Asymptotic commutativity of quantized spaces: The case of $\mathbb{CP}^{p,q}$

Fedele Lizzi,[1,2,3,*] Aleksandr Pinzul,[4,†] Allen Stern,[5,‡] and Chuang Xu[5,§]

[1] *Dipartimento di Fisica Ettore Pancini, Università di Napoli Federico II, Napoli 80126, Italy*
[2] *INFN, Sezione di Napoli 80126, Italy*
[3] *Departament de Física Quàntica i Astrofísica and Institut de Ciéncies del Cosmos (ICCUB), Universitat de Barcelona, Barcelona 08007, Spain*
[4] *Universidade de Brasília, Instituto de Física 70910-900, Brasília, DF, Brasil and International Center of Physics, C.P. 04667, Brasília, DF, Brazil*
[5] *Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA*

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We present a procedure for quantizing complex projective spaces $\mathbb{CP}^{p,q}$, $q \geq 1$, as well as construct relevant star products on these spaces. The quantization is made unique with the demand that it preserves the full isometry algebra of the metric. Although the isometry algebra, namely, $su(p+1,q)$, is preserved by the quantization, the Killing vectors generating these isometries pick up quantum corrections. The quantization procedure is an extension of one applied recently to Euclidean two-dimensional anti–de Sitter space (AdS_2), where it was found that all quantum corrections to the Killing vectors vanish in the asymptotic limit, in addition to the result that the star product trivializes to pointwise product in the limit. In other words, the space is asymptotically anti–de Sitter, making it a possible candidate for the AdS/CFT correspondence principle. In this article, we find indications that the results for quantized Euclidean AdS_2 can be extended to quantized $\mathbb{CP}^{p,q}$, i.e., noncommutativity is restricted to a limited neighborhood of some origin, and these quantum spaces approach $\mathbb{CP}^{p,q}$ in the asymptotic limit.

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I. INTRODUCTION

The AdS/CFT correspondence principle posits strong-weak duality between the quantum gravity in the bulk of an asymptotically anti–de Sitter (AdS) space and a conformal field theory (CFT) on the boundary of this space [1,2]. For obvious reasons, however, most practical applications of the correspondence principle utilize *classical* gravity in the bulk. Even though a fully consistent quantum theory of gravity remains out of reach, there are model independent indications that any theory of quantum gravity will require a quantization of spacetime [3–5]. The quantization of AdS, or more generally, asymptotically AdS, spacetimes has been examined in two dimensions [6–10] and four dimensions [11]. Its application to the correspondence principle has received only some initial work in two dimensions [12,13].

*fedele.lizzi@na.infn.it
†apinzul@unb.br
‡astern@ua.edu
§cxu24@crimson.ua.edu

While in this article we do not directly address the quantization of general AdS spaces of dimension larger than 2, we do present a procedure for quantizing another set of nontrivial noncompact geometries generalizing the two dimensional case, namely, indefinite complex projective spaces in arbitrary dimensions $\mathbb{CP}^{p,q}$, $q \geq 1$. We also introduce relevant star products for these spaces. $\mathbb{CP}^{p,q}$ is a noncompact version of $\mathbb{CP}^n$. The simplest example of an indefinite complex projective space is $\mathbb{CP}^{0,1}$, which is equivalent to two-dimensional anti–de Sitter space or, more precisely, Euclidean anti–de Sitter space (EAdS_2). Another example is $\mathbb{CP}^{1,2}$, which is an S^2 bundle over AdS_4 [11]. While the noncommutative generalization of the compact $\mathbb{CP}^n$ has received some attention [14], the same cannot be said about the noncompact case or other nontrivial noncompact spaces. Hasebe has done a study of quantized, or "fuzzy," hyperboloids [15], while Steinacker and Sperling [11] have applied such spaces, more specifically, the fuzzy four-hyperboloid, or noncommutative AdS_4, to quantum cosmology. The quantization in [11] is made unique with the demand that it preserves the full isometry algebra of the metric of the four-hyperboloid. An isometry preserving quantization and star product can also be constructed for a general $\mathbb{CP}^{p,q}$, as we demonstrate here. Although the isometry algebra, namely, $su(p+1,q)$, is preserved by the quantization, the isometry generators, i.e., the Killing vectors, can pick up quantum corrections.

As stated above, the simplest example of an indefinite complex projective space is $\mathbb{CP}^{0,1}$, or EAdS_2. Its isometry preserving quantization, which we denote by $nc\mathrm{EAdS}_2$, has been examined previously [6–10,12,13]. Among the results found in this case is the fact that the star product (when expressed in a suitable set of coordinates) approaches the pointwise product in the asymptotic limit (which corresponds to the boundary limit of anti–de Sitter space) [12]. It was also argued that the quantum corrections to the Killing vectors vanish in this limit. Thus, $nc\mathrm{EAdS}_2$ asymptotically approaches commutative anti–de Sitter space. In other words, the quantum features of $nc\mathrm{EAdS}_2$ occur, for all practical purposes, in a limited neighborhood of some origin. Since $nc\mathrm{EAdS}_2$ is an asymptotically anti–de Sitter space, it can then be of relevance with regard to the AdS/CFT correspondence principle, which posits that for every asymptotically anti–de Sitter space there is a strong-weak duality correspondence between a bulk theory and a conformal field theory living on the conformal boundary. According to the correspondence principle, the isometries of anti–de Sitter space are mapped to conformal symmetries of the CFT on the AdS boundary. It is then reasonable to speculate that it has a conformal dual, barring known difficulties of the correspondence principle for two-dimensional anti–de Sitter space (see, for example, [16,17]). This was pursued in [12,13] where correlation functions were computed on the boundary.

As we argue in this article, the quantization procedure for EAdS_2 can be extended to any $\mathbb{CP}^{p,q}$, $q \geq 1$. We can ask whether analogous conclusions can be reached regarding their asymptotic behavior. The question therefore is whether there is a quantized version of $\mathbb{CP}^{p,q}$ that asymptotically becomes commutative. In other words, (1) does the star product between two functions with support "near the boundary" reduce the commutative one, and (2) do the noncommutative corrections to the Killing vectors vanish in the boundary limit? Of course, "the boundary" refers here to the asymptotic $\mathbb{CP}^{p,q}$ region, rather than a sharp edge of the manifold. The results obtained here do indeed support the affirmative answer to these questions. For the examples we consider, we find that, in the asymptotic limit, the relevant star product trivializes to the commutative product and noncommutative corrections to the Killing vectors vanish.

In Sec. II we review the quantization of Euclidean AdS_2. We parametrize the manifold in terms of two different sets of coordinates (which differ from those used in [12,13]), specifically, local affine coordinates and canonical coordinates. The former have the advantage that they can be applied to any complex projective space. The canonical coordinates, on the other hand, are useful for the purpose of quantization and satisfy three requirements: The first is, of course, the requirement that they obey the canonical Poisson brackets. The second, which is surprisingly nontrivial to ensure, is that they cover the entire complex plane. Dropping this condition would necessitate a careful treatment of the boundary of the domain in the quantum theory [18,19]. The boundary is never a sharp one, the domain of definition is always an open set, but when the coordinates are such that the boundary is at the finite value of these coordinates, the quantization scheme we are using cannot be applied. The third requirement is that the geometric measure is identical, up to a factor, to the integration measure of standard coherent states in the resulting quantum theory. In this regard, the quantum theory, and corresponding coherent states, naturally follow from canonical quantization of the canonical Poisson brackets. We quantize the space with the introduction of a noncommutative star product of the Wick-Voros type, constructed from coherent states. We show that the product asymptotically goes to the pointwise product after reexpressing it in terms of local affine coordinates. A crucial point concerns the symmetries, implemented by the analogs of the Killing vectors, which as stated above, preserve the full isometry algebra, here $su(1,1)$. We perform a perturbative expansion (with respect to the quantization parameter) for the symmetry generators and compute the leading order corrections to the Killing vectors. In agreement with results in [12,13], these corrections are seen to vanish in the asymptotic limit. The Wick-Voros product lends itself naturally to a matrix approximation, and considering finite matrices is tantamount to the imposition of a cutoff geometry [20,21], which provides both an ultraviolet and an infrared cutoff. We do not do the finite matrix approximation here.

We review $\mathbb{CP}^{p,q}$ in Sec. III, along with its parametrization in terms of local affine coordinates and canonical coordinates. The quantization procedure outlined above for EAdS_2 naturally extends to $\mathbb{CP}^{p,q}$. We do not have a universal expression for the Darboux map from local affine coordinates that is valid for all p and q and, instead, present the map for specific examples. The examples are the two four-(real)-dimensional indefinite complex projective spaces, $\mathbb{CP}^{1,1}$ and $\mathbb{CP}^{0,2}$, in Secs. IV and V, respectively, along with their higher-dimensional analogs given in Sec. VI. Like with EAdS_2, the canonical coordinates obey the canonical Poisson brackets, cover all of $\mathbb{C}^{p,q}$, and the resulting geometric measure is proportional to the integration measure of standard coherent states in the quantum theory. We carry out the quantization explicitly for the examples in Secs. IV and V, and show, like with $nc\mathrm{EAdS}_2$, that, upon taking the asymptotic limit, the star product trivializes to the commutative product and quantum corrections to the Killing vectors vanish. These quantum spaces are thus asymptotically $\mathbb{CP}^{1,1}$ and $\mathbb{CP}^{0,2}$, respectively. Some concluding remarks are given in Sec. VI.

II. QUANTIZATION OF EUCLIDEAN AdS_2

A. Euclidean AdS_2

To define AdS_2 or its Euclidean counterpart $EAdS_2$, it is convenient to first introduce a three-dimensional Minkowski background $\mathbb{R}^{2,1}$, which we shall coordinatize with $x_\alpha, \alpha = 1, 2, 3$, using the metric diag$(+,+,-)$. The spaces AdS_2, or $EAdS_2$, results from constraining the $SO(2,1)$ invariant $x_1^2 + x_2^2 - x_3^2$ to be a constant, associated with the scale. The AdS_2 surface corresponds to a positive constant, while $EAdS_2$ corresponds to a negative constant. We shall restrict our attention in this section to the Euclidean case, as this has been of traditional interest for the AdS/CFT correspondence. Therefore, we take

$$x_1^2 + x_2^2 - x_3^2 = -1, \tag{2.1}$$

where for convenience we fixed the scale to be one.[1] The surface identified by this relation is a two sheeted hyperboloid. The reason why it is called Euclidean AdS_2 is that the induced metric has a Euclidean signature.[2] Later we shall restrict to a single component of the hyperboloid H^2. This space is maximally isotropic, and the three Killing vectors, which we denote by K_α, $\alpha = 1, 2, 3$, form a basis for an $so(2,1)$ algebra

$$[K_1, K_2] = -2K_3, \quad [K_2, K_3] = 2K_1, \quad [K_3, K_1] = 2K_2. \tag{2.2}$$

Because H^2 could be thought of as a coadjoint orbit, a natural Lie-Poisson structure exists on it. It is easily defined by setting the Poisson brackets of the embedding coordinates to satisfy the $so(2,1)$ algebra

$$\{x_1, x_2\} = -2x_3, \qquad \{x_2, x_3\} = 2x_1, \qquad \{x_3, x_1\} = 2x_2. \tag{2.3}$$

With such a choice, one can then use Lie-Poisson structure to implement the action of the Killing vectors on arbitrary functions f on H^2. Specifically, if one defines K_α acting on f by

$$[K_\alpha f](x) = \{x_\alpha, f\}, \tag{2.4}$$

then from the Jacobi identity, one recovers $so(2,1)$ algebra of the Killing vectors (2.2).

[1] Alternatively, we can introduce a nonunit length scale ℓ_0 for $EAdS_2$ by replacing the dimensionless coordinates x_α by $x_\alpha^{(\ell_0)} = \ell_0 x_i$.

[2] For example, in the so-called global coordinates, the induced metric takes the form

$$ds^2|_{\mathrm{EAdS}} = \cosh^2\rho dt^2 + d\rho^2.$$

B. Local coordinates

A number of coordinatizations have been introduced to $EAdS_2$. A popular choice has been Fefferman-Graham coordinates [22] because of its convenience in the AdS/CFT correspondence principle. Here, we shall instead work with two other sets of coordinates, local affine coordinates and canonical coordinates. The former has the advantage that it can be applied to any noncompact projective space, while the latter provides a useful step for quantization. Although the local affine coordinates for $EAdS_2$ are not defined on the entire complex plane, it is expedient, for the purpose of quantization, that the canonical coordinates span all of $\mathbb{C}$. We shall make this requirement below. Note that the canonical coordinates we use here differ from those used in [12,13], because the latter are not very useful for the higher-dimensional generalizations. Both sets of coordinates are, of course, related by a canonical transformation. We also require that the new coordinates (which put the boundary at finite values) do not add a spurious compactification of space. This is achieved requiring that continuous functions that vanish at infinity in the original coordinates still do so at the point (or points) to which infinity is mapped.

1. Local affine coordinates

We denote the local affine coordinate of H^2 by ζ, and its complex conjugate is ζ^*. The map from the (ζ, ζ^*) to the embedding coordinates (x_1, x_2, x_3) corresponds to the noncompact analog of a stereographic projection of S^2. It is

$$x_1 - ix_2 = \frac{2\zeta}{|\zeta|^2 - 1}, \qquad x_3 = \frac{|\zeta|^2 + 1}{|\zeta|^2 - 1}. \tag{2.5}$$

By imposing the condition $|\zeta| > 1$, we restrict to the "upper" hyperboloid, $x_3 \geq 1$. $|\zeta| \to \infty$ maps the point $(x_1, x_2, x_3) = (0, 0, 1)$ on the hyperboloid, while $|\zeta| \to 1$ corresponds to the asymptotic limit. Starting with the Lorentz metric on $\mathbb{R}^{2,1}$, and using (2.5), we obtain the following induced metric on H^2:

$$ds^2 = \frac{4|d\zeta|^2}{(|\zeta|^2 - 1)^2}. \tag{2.6}$$

This is the Fubini-Study metric, and as was indicated above, it has Euclidean signature. The metric tensor $g_{\zeta,\zeta^*} = \frac{2}{(|\zeta|^2-1)^2}$ can be expressed in terms of the Kähler potential $g_{\zeta,\zeta^*} = \frac{\partial^2}{\partial\zeta\partial\zeta^*} V$, $V = -2\ln(|\zeta|^2 - 1)$. The geometric measure resulting from this metric is

$$d\mu_{\mathrm{geom}}(\zeta, \zeta^*) = \frac{2}{(|\zeta|^2 - 1)^2} d\zeta \wedge d\zeta^*. \tag{2.7}$$

Using (2.5), the $so(2,1)$ Poisson brackets algebra of the embedding coordinates (2.3) results from the following fundamental Poisson bracket on H^2:

$$\{\zeta, \zeta^*\} = i(|\zeta|^2 - 1)^2. \tag{2.8}$$

Then from (2.4) we get explicit expressions for the Killing vectors in terms of the local affine coordinates

$$K_1 - iK_2 = 2i\left(\zeta^2 \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \zeta^*}\right),$$
$$K_3 = 2i\left(\zeta \frac{\partial}{\partial \zeta} - \zeta^* \frac{\partial}{\partial \zeta^*}\right). \tag{2.9}$$

2. Canonical coordinates

We next apply a Darboux transformation from the local affine coordinates to canonical coordinates (y, y^*), satisfying

$$\{y, y^*\} = -i. \tag{2.10}$$

As stated above, for the purpose of quantization, it is necessary to have y span all of the complex plane, unlike ζ, which is defined only outside the unit disk $|\zeta| > 1$. This fixes (y, y^*) up to canonical transformations. For the natural *Ansatz* $y = f(|\zeta|)\zeta$, one obtains the following condition on the function $f(x)$:

$$f^2 + \frac{x}{2}(f^2)' = -\frac{1}{(x^2-1)^2}, \tag{2.11}$$

which has the general solution

$$f(x)^2 = \frac{C}{x^2} + \frac{1}{x^2(x^2-1)}, \tag{2.12}$$

where C is an arbitrary non-negative constant. From here it follows that $|y|^2 = C + \frac{1}{|\zeta|^2-1}$, and it spans the entire positive real axis (including $|y| = 0$) only when $C = 0$. Then, for this *Ansatz*, we have

$$y = \frac{\zeta}{|\zeta|\sqrt{|\zeta|^2 - 1}}. \tag{2.13}$$

Another desirable feature, from the point of view of quantization, is that the geometric measure reduces to a flat measure when expressed in terms of the canonical coordinates. This easily follows from the Jacobian of the transformation, which is $|\frac{\partial(\zeta,\zeta^*)}{\partial(y,y^*)}| \equiv |\{\zeta, \zeta^*\}| = (|\zeta|^2 - 1)^2$. So (2.7) is transformed to

$$d\mu_{\mathrm{geom}}(y, y^*) = 2dy \wedge dy^*. \tag{2.14}$$

When reexpressed in terms of (y, y^*), the expression (2.5) for the embedding coordinates becomes

$$x_1 - ix_2 = 2y\sqrt{|y|^2 + 1}, \qquad x_3 = 2|y|^2 + 1. \tag{2.15}$$

Therefore, the origin of the complex plane spanned by the canonical coordinates is the image of the point $(x_1, x_2, x_3) = (0, 0, 1)$ on the hyperboloid, while $|y| \to \infty$ corresponds to the asymptotic limit. The Killing vectors (2.9) when expressed in terms of the canonical coordinates become

$$K_1 - iK_2 = \frac{i}{\sqrt{|y|^2+1}}\left(y^2 \frac{\partial}{\partial y} - (2 + 3|y|^2)\frac{\partial}{\partial y^*}\right),$$
$$K_3 = 2i\left(y\frac{\partial}{\partial y} - y^* \frac{\partial}{\partial y^*}\right). \tag{2.16}$$

C. Quantization

One can now perform canonical quantization by replacing the coordinates (y, y^*) by operators $(\hat{y}, \hat{y}^\dagger)$ satisfying commutation relations

$$[\hat{y}, \hat{y}^\dagger] = \bar{k}\mathbb{1}, \tag{2.17}$$

$\bar{k}$ being the noncommutative parameter, and $\mathbb{1}$ being the identity operator. Equivalently, we have raising and lowering operators, $\hat{a}^\dagger = \hat{y}^\dagger/\sqrt{\bar{k}}$ and $\hat{a} = \hat{y}/\sqrt{\bar{k}}$, satisfying $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$. Note that, apart from the commutation relation, it is equally fundamental that the canonical coordinates y, y^* were defined on the whole plane (unlike the case with ζ, ζ^*). Otherwise, one would require a delicate treatment of the domain with a boundary [18,19].

The operators $\hat{y}$ and $\hat{y}^\dagger$ act on the infinite-dimensional harmonic oscillator Hilbert space $\mathcal{H}$ spanned by orthonormal states $|n\rangle, n = 0, 1, 2\ldots$

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{2.18}$$

where $\hat{a}|0\rangle = 0$, and $\langle 0|0\rangle = 1$. Alternatively, one can introduce standard coherent states $\{|\alpha\rangle \in \mathcal{H}, \alpha \in \mathbb{C}\}$ written on $\mathbb{C}$,

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger}|0\rangle, \tag{2.19}$$

where α is the eigenvalue of $\hat{a}$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Coherent states form an overcomplete set with unit norm. The completeness relation and normalization condition are

$$\int d\mu(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| = \mathbb{1},$$
$$\langle\alpha|\alpha'\rangle = \exp\left\{\alpha^*\alpha' - \frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2}\right\}. \tag{2.20}$$

The integration measure for coherent states $d\mu(\alpha, \alpha^*)$ is

$$d\mu(\alpha,\alpha^*) = \frac{i}{2\pi} d\alpha \wedge d\alpha^* = \frac{i}{2\pi \bar{k}} dy \wedge dy^*, \qquad (2.21)$$

which is, up to a factor, identical to the geometric measure (2.14). Here we have reintroduced the canonical coordinates (y, y^*) using $y = \sqrt{\bar{k}}\alpha$ and $y^* = \sqrt{\bar{k}}\alpha^*$.

The Wick-Voros star product $\star$ is constructed from the standard coherent states. Here we briefly review it. For details of the construction, see, e.g., [23–25]. One first defines symbols $\mathcal{A}(\alpha,\alpha^*)$ on the complex plane associated with operator functions A of $\hat{a}$ and $\hat{a}^\dagger$ using

$$\mathcal{A}(\alpha,\alpha^*) = \langle\alpha|A|\alpha\rangle. \qquad (2.22)$$

Then given any two functions A and B of $\hat{a}$ and $\hat{a}^\dagger$, with symbols $\mathcal{A}$ and $\mathcal{B}$, respectively, the symbol of their product is

$$[\mathcal{A}\star\mathcal{B}](\alpha,\alpha^*) = \langle\alpha|AB|\alpha\rangle, \qquad (2.23)$$

which gives the Wick-Voros star product of the two symbols. It is given explicitly in terms of the canonical coordinates by

$$[\mathcal{A}\star\mathcal{B}](y,y^*) = \mathcal{A}(y,y^*)\exp\left\{\bar{k}\frac{\overleftarrow{\partial}}{\partial y}\frac{\overrightarrow{\partial}}{\partial y^*}\right\}\mathcal{B}(y,y^*). \qquad (2.24)$$

This expression realizes the fundamental commutation relation $[y, y^*]_\star = \bar{k}$, where $[\mathcal{A},\mathcal{B}]_\star = \mathcal{A}\star\mathcal{B} - \mathcal{B}\star\mathcal{A}$ denotes the star commutator, and gives the desired commutative limit

$$\mathcal{A}\star\mathcal{B} = \mathcal{A}\mathcal{B} + \mathcal{O}(\bar{k}),$$
$$[\mathcal{A},\mathcal{B}]_\star = i\bar{k}\{\mathcal{A},\mathcal{B}\} + \mathcal{O}(\bar{k}^2). \qquad (2.25)$$

The star product can be reexpressed in terms of the local affine coordinates using (2.13). One gets

$$[\mathcal{A}\star\mathcal{B}](\zeta,\zeta^*) = \mathcal{A}(\zeta,\zeta^*)\exp\{\bar{k}\overleftarrow{\mathcal{D}}\overrightarrow{\mathcal{D}}^*\}\mathcal{B}(\zeta,\zeta^*), \qquad (2.26)$$

where

$$\mathcal{D} = \frac{\sqrt{|\zeta|^2-1}}{2|\zeta|}\left(\frac{\partial}{\partial\zeta} - (2|\zeta|^2-1)\frac{\zeta^*}{\zeta}\frac{\partial}{\partial\zeta^*}\right),$$
$$\mathcal{D}^* = \frac{\sqrt{|\zeta|^2-1}}{2|\zeta|}\left(\frac{\partial}{\partial\zeta^*} - (2|\zeta|^2-1)\frac{\zeta}{\zeta^*}\frac{\partial}{\partial\zeta}\right). \qquad (2.27)$$

The presence of the $\sqrt{|\zeta|^2-1}$ factor is crucial. As we mentioned earlier, the conformal boundary is obtained in the limit $|\zeta| \to 1$, and therefore this shows that the value of the product of two functions asymptotically is not different from the one obtained with the usual commutative multiplication. It also means that the star commutator reduces to $i\bar{k}$ times the Poisson bracket in the asymptotic limit. The advantage of the ζ coordinates is to put infinity at a finite distance. In view of the above property, we must also require the functions and their derivatives do not diverge too much, so that the noncommutative corrections vanish at the conformal boundary. These properties could also have been imposed for the y coordinates. We will not go into the details of the class of functions allowed.

We will characterize noncommutative EAdS_2 in terms of the noncommutative analogs of the embedding coordinates (x_1, x_2, x_3) [6–10]. We need a set of noncommutative coordinates, which we call X_α, that satisfy the $\star$ analog of the conditions (2.1) and (2.3),

$$X_1\star X_1 + X_2\star X_2 - X_3\star X_3 = -\mathcal{C} \qquad (2.28)$$

and

$$[X_1, X_2]_\star = -2i\bar{k}X_3, \qquad [X_2, X_3]_\star = 2i\bar{k}X_1,$$
$$[X_3, X_1]_\star = 2i\bar{k}X_2, \qquad (2.29)$$

with $\mathcal{C} > 0$, a dimensionless constant that, along with $\bar{k}$, defines the Euclidean version of noncommutative AdS_2. As in the commutative case, we can introduce a length scale ℓ by replacing the dimensionless coordinates X_α by $X_\alpha^{(\ell)} = \ell X_\alpha$. We note that the length scale does not get quantized in the noncommutative theory. Rather, it is the quantization parameter $\bar{k}$ that gets quantized after restricting to the relevant unitary representations. For this, note that, if we do another rescaling of the coordinates and define $X'_\alpha = X_\alpha/(2\bar{k})$, then the commutator algebra for X'_α is $su(1,1)$ (with no scale factors). The Casimir $X'_\alpha X'^\alpha$ is $j(j+1)$, where j is an integer for the discrete series representations, which were shown to be the appropriate unitary irreducible representations for noncommutative EAdS_2 [12,13]. Using (2.28), this gives $\mathcal{C}/(4\bar{k})^2 = j(j+1)$. From this we see that the commutative limit $\bar{k}\to 0$ corresponds to j going to infinity. We note that, in general, $\mathcal{C}$ can pick up quantum correction, and so one gets a nontrivial result for the allowed values of $\bar{k}$. In order to recover (2.1) in the commutative limit, we need $\mathcal{C} = 1 + \mathcal{O}(\bar{k})$. The X's should be functions of the embedding coordinates (x_1, x_2, x_3) of the commutative theory and must reduce to them in the limit [or, in terms of the local coordinates, they should be functions of (ζ,ζ^*) or (y, y^*) and must reduce to (2.5) or (2.15), respectively]. Relation (2.29) for the X_α's then defines the $so(2,1)$ algebra, and $\mathcal{C}$ fixes the Casimir. We thereby obtain irreducible representations of $so(2,1)$.

Given the noncommutative analogs of the embedding coordinates, one can introduce noncommutative analogs of the Killing vectors of EAdS_2. Denote them by $K^\star_\alpha$. They are defined in an analogous way to K_α, by essentially replacing the Poisson bracket in (2.4) by the star commutator,

$$[K^{\star}_{\alpha} f](X) = \frac{1}{i\bar{k}}[X_{\alpha}, f]_{\star}, \quad (2.30)$$

where $f(X)$ denotes a function on $nc\mathrm{EAdS}_2$. Like the Killing vectors K_{α} of EAdS_2, $K^{\star}_{\alpha}$ satisfy the $so(2,1)$ algebra. Furthermore, from (2.25), we see that $K^{\star}_{\alpha}$ reduce to K_{α} in the commutative limit. On the other hand, the expressions (2.9) for K_{α} do not hold for the noncommutative analogs of the Killing vectors (except for $\alpha = 3$, and except for the asymptotic limit, as we shall see below). Thus, quantization leads to deformations of the Killing vectors, although the algebra they generate is not deformed.

We next write X_{α} in terms of the canonical coordinates y and y^*. For this we will need several simple properties of the star product (2.24).

(1) The symbol of the operator $\hat{y}^{\dagger}\hat{y}$ is $|y|^2$. In general, any function $\mathcal{F}(|y|^2)$ is a symbol of some operator $F(\hat{y}^{\dagger}\hat{y})$ and, vice versa, any operator $F(\hat{y}^{\dagger}\hat{y})$ has a symbol depending only on $|y|^2$,

$$\mathcal{F}(|y|^2) = \exp\left(-\frac{|y|^2}{\bar{k}}\right)\sum_{n=0}^{\infty}\frac{|y|^{2n}}{\bar{k}^n n!}F(\bar{k}n). \quad (2.31)$$

This can be readily seen by noting that $\hat{y}^{\dagger}\hat{y} = \bar{k}\hat{a}^{\dagger}\hat{a} \equiv \bar{k}\hat{n}$ and using in the definition of a symbol 2.22)) the coherent states (2.19) in the form $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$.

(2) For any function $\mathcal{F}(y, y^*)$, we have

$$\mathcal{F}(y,y^*) \star y = y\mathcal{F}(y,y^*),$$
$$y^* \star \mathcal{F}(y,y^*) = y^*\mathcal{F}(y,y^*). \quad (2.32)$$

(The ordering on the left-hand side of the equations is important.)

(3) For any two functions of $|y|^2$, $\mathcal{F}(|y|^2)$ and $\mathcal{G}(|y|^2)$, we have

$$\mathcal{F}(|y|^2) \star \mathcal{G}(|y|^2) = \sum_{n=0}^{\infty}\frac{\bar{k}^n|y|^{2n}}{n!}\mathcal{F}^{(n)}(|y|^2)\mathcal{G}^{(n)}(|y|^2), \quad (2.33)$$

where the derivative is taken with respect to $|y|^2$.

Motivated by (0.15), we look for the noncommutative coordinates X_{α} satisfying (0.29) in the form

$$X_- = X_1 - iX_2 = 2\mathcal{S} \star y \overset{(0.32)}{\equiv} 2y\mathcal{S},$$
$$X_+ = X_1 + iX_2 = 2y^* \star \mathcal{S} \overset{(0.32)}{\equiv} 2y^*\mathcal{S}, \quad (2.34)$$

where $\mathcal{S} = \mathcal{S}(|y|^2)$ is some real function to be determined below. Using the properties of the star product (2.31)–(2.33), one can easily find

$$[X_-, X_+]_{\star} = 4\bar{k}(\mathcal{S} \star \mathcal{S} + |y|^2(\mathcal{S} \star \mathcal{S})'), \quad (2.35)$$

where the prime denotes a derivative with respect to $|y|^2$. According to (2.29), this should be equal to $4\bar{k}X_3$. So we have that $X_3 = X_3(|y|^2)$, and in terms of S, is given by

$$X_3 = \mathcal{S} \star \mathcal{S} + |y|^2(\mathcal{S} \star \mathcal{S})'. \quad (2.36)$$

Using (2.29) one more time

$$4\bar{k}\mathcal{S} \star y \equiv 2\bar{k}X_- = [X_-, X_3]_{\star} = 2\mathcal{S} \star [y, X_3]_{\star} \quad (2.37)$$

and taking into account that there exists $\mathcal{S}^{-1}$ such that $\mathcal{S}^{-1} \star \mathcal{S} = 1$ (since it exists to zeroth order in $\bar{k}$, and we assume that the expansion in $\bar{k}$ is valid), we arrive at the equation for X_3

$$[y, X_3]_{\star} = 2\bar{k}y \quad \text{or} \quad X_3' = 2, \quad (2.38)$$

which leads to

$$X_3 = 2|y|^2 + c, c = \text{constant}. \quad (2.39)$$

Using this in (2.36) we arrive at the differential equation for $\mathcal{S} \star \mathcal{S}$

$$\mathcal{S} \star \mathcal{S} + |y|^2(\mathcal{S} \star \mathcal{S})' = 2|y|^2 + c, \quad (2.40)$$

which is easily solved to give

$$\mathcal{S} \star \mathcal{S} = |y|^2 + c + \frac{a}{|y|^2}, \quad (2.41)$$

where a is another integration constant. It is clear that one should set $a = 0$ in order to have nonsingular noncommutative corrections for $|y| \to 0$ (and to recover that $X_1, X_2 \to 0$ in this limit). So, we have

$$\mathcal{S} \star \mathcal{S} = |y|^2 + c. \quad (2.42)$$

The Casimir in (2.28) is now easily computable

$$\mathcal{C} = -\frac{1}{2}(X_- \star X_+ + X_+ \star X_-) + X_3 \star X_3 = c^2 - 2\bar{k}c. \quad (2.43)$$

In general, the constant c should have the form $c = 1 + \mathcal{O}(\bar{k})$. We fix this freedom in quantization by requiring that the symbol X_3 remains undeformed, i.e., by setting $c = 1$. Then (2.42) looks exactly as in the commutative case (2.15)

$$\mathcal{S} \star \mathcal{S} = |y|^2 + 1; \quad (2.44)$$

i.e., $\mathcal{S}$ is a symbol of the operator $\sqrt{1 + \hat{y}^{\dagger}\hat{y}}$, which can be formally written using (2.31) as

$$\mathcal{S}(|y|^2) = \exp\left(-\frac{|y|^2}{\hbar}\right)\sum_{n=0}^{\infty}\frac{|y|^{2n}}{\hbar^n n!}\sqrt{\hbar n+1}. \tag{2.45}$$

Though we do not have the closed answer for the series (2.45), we can systematically calculate $\mathcal{S}$ to any order in $\hbar$. Let $\mathcal{S}_n$ be the functions independent of $\hbar$ and defined by

$$\mathcal{S}(|y|^2) = \sum_{n=0}^{\infty}\hbar^n\mathcal{S}_n(|y|^2). \tag{2.46}$$

Plugging this into (2.44) and using (2.33) we have after some trivial index relabeling

$$1+|y|^2 = \mathcal{S}\star\mathcal{S} = \sum_{n=0}^{\infty}\hbar^n\left(\sum_{m=0}^{n}\frac{|y|^{2m}}{m!}\sum_{r=0}^{n-m}\mathcal{S}^{(m)}_{n-m-r}\mathcal{S}^{(m)}_r\right). \tag{2.47}$$

From (2.47) we obtain the recursion relations defining $\mathcal{S}_n$ for any n,

$$n=0,\qquad \mathcal{S}_0=\sqrt{1+|y|^2}$$
$$n\geq 1,\qquad \sum_{m=0}^{n}\frac{|y|^{2m}}{m!}\sum_{r=0}^{n-m}\mathcal{S}^{(m)}_{n-m-r}\mathcal{S}^{(m)}_r = 0. \tag{2.48}$$

For example, for $n=1$ we have

$$\mathcal{S}_1\mathcal{S}_0+\mathcal{S}_0\mathcal{S}_1+|y|^2\mathcal{S}_0'\mathcal{S}_0' = 0 \Rightarrow \mathcal{S}_1 = -\frac{|y|^2}{8(1+|y|^2)^{3/2}}. \tag{2.49}$$

In general, it is not hard to see from (2.48) that, for an arbitrary n, $\mathcal{S}_n$ will have the following form:

$$\mathcal{S}_n = \sqrt{1+|y|^2}\,\frac{P_n(|y|^2)}{(1+|y|^2)^{2n}} =: \sqrt{1+|y|^2}\,\mathcal{L}_n, \tag{2.50}$$

where $P_n(x)$ is some polynomial of degree n, with $P_0=1$. Then we can write our noncommutative coordinates X_α in terms of the commutative ones as

$$X_\pm = x_\pm\sum_{n=0}^{\infty}\hbar^n\mathcal{L}_n,\qquad X_3=x_3, \tag{2.51}$$

$x_\pm = x_1\pm x_2$ being the commutative counterparts to $X_\pm$. We conclude that $X_\pm\to x_\pm$ in the asymptotic limit $|y|^2\to\infty$,

$$X_\pm = x_\pm\left(1+\mathcal{O}\left(\frac{\hbar}{|y|^2}\right)\right),\qquad X_3=x_3. \tag{2.52}$$

Using (2.51) and its asymptotics (2.52) we can easily study the behavior of the noncommutative Killing vectors, defined by (2.30), near the conformal boundary. Let us denote by $\mathcal{L}$ the sum in (2.51), $\mathcal{L}=\sum_{n=0}^{\infty}\hbar^n\mathcal{L}_n$. Since $X_3=x_3$, $K_3^\star$ has exactly the same form as its commutative counterpart K_3 in (2.16). Trivial analysis shows that when $|y|\to\infty$, $K_\pm^\star$ behave as

$$\begin{aligned}K^\star_\pm f(y,y^*) &\equiv \frac{1}{i\hbar}[X_\pm,f]_\star\\ &= \mathcal{L}K_\pm f+\frac{1}{2}x_\pm\mathcal{L}'K_3 f\\ &\quad+\sum_{n=2}^{\infty}\frac{(i\hbar)^{n-1}}{n!}[\partial^n_y(x_\pm\mathcal{L})\partial^n_{y^*}f-\partial^n_{y^*}(x_\pm\mathcal{L})\partial^n_y f]\\ &= \left(1+\mathcal{O}\left(\frac{\hbar}{|y|^2}\right)\right)K_\pm f,\end{aligned} \tag{2.53}$$

where we naturally assumed that $K_3 f$ has the same asymptotic behavior as $K_\pm f$. This shows that the noncommutative corrections to $K^\star_\alpha$ vanish in the asymptotic limit. Of course, the same is true for the case of the local affine coordinates (ζ,ζ^*). In this case, the commutative limit for both, the coordinates X_α and Killings $K^\star_\alpha$, will be recovered as $|\zeta|^2\to 1$.

Thus, upon expressing the system in terms of the canonical or local affine coordinates, we see that the noncommutative coordinates X_α, as well as the $so(2,1)$ isometry generators of $nc\mathrm{EAdS}_2$, approach the standard EAdS_2 expressions, while the star product approaches the ordinary product, which is seen in local affine coordinates. We can then argue that $nc\mathrm{EAdS}_2$ reduces to EAdS_2 in the asymptotic limit.

III. $\mathbb{CP}^{p,q}$

The natural generalization of $nc\mathrm{EAdS}_2$ is the quantization of the indefinite complex projective space, denoted by $\mathbb{CP}^{p,q}$, where p and q are positive integers; p can be zero, while $q\geq 1$. EAdS_2 corresponds to $p=0$, $q=1$. In this section, we review $\mathbb{CP}^{p,q}$, writing down the Killing vectors and analogs of embedding coordinates in terms of appropriate Fubini-Study coordinates (ζ^i,ζ_i^*), $i=1,\ldots,p+q$, for these spaces. In order to reproduce the quantization program of the previous section, we will need to find the Darboux transform from the Fubini-Study coordinates to canonical coordinates (y_i,y_i^*) spanning *all* of $\mathbb{C}^{p+q}$. As was mentioned for the case of EAdS_2, if the canonical coordinates do not span the entire $\mathbb{C}^{p+q}$, quantization becomes unmanageable due to the presence of boundaries. We have not found a general expression for the Darboux transformation that applies to all $\mathbb{CP}^{p,q}$ spaces. Rather, we can give the transformation for various classes of such spaces, which we shall illustrate in Secs. IV and V.

A. Definition

The space $\mathbb{CP}^{p,q}$, $q\geq 1$, is defined as the $\mathbb{H}^{2q,2p+1}$ hyperboloid mod S^1. It can be constructed starting from a $p+q+1$-dimensional complex space $\mathbb{C}^{p+1,q}$, with indefinite metric

$$\eta_{\mathbb{C}} = \mathrm{diag}(\underbrace{+\cdots+}_{p+1},\underbrace{-\cdots-}_{q}). \tag{3.1}$$

Say $\mathbb{C}^{p+1,q}$ is coordinatized by z^a, $a = 1, \ldots, p+q+1$, along with their complex conjugates z^{a*}, where the indices $a, b, \ldots$ are raised and lowered using the metric $\eta_{\mathbb{C}}$. To embed $\mathbb{H}^{2q,2p+1}$ in $\mathbb{C}^{p+1,q}$, one imposes the constraint

$$z_a^* z^a = 1. \tag{3.2}$$

To obtain $\mathbb{CP}^{p,q}$, one further makes the identification

$$z^a \sim e^{i\chi} z^a, \tag{3.3}$$

$e^{i\chi}$ being an arbitrary phase. The compact complex projective space $\mathbb{CP}^p$ corresponds to $q = 0$. We will not be concerned with it in the following. The space $\mathbb{CP}^{p,q}$ can be equivalently defined as the coset space $SU(p+1,q)/U(p,q)$.

The standard metric and Poisson bracket on complex projective spaces are the Fubini-Study metric and the canonical one, respectively. The former is given by

$$ds^2 = dz_a^* dz^a - |z_a^* dz^a|^2, \tag{3.4}$$

while the latter is

$$\{z^a, z_b^*\} = -i\delta_b^a, \qquad \{z^a, z^b\} = \{z_a^*, z_b^*\} = 0,$$
$$a, b = 1, \ldots, p+q+1. \tag{3.5}$$

Using (3.5), it follows that (3.2) is the first class constraint (in the sense of Dirac's Hamiltonian formalism) that generates the phase equivalence (3.3).

B. Coordinates

Here we are interested in generalizing the two sets of coordinates given previously for EAdS_2, i.e., local affine coordinates and canonical coordinates. While here we give explicit expressions for the former, we just discuss qualitative features of the latter. We shall postpone giving explicit expressions for the Darboux transformation to sections that follow.

1. Local affine coordinates

The local affine coordinates $(\zeta^i, \zeta_i^*), i = 1, \ldots, p+q$, are defined in terms of the coordinates z^a by[3]

$$\zeta^i = \frac{z^i}{z^{p+q+1}}, \qquad z^{p+q+1} \neq 0. \tag{3.6}$$

They are invariant under the phase equivalence transformation (3.3). The ζ_i^* are obtained by taking the complex conjugate of (3.6) and lowering the index using the background metric tensor on the $p+q$-dimensional subspace (3.1). We note that it is the Euclidean metric for the special case of $q = 1$. From the constraint (3.2), one has

$$\zeta^i \zeta_i^* = 1 + \frac{1}{|z^{p+q+1}|^2}, \tag{3.7}$$

and it follows that $\zeta^i\zeta_i^* > 1$, which further implies that $|\zeta_1|^2 + \cdots + |\zeta_{p+1}|^2 > 1$. Therefore, the coordinate patch spanned by (ζ^i, ζ_i^*) is $\mathbb{C}^{p+1,q-1}$ with the region $\zeta^i\zeta_i^* \leq 1$ removed. For reasons stated below we call the boundary of this region the general "asymptotic limit,"

$$\zeta^i\zeta_i^* \to 1 \quad \text{or} \quad z^{p+q+1} \to 0. \tag{3.8}$$

This is in agreement with the asymptotic limit defined previously for EAdS_2.

While (3.1) is the background metric, the metric on the surface $\mathbb{CP}^{p,q}$ is the Fubini-Study metric (3.4). Substituting $z^i = z^{p+q+1}\zeta^i$ into (3.4) gives the Fubini-Study metric tensor in terms of local affine coordinates

$$ds^2 = g_{i\bar{j}}(\zeta, \zeta^*) d\zeta^i d\zeta_j^* = \frac{d\zeta_i^* d\zeta^i}{\mathcal{Z}^2} - \frac{|\zeta_i^* d\zeta^i|^2}{\mathcal{Z}^4},$$
$$i, j, k, \ldots = 1, \ldots, p+q, \tag{3.9}$$

where we denote

$$\mathcal{Z}^2 = \zeta^i\zeta_i^* - 1. \tag{3.10}$$

For $p = 0$, $q = 1$, $g_{i\bar{j}}(\zeta, \zeta^*)$ reduces to the metric tensor (2.6) on EAdS_2 (up to an overall factor). It can be expressed in terms of the Kähler potential

$$g_{i\bar{j}} = \frac{\partial^2}{\partial\zeta^i \partial\zeta_j^*} 2\ln\mathcal{Z}. \tag{3.11}$$

The geometric measure associated with the metric (3.9) is

$$d\mu_{\mathrm{geom}}(\zeta, \zeta^*) = \frac{1}{2^{p+q}\mathcal{Z}^{2(p+q+1)}} d\zeta^1 \wedge \cdots \wedge d\zeta^{p+q} \wedge d\zeta_1^*$$
$$\wedge \cdots \wedge d\zeta_{p+q}^*, \tag{3.12}$$

which is the generalization of (2.7). To verify (3.12) we only need the identity

$$\det(\mathbb{1}_n + vw^T) = 1 + w^T v, \tag{3.13}$$

[3]As usual, one can replace z^{p+q+1} in the denominator by another complex coordinate, say $z^{\mathbf{a}}$, which would be valid for $z^a \neq 0$, thereby defining local affine coordinates on a different coordinate patch.

where $v, w \in \mathrm{Vec}_n$ and $\mathbb{1}_n$ is the n-dimensional identity matrix, which easily follows from the definition of the determinant, $\det M = \frac{1}{n!}\epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n}M_{i_1 j_1}\cdots M_{i_n j_n}$ for any $M \in \mathrm{Mat}_n$. We can write the invariant interval in (3.9) as

$$ds^2 = d\Xi^T G d\Xi,$$
$$G = \frac{\gamma^2}{2}\begin{pmatrix} 0 & \mathbb{1}_{p+q} - \gamma^2\zeta^*\zeta^T \\ \mathbb{1}_{p+q} - \gamma^2\zeta\zeta^{*T} & 0 \end{pmatrix},$$
$$\Xi = \begin{pmatrix} \zeta \\ \zeta^* \end{pmatrix}, \tag{3.14}$$

where $\zeta = \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^{p+q} \end{pmatrix}$, $\zeta^* = \begin{pmatrix} \zeta_1^* \\ \vdots \\ \zeta_{p+q}^* \end{pmatrix}$, and $\gamma = \frac{1}{\mathcal{Z}}$. The geometric measure is then

$$d\mu_{\mathrm{geom}}(\zeta, \zeta^*) = \sqrt{|\det G|}d\zeta^1 \wedge \cdots \wedge d\zeta^{p+q} \wedge d\zeta_1^* \wedge \cdots \wedge d\zeta_{p+q}^*. \tag{3.15}$$

In order to recover (3.12), we then use (3.13) to get

$$\det G = -\frac{\gamma^{4(p+q)}}{2^{2(p+q)}}(\det(\mathbb{1}_{p+q} - \gamma^2\zeta^*\zeta^T))^2 = -\frac{\gamma^{4(p+q+1)}}{2^{2(p+q)}}. \tag{3.16}$$

From (3.5), the Poisson brackets on the coordinate patch spanned by (ζ^i, ζ_i^*) are

$$\{\zeta^i, \zeta_j^*\} = i\mathcal{Z}^2(\zeta^i\zeta_j^* - \delta_j^i), \{\zeta^i, \zeta^j\} = \{\zeta_i^*, \zeta_j^*\} = 0, \tag{3.17}$$

generalizing the Poisson bracket (2.8) for the case of EAdS_2.

The isometry group of $\mathbb{CP}^{p,q}$ is $SU(p+1, q)$. There are then a total of $(p+q)(p+q+2)$ Killing vectors associated with the metric tensor (3.9). In terms of the local affine coordinates, they are given by

$$\kappa_i{}^j = \zeta^j\frac{\partial}{\partial\zeta^i} - \zeta_i^*\frac{\partial}{\partial\zeta_j^*},$$
$$\kappa_i{}^{p+q+1} = \frac{\partial}{\partial\zeta^i} - \zeta_i^*\zeta_j^*\frac{\partial}{\partial\zeta_j^*},$$
$$\kappa_{p+q+1}{}^i = \frac{\partial}{\partial\zeta_i^*} - \zeta^i\zeta^j\frac{\partial}{\partial\zeta^j}, \tag{3.18}$$

generalizing (2.9). κ_{ij}, κ_{ip+q+1} and κ_{p+q+1i} form a basis for $su(p+1, q)$

$$[\kappa_i{}^j, \kappa_k{}^\ell] = \delta_i^\ell\kappa_k{}^j - \delta_k^j\kappa_i{}^\ell,$$
$$[\kappa_i{}^{p+q+1}, \kappa_j{}^k] = \delta_i^k\kappa_j{}^{p+q+1},$$
$$[\kappa_i{}^j, \kappa_{p+q+1}{}^k] = \delta_i^k\kappa_{p+q+1}{}^j,$$
$$[\kappa_i{}^{p+q+1}, \kappa_{p+q+1}{}^j] = -\kappa_i{}^j - \delta_i^j\kappa_k{}^k. \tag{3.19}$$

To recover the Killing vectors K_1, K_2, K_3 defined previously for EAdS_2, we need $K_1 - iK_2 = -2i\kappa_1{}^2$ and $K_3 = 2i\kappa_1{}^1$.

By generalizing the notion of the real embedding coordinates x_i for EAdS_2 (2.5), we can implement the action of the Killing vectors (3.18) using the Poisson bracket (3.17). Call $x_a{}^b, a, b = 1, \ldots, p+q+1$, real embedding coordinates for $\mathbb{CP}^{p,q}$ (in contrast to the complex embedding coordinates z^a). Their Poisson bracket algebra should correspond to $su(p+1, q)$. For this we define $x_a{}^b$ in terms of z^a's and then on the coordinate patch spanned by the local affine coordinates (ζ^i, ζ_i^*). In terms of the complex embedding coordinates we have

$$x_a{}^b = z_a^* z^b, \quad a, b = 1, \ldots, p+q+1. \tag{3.20}$$

Using (3.5) one can easily see that the Poisson brackets of x_{ab} close to give the $su(p+1, q)$ isometry algebra

$$\{x_{ab}, x_{cd}\} = i(\eta_{\mathbb{C}ad}x_{cb} - \eta_{\mathbb{C}cb}x_{ad}). \tag{3.21}$$

Then, as usual, we can write the action of $SU(p+1, q)$ Killing vectors in terms of these Poisson brackets

$$\kappa_a{}^b f = -i\{x_a{}^b, f\}. \tag{3.22}$$

The appearance of an extra Killing vector due to x_{p+q+1}^{p+q+1} is apparent, which could be seen by noticing that not all $x_a{}^b$'s are independent due to the constraint (3.2), which leads to

$$\mathrm{tr}x = x_a{}^a = 1,$$

as well as the higher order conditions

$$\mathrm{tr}x^2 = x_a{}^b x_b{}^a = 1,$$
$$\mathrm{tr}x^3 = x_a{}^b x_b{}^c x_c{}^a = 1,$$
$$\ldots$$
$$\mathrm{tr}x^n = x_{a_1}{}^{a_2}x_{a_2}{}^{a_3}\cdots x_{a_n}{}^{a_1} = 1. \tag{3.23}$$

Since $[x_a{}^b]$ is a finite-dimensional matrix, there is a finite number of independent such conditions on $x_a{}^b$. More specifically, there is a maximum number of $n = (p+q)^2$ independent conditions on the $(p+q+1)\times(p+q+1)$ on $[x_a{}^b]$ (excluding $\mathrm{tr}x = 1$). So, in particular, from

$\mathrm{tr}x = x_a{}^a = 1$ it follows that $\kappa_a{}^b$ is traceless, i.e., κ^{p+q+1}_{p+q+1} is not independent: $\kappa^{p+q+1}_{p+q+1} = -\kappa_i{}^i$.

Now we can trivially repeat this construction on the coordinate patch spanned by the local affine coordinates (ζ^i, ζ^*_i). Using (3.6) we have

$$x_i{}^j = \frac{\zeta^*_i \zeta^j}{\mathcal{Z}^2}, \qquad x^i_{p+q+1} = -\frac{\zeta^i}{\mathcal{Z}^2},$$
$$x_i{}^{p+q+1} = \frac{\zeta^*_i}{\mathcal{Z}^2}, \qquad x^{p+q+1}_{p+q+1} = -\frac{1}{\mathcal{Z}^2}. \tag{3.24}$$

It is because the embedding coordinates are, in general, divergent in the limit (3.8), that we call this the asymptotic limit. (Components of $x_a{}^b$ may vanish in the limit in the special cases where $\zeta_i = 0$.) The action of the Killing vectors $\kappa_i{}^j$ on functions f on the coordinate patch is written exactly as in (3.22)

$$\kappa_a{}^b f = -i\{x_a{}^b, f\}. \tag{3.25}$$

Upon using (3.17) we can explicitly verify that $\kappa_a{}^b$ has the form (3.18) [though, of course, this should be obvious from the derivation of (3.17) from (3.5)].

For the case of EAdS_2, the three real embedding coordinates x_1, x_2, x_3 of the section II A are recovered from $x_a{}^b$ by setting

$$x_1 = x_1{}^2 - x_2{}^1, \quad x_2 = -i(x_1{}^2 + x_2{}^1), \quad x_3 = x_1{}^1 - x_2{}^2. \tag{3.26}$$

There is only one independent constraint in this case, namely,

$$x_1^2 + x_2^2 - x_3^2 = -2x_a{}^b x_b{}^a + (x_a{}^a)^2 = -1. \tag{3.27}$$

2. *Canonical coordinates*

Following the previous section, the next step is to perform the Darboux transformation. As was mentioned above we have not found a single expression for the Darboux transformation that applies for all $\mathbb{CP}^{p,q}$ spaces. The difficulty is due to our restriction that the resulting canonical coordinates (y_i, y^*_i) are valid for the whole of $\mathbb{C}^{p+q}$, so that there are no boundaries on our domain in the corresponding quantized theory. As stated above, we shall give the Darboux transformation for various examples in the sections that follow. As in the previous case of EAdS_2, we find that the Jacobian of the Darboux transformation goes like

$$\left|\frac{\partial(\zeta, \zeta^*)}{\partial(y, y^*)}\right| = \mathcal{Z}^{2(p+q+1)}, \tag{3.28}$$

and hence in terms of the canonical coordinates, the geometric measure is proportional to the flat measure

$$d\mu_{\mathrm{geom}}(\zeta, \zeta^*) = \frac{1}{2^{p+q}} dy^1 \wedge \cdots \wedge dy^{p+q} \wedge dy^*_1 \wedge \cdots \wedge dy^*_{p+q}. \tag{3.29}$$

In order to proceed further, we need to assume a Darboux transformation for $\mathbb{CP}^{p,q}$ that takes the local affine coordinates (ζ^i, ζ^*_i) to coordinates (y_i, y^*_i) spanning all of $\mathbb{C}^{p+q}$, which satisfies the canonical Poisson bracket relations

$$\{y_i, y^*_j\} = -i\delta_{ij}, \qquad \{y_i, y_j\} = \{y^*_i, y^*_j\} = 0 \tag{3.30}$$

for all $i, j = 1, \ldots, p+q$. We do not have a general proof of this existence, nor that (3.28), and hence (3.29), in general, hold, but we are able to find such transformations for the examples in Secs. IV and V.

C. Quantization

Generalizing the procedure that was adapted for EAdS_2, we perform canonical quantization, replacing the coordinates (y_i, y^*_i) by the set of operators $(\hat{y}_i, \hat{y}^\dagger_i)$, satisfying commutation relations

$$[\hat{y}_i, \hat{y}^\dagger_j] = \hbar\delta_{ij}, \qquad [\hat{y}_i, \hat{y}_j] = [\hat{y}^\dagger_i, \hat{y}^\dagger_j] = 0, \tag{3.31}$$

$\hbar$ once again being the noncommutative parameter. This is the algebra for $p+q$ harmonic oscillators. The lowering and raising operators $\hat{a}_i$ and $\hat{a}^\dagger_i$ are obtained by rescaling $\hat{y}_i$ and $\hat{y}^\dagger_i$, respectively,

$$\hat{a}_i = \frac{1}{\sqrt{\hbar}}\hat{y}_i, \qquad \hat{a}^\dagger_i = \frac{1}{\sqrt{\hbar}}\hat{y}^\dagger_i. \tag{3.32}$$

Then $[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij}$ and $[\hat{a}_i, \hat{a}_j] = [\hat{a}^\dagger_i, \hat{a}^\dagger_j] = 0$ for all $i, j = 1, \ldots, p+q$. $\hat{a}_i$ and $\hat{a}^\dagger_i$ act on the infinite-dimensional Hilbert space $\mathcal{H}$, now spanned by orthonormal states

$$|n\rangle = |n_1, \ldots, n_{p+q}\rangle = \frac{(\hat{a}^\dagger_1)^{n_1} \cdots (\hat{a}^\dagger_{p+q})^{n_{p+q}}}{\sqrt{n_1! \cdots n_{p+q}!}}|0\rangle, \tag{3.33}$$

where n_i are non-negative integers. The bottom state $|0\rangle = |0, \ldots, 0\rangle$ is annihilated by any $\hat{a}_i$, and has unit norm $\langle 0|0\rangle = 1$.

It is straightforward to generalize the coherent states (2.19) and Wick-Voros star product (2.24) to $\mathbb{C}^{p+q}$. The former are given by

$$|\vec{\alpha}\rangle = |\alpha_1, \ldots, \alpha_{p+q}\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha_i \hat{a}^\dagger_i}|\vec{0}\rangle \in \mathcal{H}, \tag{3.34}$$

where α_i are complex eigenvalues of $\hat{a}_i$, $\hat{a}_i|\vec{\alpha}\rangle = \alpha_i|\vec{\alpha}\rangle$, and $|\alpha|^2 = \alpha^*_i\alpha_i$. The completeness relation and normalization condition are now

$$\int d\mu(\vec{\alpha},\vec{\alpha}^*)|\vec{\alpha}\rangle\langle\vec{\alpha}| = \mathbb{1},$$

$$\langle\vec{\alpha}|\vec{\alpha}'\rangle = \exp\left\{\alpha_i^*\alpha_i' - \frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2}\right\}, \qquad (3.35)$$

where the integration measure for coherent states $d\mu(\vec{\alpha},\vec{\alpha}^*)$ is

$$d\mu(\vec{\alpha},\vec{\alpha}^*) = \left(\frac{i}{2\pi}\right)^{p+q} d\alpha_1 \wedge d\alpha_1^* \wedge \cdots \wedge d\alpha_{p+q} \wedge d\alpha_{p+q}^*. \qquad (3.36)$$

Upon doing the rescaling back to canonical coordinates, $y_i = \sqrt{\hbar}\alpha_i$, we see that it agrees, up to a constant factor, with the geometric measure (3.29). Symbols of operators are defined as in (2.22), while the Wick-Voros product of symbols is

$$[\mathcal{A} \star \mathcal{B}](\vec{y},\vec{y}^*) = \mathcal{A}(\vec{y},\vec{y}^*) \exp\left\{\hbar \sum_{i=1}^{p+q} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\vec{\partial}}{\partial y_i^*}\right\} \mathcal{B}(\vec{y},\vec{y}^*). \qquad (3.37)$$

Then the star commutator gives a realization of the fundamental commutation relations (3.31), and the requirements (2.25) for the commutative limit are satisfied. The star product can be reexpressed in terms of local affine coordinates. For the examples that follow, as well as the one in Sec. II, we find that the star product reduces to the ordinary product in the asymptotic limit (3.8).

To define the noncommutative version of $\mathbb{CP}^{p,q}$, we should construct the noncommutative analogs of the matrix elements $x_a{}^b$. Denoting them by $X_a{}^b$, we demand that they satisfy $su(p+1,q)$ commutation relations

$$[X_a{}^b, X_c{}^d]_\star = \hbar(\delta_a^d X_c{}^b - \delta_c^b X_a{}^d), \qquad (3.38)$$

as well as the analogs of the conditions (3.23). The analogs of these conditions fix the Casimirs of the algebra, restricting the allowable representations of $su(p+1,q)$ of the noncommutative theory. We, of course, demand that $X_a{}^b \to x_a{}^b$ when $\hbar \to 0$. In Secs. IV and V, we shall provide perturbative expansions in $\hbar$ for $X_a{}^b$ as functions of local coordinates for the examples $\mathbb{CP}^{1,1}$ and $\mathbb{CP}^{0,2}$, respectively.

Given $X_a{}^b$, it is then easy to define noncommutative analogs $\kappa_a^{\star b}$ of the Killing vectors. Generalizing (2.30), the action of $\kappa_a^{\star b}$ on functions f on noncommutative $\mathbb{CP}^{p,q}$, we have

$$[\kappa_a^{\star b} f](X) = -\frac{1}{\hbar}[X_a{}^b, f]_\star. \qquad (3.39)$$

Then $\kappa_a^{\star b}$ are deformations of the Killing vectors $\kappa_a{}^b$, with the deformation vanishing in the commutative limit $\hbar \to 0$.

In order to extract the leading order corrections to $\kappa_a{}^b$, we need to obtain $[X_a{}^b, f]_\star$ up to second order in $\hbar$. Even though $\kappa_a^{\star b}$ are deformations of the Killing vectors, they satisfy the same algebra as $\kappa_a{}^b$, namely, the $su(p+1,q)$ isometry algebra

$$[\kappa_a^{\star b}, \kappa_c^{\star d}] = \delta_a^d \kappa_c^{\star b} - \delta_c^b \kappa_a^{\star d}. \qquad (3.40)$$

For the two examples that follow, as well as the one in Sec. II, we get that the deformation of the Killing vectors vanishes in the asymptotic limit (3.8).

IV. $\mathbb{CP}^{1,1}$

In this section and the next one, we write down the explicit Darboux transformation from local affine coordinates and perform the quantization procedure as outlined previously.

Here the example is $\mathbb{CP}^{1,1} \simeq H^{2,3}/S^1 \simeq SU(2,1)/U(1,1)$. It can be constructed from $\mathbb{C}^{2,1}$, spanned by z^a, $a = 1, 2, 3$. $\mathbb{CP}^{1,1}$ is then defined by the constraint (3.2), which becomes $|z^1|^2 + |z^2|^2 - |z^3|^2 = 1$, along with the equivalence relation (3.3).

There are two complex affine coordinates ζ_i, $i = 1, 2$, along with their complex conjugates. In this case, the background metric on the reduced space is Euclidean, diag$(+,+)$. The condition (3.7) leads to the restriction that the local affine coordinates are defined on a real four-dimensional space with a solid three sphere removed,

$$\mathcal{Z}^2 = |\zeta_1|^2 + |\zeta_2|^2 - 1 > 0. \qquad (4.1)$$

The quantity $\mathcal{Z}^2$ spans the positive real line, excluding the origin that corresponds to the asymptotic limit, (3.8) or $\mathcal{Z}^2 \to 0$. While the background metric for the coordinates is Euclidean, the Fubini-Study metric (3.9) has a Lorentzian signature. The latter solves the sourceless Einstein equations with $\Lambda = 3$ [26].

There are eight real embedding coordinates (3.24), $x_a{}^b$, with tr$x = 1$. Since $\mathbb{CP}^{1,1}$ has four real dimensions, $x_a{}^b$ are subject to four additional independent conditions (3.23).

A. Darboux map

Here we give the transformation from local affine coordinates to canonical coordinates (y_i, y_i^*), $i = 1, 2$, satisfying (3.30). As stated previously, we require the domain of the latter to be all of $\mathbb{C}^2$, unlike the domain of local affine coordinates. Up to canonical transformations, the Darboux transformation is given by

$$y_i = \begin{cases} \sqrt{\frac{|\zeta_i|^2}{\mathcal{Z}^2} - \frac{1}{2}}\,\frac{\zeta_i}{|\zeta_i|}, & \frac{|\zeta_i|^2}{\mathcal{Z}^2} > \frac{1}{2} \\ \sqrt{\frac{|\zeta_i|^2}{\mathcal{Z}^2} - \frac{1}{2}}\,\frac{\zeta_i^*}{|\zeta_i|}, & \frac{|\zeta_i|^2}{\mathcal{Z}^2} < \frac{1}{2} \end{cases}. \qquad (4.2)$$

Note that the square root is not necessarily real. To see that the coordinates cover the full complex plane once, let us express them as

$$y_1 = \sqrt{\frac{1}{2\mathcal{Z}^2}||\zeta_1|^2 - |\zeta_2|^2 + 1|} \times \begin{cases} \exp\{i\arg\zeta_1\}, & |\zeta_1|^2 - |\zeta_2|^2 + 1 > 0 \\ \exp\{-i\arg\zeta_1\}, & |\zeta_1|^2 - |\zeta_2|^2 + 1 < 0 \end{cases},$$
$$y_2 = \sqrt{\frac{1}{2\mathcal{Z}^2}||\zeta_2|^2 - |\zeta_1|^2 + 1|} \times \begin{cases} \exp\{i\arg\zeta_2\}, & |\zeta_2|^2 - |\zeta_1|^2 + 1 > 0 \\ \exp\{-i\arg\zeta_2\}, & |\zeta_2|^2 - |\zeta_1|^2 + 1 < 0 \end{cases}. \tag{4.3}$$

One can see that, by fixing ζ_2, and letting ζ_1 be arbitrary, y_1 covers the complex plane, and of course the same holds exchanging 1 with 2. The asymptotic limit is

$$r^2 = |y_1|^2 + |y_2|^2 = \frac{1}{\mathcal{Z}^2} \to \infty. \tag{4.4}$$

The Jacobian of the Darboux transformation is $|\frac{\partial(\zeta,\zeta^*)}{\partial(y,y^*)}| = \mathcal{Z}^6$ in agreement with (3.28), and so we recover the flat measure (3.29).

Substituting the Darboux transformation in the expressions for the embedding coordinates (3.24) gives

$$x_i^j = \sqrt{\left(|y_i|^2 + \frac{1}{2}\right)\left(|y_j|^2 + \frac{1}{2}\right)}\frac{y_i^* y_j}{|y_i||y_j|}, \qquad x_3^i = -\sqrt{|y_i|^2 + \frac{1}{2}}\frac{ry_i}{|y_i|},$$
$$x_i^3 = \sqrt{|y_i|^2 + \frac{1}{2}}\frac{ry_i^*}{|y_i|}, \qquad x_3^3 = -r^2, \tag{4.5}$$

r being the positive square root of r^2. We can then check that the constraints (3.23) and the $su(2,1)$ Poisson bracket algebra (3.21) hold. Substituting (4.5) into (3.25) gives the Killing vectors in terms of canonical coordinates.

B. Quantization

Quantization proceeds as in Sec. III, with the Hilbert space $\mathcal{H}$ being that of a two-dimensional harmonic oscillator. The Wick-Voros star product is given in (3.37) and can be reexpressed in terms of local affine coordinates by making the replacement

$$\frac{\partial}{\partial y_1} \to \frac{\mathcal{Z}}{2\sqrt{2}\zeta_1}\left\{-|\zeta_1|\sqrt{|\zeta_1|^2 - |\zeta_2|^2 + 1}\left(\zeta_2^*\frac{\partial}{\partial\zeta_2^*} + \zeta_2\frac{\partial}{\partial\zeta_2}\right)\right.$$
$$\left. + \frac{-|\zeta_2|^2 + 1}{|\zeta_1|\sqrt{|\zeta_1|^2 - |\zeta_2|^2 + 1}}\left(\zeta_1^*\frac{\partial}{\partial\zeta_1^*} + \zeta_1\frac{\partial}{\partial\zeta_1}\right) + \frac{|\zeta_1|(|\zeta_1|^2 - |\zeta_2|^2 + 2)}{\sqrt{|\zeta_1|^2 - |\zeta_2|^2 + 1}}\kappa_1{}^1\right\}, \tag{4.6}$$

along with the corresponding replacement for $\frac{\partial}{\partial y_2}$, obtained by switching the coordinate indices 1 and 2 in (4.6). Since they both contain the overall factor of $\mathcal{Z}$, it follows that the star product reduces to the ordinary product in the asymptotic limit, $\mathcal{Z} \to 0$.

Next we construct the noncommutative analogs $X_a{}^b$ of the embedding coordinates (4.5). We take the following *Ansätse*

$$[X_a{}^b] = \begin{pmatrix} |y_1|^2 + \frac{1}{2} & \mathcal{R}_1\frac{y_1^*}{|y_1|} \star \mathcal{R}_2\frac{y_2}{|y_2|} & \mathcal{R}_1\frac{y_1^*}{|y_1|} \star \mathcal{S} \\ \mathcal{R}_2\frac{y_2^*}{|y_2|} \star \mathcal{R}_1\frac{y_1}{|y_1|} & |y_2|^2 + \frac{1}{2} & \mathcal{R}_2\frac{y_2^*}{|y_2|} \star \mathcal{S} \\ -\mathcal{S} \star \mathcal{R}_1\frac{y_1}{|y_1|} & -\mathcal{S} \star \mathcal{R}_2\frac{y_2}{|y_2|} & -r^2 \end{pmatrix}, \tag{4.7}$$

where we assume that $\mathcal{R}_i$ is a real function of $|y_i|^2$, and $\mathcal{S}$ is a real function of r^2.

In order to recover (4.5) in the commutative limit, we need that $\mathcal{R}_i \to \mathcal{R}_i^{(0)} = \sqrt{|y_i|^2 + \frac{1}{2}}$, and $\mathcal{S} \to \mathcal{S}^{(0)} = r$ when $\hbar \to 0$. Away from the commutative limit, $\mathcal{R}_i$ and $\mathcal{S}$ can be obtained as a perturbative expansion is $\hbar$

$$\mathcal{R}_i = \mathcal{R}_i^{(0)} + \hbar\mathcal{R}_i^{(1)} + \hbar^2\mathcal{R}_i^{(2)} + \mathcal{O}(\hbar^3),$$
$$\mathcal{S} = \mathcal{S}^{(0)} + \hbar\mathcal{S}^{(1)} + \hbar^2\mathcal{S}^{(2)} + \mathcal{O}(\hbar^3). \tag{4.8}$$

For this we require that $X_a{}^b$ satisfy the $su(2,1)$ star commutator algebra (3.38). For the leading two corrections, we find

$$\mathcal{R}_i^{(1)} = -\frac{1}{32|y_i|^2(|y_i|^2+\frac{1}{2})^{3/2}} + \frac{c_1}{8\sqrt{|y_i|^2+\frac{1}{2}}},$$
$$\mathcal{R}_i^{(2)} = -\frac{7+48|y_i|^2+128|y_i|^4}{2048|y_i|^4(|y_i|^2+\frac{1}{2})^{7/2}} - \frac{3c_1}{128(|y_i|^2+\frac{1}{2})^{5/2}} - \frac{c_1+c_1^2|y_i|^2}{128|y_i|^2(|y_i|^2+\frac{1}{2})^{3/2}} + \frac{c_2}{8\sqrt{|y_i|^2+\frac{1}{2}}}, \tag{4.9}$$

and

$$\mathcal{S}^{(1)} = -\frac{1+c_1}{8r}, \quad \mathcal{S}^{(2)} = -\frac{c_1^2+6c_1+7}{128r^3} - \frac{c_2}{8r}, \tag{4.10}$$

where c_1 and c_2 are arbitrary real constants. While $\mathrm{tr}X = X_a{}^a = 1$, as in the commutative theory, there are noncommutative corrections to the constraints (3.23). For example,

$$\mathrm{tr}X^2 = X_a{}^b \star X_b{}^a = 1 + (c_1+2)\hbar + \left(c_2 + \frac{3}{2}c_1 + \frac{3}{8}c_1^2\right)\hbar^2 + \mathcal{O}(\hbar^3),$$
$$\mathrm{tr}X^3 = X_a{}^b \star X_b{}^c \star X_c{}^a = 1 + \left(\frac{3}{2}c_1 + 4\right)\hbar + \frac{3}{2}\left(c_2 + 3c_1 + \frac{1}{2}c_1^2 + \frac{8}{3}\right)\hbar^2 + \mathcal{O}(\hbar^3). \tag{4.11}$$

They correspond to the quadratic and cubic Casimir operators for $su(2,1)$. We note that there is no choice of c_1 and c_2 for which the noncommutative corrections in both $\mathrm{tr}X^2$ and $\mathrm{tr}X^3$ disappear.

Upon writing the result for the expansion (4.8) in terms of local affine coordinates, one gets

$$\frac{y_i}{|y_i|}\mathcal{R}_i = \frac{\zeta_i}{\mathcal{Z}}\left\{1 + \frac{\hbar}{16|\zeta_i|^4}\frac{\mathcal{Z}^6}{(\mathcal{Z}^2-2|\zeta_i|^2)} - \frac{\hbar^2}{512}\frac{\mathcal{Z}^8}{|\zeta_i|^8}\frac{(63|\zeta_i|^4 - 50|\zeta_i|^2(\mathcal{Z}^2-|\zeta_i|^2) + 15(\mathcal{Z}^2-|\zeta_i|^2)^2}{(\mathcal{Z}^2-2|\zeta_i|^2)^2} + \mathcal{O}(\hbar^3)\right\},$$
$$\mathcal{S} = \frac{1}{\mathcal{Z}}\left\{1 - \frac{\hbar}{8}\mathcal{Z}^2 - \frac{7\hbar^2}{128}\mathcal{Z}^4 + \mathcal{O}(\hbar^3)\right\}, \tag{4.12}$$

where for simplicity we set $c_1 = c_2 = 0$. The zeroth order terms in $\hbar$ correspond to the commutative result. When substituted into (4.7), and extracting the zeroth order terms, we recover the formulas (3.24) for embedding coordinates. The noncommutative corrections to $\frac{y_i}{|y_i|}\mathcal{R}_i$ are not valid near $\zeta_i = 0$. The noncommutative corrections to $\frac{y_i}{|y_i|}\mathcal{R}_i$ and $\mathcal{S}$, and hence $X_a{}^b$, contain factors of $\mathcal{Z}$, and so, away from $\zeta_i = 0$, these corrections vanish in the asymptotic limit $\mathcal{Z} \to 0$. For this we also use the above result that the star product, when expressed in terms of local affine coordinates, reduces to the ordinary product in the asymptotic limit. Finally, we can construct the series expansion for the noncommutative analog $\kappa_a^{\star b}$ of the Killing vector on $CP^{1,1}$ using (3.39). The above arguments show that they too reduce to the commutative Killing vectors (3.18) in the asymptotic limit.

V. $\mathbb{CP}^{\cdot 0,2}$

Like $\mathbb{CP}^{1,1}$, $\mathbb{CP}^{\cdot 0,2}$ has four real dimensions. $\mathbb{CP}^{\cdot 0,2} \simeq H^{4,1}/S^1 \simeq SU(2,1)/U(2)$ can be built from $\mathbb{C}^{1,2}$, spanned by z^a, $a = 1, 2, 3$, using the constraint (3.2), which now becomes $|z^1|^2 - |z^2|^2 - |z^3|^2 = 1$, along with the equivalence relation (3.3). This means that $|z^1| \geq 1$, and also that $|z^1| > |z^2|$ or $|z^3|$.

Once again there are two complex affine coordinates ζ_i, $i = 1, 2$, along with their complex conjugates. They are defined by $\zeta^i = \frac{z^i}{z^3}$, $z^3 \neq 0$. Unlike the case with $\mathbb{CP}^{1,1}$, here the indices $i, j, \ldots$ are raised and lowered with the Lorentzian metric, $\mathrm{diag}(+,-)$. So here (3.7) implies that

$$\mathcal{Z}^2 = |\zeta_1|^2 - |\zeta_2|^2 - 1 > 0, \tag{5.1}$$

and so $|\zeta_1| > 1$. This restriction means that the local affine coordinates are defined on a real four-dimensional space with a solid three hyperboloid removed. The boundary of this region once again corresponds to the asymptotic limit (3.8), $\mathcal{Z}^2 \to 0$. While the background metric is Lorentzian, the Fubini-Study metric (3.9) for $\mathbb{CP}^{\cdot 0,2}$ has a Euclidean signature. This is opposite the situation with $\mathbb{CP}^{1,1}$. As with $\mathbb{CP}^{1,1}$, the Fubini-Study metric solves the sourceless Einstein equations with $\Lambda = 3$ [26].

A. Darboux map

We now give the transformation from the local affine coordinates (ζ^i, ζ_i^*), $i = 1, 2$, to canonical coordinates

(y_i, y_i^*), satisfying Poisson brackets (3.30). We note that the indices for the former are raised and lowered using the Lorentzian metric, but the latter coordinates are defined on a two-dimensional complex Euclidean space. Because of this fact, it is helpful to perform an intermediate step. For this we recognize that local affine coordinates are not unique. Instead of using the coordinates (ζ^i, ζ_i^*), as defined in (3.6), we can choose to work with the alternative set of coordinates (ξ^n, ξ_n^*), $n = 1, 2$, where $\xi^n = \frac{z^{n+1}}{z^1}$, $z^1 \neq 0$. In contrast with (ζ^i, ζ_i^*), for these coordinates, the indices $n, m, \ldots$ are raised and lowered with the *Euclidean* metric, diag$(-,-)$. The transformation between the two sets of local affine coordinates (in the overlapping region) is therefore something like a Wick rotation of the parameter space, although the signature of the Fubini-Study metric, of course, remains Euclidean. The transformation between the two sets of local affine coordinates is given by

$$\xi^1 = \frac{\zeta^2}{\zeta^1} \qquad \xi^2 = \frac{1}{\zeta^1}, \qquad \zeta^1, \xi^2 \neq 0. \tag{5.2}$$

The two sets of coordinates are valid on different domains and the transformation applies in the overlapping region. From (5.2),

$$1 - |\xi_1|^2 - |\xi_2|^2 = \frac{\mathcal{Z}^2}{|\zeta_1|^2} = \frac{1}{|z^1|^2} > 0, \tag{5.3}$$

and hence (ξ^n, ξ_n^*) span the interior of a three sphere of radius one, $|\xi_1|^2 + |\xi_2|^2 < 1$. As usual, the boundary corresponds to the asymptotic limit $|\xi_1|^2 + |\xi_2|^2 \to 1$. The Fubini-Study metric and Poisson brackets can be reexpressed in terms of the new local affine coordinates (ξ^n, ξ_n^*).

It is now not difficult to find the map from the affine coordinates (ξ^n, ξ_n^*) to canonical coordinates (y^i, y_i^*), $i = 1, 2$, having the desired properties. Up to canonical transformations, it is

$$y_1 = \frac{i\xi_1^*}{\sqrt{1 - |\xi_1|^2 - |\xi_2|^2}}, \qquad y_2 = \frac{-i\xi_2^*}{\sqrt{1 - |\xi_1|^2 - |\xi_2|^2}}. \tag{5.4}$$

There are no restrictions on the domain of (y_i, y_i^*), i.e., they span all of $\mathbb{C}^2$. To see this, note that

$$r^2 = \frac{|\xi_1|^2 + |\xi_2|^2}{1 - |\xi_1|^2 - |\xi_2|^2} \geq 0, \tag{5.5}$$

where we once again define $r^2 = |y_1|^2 + |y_2|^2$. The right-hand side of (5.5) spans the entire positive real line. Moreover, $|y_1|^2$ and $|y_2|^2$ span the entire positive real line. Just as with the case of $\mathbb{CP}^{1,1}$, $r^2 \to \infty$ is the boundary limit.

Using (5.2) and (5.4), we can write the Darboux map from the original set of affine coordinates (ζ^i, ζ_i^*). It is

$$y_1 = \frac{-i\zeta_2^*}{\mathcal{Z}} \sqrt{\frac{\zeta_1}{\zeta_1^*}}, \qquad y_2 = \frac{i}{\mathcal{Z}} \sqrt{\frac{\zeta_1}{\zeta_1^*}}. \tag{5.6}$$

This is an extension of the Darboux map for EAdS^2 (2.13), where ζ and y now correspond to ζ_1 and $-iy_2$, respectively. The Jacobian of the transformation is $|\frac{\partial(\zeta,\zeta^*)}{\partial(y,y^*)}| = \mathcal{Z}^6$, so we again recover the flat geometric measure when expressed in terms of canonical coordinates.

Writing the embedding coordinates (3.24) in terms of canonical coordinates gives

$$[x_a{}^b] = \begin{pmatrix} r^2+1 & iy_1^*\sqrt{r^2+1} & iy_2^*\sqrt{r^2+1} \\ iy_1\sqrt{r^2+1} & -|y_1|^2 & -y_2^* y_1 \\ iy_2\sqrt{r^2+1} & -y_1^* y_2 & -|y_2|^2 \end{pmatrix}. \tag{5.7}$$

We can then check that the constraints (3.23) and the $su(1,2)$ Poisson bracket algebra (3.21) hold. Substituting (5.7) into (3.25) gives the Killing vectors in terms of canonical coordinates.

B. Quantization

Quantization proceeds as in the previous section. The algebra of observables is again that of a two-dimensional harmonic oscillator, which is realized with the Wick-Voros star product (3.37). The star product can again be reexpressed in terms of the original local affine coordinates (ζ^i, ζ_i^*), now by making the replacement

$$\frac{\partial}{\partial y_1} \to \frac{i\mathcal{Z}}{2\zeta_1|\zeta_1|}\left\{\zeta_2\left(\zeta_1\frac{\partial}{\partial\zeta_1} + \zeta_1^*\frac{\partial}{\partial\zeta_1^*}\right) + 2|\zeta_1|^2\frac{\partial}{\partial\zeta_2^*}\right\},$$
$$\frac{\partial}{\partial y_2} \to \frac{-i\mathcal{Z}}{2\zeta_1|\zeta_1|}\left\{\zeta_1\frac{\partial}{\partial\zeta_1} + (1-2|\zeta_1|^2)\zeta_1^*\frac{\partial}{\partial\zeta_1^*} - 2|\zeta_1|^2\zeta_2^*\frac{\partial}{\partial\zeta_2^*}\right\}. \tag{5.8}$$

Because of the overall factor of $\mathcal{Z}$, it follows that the star product reduces to the ordinary product in the asymptotic limit, $\mathcal{Z} \to 0$.

Next we construct the noncommutative analogs $X_a{}^b$ of the embedding coordinates (5.7). We try writing

$$[X_a{}^b] = \begin{pmatrix} r^2+1 & iy_1^* \star \mathcal{S} & iy_2^* \star \mathcal{S} \\ i\mathcal{S} \star y_1 & -|y_1|^2 & -y_2^* y_1 \\ i\mathcal{S} \star y_2 & -y_1^* y_2 & -|y_2|^2 \end{pmatrix}, \tag{5.9}$$

where we assume that $\mathcal{S}$ is a real function of r^2. We need that $\mathcal{S} \to \mathcal{S}_0 = \sqrt{r^2+1}$ when $\hbar \to 0$, in order to recover

(5.7) in the commutative limit. In order to obtain $\mathcal{S}$ away from the commutative limit, we require that $X_a{}^b$ satisfy the $su(1,2)$ star commutator algebra (3.38). We can then get $\mathcal{S}$ in a perturbative expansion in $\bar{k}$. So, as before, we write $\mathcal{S} = \mathcal{S}_0 + \bar{k}\mathcal{S}_1 + \bar{k}^2\mathcal{S}_2 + \cdots$. For the leading two corrections, we get

$$\mathcal{S}_1 = -\frac{r^2}{8(r^2+1)^{3/2}}, \qquad \mathcal{S}_2 = \frac{r^2(8-7r^2)}{128(r^2+1)^{7/2}}. \tag{5.10}$$

Once again, while $\mathrm{tr}X = X_a{}^a = 1$, as in the commutative theory, there are noncommutative corrections to the constraints (3.23). For example,

$$\mathrm{tr}X^2 = X_a{}^b \star X_b{}^a = 1 - 2\bar{k} + \mathcal{O}(\bar{k}^3),$$
$$\mathrm{tr}X^3 = X_a{}^b \star X_b{}^c \star X_c{}^a = 1 - 2\bar{k} - 2\bar{k}^2 + \mathcal{O}(\bar{k}^3). \tag{5.11}$$

In comparing the expansion found here with the one found for $\mathbb{CP}^{1,1}$, we note that the latter was expressed in terms of undetermined integration constants c_1 and c_2. Integration constants may appear for $\mathbb{CP}^{0,2}$ as well upon generalizing the *Ansatz* (5.9).

From (5.9), noncommutative corrections to the embedding coordinates only appear for $X_1{}^2$, $X_1{}^3$, $X_2{}^1$, and $X_3{}^1$. After writing the leading order terms for these four matrix elements in the original affine coordinates (ζ^i, ζ_i^*), we get

$$X_a{}^b = x_a{}^b\left(1 - \frac{\mathcal{Z}^2(1+|\zeta_2|^2)}{8|\zeta_1|^4}\bar{k} + \frac{\mathcal{Z}^4(1+|\zeta_2|^2)(8|\zeta_1|^2 - 15|\zeta_2|^2 - 15)}{128|\zeta_1|^8}\bar{k}^2 + \mathcal{O}(\bar{k}^3)\right), \tag{5.12}$$

where again this only applies for $(a,b) = (1,2), (1,3), (2,1), (3,1)$. We find that the corrections contain factors of $\mathcal{Z}^2$, and so they vanish in the asymptotic limit, $\mathcal{Z}^2 \to 0$. Finally, we can obtain the leading corrections to the Killing vectors, specifically $\kappa_1{}^2$, $\kappa_1{}^3$, $\kappa_2{}^1$, and $\kappa_3{}^1$, using the definition (3.39) for their noncommutative analog. Since they involve taking a star product, which reduces to the ordinary product in the commutative limit, we once again see that all noncommutative corrections to the Killing vectors vanish in the asymptotic limit.

VI. CONCLUDING REMARKS

In this article we have shown how to perform a unique quantization of $\mathbb{CP}^{p,q}$ which preserves the full $su(p+1,q)$ isometry algebra. For the specific examples considered here we found that noncommutativity is effectively restricted to a limited neighborhood of some origin, and that these quantum spaces approach $\mathbb{CP}^{p,q}$ in the asymptotic limit. It is likely that this is a universal result that applies for all $\mathbb{CP}^{p,q}$, $q \geq 1$ quantized in a isometry preserving manner. Just as a strong-weak duality is postulated to exist between gravity on asymptotically AdS spaces and a CFT on the boundary, it is tempting to speculate that a similar duality could exist between gravity on asymptotically $\mathbb{CP}^{p,q}$ spaces and some boundary field theory. Adapting the standard techniques to this case, it should be possible to compute n-point correlation on the boundary, which are expected to be consistent with the $su(p+1,q)$ algebra, rather than the full conformal algebra. So then, if we have that noncommutative $\mathbb{CP}^{p,q}$ is asymptotically $\mathbb{CP}^{p,q}$, there could exist a dual $SU(p+1,q)$ invariant boundary theory.

As was stated in the text, the main reason we do not have an explicit construction for all quantized $\mathbb{CP}^{p,q}$, $q \geq 1$, and cannot prove asymptotic commutativity in general, is that we do not have a universal construction of the Darboux map. The Darboux map from local affine coordinates needed to satisfy three requirements, one of which was that the resulting canonical coordinates cover the entire complex plane. We found explicit constructions of the map for all examples in two and four dimensions. Straightforward higher-dimensional generalizations of these constructions exist, but they cannot be applied to all cases. There are two types of higher-dimensional extensions: (1) $\mathbb{CP}^{p,1}$ and (2) $\mathbb{CP}^{0,q}$.

(1) $\mathbb{CP}^{p,1}$, the coordinate patch spanned by the local affine coordinates (ζ^i, ζ_i^*) is $\mathbb{C}^{p+1}$ with the region $|\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_{p+1}|^2 \leq 1$ removed. The Darboux transformation to canonical coordinates (y_i, y_i^*), $i = 1, 2, \ldots, p+1$, can again be given by (4.2). The latter are defined on all of $\mathbb{C}^{p+1}$. The expressions for the $su\ (p+1,\ 1)$ embedding coordinates x_{ab} have the form (4.5), and their quantum corrections can be computed as in Sec. IV.

(2) $\mathbb{CP}^{0,q}$, the coordinate patch spanned by the local affine coordinates (ζ^i, ζ_i^*) is $\mathbb{C}^{1,q-1}$ with the region $|\zeta_1|^2 - |\zeta_2|^2 - \cdots - |\zeta_q|^2 \leq 1$ removed. A Darboux transformation to canonical coordinates (y_i, y_i^*), $i = 1, 2, \ldots, q$, is

$$y_1 = \frac{-i\zeta_2^*}{\mathcal{Z}}\sqrt{\frac{\zeta_1}{\zeta_1^*}}, \qquad \ldots, \qquad y_{q-1} = \frac{-i\zeta_q^*}{\mathcal{Z}}\sqrt{\frac{\zeta_1}{\zeta_1^*}},$$
$$y_q = \frac{i}{\mathcal{Z}}\sqrt{\frac{\zeta_1}{\zeta_1^*}}, \tag{6.1}$$

which generalizes (5.6). The coordinates (y_i, y_i^*) span all of $\mathbb{C}^q$. The expressions for the $su(1,q)$ embedding coordinates x_{ab} become

$$[x_a{}^b] = \begin{pmatrix} r^2+1 & iy_1^*\sqrt{r^2+1} & iy_2^*\sqrt{r^2+1} & \cdots & iy_q^*\sqrt{r^2+1} \\ iy_1\sqrt{r^2+1} & -|y_1|^2 & -y_2^*y_1 & \cdots & -y_q^*y_1 \\ iy_2\sqrt{r^2+1} & -y_1^*y_2 & -|y_2|^2 & \cdots & -y_q^*y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ iy_q\sqrt{r^2+1} & -y_1^*y_q & -y_2^*y_q & \cdots & -|y_q|^2 \end{pmatrix}, \tag{6.2}$$

generalizing (5.7), while their quantum corrections can be computed as in Sec. V.

More work is required to obtain the Darboux map for other cases, as it appears that a universal formula does not apply. One case, in particular, that is not included in (1) and (2), and may be worth pursuing, is $\mathbb{CP}^{1,2}$, as it contains Euclidean AdS_4 as a submanifold, and its noncommutative version is of possible interest for quantum cosmology [11]. The noncommutative analog of Euclidean AdS_4 is constructed from quantized $\mathbb{CP}^{1,2}$. Therefore, if, as expected, quantized $\mathbb{CP}^{1,2}$ is asymptotically commutative, it should naturally follow that noncommutative AdS_4 is asymptotically anti–de Sitter, having a dual three-dimensional conformal theory at the boundary.

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