(\mathbf{r}), t_0]$ to observe the system with the profile $\rho(\mathbf{r})$ at time t satisfies the functional FP equation (C7). Therefore, Eq. (C12) provides the exact solution of Eq. (C7).

In the weak noise limit, the typical paths explored by the system are concentrated close to the most probable path.

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[32]For systems with long-range interactions, the noise is also weak when the number of particles $N \to +\infty$.

In that case, a steepest-descent evaluation of the path integral is possible. The path integral is dominated by the most probable path. To determine the most probable path, we have to minimize the OM functional $S[\rho(\mathbf{r}, t)]$, i.e., we have to solve the minimization problem

$$\min_{\rho(\mathbf{r},t)} \quad \{S[\rho(\mathbf{r}, t)]\}. \tag{C13}$$

The equation for the most probable path $\rho_c(\mathbf{r}, t)$ that connects two attractors is called an "instanton" [291]. It is obtained by cancelling the first order variations of the action

$$\delta S = 0. \tag{C14}$$

In the weak noise limit, the transition probability from one state to the other is dominated by the most probable path:

$$P[\rho_2(\mathbf{r}), t_2|\rho_1(\mathbf{r}), t_1] \simeq e^{-S[\rho_c]/k_B T}. \tag{C15}$$

This formula can be interpreted as a large deviation result. It provides an approximate solution of the functional FP equation (C7). On the other hand, it can be shown that the escape rate of the system over the barrier of free energy is given by

$$\Gamma \propto e^{-S[\rho_c]/k_B T}, \tag{C16}$$

where $S[\rho_c]$ is the action of the most probable path (instanton) that connects the metastable state to the stable state. In the limit of weak noise, it can be shown that the most probable path between the metastable state and the stable state must necessarily pass through the saddle point $\rho_U(\mathbf{r})$ (playing the role of a "critical droplet" in problems of nucleation). Once the system reaches the saddle point it may either return to the initial metastable state or reach the stable state. In the latter case, it has crossed the barrier of free energy.

To determine the instanton which solves the variational problem (C13), we can proceed as follows. The Lagrangian associated with the OM functional (C11) is

$$L = \frac{1}{4\chi} \int d\mathbf{r} \left(\frac{\partial \rho}{\partial t} + \chi \frac{\delta F}{\delta \rho} \right)^2. \tag{C17}$$

The corresponding Hamiltonian is defined by

$$H = \int \dot{\rho} \frac{\delta L}{\delta \dot{\rho}} d\mathbf{r} - L. \tag{C18}$$

Since the Lagrangian does not explicitly depend on time, the Hamiltonian is conserved. Therefore, using Eq. (C17), we get

$$H = \frac{1}{4\chi} \int d\mathbf{r} \left(\frac{\partial \rho}{\partial t} - \chi \frac{\delta F}{\delta \rho} \right) \left(\frac{\partial \rho}{\partial t} + \chi \frac{\delta F}{\delta \rho} \right), \tag{C19}$$

where H is a constant. Since the attractors satisfy $\partial \rho/\partial t = 0$ and $\delta F/\delta \rho = 0$, the constant H is equal to zero ($H = 0$). Therefore, the instanton satisfies the equations

$$\frac{\partial \rho_c}{\partial t} = \mp \chi \frac{\delta F}{\delta \rho_c} \tag{C20}$$

with the boundary conditions $\rho_c(\mathbf{r}, -\infty) = \rho_M(\mathbf{r})$ and $\rho_c(\mathbf{r}, +\infty) = \rho_S(\mathbf{r})$. We note that the most probable path corresponds to the deterministic dynamics (C4) with a sign $\mp$.[33] The physical interpretation of Eq. (C20) is the following. Starting from the metastable state $\rho_M(\mathbf{r})$, the most probable path follows the time-reversed deterministic dynamics against the free energy gradient up to the saddle point $\rho_U(\mathbf{r})$; beyond the saddle point, it follows the forward-time deterministic dynamics down to the stable state $\rho_S(\mathbf{r})$. According to Eqs. (C11) and (C20), the action of the most probable path corresponding to the transition from the saddle point to the stable state (downhill solution corresponding to Eq. (C20) with the sign $-$) is zero while the action of the most probable path corresponding to the transition from the metastable state to the saddle point (uphill solution corresponding to Eq. (C20) with the sign $+$) is nonzero. This is to be expected since the descent from the saddle point to the stable state is a "free" descent that does not require thermal noise; it thus gives the smallest possible value of zero of the action. By contrast, the rise from the metastable state to the saddle point is a rare event that requires thermal noise. The action for the uphill solution is

$$S[\rho_c(\mathbf{r}, t)] = \int dt \int d\mathbf{r} \frac{\partial \rho_c}{\partial t} \frac{\delta F}{\delta \rho_c} = \int dt \frac{dF}{dt} = \Delta F, \tag{C21}$$

where $\Delta F = F[\rho_U] - F[\rho_M]$ is the barrier of free energy between the metastable state and the unstable state. The total action for the most probable path connecting the attractors is therefore $S_c = S[\rho_c^+] + S[\rho_c^-] = \Delta F + 0 = \Delta F$. It is determined solely by the uphill path. The instanton solution gives the dominant contribution to the transition rate for a weak noise. Therefore, the rate for the system to pass from the metastable state to the stable state (escape rate) is

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[33]Considering the solution with the sign $-$, which corresponds to the downhill solution (see below), we see that the most probable path (instanton) coincides with the ensemble average path, i.e., the deterministic GL equation (C4) obtained by averaging the stochastic GL equation (C1) over the noise. It has a zero action ($S = 0$). As a result, the deterministic GL equation (C4)—the average path—can be obtained by minimizing the OM functional (C11).

$$\Gamma \propto e^{-\Delta F/k_B T}. \tag{C22}$$

This is the celebrated Arrhenius (or Kramers) formula stating that the transition rate is inversely proportional to the exponential of the barrier of free energy divided by $k_B T$.[34] The typical lifetime of a metastable state is $t_{\text{life}} \sim \Gamma^{-1}$. For systems with long-range interactions, the free energy scales as N so the typical lifetime of a metastable state scales as

$$t_{\text{life}} \propto e^{N\Delta f/k_B T}. \tag{C23}$$

For systems with long-range interactions, the metastable states are very relevant since their lifetime scales as e^N with $N \gg 1$. Therefore, metastable states are stable in practice. Only very close to the critical point where $\Delta f \to 0$ does their lifetime decrease substantially.

APPENDIX D: MAXIMUM MASS OF GENERAL RELATIVISTIC SELF-INTERACTING BOSON STARS

We consider a relativistic complex SF φ with a self-interaction potential $V(|\varphi|^2)$ like in Refs. [57,60,63,64, 78,80,137,138,170]. In the TF (or semiclassical) limit where the quantum kinetic energy can be neglected, the energy density and the pressure are given by [78]

$$\epsilon = \rho c^2 + V(\rho) + \rho V'(\rho), \tag{D1}$$

$$P = \rho V'(\rho) - V(\rho) = \rho^2 \left(\frac{V}{\rho}\right)', \tag{D2}$$

where ρ is the pseudo rest-mass density

$$\rho = \frac{m^2}{\hbar^2} |\varphi|^2. \tag{D3}$$

Therefore, in this approximation, a self-interacting boson star is equivalent to a relativistic fluid described by a barotropic equation of state $P(\epsilon)$ defined in implicit form by Eqs. (D1) and (D2). We note that Eq. (D2) has the same form as in the nonrelativistic limit where $\rho = |\psi|^2$ represents the mass density (see [79] for detail).

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[34]This formula can be simply obtained as follows. The equilibrium probability of observing the density $\rho(\mathbf{r})$ is $\propto e^{-\beta F[\rho]}$. Therefore, the probability for the system initially prepared in the metastable state to form a "critical droplet" (unstable state ρ_U) and then reach the stable state ρ_S is $\propto e^{-\beta(F[\rho_U]-F[\rho_M])}$. The typical lifetime of a metastable state may then be estimated by $t_{\text{life}} \sim e^{\beta\Delta F}$, where $\Delta F = F[\rho_U] - F[\rho_M]$ is the barrier of free energy between the metastable state and the unstable state.

Let us consider a power-law potential

$$V(|\varphi|^2) = A|\varphi|^{2\gamma} \tag{D4}$$

with $\gamma > 1$. Using Eq. (D3), we get

$$V(\rho) = \frac{K}{\gamma - 1} \rho^\gamma \tag{D5}$$

with

$$K = (\gamma - 1)A \left(\frac{\hbar}{m}\right)^{2\gamma}. \tag{D6}$$

According to Eq. (D2), the pressure is given by

$$P = K\rho^\gamma. \tag{D7}$$

This is a polytropic equation of state with polytropic constant K and polytropic index $\gamma = 1 + 1/n$. On the other hand, according to Eq. (D1), the energy density is given by

$$\epsilon = \rho c^2 + \frac{K(\gamma + 1)}{\gamma - 1}\rho^\gamma = \rho c^2 + (2n + 1)P. \tag{D8}$$

This leads to the equation of state

$$\epsilon = \left(\frac{P}{K}\right)^{1/\gamma} c^2 + \frac{\gamma + 1}{\gamma - 1} P \tag{D9}$$

given in the reversed form $\epsilon(P)$. At low densities $\rho \to 0$, we get $\epsilon \sim \rho c^2$ so that the energy density is dominated by the rest-mass energy. This corresponds to the nonrelativistic limit. At high densities $\rho \to +\infty$, we obtain $\epsilon \sim (2n + 1)P$ or, equivalently,

$$P \sim \frac{1}{2n + 1}\epsilon. \tag{D10}$$

This corresponds to the ultrarelativistic limit. Since the relation between the pressure and the energy density is linear ($P = q\epsilon$), the mass-radius relation $M(R)$, parametrized by ϵ, forms a spiral at high densities as in the case of neutron stars [228]. Furthermore, the series of equilibria becomes unstable at the maximum mass M_{max} corresponding to the first turning point of the spiral.[35] The square of the

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[35]More precisely, according to Wheeler's theorem, a mode of stability is lost at a turning point of mass if the $M(R)$ curve rotates anticlockwise and gained if it rotates clockwise [228]. Now, we know that nonrelativistic polytropic gaseous spheres are stable for $n < 3$ and unstable for $n > 3$ [292]. Therefore, when $n < 3$, the series of equilibria is stable before the first turning point of mass and becomes unstable afterwards. When $n > 3$, the whole series of equilibria is unstable.

speed of sound is $c_s^2 = P'(\epsilon)c^2 = c^2/(2n+1)$. Since $n > 0$, the speed of sound is always less than the speed of light ($c_s < c$).

(i) We first consider a $|\varphi|^4$ potential with a repulsive self-interaction ($a_s > 0$) of the form [57,60,63,64,78,80, 137,138]

$$V(|\varphi|^2) = \frac{2\pi a_s m}{\hbar^2} |\varphi|^4. \tag{D11}$$

Using Eq. (D3) we get

$$V(\rho) = \frac{2\pi a_s \hbar^2}{m^3} \rho^2. \tag{D12}$$

According to Eq. (D2) the pressure is given by

$$P = \frac{2\pi a_s \hbar^2}{m^3} \rho^2. \tag{D13}$$

This is a polytropic equation of state of polytropic constant $K = 2\pi a_s \hbar^2/m^3$ and polytropic index $\gamma = 2$ (i.e., $n = 1$). According to Eq. (D1) the energy density is given by

$$\epsilon = \rho c^2 + 3P = \rho c^2 + \frac{6\pi a_s \hbar^2}{m^3} \rho^2. \tag{D14}$$

This is a quadratic equation for ρ. Solving this equation and substituting the result into Eq. (D13), we obtain the equation of state

$$P = \frac{m^3 c^4}{72\pi a_s \hbar^2} \left(\sqrt{1 + \frac{24\pi a_s \hbar^2}{m^3 c^4} \epsilon} - 1 \right)^2. \tag{D15}$$

It coincides with the result of [137]. For $\rho \to +\infty$, the equation of state reduces to $P \sim \epsilon/3$ like for the ordinary radiation due to photons. The mass-radius relation corresponding to the equation of state (D14) has been obtained in [60,138]. It displays a maximum mass

$$M_{\mathrm{max,GR}} = 0.307 \frac{\hbar c^2 \sqrt{a_s}}{(Gm)^{3/2}} \tag{D16}$$

at a radius

$$R_{*,\mathrm{GR}} = 1.923 \left(\frac{a_s \hbar^2}{Gm^3} \right)^{1/2}, \tag{D17}$$

and forms a spiral at high densities as explained previously.

(ii) We now consider axion boson stars (in the sense of [170]) with the axion boson potential $V(|\varphi|^2)$ truncated at the order $|\varphi|^6$ as in [86]:

$$V(|\varphi|^2) = \frac{2\pi a_s m}{\hbar^2} |\varphi|^4 + \frac{32\pi^2 a_s^2}{9\hbar^2 c^2} |\varphi|^6. \tag{D18}$$

The $|\varphi|^4$ term is attractive ($a_s < 0$) while the $|\varphi|^6$ is repulsive. We are interested in describing the branch of dense axion boson stars for large mass M where general relativistic effects are important. Since we are considering a complex SF, the number of bosons is conserved. As a result, dense axion boson stars should be stable with respect to the decay via emission of relativistic axions contrary to the case where the SF is real (see the introduction). Using Eq. (D3) we get

$$V(\rho) = \frac{2\pi a_s \hbar^2}{m^3} \rho^2 + \frac{32\pi^2 a_s^2 \hbar^4}{9m^6 c^2} \rho^3. \tag{D19}$$

At high densities the repulsive $|\varphi|^6$ term dominates over the attractive $|\varphi|^4$ term. If we just keep the repulsive $|\varphi|^6$ potential, we obtain

$$V(\rho) = \frac{32\pi^2 a_s^2 \hbar^4}{9m^6 c^2} \rho^3. \tag{D20}$$

According to Eq. (D2) the pressure is given by

$$P = \frac{64\pi^2 a_s^2 \hbar^4}{9m^6 c^2} \rho^3. \tag{D21}$$

This is the equation of state of a polytrope with polytropic constant $K = 64\pi^2 a_s^2 \hbar^4/9m^6 c^2$ and polytropic index $\gamma = 3$ (i.e., $n = 1/2$). According to Eq. (D1) the energy density is given by

$$\epsilon = \rho c^2 + \frac{128\pi^2 a_s^2 \hbar^4}{9m^6 c^2} \rho^3 = \rho c^2 + 2P. \tag{D22}$$

This is a third degree equation for ρ. For $\rho \to +\infty$, the equation of state reduces to $P \sim \epsilon/2$. The corresponding mass-radius relation will be studied in a specific paper [293]. We just provide below preliminary results.

At low densities, the system is nonrelativistic. The general mass-radius relation of polytropic spheres is

$$M^{(n-1)/n} R^{(3-n)/n} = \frac{K(1+n)}{(4\pi)^{1/n} G} \omega_n^{(n-1)/n}, \tag{D23}$$

where ω_n is a constant that can be obtained from the Lane-Emden equation [292]. Specializing on the equation of state (D21), we obtain

$$M = \frac{3Gm^6 c^2}{2\hbar^4 a_s^2} \omega_{1/2} R^5 = 0.0323 \frac{Gm^6 c^2}{\hbar^4 a_s^2} R^5, \tag{D24}$$

where we have used $\omega_{1/2} = 0.02156\ldots$. We note that the mass increases with the radius.

At high densities, the system is ultrarelativistic. Since the equation of state is linear at high densities ($P \sim \epsilon/2$), we expect that the mass-radius relation $M(R)$ will form a spiral

and display a maximum mass M_{max}. An estimate of the maximum mass of general relativistic dense boson axion stars in the $|\varphi|^6$ approximation can be obtained by combining the Newtonian mass-radius relation (D24) with the constraint $R \geq R_S$, where $R_S = 2GM/c^2$ is the Schwarzschild radius. This gives a maximum general relativistic mass

$$M_{\text{max,GR}}^{\text{dense}} = 0.991 \left(\frac{|a_s| \hbar^2 c^4}{G^3 m^3} \right)^{1/2} \qquad \text{(D25)}$$

and a corresponding radius

$$R_{*,\text{GR}}^{\text{dense}} = 1.98 \left(\frac{|a_s| \hbar^2}{G m^3} \right)^{1/2}. \qquad \text{(D26)}$$

We can also express these results in terms of the axion decay constant

$$f = \left(\frac{\hbar c^3 m}{32 \pi |a_s|} \right)^{1/2}. \qquad \text{(D27)}$$

We get

$$M_{\text{max,GR}}^{\text{dense}} = 0.0988 \left(\frac{\hbar^3 c^7}{G^3} \right)^{1/2} \frac{1}{fm}, \qquad \text{(D28)}$$

$$R_{*,\text{GR}}^{\text{dense}} = 0.197 \left(\frac{\hbar^3 c^3}{G} \right)^{1/2} \frac{1}{fm}. \qquad \text{(D29)}$$

If we measure the axion decay constant f in units of 10^{15} GeV and the axion mass m in units of 10^{-22} eV/c^2 we obtain $M_{\text{max,GR}}^{\text{dense}} = 1.61 \times 10^{15} (fm)^{-1} M_\odot$ and $R_{*,\text{GR}}^{\text{dense}} = 154 (fm)^{-1}$ pc.

For QCD axions with $m = 10^{-4}$ eV/c^2, $a_s = -5.8 \times 10^{-53}$ m and $f = 5.82 \times 10^{19}$ eV $= 4.77 \times 10^{-9} M_P c^2$, we find that $M_{\text{max,GR}}^{\text{dense}} = 27.7 M_\odot$ and $R_{*,\text{GR}}^{\text{dense}} = 81.9$ km.

For ULAs with $m = 2.19 \times 10^{-22}$ eV/c^2, $a_s = -1.11 \times 10^{-62}$ fm and $f = 1.97 \times 10^{23}$ eV $= 1.61 \times 10^{-5} M_P c^2$, we find that $M_{\text{max,GR}}^{\text{dense}} = 3.74 \times 10^{15} M_\odot$ and $R_{*,\text{GR}}^{\text{dense}} = 358$ pc.

Remark: For QCD axions, the product $mf \equiv (\Lambda_{\text{QCD}}/c)^2$ of the mass and decay constant is fixed to the value $\Lambda_{\text{QCD}} = 7.6 \times 10^7$ eV [125]. This gives a universal maximum mass and maximum stable radius $M_{\text{max,GR}}^{\text{dense}} = 27.7 M_\odot$ and $R_{*,\text{GR}}^{\text{dense}} = 81.9$ km [see Eqs. (D28) and (D29)]. We stress that this result is valid only for the $|\varphi|^6$ potential given by Eq. (D20) in the TF limit in which $M_{\text{max,GR}}^{\text{dense}} \propto 1/(fm)$. The fact that the maximum mass of dense axion boson stars obtained numerically by Guerra *et al.* [170] depends on f when mf is fixed shows that the exact description of dense axion boson stars [170] is more complicated than the present analysis.

APPENDIX E: OTHER TYPES OF POTENTIALS

In Appendix D, we have considered the case of power-law potentials. Here, we briefly introduce other types of potentials that will be specifically studied in future works.

For a potential of the form

$$V(\rho) = T\rho \ln \left(\frac{\rho}{\rho_*} \right), \qquad \text{(E1)}$$

the pressure is given by [see Eq. (D2)]

$$P = \rho T. \qquad \text{(E2)}$$

This is an isothermal equation of state with temperature T. According to Eq. (D1), the energy density is given by

$$\epsilon = \rho c^2 + 2T\rho \ln \left(\frac{\rho}{\rho_*} \right) + \rho T. \qquad \text{(E3)}$$

This leads to the equation of state

$$\epsilon = \frac{Pc^2}{T} + 2P \ln \left(\frac{P}{T\rho_*} \right) + P \qquad \text{(E4)}$$

given in the reversed form $\epsilon(P)$. The square of the speed of sound is $c_s^2 = c^2/[c^2/T + 2\ln(\rho/\rho_*) + 3]$.

For a potential of the form

$$V(\rho) = -A \ln \left(\frac{\rho}{\rho_P} \right) - A, \qquad \text{(E5)}$$

the pressure is given by [see Eq. (D2)]

$$P = A \ln \left(\frac{\rho}{\rho_P} \right). \qquad \text{(E6)}$$

This is a logotropic equation of state with generalized temperature A [294]. According to Eq. (D1), the energy density is given by

$$\epsilon = \rho c^2 - A \ln \left(\frac{\rho}{\rho_P} \right) - 2A. \qquad \text{(E7)}$$

This leads to the equation of state

$$\epsilon = \rho_P c^2 e^{P/A} - P - 2A \qquad \text{(E8)}$$

given in the reversed form $\epsilon(P)$. The square of the speed of sound is $c_s^2 = c^2/(\rho c^2/A - 1)$.

For a potential of the form

$$V(\rho) = \frac{A}{\gamma - 1} \rho^\gamma \left[\ln \left(\frac{\rho}{\rho_P} \right) - \frac{1}{\gamma - 1} \right], \qquad \text{(E9)}$$

the pressure is given by [see Eq. (D2)]

$$P = A\rho^\gamma \ln\left(\frac{\rho}{\rho_P}\right). \tag{E10}$$

This is a generalized logotropic equation of state. According to Eq. (D1), the energy density is given by

$$\epsilon = \rho c^2 + A\frac{\gamma+1}{\gamma-1}\rho^\gamma \ln\left(\frac{\rho}{\rho_P}\right) - \frac{2A}{(\gamma-1)^2}\rho^\gamma. \tag{E11}$$

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