Simplifying 4D $\mathcal{N}=3$ harmonic superspace

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We quantize super Yang-Mills action in $\mathcal{N}=3$ harmonic superspace using “Fermi-Feynman” gauge and also develop the background field formalism. This leads to simpler propagators and Feynman rules that are useful in performing explicit calculations. The superspace rules are used to show that divergences do not appear at one loop and beyond. We also compute a finite contribution to the effective action from a four-point diagram at one loop, which matches the expected covariant result.

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I. INTRODUCTION

$\mathcal{N}=3$ harmonic superspace in four dimensions was developed by GIKOS around three and a half decades ago [1,2], and it provided the first successful off-shell formulation of four-dimensional (4D) $\mathcal{N}=3$ super Yang-Mills (SYM) theory. This theory was quantized in “Landau” gauge a few years later by Delduc and McCabe [3]; however, the propagators obtained did not lend themselves to easier calculations. It is well known that the field content of a $\mathcal{N}=3$ vector multiplet is the same as that of a $\mathcal{N}=4$ one, and Zupnik explicitly showed this hidden supersymmetry of the $\mathcal{N}=3$ SYM in [4]. The $\mathcal{N}=3$ superspace also manifests the full superconformal symmetry, and using such symmetry arguments, low-energy effective action for $\mathcal{N}=3$ and $\mathcal{N}=4$ were considered by Zupnik and collaborators in [5–8]. A “twistorial” perspective on the $\mathcal{N}=3$ SYM action was presented in [9] a few years ago, but no concrete progress has yet been made “to bring the quantization scheme into a form suitable for computations” [2].

We present evidence of some progress in the direction of simplifying computations here. We choose “Fermi-Feynman” gauge to drastically reduce the number ($9\to1$) and simplify the form ({chiral, antichiral, linear}-analytic → just analytic) of propagators when compared to [3]. This simplifies the proof of the nonrenormalization theorem as one might expect. Moreover, we also introduce the background field formalism in the $\mathcal{N}=3$ harmonic superspace to simplify computations further.

In Sec. II, we review the basic ingredients of the $\mathcal{N}=3$ harmonic superspace and the $\mathcal{N}=3$ SYM action. In Sec. III, we introduce the Fermi-Feynman gauge to gauge fix this SYM action and derive the propagators. As an application, we prove the nonrenormalization theorem. In Sec. IV, we introduce the background field gauge to simplify the diagrammatic computations and present a sample calculation. Finally, we conclude with some discussion in Sec. V.

II. REVIEW

Our notation will closely follow [3], and we review it here for orientation purposes. The full 4D $\mathcal{N}=3$ superspace has the usual set of ordinary bosonic ($x^{\alpha\dot{\alpha}}$) and fermionic ($\theta_i^\alpha, \bar{\theta}^{i\dot{\alpha}}$) coordinates with $i=1,2,3$. The harmonic superspace augments these with six internal bosonic coordinates of the R-symmetry coset $SU(3)/U(1)\times U(1)$, denoted collectively as u. Using these internal coordinates, an “analytic” subspace with eight out of the twelve θ's of the full superspace is identified, which allows one to

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construct an off-shell action for the $\mathcal{N}=3$ SYM and proceed with its quantization. In this section, we discuss the internal coordinates in some detail first, then the fermionic ones, and finally the superspace action of $\mathcal{N}=3$ SYM. The quantization is dealt with in subsequent sections.

A. Internal coordinates

A $SU(3)/U(1)\times U(1)$ coset element can be parametrized in matrix form as follows:

$$\mathcal{U}=(u_i^1,u_i^2,u_i^3)\equiv(u_i^{(1,1)},u_i^{(-1,1)},u_i^{(0,-2)}),$$
$$\mathcal{U}^\dagger=\begin{pmatrix}\bar{u}_1^i\\ \bar{u}_2^i\\ \bar{u}_3^i\end{pmatrix}\equiv\begin{pmatrix}\bar{u}^{i(-1,-1)}\\ \bar{u}^{i(1,-1)}\\ \bar{u}^{i(0,2)}\end{pmatrix}, \qquad (2.1)$$

$$\text{Constraints: } \mathcal{U}^\dagger\mathcal{U}=\mathcal{U}\mathcal{U}^\dagger=1, \qquad \det\mathcal{U}=1,$$
$$\Rightarrow \bar{u}_a\cdot u^b=\bar{u}_a^i u_i^b=\delta_a^b, \qquad u_i^a\bar{u}_a^j=\delta_i^j,$$
$$\epsilon^{ijk}u_i^1u_j^2u_k^3=1. \qquad (2.2)$$

The notation (q_1,q_2) denotes the charges corresponding to the two Cartan $U(1)$ generators Q_1,Q_2 of $SU(3)$. Given the constraints in Eq. (2.2), we have eight independent coordinates in the $\mathcal{U}$ matrix, as expected for $SU(3)$. However, we also require the two $U(1)$ charges to be fixed (i.e., $Q_iu=q_iu$), which effectively implements the $U(1)^2$ quotient, and we are left with six independent coordinates.

Furthermore, the six harmonic covariant derivatives acting on these coordinates are

$$D_b^a=u_i^a\frac{\partial}{\partial u_i^b}-\bar{u}_b^i\frac{\partial}{\partial\bar{u}_a^i} \quad (a\neq b). \qquad (2.3)$$

These derivatives satisfy the $SU(3)$ Lie algebra given by

$$[D_b^a,D_d^c]=\delta_b^cD_d^a-\delta_d^aD_b^c. \qquad (2.4)$$

We note here that the two Cartan generators are given in terms of D_a^a (no sum over a) as follows:

$$\begin{aligned}Q_1&=D_1^1-D_2^2\\ Q_2&=D_1^1+D_2^2-2D_3^3\end{aligned} \quad\Rightarrow\quad \begin{aligned}D_1^1-D_3^3&=\tfrac{1}{2}(Q_1+Q_2)\\ D_2^2-D_3^3&=\tfrac{1}{2}(Q_2-Q_1).\end{aligned} \qquad (2.5)$$

Their commutators with the harmonic derivatives are $[Q_i,D_b^a]=q_iD_b^a$ with the charges given by

$U(1)^2$	D_3^1	D_2^3	D_2^1	D_1^3	D_3^2	D_1^2
q_1	1	1	2	−1	−1	−2
q_2	3	−3	0	−3	3	0

In what follows, we will mostly be dealing with functions defined at two different points in this coset space, labeled as u and v. We denote their products by the notation $U^a{}_b=u^a\cdot\bar{v}_b$ and $\bar{U}_a{}^b=\bar{u}_a\cdot v^b$ such that the covariant derivatives in this basis simply read

$$D_b^a=U^a{}_c\frac{\partial}{\partial U^b{}_c}-\bar{U}_b{}^c\frac{\partial}{\partial\bar{U}_a{}^c} \quad (a\neq b). \qquad (2.6)$$

Finally, the integration over this coset space is defined such that only a $SU(3)$ singlet integrand gives a nonvanishing result, i.e.,

$$\int du1=1, \qquad \int duD_b^af(u)=0. \qquad (2.7)$$

The latter integral allows one to integrate by parts in the u space.

B. Fermionic coordinates

We make a coordinate transformation of the usual θ's with $SU(3)$ indices to θ's having definite $U(1)$ charges as follows:

$$\theta_a^\alpha=\bar{u}_a^i\theta_i^\alpha, \qquad \bar{\theta}^{a\dot{\alpha}}=u_i^a\bar{\theta}^{i\dot{\alpha}}. \qquad (2.8)$$

The index a identifies the $U(1)$ charges straightforwardly via Eq. (2.1). Then the corresponding spinorial covariant derivatives satisfy the following commutators:

$$\{D_\alpha^a,D_\beta^b\}=0, \quad \{\bar{D}_{a\dot{\alpha}},\bar{D}_{b\dot{\beta}}\}=0, \quad \{D_\alpha^a,\bar{D}_{b\dot{\beta}}\}=i\delta_b^a\partial_{\alpha\dot{\beta}}, \qquad (2.9)$$

$$[D_b^a,D_\alpha^c]=\delta_b^cD_\alpha^a, \qquad [D_b^a,\bar{D}_{c\dot{\alpha}}]=-\delta_c^a\bar{D}_{b\dot{\alpha}}. \qquad (2.10)$$

Explicitly, the $U(1)^2$ charges of the spinorial derivatives are

$U(1)^2$	D_α^1	D_α^2	D_α^3	$\bar{D}_{1\dot{\alpha}}$	$\bar{D}_{2\dot{\alpha}}$	$\bar{D}_{3\dot{\alpha}}$
q_1	1	−1	0	−1	1	0
q_2	1	1	−2	−1	−1	2

The harmonic superspace is an analytic subspace of the full superspace, where the coordinates θ_1^α and $\bar{\theta}^{2\dot{\alpha}}$ do not appear explicitly in a given harmonic superfield $\Phi^{(q_1,q_2)}(x,\theta,u)$, i.e.,

$$D_\alpha^1\Phi^{(q_1,q_2)}=\bar{D}_{2\dot{\alpha}}\Phi^{(q_1,q_2)}=0. \qquad (2.11)$$

Note that these analytic constraints are preserved by the three harmonic derivatives D_2^1,D_3^1,D_2^3.

Finally, we can define an analytic measure $\int dud\zeta$ on harmonic superspace via the full superspace as follows:

$$\int d^4xd^{12}\theta du\equiv\int dud\zeta_{11}^{22}(D^1)^2(\bar{D}_2)^2$$
$$\Rightarrow\int d\zeta_{11}^{22}=\int d^4x_A(D^2)^2(D^3)^2(\bar{D}_1)^2(\bar{D}_3)^2. \qquad (2.12)$$

We frequently use the notation $[D_\vartheta^4]_{22}^{11}\equiv(D^1)^2(\bar{D}_2)^2$ to denote the four θ's that are not part of the harmonic superspace. The $[D^4_\vartheta]^{11}_{22}$ has $U(1)^2$ charge (4,0), the negative of that for the measure $d\zeta^{22}_{11}$.

C. SYM action

We do not review here the procedure for finding prepotentials of the $\mathcal{N}=3$ SYM in harmonic superspace but instead simply state the results. The $\mathcal{N}=3$ prepotentials are the gauge connections of the analyticity-preserving harmonic derivatives, i.e., we have three connections defined by $\nabla = D + iA$. The gauge transformations read as usual: $\delta A = -\nabla\lambda$. The field strengths are introduced via the "flat" commutation relations as follows:

$$[\nabla^1_2, \nabla^1_3] = F^{11}_{23}, \quad [\nabla^3_2, \nabla^1_2] = F^{31}_{22}, \quad [\nabla^1_3, \nabla^3_2] = \nabla^1_2 + F^1_2. \tag{2.13}$$

The equations of motion are, of course, all $F=0$. These are generated simply by the following Chern-Simons-like action:

$$\mathcal{S} = \mathrm{tr}\int du d\zeta^{22}_{11}(A^1_2 F^1_2 + A^3_2 F^{11}_{23} + A^1_3 F^{31}_{22} - iA^1_2[A^1_3, A^3_2]). \tag{2.14}$$

Also, notice that one of the three prepotentials is related algebraically to the other two on shell,

$$D^1_3 A^3_2 - D^3_2 A^1_3 + i[A^1_3, A^3_2] = A^1_2, \tag{2.15}$$

from which we start the quantization procedure in the next section.

III. QUANTIZING SYM IN FERMI-FEYNMAN GAUGE

The $\mathcal{N}=3$ SYM action (2.14), after substituting the algebraic equation defining A^1_2 (2.15), depends only on two harmonic connections and reads

$$\mathcal{S} = \mathrm{tr}\int du\, d\zeta^{22}_{11}\{(D^1_3 A^3_2)^2 + (D^3_2 A^1_3)^2 + 2A^1_3(D^3_2 D^1_3 A^3_2) - 2A^1_3 D^1_2 A^3_2 + 2i[A^1_3, A^3_2](D^1_3 A^3_2 - D^3_2 A^1_3) - [A^1_3, A^3_2]^2\}. \tag{3.1}$$

We choose the following gauge-fixing function:

$$\mathcal{S}_{gf} = -\mathrm{tr}\int du d\zeta^{22}_{11}(D^1_3 A^3_2 + D^3_2 A^1_3)^2 = -\mathrm{tr}\int du d\zeta^{22}_{11}\{(D^1_3 A^3_2)^2 + (D^3_2 A^1_3)^2 - 2A^1_3(D^3_2 D^1_3 A^3_2)\}, \tag{3.2}$$

such that the gauge-fixed action for SYM in Fermi-Feynman gauge becomes

$$\mathcal{S} + \mathcal{S}_{gf} = \mathrm{tr}\int du d\zeta^{22}_{11}\{2A^1_3(D^1_3 D^3_2 + D^3_2 D^1_3 - 2D^1_2)A^3_2 + 2i[A^1_3, A^3_2](D^1_3 A^3_2 - D^3_2 A^1_3) - [A^1_3, A^3_2]^2\}, \tag{3.3}$$

where we use $[D^1_3, D^3_2] = D^1_2$ once. The ghost action follows from the Becchi-Rouet-Stora-Tyutin formalism (BRST) straightforwardly by using $\delta A = -D\lambda - i[A, \lambda]$:

$$\mathcal{S}_{gh} = -\mathrm{tr}\int du d\zeta^{22}_{11}\{b^1_2(D^1_3 D^3_2 + D^3_2 D^1_3)c + i(D^1_3 b^1_2[A^3_2, c] + D^3_2 b^1_2[A^1_3, c])\}. \tag{3.4}$$

Having just introduced new fields, let us recap the $U(1)^2$ charges of all the fields here (which are straightforwardly deduced from the covariant derivatives):

$U(1)^2$	A^1_3	A^3_2	A^1_2	λ	c	b^1_2
q_1	1	1	2	0	0	2
q_2	3	−3	0	0	0	0

A. Propagators

From the gauge-fixed SYM and ghost actions given above, we can derive the equations to solve for the Green's functions for vector and ghost superfields:

$$\begin{aligned}
(K^1_2)_0 G^{0|1}_{0|2}(1,2) &= [\delta_A]^{1|1}_{2|2}(1,2),\\
(K^1_2)_0 G^{1|0}_{2|0}(1,2) &= [\delta_A]^{11|0}_{22|0}(1,2),\\
(K^1_2)_{+1} G^{1|3}_{3|2}(1,2) &= [\delta_A]^{11|3}_{23|2}(1,2),\\
(K^1_2)_{-1} G^{3|1}_{2|3}(1,2) &= [\delta_A]^{13|1}_{22|3}(1,2),
\end{aligned} \tag{3.5}$$

where $(K^1_2)_a = (1+|a|)(\{D^1_3, D^3_2\} + 2aD^1_2)$ and the analytic delta functions explicitly read

$$\begin{aligned}
[\delta_A]^{1|1}_{2|2}(1,2) &= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})\delta^{1|2}_{2|1}(u,v)\\
&= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})(U^1{}_1\bar{U}_2{}^2)\delta^6(u,v),\\
[\delta_A]^{11|0}_{22|0}(1,2) &= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})\delta^{11|22}_{22|11}(u,v)\\
&= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})(U^1{}_1\bar{U}_2{}^2)^2\delta^6(u,v),\\
[\delta_A]^{11|3}_{23|2}(1,2) &= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})\delta^{11|23}_{23|11}(u,v)\\
&= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})(U^1{}_1)^3\delta^6(u,v),\\
[\delta_A]^{13|1}_{22|3}(1,2) &= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})\delta^{13|22}_{22|13}(u,v)\\
&= \delta(x_{12})[D^4_{v\vartheta}]^{11}_{22}\delta^{12}(\theta_{12})(\bar{U}_2{}^2)^3\delta^6(u,v).
\end{aligned} \tag{3.6}$$

The general form of G's which satisfy the Green's function equations then looks like

$$\langle c(1)b_2^1(2)\rangle \equiv G_{0|2}^{0|1}(1,2) = \mathcal{F}_{11|1}^{22|2}(u,v)\frac{1}{\Box}[D_{u\vartheta}^4]_{22}^{11}[D_{v\vartheta}^4]_{22}^{11}\delta^{12}(\theta_{12})\delta(x_{12}), \tag{3.7}$$

$$\langle b_2^1(1)c(2)\rangle \equiv G_{2|0}^{1|0}(1,2) = \mathcal{F}_{1|11}^{2|22}(u,v)\frac{1}{\Box}[D_{u\vartheta}^4]_{22}^{11}[D_{v\vartheta}^4]_{22}^{11}\delta^{12}(\theta_{12})\delta(x_{12}), \tag{3.8}$$

$$\langle A_3^1(1)A_2^3(2)\rangle \equiv G_{3|2}^{1|3}(1,2) = \mathcal{F}_{13|11}^{22|23}(u,v)\frac{1}{\Box}[D_{u\vartheta}^4]_{22}^{11}[D_{v\vartheta}^4]_{22}^{11}\delta^{12}(\theta_{12})\delta(x_{12}), \tag{3.9}$$

$$\langle A_2^3(1)A_3^1(2)\rangle \equiv G_{2|3}^{3|1}(1,2) = \mathcal{F}_{11|13}^{23|22}(u,v)\frac{1}{\Box}[D_{u\vartheta}^4]_{22}^{11}[D_{v\vartheta}^4]_{22}^{11}\delta^{12}(\theta_{12})\delta(x_{12}), \tag{3.10}$$

such that

$$(K_2^1)_a\mathcal{F}_{\cdot|\cdot}^{\cdot|\cdot}(u,v) = \frac{1}{2}(D_1^2)^2\delta_{\cdot|\cdot}^{\cdot|\cdot}(u,v). \tag{3.11}$$

The $\delta_{\cdot|\cdot}^{\cdot|\cdot}(u,v)$ functions are the same δ-functions appearing in the corresponding $[\delta_A]_{\cdot|\cdot}^{\cdot|\cdot}(1,2)$ defined in Eq. (3.6). Equation (3.11) is motivated by the identity[1]

$$(D_1^2)^2D_{u\vartheta}^4D_{v\vartheta}^4\delta^6(u,v) = 2\Box D_{v\vartheta}^4\delta^6(u,v), \tag{3.12}$$

which can be used to prove that Eqs. (3.7)–(3.10) indeed satisfy Eq. (3.5).

In order to make the above equations simpler and more tractable, we choose the following "independent" internal coordinates in the $\{U^a{}_b, \bar{U}_a{}^b\}$ basis:

$$U^1{}_1, U^1{}_3, U^2{}_1, U^3{}_1, \bar{U}_2{}^1, \bar{U}_2{}^2, \bar{U}_2{}^3, \bar{U}_3{}^2, \tag{3.13}$$

and the "zero charge" δ-function in these coordinates reads

$$\delta^6(u,v) = \pi U^1{}_1\bar{U}_2{}^2\delta(U^1{}_3)\delta(U^3{}_1)\delta(\bar{U}_2{}^3) \times\delta(\bar{U}_3{}^2)\delta(U^2{}_1)\delta(\bar{U}_2{}^1). \tag{3.14}$$

The rest of the U coordinates can be written in terms of the chosen ones as follows:

$$\begin{aligned}
U^1{}_2 &= -\frac{U^1{}_1\bar{U}_2{}^1 + U_3{}^1\bar{U}_2{}^3}{\bar{U}_2{}^2},\\
U^2{}_2 &= -\frac{U^1{}_1U^2{}_1\bar{U}_2{}^1 + U_3{}^1U^2{}_1\bar{U}_2{}^3 - \bar{U}_2{}^3\bar{U}_3{}^2 - U^1{}_1}{U^1{}_1\bar{U}_2{}^2},\\
U^2{}_3 &= \frac{U^1{}_3U^2{}_1 - \bar{U}_3{}^2}{U^1{}_1},\\
U^3{}_2 &= -\frac{U^1{}_1U^3{}_1\bar{U}_2{}^1 + U^1{}_3U^3{}_1\bar{U}_2{}^3 + \bar{U}_2{}^2\bar{U}_2{}^3}{U^1{}_1\bar{U}_2{}^2},\\
U^3{}_3 &= \frac{U^1{}_3U^3{}_1 + \bar{U}_2{}^2}{U^1{}_1},\\
\bar{U}_1{}^1 &= \frac{-U^2{}_1\bar{U}_2{}^1\bar{U}_2{}^2 - U^3{}_1\bar{U}_2{}^1\bar{U}_3{}^2 + U^1{}_3U^3{}_1 + \bar{U}_2{}^2}{U^1{}_1\bar{U}_2{}^2},\\
\bar{U}_1{}^2 &= -\frac{U^2{}_1\bar{U}_2{}^2 + U^3{}_1\bar{U}_3{}^2}{U^1{}_1},\\
\bar{U}_1{}^3 &= -\frac{U^2{}_1\bar{U}_2{}^2\bar{U}_2{}^3 + U^3{}_1\bar{U}_2{}^3\bar{U}_3{}^2 + U^1{}_1U^3{}_1}{U^1{}_1\bar{U}_2{}^2},\\
\bar{U}_3{}^1 &= \frac{\bar{U}_2{}^1\bar{U}_3{}^2 - U^1{}_3}{\bar{U}_2{}^2}, \quad \bar{U}_3{}^3 = \frac{\bar{U}_2{}^3\bar{U}_3{}^2 + U^1{}_1}{\bar{U}_2{}^2}.
\end{aligned} \tag{3.15}$$

The differential operators get modified as well, leading to the following expressions:

$$\begin{aligned}
D^1{}_2 &= U^1{}_1\frac{\partial}{\partial U^2{}_1}, \qquad D^1{}_3 = U^1{}_1\frac{\partial}{\partial U^3{}_1},\\
D^3{}_2 &= U^3{}_1\frac{\partial}{\partial U^2{}_1} - \bar{U}_2{}^2\frac{\partial}{\partial \bar{U}_3{}^2},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
D^2{}_1 &= U^2{}_1\frac{\partial}{\partial U^1{}_1} + U^2{}_3\frac{\partial}{\partial U^1{}_3} - \bar{U}_1{}^2\frac{\partial}{\partial \bar{U}_2{}^1} - \bar{U}_1{}^2\frac{\partial}{\partial \bar{U}_2{}^2} - \bar{U}_1{}^3\frac{\partial}{\partial \bar{U}_2{}^3}\\
&= U^2{}_1\frac{\partial}{\partial U^1{}_1} + \left(\frac{U^1{}_3U^2{}_1 - \bar{U}_3{}^2}{U^1{}_1}\right)\frac{\partial}{\partial U^1{}_3}\\
&\quad - \left(\frac{-U^2{}_1\bar{U}_2{}^1\bar{U}_2{}^2 - U^3{}_1\bar{U}_2{}^1\bar{U}_3{}^2 + U^1{}_3U^3{}_1 + \bar{U}_2{}^2}{U^1{}_1\bar{U}_2{}^2}\right)\\
&\quad \times\frac{\partial}{\partial \bar{U}_2{}^1} + \left(\frac{U^2{}_1\bar{U}_2{}^2 + U^3{}_1\bar{U}_3{}^2}{U^1{}_1}\right)\frac{\partial}{\partial \bar{U}_2{}^2}\\
&\quad + \left(\frac{U^2{}_1\bar{U}_2{}^2\bar{U}_2{}^3 + U^3{}_1\bar{U}_2{}^3\bar{U}_3{}^2 + U^1{}_1U^3{}_1}{U^1{}_1\bar{U}_2{}^2}\right)\frac{\partial}{\partial \bar{U}_2{}^3},
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
D^3{}_1 &= U^3{}_1\frac{\partial}{\partial U^1{}_1} + U^3{}_3\frac{\partial}{\partial U^1{}_3} - \bar{U}_1{}^2\frac{\partial}{\partial \bar{U}_3{}^2}\\
&= U^3{}_1\frac{\partial}{\partial U^1{}_1} + \left(\frac{U^1{}_3U^3{}_1 + \bar{U}_2{}^2}{U^1{}_1}\right)\frac{\partial}{\partial U^1{}_3}\\
&\quad + \left(\frac{U^2{}_1\bar{U}_2{}^2 + U^3{}_1\bar{U}_3{}^2}{U^1{}_1}\right)\frac{\partial}{\partial \bar{U}_3{}^2},
\end{aligned} \tag{3.18}$$

[1] We will suppress the $SU(3)$ "indices" on $[D_\vartheta^4]_{22}^{11}$ from now on.

$$D^2{}_3 = U^2{}_1\frac{\partial}{\partial U^3{}_1} - \bar{U}_3{}^1\frac{\partial}{\partial \bar{U}_2{}^1} - \bar{U}_3{}^2\frac{\partial}{\partial \bar{U}_2{}^2} - \bar{U}_3{}^3\frac{\partial}{\partial \bar{U}_2{}^3}$$
$$= U^2{}_1\frac{\partial}{\partial U^3{}_1} + \left(\frac{U^1{}_3 - \bar{U}_2{}^1\bar{U}_3{}^2}{\bar{U}_2{}^2}\right)\frac{\partial}{\partial \bar{U}_2{}^1} - \bar{U}_3{}^2\frac{\partial}{\partial \bar{U}_2{}^2} - \left(\frac{U^1{}_1 + \bar{U}_2{}^3\bar{U}_3{}^2}{\bar{U}_2{}^2}\right)\frac{\partial}{\partial \bar{U}_2{}^3}. \quad (3.19)$$

Using all these expressions, we can write Eq. (3.11) explicitly in the following form:

$$(1+|a|)\left[U^3{}_1\frac{\partial^2}{\partial U^2{}_1\partial U^3{}_1} - \bar{U}_2{}^2\frac{\partial^2}{\partial \bar{U}_3{}^2\partial U^3{}_1} + \left(a+\frac{1}{2}\right)\frac{\partial}{\partial U^2{}_1}\right]\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot} = \Delta^{\cdot|\cdot}_{\cdot|\cdot}\delta(U^3{}_1)\delta(U^2{}_1)\delta(\bar{U}_3{}^2), \quad (3.20)$$

where

$$\Delta^{\cdot|\cdot}_{\cdot|\cdot} = \frac{\pi}{2}(U^1{}_1)^{n_1}(\bar{U}_2{}^2)^{n_2}\delta(U^1{}_3)\delta(\bar{U}_2{}^3)\delta''(\bar{U}_2{}^1) \quad \text{with}$$

$$(n_1,n_2) = \begin{cases} (-1,2) & \text{for } \langle c(1)b_2^1(2)\rangle, \\ (0,3) & \text{for } \langle b_2^1(1)c(2)\rangle, \\ (1,1) & \text{for } \langle A_3^1(1)A_2^3(2)\rangle, \\ (-2,4) & \text{for } \langle A_2^3(1)A_3^1(2)\rangle. \end{cases}$$

Relabeling $U^3{}_1 = x$, $U^1{}_3 = \hat{x}$, $U^2{}_1 = y$, $\bar{U}_2{}^1 = \bar{y}$, $\bar{U}_3{}^2 = z$, $\bar{U}_2{}^3 = \hat{z}$, $\bar{U}_2{}^2 = A$, $U^1{}_1 = B$, and $a+\frac{1}{2} = b$, we get a simple looking partial differential equation

$$(x\partial_x\partial_y - A\partial_x\partial_z + b\partial_y)\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot} = \frac{\pi}{2(1+|a|)}B^{n_1}A^{n_2}\delta(x)\delta(\hat{x})\delta(y)\delta''(\bar{y})\delta(z)\delta(\hat{z}). \quad (3.21)$$

We solve it by choosing an ansatz of the form

$$\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot} = C_aA^pB^q\frac{y^r\bar{y}^s}{(\bar{y}+\frac{\epsilon^2}{y})^3}\left(\frac{Ay+xz}{B}\right)^t\left(\frac{B\bar{y}+\hat{x}\hat{z}}{A}\right)^{-1}, \quad (3.22)$$

where ϵ is an infinitesimal parameter and the exponents p, q, r, s, t along with the normalization factor C_a are to be determined. Plugging Eq. (3.22) into the lhs of Eq. (3.21), we find that the values $t=-b$, $r=b-1$, $s=0$ simplify the expression to a single term as follows:

$$\begin{aligned}\text{lhs of Eq. (3.21)} &= \frac{3C_abA^{p+2}B^{q+b}y^{b+2}(Ay+xz)^{-b}\epsilon^2}{(Ay+xz)(B\bar{y}+\hat{x}\hat{z})(y\bar{y}+\epsilon^2)^4} \\ &= \frac{C_abA^{p+2}B^{q+b}(A+\frac{xz}{y})^{-b}}{2(Ay+xz)(B\bar{y}+\hat{x}\hat{z})}\pi\delta(y)\delta''(\bar{y}) \\ &= \frac{\pi}{2}C_abA^{p+2}B^{q+b}\left(A+\frac{xz}{y}\right)^{-b}\delta(x)\delta(z)\delta(\hat{x})\delta(\hat{z})\delta(y)\delta''(\bar{y}) \\ &= \frac{\pi}{2}C_abB^{q+b}A^{p+2-b}\delta(x)\delta(\hat{x})\delta(y)\delta''(\bar{y})\delta(z)\delta(\hat{z}), \end{aligned} \quad (3.23)$$

where we use the fact that δ's for y, $\bar{y}$ are produced in the limit of $\epsilon\to 0$ and $y\to 0$ via the following identity:

$$\frac{(m+1)!(-y)^m\epsilon^2}{(y\bar{y}+\epsilon^2)^{m+2}} \to \pi\delta(y)\delta^{(m)}(\bar{y}), \quad (3.24)$$

and $\delta(z) = \frac{1}{z}$, etc., for rest of the four nonconjugate complex variables. Now, comparing Eq. (3.23) to the rhs of Eq. (3.21), we can deduce that $p = n_2+b-2$, $q = n_1-b$, and $C_a = \frac{1}{b(1+|a|)}$. Thus, the final form of $\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot}$ that solves Eq. (3.21) reads

$$\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot} = \frac{1}{b(1+|a|)}A^{n_2+b-2}B^{n_1-b} \times \frac{y^{b-1}}{(\bar{y}+\frac{\epsilon^2}{y})^3}\left(\frac{Ay+xz}{B}\right)^{-b}\left(\frac{B\bar{y}+\hat{x}\hat{z}}{A}\right)^{-1}. \quad (3.25)$$

Finally, the complete propagators in terms of the U variables read as follows:

$$G^{\cdot|\cdot}_{\cdot|\cdot}(1,2) = \frac{(\bar{U}_2^2)^{n_2+b-2}(U_1^1)^{n_1-b}}{b(1+|a|)}\frac{(U^2{}_1)^{b-1}}{(\bar{U}_2{}^1+\frac{\epsilon^2}{U^2{}_1})^3} \times \frac{(-\bar{U}_1{}^2)^{-b}}{(-U^1{}_2)}\frac{1}{\Box}D^4_{u\vartheta}D^4_{v\vartheta}\delta^{12}(\theta_{12})\delta(x_{12}). \quad (3.26)$$

B. Feynman rules

The Feynman rules are now derived as usual. The vector and ghost propagators are given in Eq. (3.26), but we reproduce them here individually with explicit harmonic factors in momentum space (replace $\frac{1}{\Box}\delta(x_{12}) \to \frac{1}{-k^2}$):

$$\langle c(1)b_2^1(2)\rangle = \frac{2(\bar{U}_2^2)^{\frac{1}{2}}}{(U^1{}_1)^{\frac{3}{2}}}\frac{(U^2{}_1)^{-\frac{1}{2}}}{(\bar{U}_2{}^1+\frac{\epsilon^2}{U^2{}_1})^3}\frac{(-\bar{U}_1{}^2)^{-\frac{1}{2}}}{U^1{}_2} \times \frac{1}{k^2}D^4_{u\vartheta}D^4_{v\vartheta}\delta^{12}(\theta_{12}), \tag{3.27}$$

$$\langle b_2^1(1)c(2)\rangle = \frac{2(\bar{U}_2{}^2)^{\frac{3}{2}}}{(U^1{}_1)^{\frac{1}{2}}}\frac{(U^2{}_1)^{-\frac{1}{2}}}{(\bar{U}_2{}^1+\frac{\epsilon^2}{U^2{}_1})^3}\frac{(-\bar{U}_1{}^2)^{-\frac{1}{2}}}{U^1{}_2} \times \frac{1}{k^2}D^4_{u\vartheta}D^4_{v\vartheta}\delta^{12}(\theta_{12}), \tag{3.28}$$

$$\langle A_3^1(1)A_2^3(2)\rangle = \frac{(\bar{U}_2{}^2)^{\frac{1}{2}}}{3(U^1{}_1)^{\frac{1}{2}}}\frac{(U^2{}_1)^{\frac{1}{2}}}{(\bar{U}_2{}^1+\frac{\epsilon^2}{U^2{}_1})^3}\frac{(-\bar{U}_1{}^2)^{-\frac{3}{2}}}{U^1{}_2} \times \frac{1}{k^2}D^4_{u\vartheta}D^4_{v\vartheta}\delta^{12}(\theta_{12}), \tag{3.29}$$

$$\langle A_2^3(1)A_3^1(2)\rangle = -\frac{(\bar{U}_2{}^2)^{\frac{3}{2}}}{(U^1{}_1)^{\frac{3}{2}}}\frac{(U^2{}_1)^{-\frac{3}{2}}}{(\bar{U}_2{}^1+\frac{\epsilon^2}{U^2{}_1})^3}\frac{(-\bar{U}_1{}^2)^{\frac{1}{2}}}{U^1{}_2} \times \frac{1}{k^2}D^4_{u\vartheta}D^4_{v\vartheta}\delta^{12}(\theta_{12}), \tag{3.30}$$

where we have kept the ϵ prescription explicit. The vertices can be read from Eqs. (3.3) and (3.4):

$$\langle (A_3^1)^a(A_2^3)^b(A_2^3)^c\rangle/\langle (A_2^3)^a(A_3^1)^b(A_3^1)^c\rangle \to 2\int du d^8\theta f^{abc}[D_3^1/D_2^3],$$
$$\langle (A_3^1)^a(A_2^3)^b(A_3^1)^c(A_2^3)^d\rangle \to i\int du d^8\theta f^{abe}f^{ecd},$$
$$\langle (A_2^3)^a c^b(b_2^1)^c\rangle/\langle (A_3^1)^a c^b(b_2^1)^c\rangle \to -\int du d^8\theta f^{abc}[D_3^1/D_2^3], \tag{3.31}$$

where $\int d^8\theta \equiv (D^2)^2(D^3)^2(\bar{D}_1)^2(\bar{D}_3)^2$, f^{abc} are structure constants of the gauge group and the harmonic derivatives act on the leg corresponding to the group index 'c'.

Evaluating loop graphs

Let us first focus on one-loop graphs. We can generate $\int d^{12}\theta$ from the analytic measure at vertices by taking off one factor of D^4_ϑ from the propagators. After this, we are left with $(v_3+v_4)\int d^{12}\theta$ integrals from three- and four-point vertices, $p\, D^4_\vartheta\delta^{12}(\theta)$'s from propagators. As usual, we need to saturate all but one $\int d^{12}\theta$, which means that we need to kill one of the $\delta^{12}(\theta)$. This is achieved by using three D^4_ϑ's and the identity

$$\begin{aligned} D^4_{w\vartheta}D^4_{v\vartheta}D^4_{u\vartheta} = {}& [((W^1{}_3)^2(\bar{W}_2{}^3)^2(V^1{}_2)^2(\bar{V}_2{}^1)^2 + (W^1{}_3)^2(\bar{W}_2{}^1)^2(V^1{}_2)^2(\bar{V}_2{}^3)^2 + (W^1{}_2)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^1)^2 \\ & + (W^1{}_2)^2(\bar{W}_2{}^1)^2(V^1{}_3)^2(\bar{V}_2{}^3)^2)D^8_{u\theta} - i(W^1{}_2)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^1)^2 D^3\cdot\partial\cdot\bar{D}_3(D^2)^2(\bar{D}_1)^2 \\ & - \frac{1}{2}\Box((W^1{}_2)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^1)^2(D^2)^2(\bar{D}_1)^2 + (W^1{}_3)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^1)^2(D^3)^2(\bar{D}_1)^2 \\ & + (W^1{}_2)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^3)^2(D^2)^2(\bar{D}_3)^2 + (W^1{}_3)^2(\bar{W}_2{}^3)^2(V^1{}_3)^2(\bar{V}_2{}^3)^2(D^3)^2(\bar{D}_3)^2)]D^4_{u\vartheta}, \end{aligned} \tag{3.32}$$

where $D^8_\theta \equiv (D^2)^2(D^3)^2(\bar{D}_1)^2(\bar{D}_3)^2$, $V^a{}_b \equiv v^a\bar{u}_b$, $\bar{W}_a{}^b \equiv \bar{w}_a u^b$, etc. The first term which contains $D^8_\theta D^4_\vartheta$ [recall that $D^4_\vartheta \equiv (D^1)^2(\bar{D}_2)^2$] can be used to kill one $\delta^{12}(\theta)$. This means that a two-point function trivially vanishes, as it does not have enough D^4_ϑ's, and a three-point function cannot have any divergent piece due to the presence of three $\Box$'s in the denominator, i.e., $\int \frac{d^4k}{(k^2)^3}$ is finite. In fact, no higher-point function can have any divergent piece at one loop because the numerator can generate at most $(p-3)$ $\Box$'s (together with a D^8_θ) compared to $p\,\Box$'s in the denominator, so the difference is always $n \geq 3$, i.e., $\int \frac{d^4k}{(k^2)^n}$ is finite.

This power counting readily generalizes to multiloop graphs because each one-loop subgraph needs to follow this procedure of D algebra, and hence the whole graph is rendered finite. This proves the nonrenormalization theorem at all loops for $\mathcal{N}=3$ SYM or, equivalently, $\mathcal{N}=4$ SYM.

IV. QUANTIZING SYM IN BACKGROUND FIELD GAUGE

The computation of finite terms for loop graphs is still cumbersome with the Feynman rules discussed in the previous section because manifestly covariant expressions are not obtained for individual graphs. For that purpose, we develop the background field formalism in this section.

Let us gauge covariantize all the differential operators $(D \to \nabla = D + iA)$. Then we do a background splitting of the connections in a straightforward manner, $A \to \mathcal{A}_{bg} + \mathfrak{a}_q$, where the subscripts are suppressed in favor of self-explanatory fonts. Next, we choose different representations for these connections: the "real" rep for background $\mathcal{A}$'s, meaning that the three harmonic connections vanish ($\mathcal{A}_2^1 = \mathcal{A}_3^1 = \mathcal{A}_2^3 = 0$), and the "analytic" rep for the quantum $\mathfrak{a}$'s, meaning that the four fermionic connections vanish ($\mathfrak{a}_\alpha^1 = \bar{\mathfrak{a}}_{2\dot{\alpha}} = 0 \Rightarrow D_\vartheta \to \nabla_\vartheta \equiv \mathcal{D}_\vartheta$).[2] Let us write down the consequences of these choices on connections and field strengths from various commutators:

$$\begin{aligned}
\{\nabla_\alpha^a, \bar{\nabla}_{b\dot{\beta}}\} &= i\delta_b^a \nabla_{\alpha\dot{\beta}}\ (\text{"unchanged"}),\\
\{\nabla_\alpha^a, \nabla_\beta^b\} &= \epsilon_{\alpha\beta}\bar{W}^{ab},\\
\{\bar{\nabla}_{a\dot{\alpha}}, \bar{\nabla}_{b\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} W_{ab},\\
[\nabla_b^a, \nabla_\alpha^c] &= \delta_b^c \nabla_\alpha^a,\\
[\nabla_b^a, \bar{\nabla}_{c\dot{\alpha}}] &= -\delta_c^a \bar{\nabla}_{b\dot{\alpha}},\\
[\nabla_b^a, \nabla_d^c] &= \delta_b^c \nabla_d^a - \delta_d^a \nabla_b^c.
\end{aligned} \tag{4.1}$$

Of course, the W's are antisymmetric in the two indices, and they satisfy a few Bianchi identities along with some analytic plus harmonic constraints [5,8]. The most relevant identity for us is

$$D_\alpha^a W_{bc} = \frac{1}{2}(\delta_b^a D_\alpha^k W_{kc} - \delta_c^a D_\alpha^k W_{kb}). \tag{4.2}$$

Another thing to note is that the spinorial background-covariant derivatives $\mathcal{D}_\alpha^1$ and $\bar{\mathcal{D}}_{2\dot{\alpha}}$ still possess the structure of Eq. (2.9), so that the "background analytic" superfields can be defined: $\mathcal{D}_\alpha^1\Phi = \bar{\mathcal{D}}_{2\dot{\alpha}}\Phi = 0$. Moreover, the fourth and fifth equations of Eq. (4.1) tell us that the harmonic connections we are most interested in are now background analytic. We will also need the following identity defining a generalized d'Alembertian:

$$\begin{aligned}
\mathcal{D}_\vartheta^4(\nabla_1^2)^2\mathcal{D}_\vartheta^4 &= 2\hat{\Box}\mathcal{D}_\vartheta^4\\
&= 2[(\Box - 2\bar{W}^{12}W_{21}) - \bar{\nabla}_1\cdot\bar{\nabla}_a\bar{W}^{1a}\\
&\quad + 2\nabla^2\cdot\mathcal{D}^1 W_{21} + \nabla_1^2(\mathcal{D}^1)^2 W_{21}]\mathcal{D}_\vartheta^4.
\end{aligned} \tag{4.3}$$

A. Feynman rules

Since the harmonic connections appearing in the action as well as the ghosts are purely quantum superfields but background analytic, only $\Box$ changes to $\hat{\Box}$ in the propagators were derived in the previous section, whereas the vertices in the Lagrangian remain the same. However, we can expand the $\hat{\Box}$ to get vertices with explicit field strengths, which leads to covariant results in the loop calculations directly. With this structure, we can write down the background Feynman rules as follows:

$$\begin{aligned}
&\text{Propagators:} \begin{cases} \mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot}(u,v)\frac{1}{-k^2}\mathcal{D}_{u\vartheta}^4\mathcal{D}_{v\vartheta}^4\delta^{12}(\theta_{12}), & \text{ignore background vertices,}\\ \frac{1}{-k^2}\delta^8(\theta_{12})\delta^6(u,v), & \text{consider all such vertices but one,}\end{cases}\\
&\text{All but one background vertex:} \int dud^8\theta(\hat{\Box} - \Box_0),\\
&\text{One background vertex:} \int dudvd^8\theta(K_2^1)_a\delta^6(u,v),\\
&\text{All quantum vertices: same as Eq. (3.31).}
\end{aligned}$$

Note that the background d'Alembertian is expanded as $\Box \equiv \mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} = \Box_0 + \cdots$, where $\Box_0$ corresponds to $-k^2$ in the momentum space.

One-loop graph computation

Let us focus on the computation of a four-point function here. It is finite and, from the symmetry arguments of Sec. 5.5 of [8], it is known to look like $(\bar{W}^{13}W_{23})^2$ in $\mathcal{N}=3$ harmonic superspace.

We can obtain such a contribution by evaluating a bubble graph and expanding the $(\hat{\Box} - \Box_0)$-vertex factors to get the relevant W's using Eqs. (4.3) and (4.2). The above Feynman rules give the following expression for the graph with vector loop shown in Fig. 1 (after performing θ and u integrals at two of the vertices):

$$\begin{aligned}
\Gamma_4^{(A)} \sim \hat{\mathcal{A}}_4 \int dudv \int d^8\theta_{1,2}(\bar{\nabla}_1\cdot\bar{\nabla}_3\bar{W}^{13}(p_1))\\
\times(\bar{\nabla}_1\cdot\bar{\nabla}_3\bar{W}^{13}(p_2))((U^2{}_1U^1{}_2)\nabla^2\cdot\nabla^3W_{23}(p_3))\\
\times\delta^8(\theta_{12})(K_2^1)_{-1}\mathcal{F}^{22|23}_{13|11}(u,v)\mathcal{D}_{u\vartheta}^4(p_4)\mathcal{D}_{v\vartheta}^4(p_4)\\
\times\delta^{12}(\theta_{12})\delta^6(u,v),
\end{aligned} \tag{4.4}$$

[2]Such a choice was used in the case of $\mathcal{N}=2$ projective superspace to construct the background field formalism [10]. It ensures that the effective action is independent of background fields with dimension 0 (like the harmonic connections), which is required for the nonrenormalization theorems to hold [11]. However, the simple splitting of $\mathcal{N}=3$ prepotentials is reminiscent of the background field formalism for the $\mathcal{N}=2$ harmonic superspace developed in [12], with further refinements and explicit calculations appearing in [13,14].

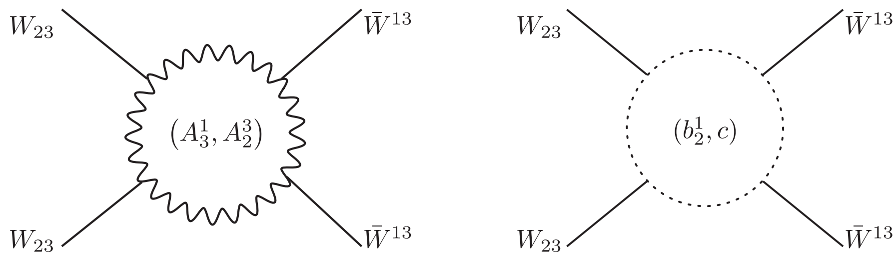

FIG. 1. One-loop four-point graph with external background field strengths.

where $\hat{\mathcal{A}}_4 \sim \int d^4k \frac{1}{(k_1^2)(k_2^2)(k_3^2)(k_4^2)}$ is the scalar box integral with the subscripts on loop momentum k denoting the external momenta (p_i) dependence. Note that we had to partially integrate $(K^1_2)_{+1}$ so that it changed to $(K^1_2)_{-1}$ acting on $\mathcal{F}^{\cdot|\cdot}_{\cdot|\cdot}$, which gives, using Eq. (3.20), $(K^1_2)_{-1}\mathcal{F}^{22|23}_{13|11}(u,v) = \frac{1}{2}(D^2_1)^2\delta^6(u,v) - 4\frac{\partial}{\partial U^2{}_1}\mathcal{F}^{22|23}_{13|11}$. The first term leads to a harmonic singularity with two $\delta^6(u,v)$ functions, but this singularity will cancel with the analogous contribution $\Gamma_4^{(bc)}$ from the ghost loop in Fig. 1. So we focus only on the second term, which has no analog from the ghost loop graph (as $a = 0$) and thus gives the complete four-point function. To get rid of the $\delta^{12}(\theta_{12})$ function, eight spinorial derivatives should be gathered off of W's in addition to $\mathcal{D}^4_\vartheta$ as follows[3]:

$$\begin{aligned}
\Gamma_4 &\sim \hat{\mathcal{A}}_4 \int du dv \int d^8\theta_{1,2} \bar{W}^{13}(p_1)\bar{W}^{13}(p_2)W_{23}(p_3)\delta^8(\theta_{12})D^1_2\mathcal{F}^{22|23}_{13|11}(u,v) \\
&\quad \times (\bar{\nabla}_1)^2(\bar{\nabla}_3)^2((U^2{}_1U^1{}_2)\nabla^2\cdot\nabla^3)(\bar{U}_2{}^1\bar{U}_2{}^1U^2{}_1U^1{}_2\nabla^2\cdot\nabla^3 W_{23}(p_4))\mathcal{D}^4_{v\vartheta}(p_4)\delta^{12}(\theta_{12})\delta^6(u,v) \\
&\sim \hat{\mathcal{A}}_4 \int du \int d^8\theta \bar{W}^{13}(p_1)\bar{W}^{13}(p_2)W_{23}(p_3)W_{23}(p_4)\left[\left(\frac{U^1{}_1}{(\bar{U}_2{}^1)^4(U^2{}_1)^2}\right)(U^2{}_1U^1{}_2)^2(\bar{U}_2{}^1)^2\right]_{|u\to v} \\
&\sim \hat{\mathcal{A}}_4 \int du \int d^8\theta \bar{W}^{13}(p_1)\bar{W}^{13}(p_2)W_{23}(p_3)W_{23}(p_4)\left[\frac{(U^1{}_2)^2}{(\bar{U}_2{}^1)^2}\right]_{|u\to v},
\end{aligned} \tag{4.5}$$

where we use $\nabla^8_\theta\mathcal{D}^4_\vartheta\delta^{12}(\theta_{12}) = 1$ in the second step. In the last step, the apparent harmonic singularity cancels as we take the limit $u \to v$, leading to the expected result for the one-loop effective action,

$$\Gamma_4 \sim \int du d\zeta^{22}_{11}\hat{\mathcal{A}}_4(\bar{W}^{13}W_{23})^2. \tag{4.6}$$

V. DISCUSSION

We have introduced a new gauge-fixing action to quantize $\mathcal{N} = 3$ SYM in $\mathcal{N} = 3$ harmonic superspace. This leads to simpler (and fewer) propagators for vector and ghost superfields in a Fermi-Feynman gauge. These are sufficient to prove the nonrenormalization theorem for $\mathcal{N} = 3, 4$ SYM at all loops. However, computation of loop graphs beyond the divergent terms can be simplified more with the background field formalism. With the background Feynman rules in hand, we have computed the one-loop four-point contribution to the effective action, which gets a finite contribution purely from the vector loop diagram.

The way that we derived the harmonic propagators here is reminiscent of how $\mathcal{N} = 2$ projective superspace [15,16] is derived from $\mathcal{N} = 2$ harmonic superspace [2,17] in [18]: by choosing a special parametrization of the harmonic R-symmetry coordinates on $SU(2)/U(1) \simeq S^2$ to obtain a single (complex) coordinate on CP^1 that forms the internal coordinate for the projective case.[4] Even though we did not take this route to its full conclusion, it should be possible to derive a $\mathcal{N} = 3$ projective superspace in such a way that it simplifies the $\int du$ integrals to something more tractable. For example, viewing the R-symmetry coset $SU(3)/U(1)^2 \simeq [SU(3)/(SU(2)\times U(1))] \times [SU(2)/U(1)] \simeq CP^2 \times CP^1$, one can expect to reduce the six $(x, \hat{x}, y, \bar{y}, z, \hat{z})$ R-symmetry coordinates of $\mathcal{N} = 3$ harmonic superspace to only three (x, y, z) for a possible $\mathcal{N} = 3$ projective superspace. This should then be followed by a projection of the harmonic

[3]The requirement of collecting eight spinorial derivatives to get ∇^8_θ [which is analogous to extracting D^8_θ from Eq. (3.32) while evaluating graphs in Fermi-Feynman gauge] is sufficient to make $\Gamma_2 = \Gamma_3 = 0$ identically at one loop.

[4]The relation between these two hyperspaces was explored from different points of view in [19,20].

gauge condition and equations of motion for the gauge and ghost fields to the projective superspace, which we leave for future work.

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