

Schwarzschild black hole surrounded by quintessential matter field as an accelerator for spinning particles

Pankaj Sheoran,^{1,*} Hemwati Nandan^{2,3,†} Eva Hackmann^{4,‡} Ulises Nucamendi^{1,5,6,§} and Amare Abebe^{3,||}

¹*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, 58040 Morelia, Michoacán, México*

²*Department of Physics, Gurukul Kangri Vishwavidyalaya, Haridwar-249 407, India*

³*Center for Space Research, North-West University, Mafikeng 2745, South Africa*

⁴*ZARM, University of Bremen, Am Fallturm 2, 28359 Bremen, Germany*

⁵*Mesoamerican Centre for Theoretical Physics, Universidad Autónoma de Chiapas, Ciudad Universitaria, Carretera Zapata Km. 4, Real del Bosque (Terán), 29040 Tuxtla Gutiérrez, Chiapas, México*

⁶*Departamento de Física, Cinvestav, Avenida Instituto Politecnico Nacional 2508, San Pedro Zacatenco, 07360 Gustavo A. Madero, Ciudad de Mexico, Mexico*



(Received 20 July 2020; accepted 31 August 2020; published 16 September 2020)

We study the collision of two massive particles with nonzero intrinsic spin moving in the equatorial plane in the background of a Schwarzschild black hole surrounded by quintessential matter field (SBHQ). For the quintessential matter equation of state (EOS) parameter, we assume three different values. It is shown that for collisions outside the event horizon, but very close to it, the centre-of-mass energy (E_{CM}) can grow without bound if exactly one of the colliding particles is what we call near-critical, i.e., if its constants of motion are fine tuned such that the time component of its four-momentum becomes very small at the horizon. In all other cases, E_{CM} only diverges behind the horizon if we respect the Møller limit on the spin of the particles. We also discuss radial turning points and constraints resulting from the requirement of subluminal motion of the spinning particles.

DOI: [10.1103/PhysRevD.102.064046](https://doi.org/10.1103/PhysRevD.102.064046)

I. INTRODUCTION

The first simplest black hole (BH) solution of Einstein's field equations was obtained by Schwarzschild in 1916 [1] immediately after the discovery of general relativity (GR) by Einstein. The BH solution found by Schwarzschild is the simplest in the sense that it has only one observable parameter (i.e., mass). Black holes (BHs) are one of the most interesting topics in gravity research, and it took almost a century to confirm that these mysterious objects do exist in our universe. Recently the LIGO and VIRGO collaborations have detected the first ever gravitational waves signals from BH merger [2]. Even more recently, the first ever direct image of a BH observed by the Event Horizon Telescope (EHT) suggests to us to strongly believe in the presence of BHs in our universe [3].

The appearance of BHs is not only limited to GR or alternative theories of gravity (ATG) like string theory [4], but they have also played a crucial role in understanding cosmology. There are two major classes of cosmological

models for dark energy. One of them is the cosmological constant Λ [5] having an equation of state (EOS) parameter $\epsilon = -1$. But in this model the fine tuning problem is yet to be resolved [6]. The other class of cosmological model mainly depends on a dynamical scalar field such as, but not confined to, quintessence [7], chameleon fields [8], K-essence [9], tachyons [10], phantom [11] and dilatons [12]. In these models, the main difference is the EOS parameter ϵ which varies from -1 to $-1/3$ for quintessence like models and less than -1 for phantom like models. A comprehensive study of various dark energy models is presented in [13]. In this paper, we restrict to three different equation of state parameters, including the cosmological constant case and two quintessence like models. In particular, we focus on particle collisions in the background of a static BH solution surrounded by quintessence like matter obtained by Kiselev in [14]. The geodesic motion and geodesic deviation around this BH spacetime is investigated in detail in [15].

A rotating BH under some specific conditions can act as a particle accelerator for two spinless particles which start from rest at infinity and collide near the event horizon of a rotating BH (Kerr BH) pointed out by Bañados, Silk, and West (BSW) [16]. They showed that the collisional energy (i.e., center-of-mass (CM) energies) of these spinless

*hukmipankaj@gmail.com

†hnandan@associates.iucaa.in

‡eva.hackmann@zarm.uni-bremen.de

§unucamendi@gmail.com

||amare.abbebe@gmail.com

particles will be infinitely high if the BH is rotating in addition to the condition that one of the particle must have attained a critical value (a very fine-tuned value) of the angular momentum. They also mentioned that if the BH is nonrotating (i.e., Schwarzschild), it is not possible to obtain an infinite amount of CM energy. After this pioneering work by BSW [16], a number of studies have been performed on the particle acceleration by all sorts of BHs in GR [17–56] and in different ATG models [57–98]. These studies conclude in their individual works that the conditions obtained by BSW to get infinite amount of high CM energy are universal and these results were also generalized by Harada in [99]. It is worth noting here that the conditions mentioned by BSW such as the BH must be extremal and one of the colliding particles should have a critical angular momentum are very rare to observe in nature. In turn, the BSW process is a very rare event to observe in nature which needs careful attention in diverse context.

The BSW mechanism is so far mainly studied for spinless test particles (i.e., particles that follow geodesics) only. However, in general a particle moving in the vicinity of a BH is an extended object having self interaction such as the case of a spinning particle. It has been shown by Matisson, Papapetrou and Dixon (MPD) [100–102] that the trajectory followed by a spinning particle is non-geodesic due to the coupling between the spin of the particle and curvature of the spacetime around a massive central object like a BH.

In 2016, it was shown by Armaza *et al.* [103] that it is still possible to obtain an infinite amount of CM energy for the Schwarzschild BH if one considers the collision of spinning particles instead of a collision of spinless particles. The study of BHs as a particle accelerator for spinning particles is further extended to the case of charged and spinning BHs in [104], where it was shown that it is possible to obtain infinitely high CM energy outside the event horizon of a nonextremal Reissner-Nordstrom (RN) BH. Zhang *et al.* [104] also concluded that the area belonging to the infinitely high CM energy in spin and total orbital angular momentum (s, l) plane of the spinning particles is very sensitive to the BH charge as it decreases as the charge of the black hole increases. They further showed that for a non-extremal Kerr BH case, we can also obtain infinitely high CM away from the event horizon and the corresponding area in the (s, l) plane increases with an increase in the spin of the BH. Combining charge and rotation in the Kerr-Newman background they finally concluded that the spin parameter and the charge of the BH affect the CM energy of the colliding particles in a completely opposite way. Recently, the universality of BSW mechanism for spinning particles, for a class of stationary axisymmetric BH, is also discussed in [105]. However, in [105], the calculations of E_{CM} in terms of 4-momentum which is a conserved quantity are not performed. Also, the timelike condition is not verified

explicitly for 4-velocity. Hence, the results are not conclusive for two spinning particles colliding in the vicinity of a BH and it is therefore worthy to discuss the collision of such particles in the vicinity of more BHs to draw the definite conclusions in this regard.

In this work, we extend the study of BH spinning particle acceleration processes and investigate two spinning particles colliding outside the event horizon of the nonextremal Schwarzschild BH which is surrounded by the quintessence like matter, which we will abbreviate as SBHQ henceforth [15]. We have observed that the CM energy of the colliding particles might be infinitely high for the collisions of the spinning particles, but the collisions must take place inside the cosmological horizon of the SBHQ. The CM energy in our case is found to be very sensitive to the value of normalization constant (λ) and the EOS parameter (ϵ) which, for quintessential matter, varies from -1 to $-1/3$.

Our paper is organized as follows. We start in Sec. II with a brief overview of the equations of motion for spinning particles in Einstein's theory of general relativity (GR). In Sec. III, we discuss the spacetime geometry of the SBHQ and its event and cosmological horizons. Following Refs. [106–108] we also derived the expressions for the four-momentum of a spinning particle. In Sec. IV, we obtain the expression for the CM energy of the colliding spinning particles in the vicinity of SBHQs and show that it reduces to the Schwarzschild black hole case [103] if the normalization constant λ vanishes and the energy e per unit mass becomes unity. We then discuss the possible scenarios where arbitrarily high E_{CM} is possible. Section V is devoted to the study of the effective potential (V_{eff}) and radial turning points for the trajectories of the spinning particles. We have divided this section into two parts: in the first part, we find the expression for V_{eff} , as it helps to characterize the path of the spinning particle moving in the background of SBHQ. Based on this, in the second part we classify the spinning particles and their trajectories according to [109] into three subclasses: usual particle, critical particle and near-critical particle, respectively. In Sec. VI, we study the superluminal constraint and the conditions to avoid the superluminal region for the spinning particles. Finally, Sec. VII is devoted to the summary and conclusions of our results and to future prospects.

Throughout our work in this paper, we set the fundamental constants to unity (i.e., $c = G = 1$), the signature of spacetime as $(-, +, +, +)$, Greek indices (i.e., α, β, \dots) run from 0 to 3 and Latin indices runs from 1 to 3 unless otherwise stated. Also, in the following sections, we chose the spin s per unit mass of the colliding particles within the Møller limit (i.e., $r_p > s$) [110,111], where r_p is the size of the spinning particle. It is important to note that size of the spinning particle is very less than the size of the BH (i.e., $r_p \ll r_{0(1)}$), therefore we have $s \ll M$ [112].

II. EQUATIONS OF MOTION OF SPINNING PARTICLES IN CURVED SPACETIME

The study of the chargeless spinning particles in GR started with the pioneering work of MPD [100–102] on spinning tops in curved spacetime. In their formulation, they showed that the trajectories followed by the chargeless spinning tops were not in accordance with the equivalence principle i.e., the above massive particles follow the non-geodesic paths. Further, Hojman [106,113] extensively studied and extended the formulation by MPD. In this section, we will present a brief overview of the equations of motion developed by Hojman with the help of Lagrangian formulation. The aforesaid equations of the motion read as

$$\frac{dx^\alpha}{d\tau} = u^\alpha, \quad (1)$$

$$\frac{DP^\alpha}{D\tau} = -\frac{1}{2}R^\alpha_{\beta\gamma\delta}u^\beta S^{\gamma\delta}, \quad (2)$$

$$\frac{DS^{\alpha\beta}}{D\tau} = S^{\alpha\gamma}\sigma_\gamma^\beta - \sigma^{\alpha\gamma}S_\gamma^\beta = P^\alpha u^\beta - P^\beta u^\alpha, \quad (3)$$

where τ , u^α , P^α , $S^{\alpha\beta}$ and $\sigma^{\alpha\beta}$ are an affine parameter, the 4-velocity, the 4-momentum vector, the spin tensor, and the antisymmetric angular velocity tensor, respectively. The antisymmetric angular velocity tensor is in turn defined as

$$\sigma^{\alpha\beta} \equiv \eta^{(\gamma\delta)} e_{(\gamma)}^\alpha \frac{De_{(\delta)}^\beta}{D\tau} = -\sigma^{\beta\alpha}. \quad (4)$$

Here, $e_{(\gamma)}^\alpha$ is an orthonormal tetrad which is used to define the orientation of the top, $De_{(\delta)}^\beta/D\tau$ is the usual covariant derivative of the orthonormal tetrad and $\eta^{(\gamma\delta)} \equiv \text{diag}(-1, 1, 1, 1) = \eta^{(\gamma\delta)}$.

As the Eqs. (1)–(3) does not form a closed set of equations (i.e., they are insufficient to determine the complete trajectory of spinning particles in a curved spacetime) and hence, spin supplementary conditions are needed. For simplicity purposes, we choose the Tulczyjew spin supplementary condition (TSSC) $S^{\alpha\beta}P_\beta = 0$ which conserves the dynamical mass of the spinning particle and choose a particular frame of the spinning particles for which only 3-components of $S^{\alpha\beta}$ are nonvanishing (i.e., $S^{0i} = 0$) [103].

Additionally, the 4-momentum P^α is not parallel to the four velocity u^α for the case of a spinning particle and a relation between P^α and u^α is essential and can be written as [114]

$$u^\alpha = \frac{\kappa}{m} \left[P^\alpha + \frac{2S^{\alpha\beta}P^\gamma R_{\beta\gamma\rho\epsilon}S^{\rho\epsilon}}{4m^2 + R_{\mu\nu\kappa\lambda}S^{\mu\nu}S^{\kappa\lambda}} \right]. \quad (5)$$

Here, κ is a normalization constant. It is worth mentioning here that the above condition on the spin tensor comes naturally from the theory if one suitably chooses the corresponding Lagrangian (for detailed analysis see [115]).

We now define the conserved quantities [115] related to the spinning top and these are the mass (m) of the spinning top

$$m^2 = -P^\alpha P_\alpha, \quad (6)$$

and its spin (S),

$$S^2 = \frac{1}{2}S^{\alpha\beta}S_{\alpha\beta}. \quad (7)$$

In addition to the above-mentioned conserved quantities, we have an extra conserved quantity D_ξ defined as below,

$$D_\xi \equiv P^\alpha \xi_\alpha - \frac{1}{2}S^{\alpha\beta} \xi_{\alpha\beta}, \quad (8)$$

which is independent of the choice of the background metric as shown in [102]. Here, ξ_α is a Killing vector associated with the spacetime metric. The motion of the tops in the background of SBHQ is presented in the next section.

III. SPINNING PARTICLES IN SBHQ BACKGROUND

The metric for SBHQ in the Schwarzschild coordinate system (t, r, θ, ϕ) reads as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2, \quad (9)$$

where

$$f(r) = \left(1 - \frac{2M}{r} - \frac{\lambda}{r^{3\epsilon+1}} \right), \quad (10)$$

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2. \quad (11)$$

Here, λ is a normalization constant whose physical interpretation depends on the specific EOS parameter value ϵ . The behavior of $f(r)$ is shown in Fig. 1 for different combinations of λ and ϵ .

In order to analyze the properties of SBHQ, we study the structure of horizon which has a two-sphere topology (except in the case $\epsilon = -1/3$ and $0 < \lambda < 1$ which has the topology of a two-sphere but a deficit solid angle [116–117]) and is calculated by the equation $g^{rr} = 0$ of the above metric. Now, using Eqs. (9) and (10) and the above definition, the horizon satisfies the following condition

$$\Delta_0 \equiv r^{3\epsilon+1} - 2Mr^{3\epsilon} - \lambda = 0. \quad (12)$$

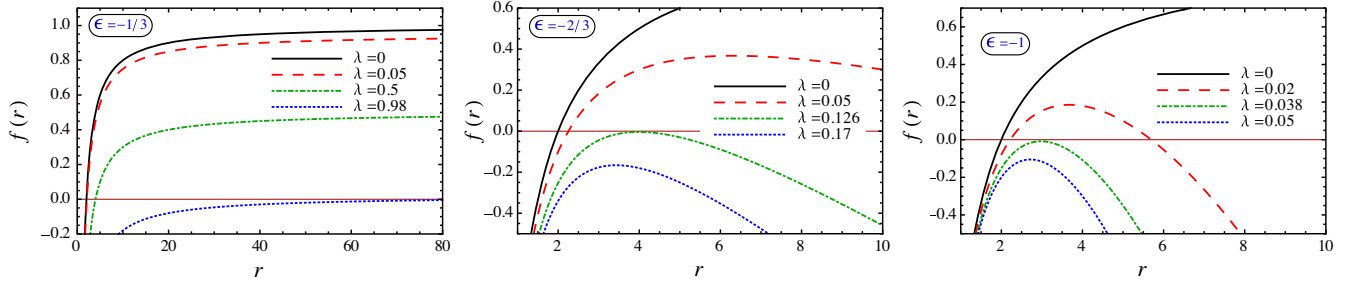


FIG. 1. The behavior of $f(r)$ with r for different values of normalization constant λ , for fixed values of ϵ ($M = 1$).

From Eq. (12), we find that the horizon of SBHQ depends upon two extra parameters, i.e., λ and ϵ respectively, besides the usual mass M of a static spherical BH as in general relativity (i.e., SBH). We consider in this work three different choices of the EOS parameter, namely $\epsilon = -1/3, -2/3, -1$. For these choices, we now analyze the possible horizons of the spacetime:

- (i) When $\epsilon = -1/3$ and $0 < \lambda < 1$, the Eq. (12) becomes linear in r and has only one root at $r = r_{0(1)} = 2M/(1 - \lambda)$, known as event horizon.
- (ii) For $\epsilon = -2/3$, the Eq. (12) becomes quadratic in r and has two roots $r_{0(1)}$ and $r_{0(2)}$, known as event and cosmological horizons, located at

$$r = r_{0(1,2)} = \frac{1 \pm \sqrt{1 - 8M\lambda}}{2\lambda}. \quad (13)$$

It is clear from above equation that for $\lambda = 1/8M$ both horizons coincide at the position $r = 4M$.

- (iii) For $\epsilon = -1$, Eq. (12) becomes a depressed cubic equation in r whose discriminant and roots are as follows:

$$\square = \frac{1}{27\lambda^3}(-1 + 27M^2\lambda), \quad (14)$$

$$\tilde{r}_1 = Y_1 + Y_2, \quad (15)$$

$$\tilde{r}_{2,3} = -\left(\frac{Y_1 + Y_2}{2}\right) \pm \frac{i\sqrt{3}}{2}(Y_1 - Y_2), \quad (16)$$

where

$$Y_{1,2} = \sqrt[3]{\frac{-M}{\lambda} \pm \sqrt{\square}}. \quad (17)$$

Depending on the values of λ we have following three subcases:

- (i) If $\lambda = 1/27M^2 \Rightarrow \square = 0$, then all roots are real, and at least two are equal (i.e., $\tilde{r}_1 < 0$, $\tilde{r}_2 \equiv r_{0(2)} = 3M$ and $\tilde{r}_3 \equiv r_{0(1)} = 3M$). This means both the event $r_{0(1)}$ and the cosmological $r_{0(2)}$ horizons coincide.

- (ii) If $0 < \lambda < 1/27M^2 \Rightarrow \square < 0$, then all roots are real and unequal (i.e., $\tilde{r}_1 < 0$, $\tilde{r}_2 \equiv r_{0(2)} > 0$ and $0 < \tilde{r}_3 \equiv r_{0(1)} < r_{0(2)}$). This means the event $r_{0(1)}$ and the cosmological $r_{0(2)}$ horizons do not coincide.
- (iii) If $\lambda > 1/27M^2 \Rightarrow \square > 0$, then one root is real and two are complex conjugates (i.e., $\tilde{r}_1 < 0$, $\tilde{r}_2 \equiv r_{0(2)} = \text{imaginary}$ and $\tilde{r}_3 \equiv r_{0(1)} = \text{imaginary}$). This means for the case $\lambda > 1/27M^2$, there are no horizons and hence corresponds to no BH spacetime. In fact, it corresponds to a naked singularity as evident from the expression for Kretschmann scalar (K) give below,

$$K = 8 \left(\frac{6M^2 r^2 + 12M\lambda r + 7\lambda^2}{r^8} \right). \quad (18)$$

Numerical values of horizon for different combination of normalization constant λ and the EOS parameter ϵ are shown in Table I.

Now, we study the motion of spinning particles in the background of the spacetime defined by Eq. (9). We restrict here to the case that the motion is planar. Due to spherical symmetry, we may assume that the particle is initially in the equatorial plane $\theta = \pi/2$. To ensure that $P^\theta = 0$, $u^\theta = 0$ we then assume that the spin vector is perpendicular to the equatorial plane [118]. We find the constants of motion with the help of Eqs. (6)–(7) which in the equatorial plane read as

$$m^2 = -\frac{(P^r)^2}{f(r)} + f(r)(P^t)^2 - r^2(P^\phi)^2, \quad (19)$$

$$S^2 = -(S^r)^2 + \frac{r^2(S^r\phi)^2}{f(r)} - f(r)r^2(S^t\phi)^2. \quad (20)$$

By using the symmetries of the background metric (9) and the Killing vectors, the energy (E) of the particles and the total angular momentum (J) orthogonal to the plane of motion as the conserved quantities are defined below,

TABLE I. Numerical values of horizons for SBHQ with $M = 1$.

λ	$\epsilon = -1/3$		$\epsilon = -2/3$		$\epsilon = -1$	
	$r_{0(1)}$	$r_{0(2)}$	$r_{0(1)}$	$r_{0(2)}$	$r_{0(1)}$	$r_{0(2)}$
0.0	2.0	2.0			2.0	
0.00001	2.00002	2.00004	99997.99	2.00008	315.22	
0.0001	2.0002	2.0004	9997.99	2.0008	98.98	
0.001	2.002	2.004	997.99	2.008	30.57	
0.01	2.02	2.04	97.96	2.09	8.78	
0.1	2.22	2.76	7.23			
0.2	2.50					
0.3	2.85					
0.4	3.33					
0.5	4.0					

$$E = f(r)P^t - \frac{f(r)'S^{tr}}{2}, \quad (21)$$

$$J = r(rP^\phi + S^{r\phi}), \quad (22)$$

where the (\prime) denotes the derivative with respect to the radial coordinate. Now, by using the Eqs. (19), (20) and the TSSC $S^{\alpha\beta}P_\beta = 0$, the components $S^{t\phi}$ and $S^{r\phi}$ come out as

$$S^{tr} = s r P^\phi, \quad S^{t\phi} = \frac{s P^r}{r f(r)} \quad \text{and} \quad S^{r\phi} = \frac{s f(r) P^t}{r}. \quad (23)$$

It is worth to note here that $s = \pm S/m$ is the spin per unit mass; the \pm signs are related to (anti) parallel spin of the particle with respect to the total angular momentum, respectively. The component of spin perpendicular to the equatorial plane may then read as

$$S_z = r S^{r\phi} = s \left(\frac{2er - jsf(r)'}{2r - s^2 f(r)'} \right). \quad (24)$$

Further, all the nonzero components of the 4-momentum vector P^α calculated with the help of Eqs. (19), (20), (21), (22), (23), and (24) as follows,

$$P^t = m \left(\frac{r^{3\epsilon+1}}{\Delta_0} \right) \mathcal{K}, \quad (25)$$

$$P^\phi = m \left(\frac{2}{r} \right) \mathcal{L}, \quad (26)$$

$$(P^r)^2 = m^2 [\mathcal{K}^2 - f(r)(1 + 4\mathcal{L}^2)]. \quad (27)$$

where

$$\mathcal{K} = \frac{2er - jsf(r)'}{2r - s^2 f(r)'}, \quad (28)$$

$$\mathcal{L} = \frac{j - es}{2r - s^2 f(r)'}. \quad (28)$$

Here, $e = E/m$ is energy per unit mass and $j = J/m$ is the total angular momentum per unit mass. Hereafter, we normalize m to unity for simplicity.

Finally, one can write the expression for $\dot{\phi}$ and \dot{r} as follows:

$$\dot{\phi} = \frac{u^\phi}{u^t} = \frac{[2r - rs^2 f(r)'] P^\phi}{[2r - s^2 f(r)'] P^t}, \quad (29)$$

$$\dot{r} = \frac{u^r}{u^t} = \frac{P^r}{P^t}. \quad (30)$$

It is worth mentioning here that the parameter corresponding to the proper time (τ) has to be fixed in order to obtain the velocity components u^t , u^ϕ , and u^r . However, for the above discussed relativistic invariants, one does not need to make any such specific choices.

IV. CENTER-OF-MASS ENERGY OF THE SPINNING PARTICLES

Let us consider two spinning massive particles (m_1 and m_2) colliding near to the horizon of the BH. The center-of-mass energy (E_{CM}) of these two particles can with the help of the formula derived as in [16] be written as

$$E_{\text{CM}}^2 = -g_{\alpha\beta} (P_1^\alpha + P_2^\alpha)(P_1^\beta + P_2^\beta), \quad (31)$$

$$= m_1^2 + m_2^2 - 2g_{\alpha\beta} P_1^\alpha P_2^\beta. \quad (32)$$

Here, with a constraint $m_1 + m_2 = \text{constant}$ along with the condition $R(r; e, l, s) > 1$ which follow under the fixed parameters (e, j, s), the and the condition $R(r; e, l, s) > 1$ (see appendix for details including the definition of the function R), the particles (spinning as well spinless) with equal masses acquire the maximum E_{CM} in comparison to the particles with unequal masses. This E_{CM} increases as the BH spin increases and diverges for the extremal rotating BH under specific conditions on the angular momentum of one of the particles. Hence, to have the maximum collisional energy, it is assumed that both the spinning particles have the same mass (i.e., $m_1 = m_2 = m$) and for simplicity we consider $m = 1$. Therefore, the Eq. (32) with these assumptions in the equatorial plane becomes

$$E_{\text{CM}}^2 = 2[1 - (g_{tt} P_1^t P_2^t + g_{rr} P_1^r P_2^r + g_{\phi\phi} P_1^\phi P_2^\phi)], \quad (33)$$

which after substituting the values of P^t , P^r and P^ϕ from Eqs. (25), (26), and (27) respectively, reduces to

$$E_{\text{CM}}^2 = \frac{2}{\Delta_0 C_1 C_2} \left[r^{3\epsilon+1} D_1 D_2 + \Delta_0 [C_1 C_2 - 4r^{6\epsilon+4} (j_1 - e_1 s_1)(j_2 - e_2 s_2)] - \sqrt{r^{3\epsilon+1} D_1^2 - \Delta_0 [C_1^2 + 4r^{6\epsilon+4} (j_1 - e_1 s_1)^2]} - \sqrt{r^{3\epsilon+1} D_2^2 - \Delta_0 [C_2^2 + 4r^{6\epsilon+4} (j_2 - e_2 s_2)^2]} \right], \quad (34)$$

where

$$C_{1,2} = 2r(r^{3\epsilon+2}) - s_{1,2}^2 [2Mr^{3\epsilon} + \lambda(3\epsilon + 1)], \\ D_{1,2} = 2r(r^{3\epsilon+2})e_{1,2} - j_{1,2}s_{1,2} [2Mr^{3\epsilon} + \lambda(3\epsilon + 1)]. \quad (35)$$

One can easily verify from Eq. (34) that E_{CM} could possibly diverge not only for $\Delta_0 = 0$ but also for $C_{1,2} = 0$. In case one substitutes $\lambda = 0$ and $e = 1$ in Eq. (34), the expression for E_{CM} reduces to

$$E_{\text{CM}}^2 = \frac{2}{\Delta \Delta_1 \Delta_2} \left[r(r^3 - Mj_1 s_1)(r^3 - Mj_2 s_2) + \Delta [\Delta_1 \Delta_2 - r^4 (j_1 - s_1)(j_2 - s_2)] - \sqrt{r(r^3 - Mj_1 s_1)^2 - \Delta [\Delta_1^2 + r^4 (j_1 - s_1)^2]} - \sqrt{r(r^3 - Mj_2 s_2)^2 - \Delta [\Delta_2^2 + r^4 (j_2 - s_2)^2]} \right]. \quad (36)$$

Here $\Delta = r - 2M$ and $\Delta_{1,2} = r^3 - Ms_{1,2}^2$. Eq. (36) matches with E_{CM} of two spinning test particles colliding near the Schwarzschild BH [103].

For Eq. (34), the case when $\Delta_0 = 0$ is not of much interest because both numerator and denominator vanish at the horizon, and the energy in this limit becomes finite. It can be generally shown from (33) that in the limit $\Delta_0 = 0$ or, equivalently, $f = 0$ we find

$$\frac{1}{2} E_{\text{CM}}^2 = 1 + \frac{1}{2} \left(\frac{\mathcal{K}_1}{\mathcal{K}_2} + \frac{\mathcal{K}_2}{\mathcal{K}_1} \right) + 2 \frac{(\mathcal{K}_1 \mathcal{L}_2 - \mathcal{K}_2 \mathcal{L}_1)^2}{\mathcal{K}_1 \mathcal{K}_2}, \quad (37)$$

where \mathcal{K}_i and \mathcal{L}_i refer to particle i . In the limit $\lambda = 0$, $s = 0$, $e = 1$ this reduces to the result in [16], and for $\lambda = 0$, $e = 1$ we recover the result in [103].

Also, the case $C_{1,2} = 0$ is not of significant interest in contrast with [103], because the radius r_d , where the divergence occurs, always is behind the horizon, when the restriction on the particle's spin is taken into consideration. This can be seen as follows: C_i is zero exactly if the denominator $(2r - s_i^2 f')$ in (28) vanishes. Note that this may only happen in the region where $f' > 0$, and then $s_i^2 = 2r/f'$. In that region, for all the cases

TABLE II. Numerical values of divergence radius r_d for SBHQ with $\epsilon = -1/3$ and $M = 1$.

s	r_d
0.2	0.341995
0.4	0.542884
0.6	0.711379
0.8	0.861774
0.99	0.993322

$\epsilon = -1/3, -2/3, -1$, the right-hand side $2r/f'$ is a monotonically increasing function of r and, if applicable, also of λ . This implies that at or outside the horizon we have $2r/f' \geq (2r/f')|_{(r=r_H, \lambda=0)} = 8M^2$, where r_H is the horizon. As s is smaller than the particle radius due to the Møller bound, and the particle radius is much smaller than M , $C_{1,2}$ can therefore not vanish at or outside the horizon. The numerical values of the divergence radius r_d are shown in Tables II, III, and IV for different combinations of λ and e .

Let us return to the case that the collision happens at or close to the horizon $f = 0$. Similar to the arguments in [22], from Eq. (37) we observe that the center of mass energy may still diverge if \mathcal{K}_1 or \mathcal{K}_2 vanishes, and that E_{CM} may become arbitrarily large if at least one of the \mathcal{K}_i becomes arbitrarily small. It might however turn out that particles with small or vanishing \mathcal{K} may not be able to reach the near horizon region. Therefore, we will now study particle motion with a particular emphasize on particles that may enable arbitrarily large center of mass energy.

TABLE III. Numerical values of divergence radius r_d for SBHQ with $\epsilon = -2/3$ and $M = 1$.

s	$\lambda = 0.00001$	$\lambda = 0.0001$	$\lambda = 0.001$	$\lambda = 0.01$
	r_d	r_d	r_d	r_d
0.2	0.341995	0.341994	0.341989	0.341929
0.4	0.542883	0.542881	0.542857	0.542617
0.6	0.711378	0.711373	0.711319	0.710779
0.8	0.861773	0.861763	0.861667	0.860709
0.99	0.993321	0.993306	0.993159	0.991691

TABLE IV. Numerical values of divergence radius r_d for SBHQ with $\epsilon = -1$ and $M = 1$.

s	$\lambda = 0.00001$	$\lambda = 0.0001$	$\lambda = 0.001$	$\lambda = 0.01$
	r_d	r_d	r_d	r_d
0.2	0.341995	0.341994	0.341991	0.341949
0.4	0.542883	0.542881	0.542855	0.542594
0.6	0.711378	0.711370	0.711293	0.710527
0.8	0.861772	0.861755	0.861590	0.859943
0.99	0.993319	0.993289	0.992998	0.990098

V. EFFECTIVE POTENTIAL AND RADIAL TURNING POINTS

The study of the effective potential and the radial turning points are very important as this help us to characterize the different trajectories of the spinning particles moving around the BHs.

A. Effective potential

The radial velocity u^r is proportional to the radial component of the conjugate momenta P^r and, therefore, we can determine the radial turning points from $P^r = 0$. We rewrite P^r in the form of an effective potential,

$$\left(\frac{P^r}{m}\right)^2 = A \left[1 - \frac{s^2 f(r)'}{2r}\right]^{-2} (e - V_{\text{eff}(+)}(r)) \times (e - V_{\text{eff}(-)}(r)),$$

$$V_{\text{eff}(\pm)}(r) = \frac{B \pm C^{1/2}}{A}, \quad (38)$$

where

$$A = 1 - \frac{f(r)s^2}{r^2},$$

$$B = \frac{js}{r^2} \left(\frac{f(r)'r}{2} - f(r) \right),$$

$$C = f(r) \left(1 - \frac{s^2 f(r)'}{2r} \right)^2 \left[1 + \frac{j^2}{r^2} - f(r) \frac{s^2}{r^2} \right]. \quad (39)$$

One needs to restrict the values of r such that $e > V_{\text{eff}(+)}(r)$ or $e < V_{\text{eff}(-)}(r)$ whenever $A > 0$, in order to have u^r to be real for the motion of the spinning particle. We can easily check for the cases $\epsilon = -1, -\frac{1}{3}, -\frac{2}{3}$ that A has some minimum outside the event horizon. For $\epsilon = -1$ we find a minimum at $r = 3M$ with $A = 1 - s^2(\frac{1}{27M^2} + \lambda)$ implying that A is positive between event and cosmological horizon for $s \ll M$. Analogously, for $\epsilon = -\frac{1}{3}$ the minimum $A = 1 - s^2 \frac{(1-\lambda)^3}{27M^2}$ is at $r = \frac{3M}{1-\lambda}$, so again A is positive for $s \ll M$. Finally, for $\epsilon = -\frac{2}{3}$ we have a minimum between event and cosmological horizon at $r = (1 - \sqrt{1 - 6M\lambda})/\lambda$, and A is always positive for $s \ll M$.

In the original paper by BSW [16] it is assumed that the colliding particles start from rest at infinity. In our case this is however not generally possible due to the presence of a cosmological horizon. Let us discuss the cases $\epsilon = -1/3, -2/3, -1$ separately: (i) If $\epsilon = -1/3$, we only have an event horizon as explained in Sec. III. We can therefore assume that the particle starts from rest at infinity. In this case, the energy of the particle is given by $e = 1 - \lambda$. (ii) For $\epsilon = -2/3$, we have an event and a cosmological horizon if we choose $\lambda < 1/(8M)$, see Sec. III. Therefore, it does not make sense to consider a particle starting from

infinity. Instead, we could choose to let the particle start from rest from the static radius, see e.g., [119,120], representing an equilibrium between gravitational attraction and cosmological expansion. A particle with $P^r = 0, P^\phi = 0$ can sit at radius $r = \sqrt{2/\lambda}$ with energy $e^2 = 1 - 2\sqrt{2\lambda}$. (iii) For $\epsilon = -1$ we may again choose the static radius as starting point, giving $r = \lambda^{-\frac{1}{3}}$ and $e^2 = 1 - 3\lambda^{\frac{1}{3}}$.

In Fig. 2, the behavior of the positive component of the effective potential $V_{\text{eff}(+)}$ is shown as a function of r for two different values of the spin (s) and several different j . We plotted here $V_{\text{eff}(+)}$ only for those values of the particle spin s and the total angular momentum j for which the particle starting from rest from infinity or the static radius, as respectively explained above, will fall into the SBHQ and does not meet the turning point first. It is shown in Fig. 2 that the maximum value of $V_{\text{eff}(+)}$ decreases with increase in j for each value of s (i.e., $s = 0.2$ and $s = 0.99$) corresponding to $\epsilon = -1/3, -2/3$ and -1 , respectively.

We showed the behavior of the positive component of P^r with r in Fig. 3 for different combinations of particle spin s , total angular momentum j , λ and ϵ as it will help in visualizing for which combinations of these parameters the spinning particle will reach the event horizon $r_{0(1)}$ first before meeting the turning point. In the figure, we fixed the normalization parameter $\lambda = 0.00001$ and increase the EOS parameter for quintessential matter ϵ from top to bottom in each column. It is easy to conclude from the

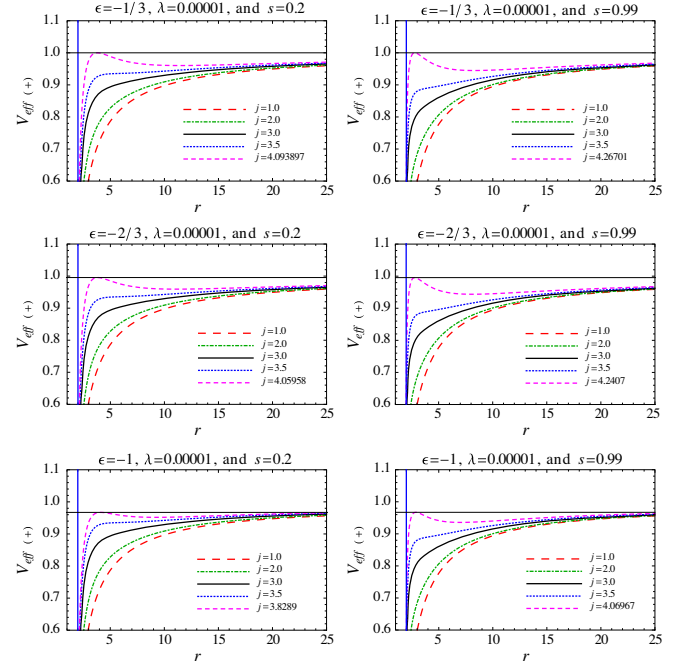


FIG. 2. Variation of the effective potential ($V_{\text{eff}(+)}$) with respect to r , for different values of spin (s) corresponding to the constant value of EOS parameter (ϵ), normalization constant (λ) and total angular momentum (j). Here, the solid (blue) vertical line indicate the location of the horizon ($M = 1$).

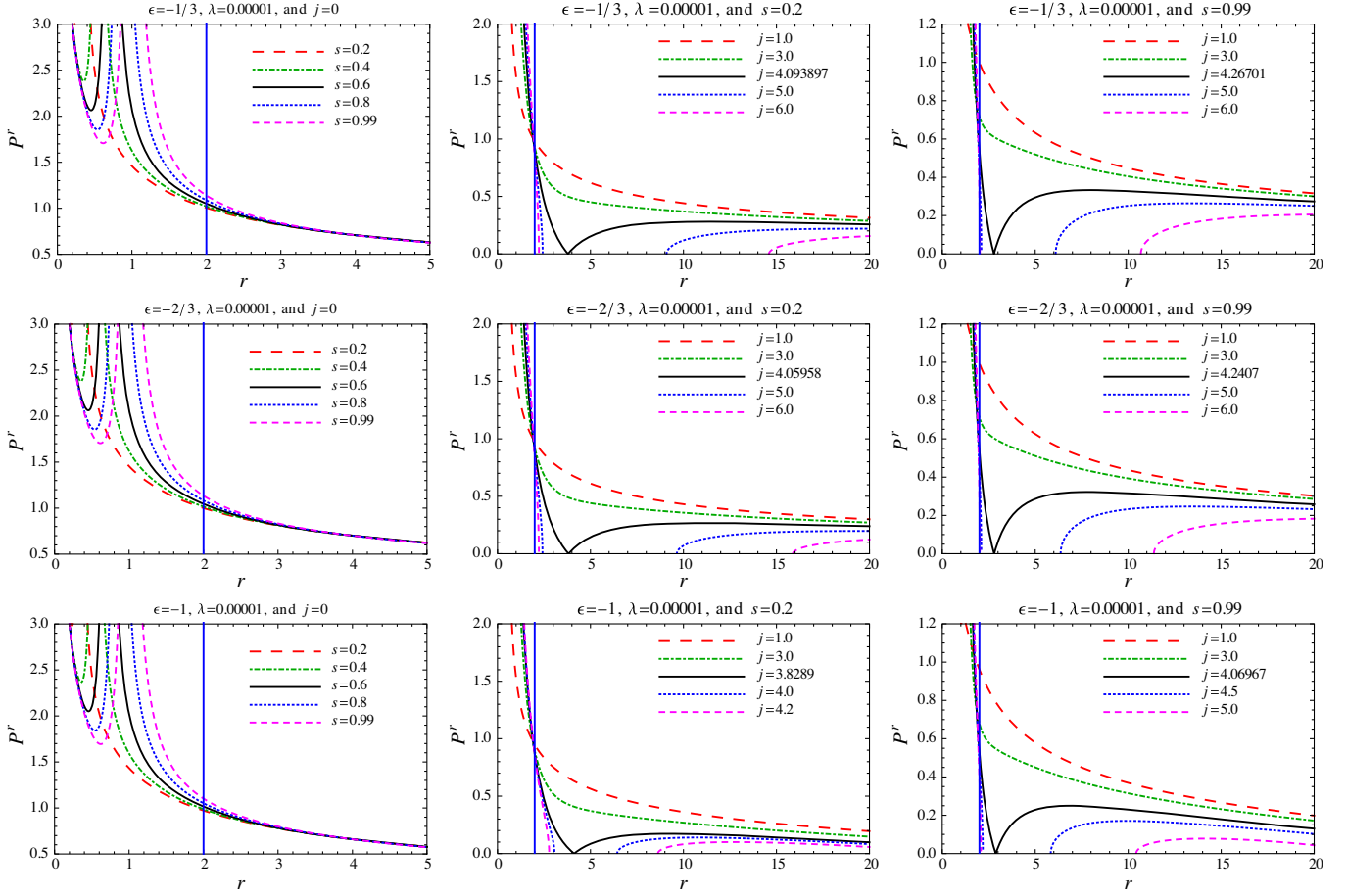


FIG. 3. The variation of P^r with r for a Schwarzschild BH surrounded by quintessential matter. Left column: shows different combinations of spin s , keeping $j = 0$. Middle column: different combinations of j , keeping $s = 0.2$. Right column: different combinations of j , keeping $s = 0.99$. In each of the rows the EOS parameter is fixed to $\epsilon = -1/3, -2/3$ and -1 , respectively, for the corresponding value of normalization constant (i.e., $\lambda = 0.00001$). Here, for the corresponding value of parameter ϵ ($-1/3, -2/3$ and -1), the value of particle energy per unit mass e is 0.9999, 0.995518 and 0.967144. The vertical (blue) solid line represents the location of event horizon ($M = 1$).

Fig. 3 (see first column) that all the spinning particles fall into the SBHQ if they obey $s \ll M$, as implied by the Møller limit, for $j = 0$. In the second and third columns the value of s is fixed to 0.2 and 0.99, respectively. It is found from the second and third columns that for each ϵ value the range of this total angular momentum j increases with increase in s . However, the radial distance for which $P^r = 0$ decreases as s increases.

B. Classification of the spinning particles and their trajectories

Let us return now to particles that might produce arbitrarily high center of mass energies. According to Eq. (37) this may happen for collisions near the horizon if \mathcal{K} of at least one of the colliding particles becomes very small.

From now onward we denote the event horizon $r_{0(1)} \equiv r_0$ until and otherwise stated. We start by classifying the spinning particles into three different classes: We call a

particle *critical* if $\mathcal{K}|_{r=r_0} = 0$, *near-critical* if $\mathcal{K}|_{r=r_0} = \mathcal{O}(\sqrt{r_c - r_0})$ with the point of collision r_c , and all other particles *usual*.

Let us start with critical particles. The condition $\mathcal{K}|_{r=r_0} = 0$ implies

$$e = \frac{jsf(r_0)'}{2r_0}. \quad (40)$$

Then, the expression for \mathcal{K} near the event horizon (in the first approximation) reads

$$\mathcal{K} \approx \frac{3js}{r_0} \left[\frac{2Mr_0^{3\epsilon} + (\epsilon + 1)(3\epsilon + 1)\lambda}{2r_0^{3\epsilon+3} - s^2(2Mr_0^{3\epsilon} + (3\epsilon + 1)\lambda)} \right] (r - r_0). \quad (41)$$

Thus, the second term in Eq. (27) becomes larger than \mathcal{K}^2 close to the horizon, where the collision should take place. Hence $(P^r)^2$ is negative there which in turn means

that the spinning particle cannot reach the event horizon and meets the turning point first.

For a near critical particle, to have $\mathcal{K}|_{r=r_0} = \mathcal{O}(\sqrt{r_c - r_0})$, we may for instance choose the energy as

$$e = \frac{jsf'(r_0) + a\sqrt{r_c - r_0}(2r_0 - s^2f'(r_0))}{2r_0}, \quad (42)$$

where a is some positive constant. At the point of collision r_c we then find

$$\begin{aligned} \frac{1}{2}E_{\text{CM}} &= 1 - 4\mathcal{L}_1\mathcal{L}_2 + \frac{\mathcal{K}_1 a_2}{\tilde{f}\sqrt{r_c - r_0}} + \mathcal{O}(\sqrt{r_c - r_0}) \\ &\quad - \frac{\sqrt{[\mathcal{K}_1^2 - (r_c - r_0)\tilde{f}(1 + 4\mathcal{L}_1^2)][a_2^2 + \mathcal{O}(\sqrt{r_c - r_0}) - \tilde{f}(1 + 4\mathcal{L}_2^2)]\sqrt{r_c - r_0}}}{\tilde{f}(r_c - r_0)} \end{aligned} \quad (45)$$

$$= 1 - 4\mathcal{L}_1\mathcal{L}_2 + \frac{\mathcal{K}_1 \left[a_2 - \sqrt{a_2^2 - \tilde{f}(1 + 4\mathcal{L}_2^2)} \right]}{\tilde{f}\sqrt{r_c - r_0}} + \mathcal{O}(1). \quad (46)$$

Here, it is important to note that in the limit $\lambda \rightarrow 0$, the Eq. (46) converges to the result found in [109]. We see that this expression is only valid if $a_2^2 - \tilde{f}(1 + 4\mathcal{L}_2^2) > 0$. In the limit $s = 0$ this condition can be fulfilled, and by continuity it should also hold for small s . If the point of collision r_c now approaches the horizon r_0 the center of mass energy (46) can grow without bound.

If both particles are near-critical, we can calculate the center of mass energy analogously,

$$\begin{aligned} \frac{1}{2}E_{\text{CM}} &= 1 - 4\mathcal{L}_1\mathcal{L}_2 + \frac{a_1 a_2}{\tilde{f}} \\ &\quad - \frac{\sqrt{[a_1^2 - \tilde{f}(1 + 4\mathcal{L}_1^2)][a_2^2 - \tilde{f}(1 + 4\mathcal{L}_2^2)]}}{\tilde{f}} \\ &\quad + \mathcal{O}(\sqrt{r_c - r_0}), \end{aligned} \quad (47)$$

which will remain finite for $r_c \rightarrow r_0$. Finally, if both particles are usual, we can directly see that the diverging parts will cancel and the center of mass energy remains finite, too.

VI. AVOIDANCE OF SUPERLUMINAL REGION

It is shown in [103,115,121–123] that the four-momentum satisfies the relation $P^\alpha P_\alpha = -1$ and hence is a conserved quantity, in contrast to the four-velocity u^α ,

$$\begin{aligned} \mathcal{K}|_{r=r_c} &= \mathcal{K}|_{r=r_0} + \mathcal{K}'|_{r=r_0}(r_c - r_0) + \dots \\ &= a\sqrt{r_c - r_0} + \mathcal{O}(r_c - r_0). \end{aligned} \quad (43)$$

Now consider the case that one particle, say particle 1, is usual and the other particle is near-critical. To calculate the center of mass energy for this case we write $f = (r - r_0)\tilde{f}$ and derive from (33),

$$\frac{1}{2}E_{\text{CM}} = 1 - 4\mathcal{L}_1\mathcal{L}_2 + \frac{1}{f} \left(\mathcal{K}_1\mathcal{K}_2 - \sqrt{R_1 R_2} \right) \quad (44)$$

where $R = \mathcal{K}^2 - f(1 + 4\mathcal{L}^2)$. If we evaluate all quantities at $r = r_c$ we find

which is not a conserved quantity for the spinning test particles moving in the curved background. Therefore, the P^α vector remains timelike throughout the motion of spinning particle around the BH, whereas the u^α vector might change from the subluminal (timelike) to superluminal (spacelike) region depending upon the invariant relation $u^\alpha u_\alpha < 0$ or $u^\alpha u_\alpha > 0$, respectively. As the four-velocity u^α of two colliding spinning test particles will not always lie in the subluminal region, it becomes important to examine closely the behavior of the square of the four-velocity in the region where E_{CM} diverges. The square of the four velocity thus reads as

$$U^2 = \frac{u^\alpha u_\alpha}{(u^t)^2} = g_{tt} + g_{rr} \left(\frac{u^r}{u^t} \right)^2 + g_{\phi\phi} \left(\frac{u^\phi}{u^t} \right)^2. \quad (48)$$

Using Eqs. (25), (26), (27), (29), and (30) in Eq. (48) leads to:

$$U^2 = -f(r)^2 \left(\frac{2r - s^2 f(r)'}{2er - jsf(r)'} \right)^2 (1 - \Sigma), \quad (49)$$

$$\Sigma = \frac{(2(j - es)s^2(\eta_-)(4r - s^2(\eta_+)))}{(2r - s^2 f(r)')^4}, \quad (50)$$

where $\eta_\pm = f(r)' \pm rf(r)''$.

The E_{CM} calculated in Eq. (34) diverges when either $C_1 = 0$ or $C_2 = 0$ as mentioned in Sec. IV. We already showed there that the point where $C_i = 0$ always lies behind the horizon and, therefore, is not of importance for our analysis. We note here that, in addition, the condition $C_i = 0$ leads to a transition of U^2 [i.e., Eq. (49)] of the colliding spinning particle from the subluminal region (physical) to the superluminal region (unphysical) as seen in the Fig 4. We have also concluded earlier from Eq. (37) that the center of mass energy remains finite when the collision takes place at the event horizon. Hence, in this work we are more interested in finding location outside the event horizon where the square of the four-velocity lies in subluminal region. This leads to the condition $\Sigma < 1$ according to Eq. (49).

We may rewrite the expression for Σ in Eq. (50) as

$$\Sigma = 4\mathcal{L}^2(G - 1), \quad (51)$$

$$G := \left(\frac{2r - rs^2 f(r)''}{2r - s^2 f(r)'} \right)^2. \quad (52)$$

We first notice that both G and \mathcal{L}^2 are monotonically decreasing functions, and that $\Sigma > 0$, for $\epsilon = -\frac{1}{3}$ and $\epsilon = -1$. For $\epsilon = -\frac{2}{3}$ this only holds in the vicinity of the horizon. We therefore find that $\Sigma < 1$ holds in the vicinity of the horizon r_0 if

$$\Sigma_{r_0} < 1. \quad (53)$$

In order to have an arbitrarily high collisional E_{CM} outside the event horizon, one of the colliding particles must be the usual particle (i.e., a particle for which $\mathcal{K}_{r_0} \neq 0$) and the other must be a near-critical one as shown in the previous section.

For the usual particle with $s = 0$, the condition (53) is satisfied automatically. If $s \neq 0$ we can always choose $(j - es)^2$ such that (53) holds, for instance, one could choose $j = es$. For near-critical particles, we fixed an energy e in (42). To achieve the inequality (53), we could then for instance choose $j = es$ again. For near-critical particles we can also explicitly solve $\Sigma = 1$ for j , using the energy e from Eq. (42). We find

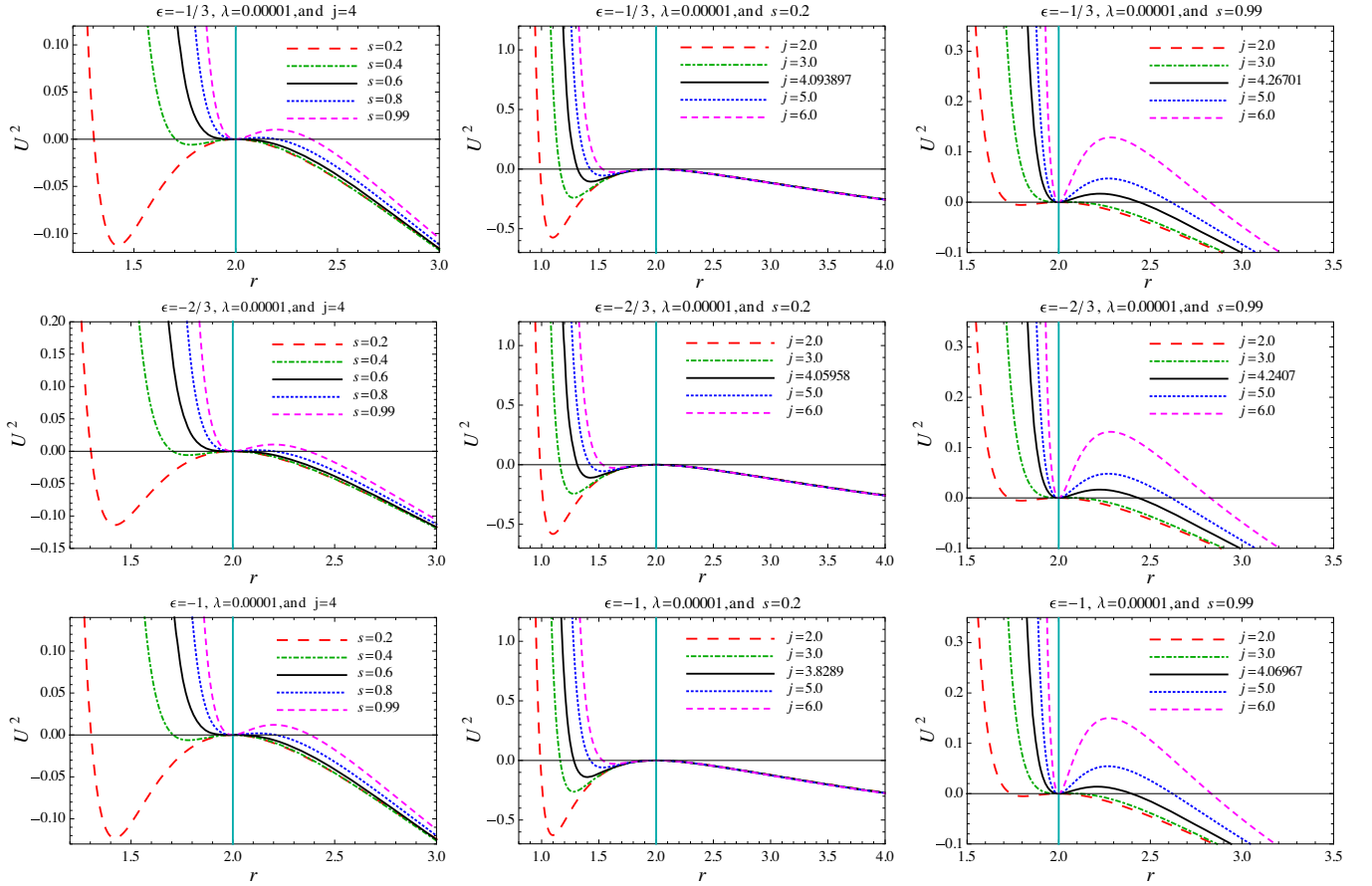


FIG. 4. Variation of U^2 as a function of r with different values of ϵ , s and j . The vertical (darker cyan) solid line is the horizon r_0 ($M = 1$). Here, for the corresponding value of parameter ϵ ($-1/3$, $-2/3$ and -1), the value of particle energy per unit mass e is 0.9999, 0.995518 and 0.967144.

$$j = as\sqrt{r_c - r_0} \pm \frac{r_0}{s} N_\epsilon \quad (54)$$

with

$$N_{-1} = \frac{Ms^2 - r_0^3(\lambda s^2 - 1)}{\sqrt{3M(Ms^2 + 2r_0^3(\lambda s^2 + 1))}}, \quad (55)$$

$$N_{-1/3} = \frac{Ms^2 - r_0^3}{\sqrt{3M(Ms^2 + 2r_0^3)}}, \quad (56)$$

$$N_{-2/3} = \frac{2Ms^2 - r_0^2(\lambda s^2 - 2r_0)}{\sqrt{(6M - \lambda r_0^2)(2Ms^2 + r_0^2(\lambda s^2 + 4r_0))}}. \quad (57)$$

Hence, we may conclude that the collision of a near-critical particle with a usual particle can produce arbitrarily high center-of-mass energy E_{CM} if we fine tune the parameters. For instance, we could choose a usual particle starting from rest from infinity or the static radius, respectively, with vanishing total angular momentum $j = \epsilon s$, and a near-critical particle with energy as given in (42) and vanishing total angular momentum starting from a radius close to the event horizon (but outside of it).

VII. SUMMARY, CONCLUSION AND FUTURE PROSPECTS

We discussed the collision of spinning particles close to the event horizon of a Schwarzschild black hole surrounded by quintessential matter. In particular, we found that the center of mass energy may grow without bound under certain conditions.

After reviewing the equations of motion for spinning particles (under the Tulczyjew spin supplementary condition), we started with an analysis of the horizon structure of the spacetime under discussion. For the equation of state parameter we assumed in this work three different values, $\epsilon = -1/3$, $\epsilon = -2/3$ and $\epsilon = -1$. In addition to the event horizon, for $\epsilon = -2/3$ and $\epsilon = -1/3$ also a cosmological horizon may be present. In the following, we restricted to values of the normalization constant λ which allow for a black hole solution.

We then focused on the center of mass energy of two spinning particles colliding in the vicinity of the event horizon. For generic particles, we showed that the center of mass energy remains finite in the limit that the collision takes place at the horizon. Moreover, potential additional points of divergence given by $C_{1,2} = 0$, see Eq. (34), are shown to always being located behind the event horizon, if we respect the Møller limit on the spin of the particles. Therefore, these points are not of further interest.

In order to determine if fine-tuned particles, that might produce unbound center of mass energies, can reach the

vicinity of the event horizon, we proceeded with a discussion of the effective potential and radial turning points. The behavior of P^r as a function of r , plotted in Fig. 3, is used to analyze and distinguish between different trajectories of the spinning particles. From this figure, it seems that the maximum allowed range of total angular momentum j , for which a spinning particle reaches the event horizon of the SBHQ, increases with increase in the value of particle's spin s . However, it is also observed that the radius at which P^r becomes zero, decreases with increase in s .

We then identify three different types of spinning particle trajectories dependent on their behavior close to the event horizon: usual, critical, and near-critical particles. From studying collisions between all different combinations of these trajectory types, we concluded that only the collision between a near-critical and a usual particle may produce arbitrarily high center of mass energies.

As for spinning particles with Tulczyjew spin supplementary condition $u_\alpha u^\alpha$ is not conserved, where u^α is the four velocity, the motion may change from subluminal to superluminal. For the combination of interest, namely usual and near-critical particle, we showed that we can always choose the angular momentum of the particles such that the motion is subluminal in the vicinity of the horizon.

Collisions of spinning particles around the Schwarzschild black hole (without surrounding quintessential matter) was studied in [103]. In that work, mainly the points where $C_{1,2} = 0$, see Eq. (34), were analyzed. However, these points are always behind the horizon if we take the Møller limit on the spin into account. The divergence of the center of mass energy discussed in the present work is of a different type. A similar setup for geodesics around the Schwarzschild black hole was shortly discussed in [44] and around general static and spherically symmetric spacetimes, which includes the present case of SBHQ, was studied recently by some of us in [124]. The physical relevance of these setups, meaning the question if the near-critical particle can be created by a foregoing collision of particles, was studied for the Schwarzschild and extremal Reissner-Nordström black holes in [125]. It is an open question if the near-critical particle becomes physically more relevant if we add spin to the particles or quintessential matter to the spacetime. Another obvious direction for further research is to include the rotation of the black hole.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referee for the constructive comments and suggestions which helped us to improve the presentation of this paper. H. N. is thankful to Prof. Philippe Jetzer for invaluable insights and suggestions during the early stage of this work. P. S. would like to thank *Programa de Desarrollo Profesional Docente* (PRODEP) of the *Secretaría de*

Educación Pública (SEP) of the Mexican government, for providing the financial support. H. N. would like to thank Science and Engineering Research Board (SERB), New Delhi, India for financial support through Grant No. EMR/2017/000339. He also thankful to Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India (where a part of the work was completed) for support in form of academic visits under its Associateship programme. E. H. is grateful for support from the research training group RTG 1620 “Models of Gravity” and the center of excellence EXC 2123 “QuantumFrontiers,” both funded by the German Research Foundation (DFG). U. N. acknowledges support from PRODEP-SEP, SNI-CONACYT and CIC-UMSNH. A. A. acknowledges that this work is based on the research supported in part by the National Research Foundation (NRF) of South Africa (Grants No. 109257 and No. 112131). He also acknowledges the hospitality of the High Energy and Astroparticle Physics Group of the Department of Physics of Sultan Qaboos University, where part of this work was completed.

APPENDIX

For two spinning particles with mass m_1 and m_2 , the center of mass energy of the collision of both can be written as

$$E_{\text{CM}}^2 = m_1^2 + m_2^2 + 2 \left(f(r) P_1^r P_2^r - \frac{1}{f(r)} P_1^\phi P_2^\phi - r^2 P_1^\theta P_2^\theta \right), \quad (\text{A1})$$

which, after substituting the values of P^t , P^r and P^ϕ from Eqs. (25), (26), and (27) for every particle respectively, reduces to

$$E_{\text{CM}}^2 = m_1^2 + m_2^2 + 2m_1 m_2 R(r; e, j, s), \quad (\text{A2})$$

where the function $R(r; e, j, s)$ is given as

$$\begin{aligned} R(r; e, j, s) \equiv & f(r) \left[\frac{r^{3e+1}}{\Delta_0} \right]^2 \mathcal{K}_1 \mathcal{K}_2 - 4\mathcal{L}_1 \mathcal{L}_2 \\ & - \frac{1}{f(r)} \sqrt{\mathcal{K}_1^2 - f(r)(1 + 4\mathcal{L}_1^2)} \\ & \times \sqrt{\mathcal{K}_2^2 - f(r)(1 + 4\mathcal{L}_2^2)}. \end{aligned} \quad (\text{A3})$$

In order to obtain the maximum of the E_{CM} as given by Eq. (A2), we propose the following:

Statement: Assuming the constraint $\mu \equiv m_1 + m_2 =$ constant with the fixed parameters (e, j, s) , the maximum of the center of mass energy given by (A2) can be obtained when both masses are equal $m_1 = m_2$ and $R(r; e, j, s) > 1$.

Proof: Once the parameter m_2 is replaced in terms of m_1 , one can obtain the first and second derivatives of (A2) with respect to the parameter m_1 . In order to find the critical points and to obtain the maximum, one need to put them zero and lesser than zero respectively as below,

$$\frac{dE_{\text{CM}}^2}{dm_1} = 2(m_1 - m_2)[1 - R(r; e, j, s)] = 0, \quad (\text{A4})$$

$$\frac{d^2 E_{\text{CM}}^2}{dm_1^2} = 4[1 - R(r; e, j, s)] < 0, \quad (\text{A5})$$

from the above Eqs. (A4) and (A5), it is clear that the maximum of E_{CM} is achievable while both masses are equal and $R(r; e, j, s) > 1$ (also see Figs. 5 and 6 for reference). The case for spinless particles can be obtained setting the spin parameter per unit mass $s = 0$ in the function $R(r; e, j, s)$. Further $R(r; e, j, s) > 1$ also holds which is evident from Figs. 5 and 6.

One could also conclude the same result when introduce $\mu = m_2/m_1 \leq 1$ and then

$$\frac{E_{\text{cm}}^2}{m_1 m_2} = \mu + \frac{1}{\mu} - 2g_{ab} v_1^a v_2^b, \quad (\text{A6})$$

where $v_i^a = P_i^a/m_i$ is independent of the mass. The left-hand side of the above equation has a maximum at $\mu = 1$, which in turn clearly implies the minimum of E_{cm} at $\mu = 1$.

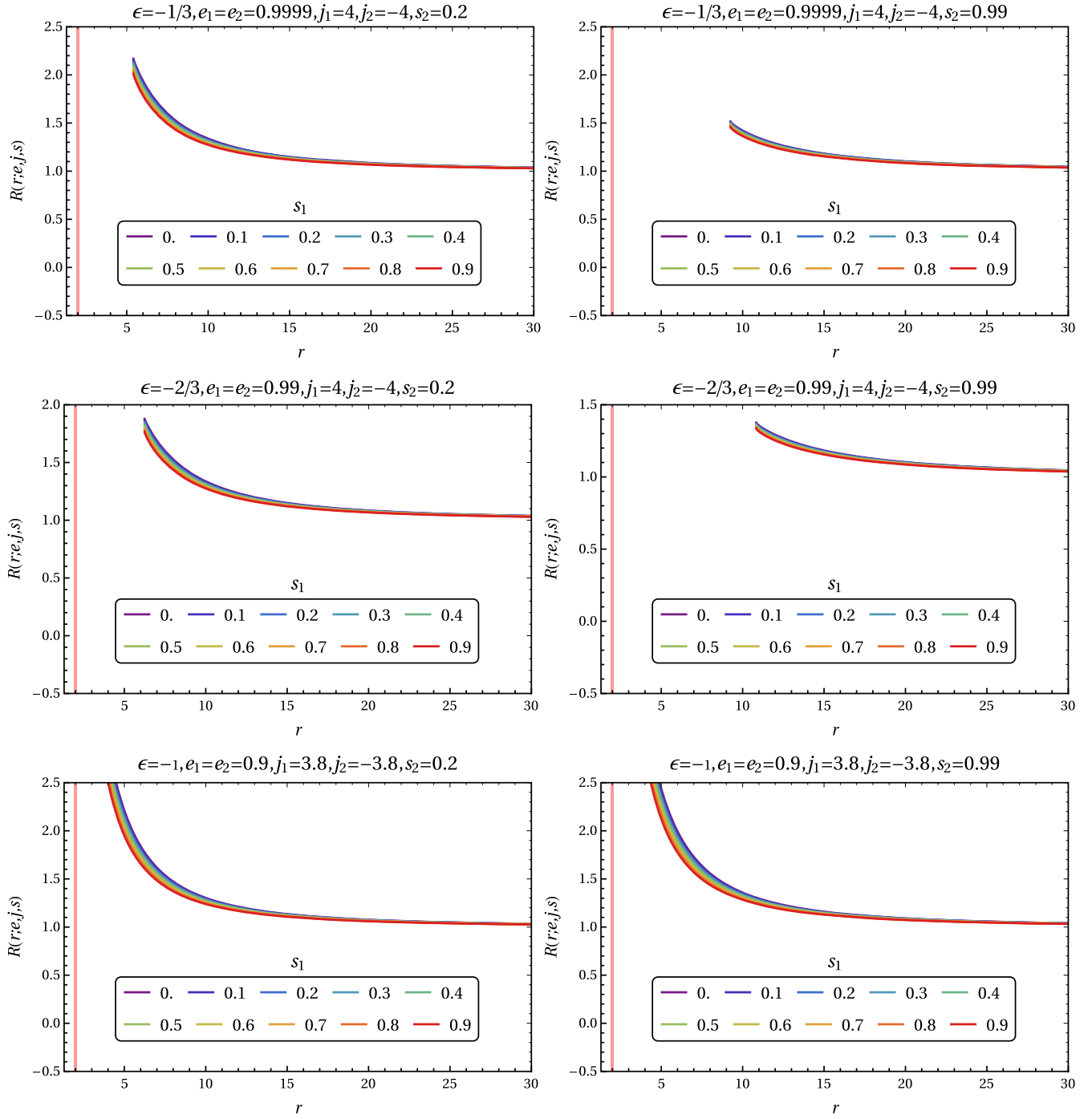


FIG. 5. Variation of $R(r; e, j, s)$ as a function of r with different values of ϵ, s and j . Here, the vertical (red) solid line is the horizon r_0 ($M = 1$) and $\lambda = 0.00001$.

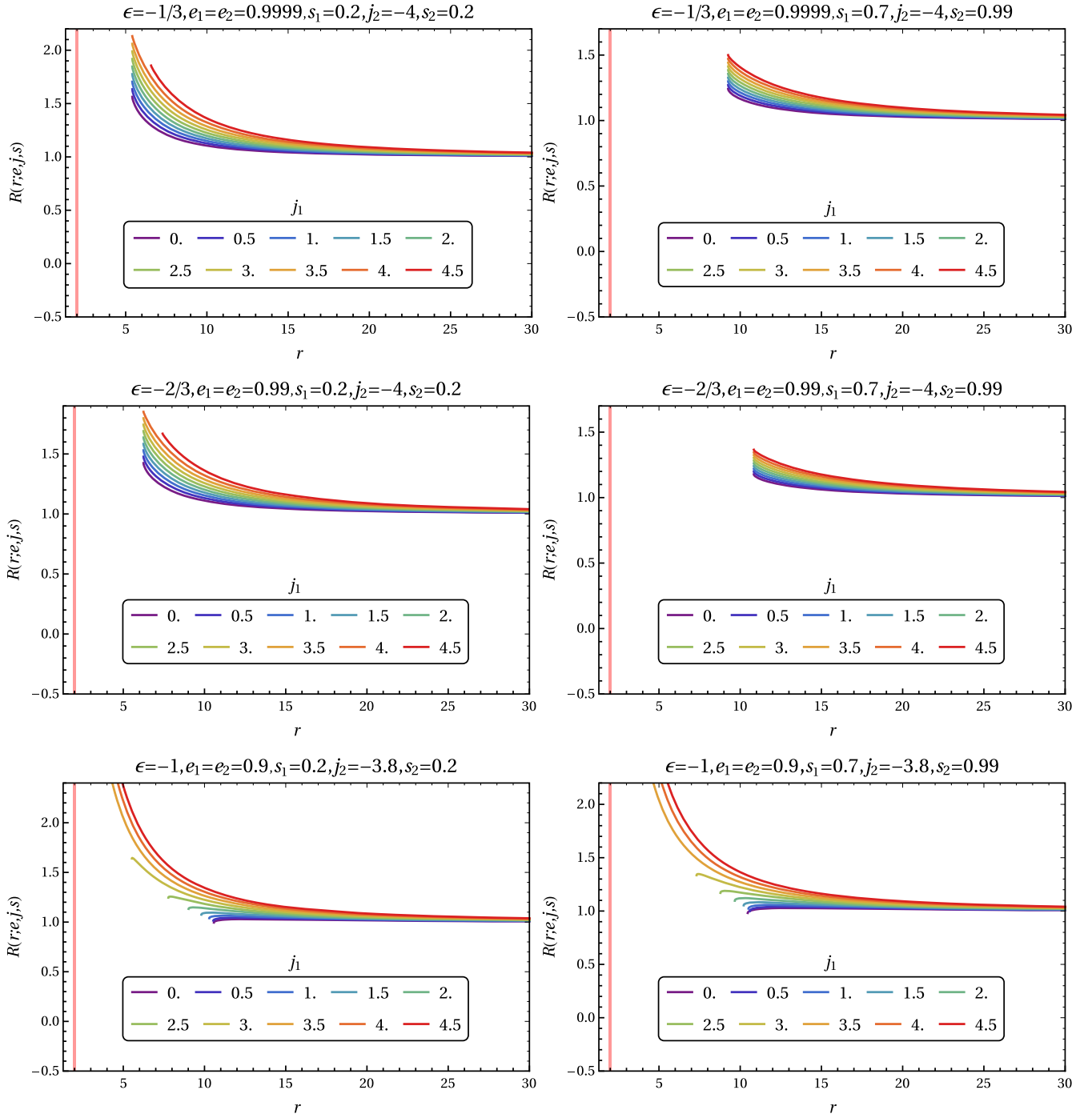


FIG. 6. Variation of $R(r; e, j, s)$ as a function of r with different values of ϵ , s and j . Here, the vertical (red) solid line is the horizon r_0 ($M = 1$) and $\lambda = 0.00001$.

[1] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189 (1916).
 [2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. **116**, 061102 (2016).

[3] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), *Astrophys. J.* **875**, L1 (2019).
 [4] J. M. Maldacena, Ph.D. thesis, Princeton University, 1996, <http://wwwlib.umi.com/dissertations/fullcit?p9627605>.

- [5] T. Padmanabhan, *Phys. Rep.* **380**, 235 (2003).
- [6] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
- [7] S. M. Carroll, *Phys. Rev. Lett.* **81**, 3067 (1998).
- [8] J. Khoury and A. Weltman, *Phys. Rev. Lett.* **93**, 171104 (2004).
- [9] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, *Phys. Rev. Lett.* **85**, 4438 (2000).
- [10] T. Padmanabhan, *Phys. Rev. D* **66**, 021301 (2002).
- [11] R. R. Caldwell, *Phys. Lett. B* **545**, 23 (2002).
- [12] M. Gasperini, F. Piazza, and G. Veneziano, *Phys. Rev. D* **65**, 023508 (2001).
- [13] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
- [14] V. V. Kiselev, *Classical Quantum Gravity* **20**, 1187 (2003).
- [15] R. Uniyal, N. Chandrachani Devi, H. Nandan, and K. D. Purohit, *Gen. Relativ. Gravit.* **47**, 16 (2015).
- [16] M. Banados, J. Silk, and S. M. West, *Phys. Rev. Lett.* **103**, 111102 (2009).
- [17] T. Jacobson and T. P. Sotiriou, *Phys. Rev. Lett.* **104**, 021101 (2010).
- [18] A. A. Grib and Yu. V. Pavlov, *Astropart. Phys.* **34**, 581 (2011).
- [19] K. Lake, *Phys. Rev. Lett.* **104**, 211102 (2010); **104**, 259903(E) (2010).
- [20] S.-W. Wei, Y.-X. Liu, H. Guo, and C.-E. Fu, *Phys. Rev. D* **82**, 103005 (2010).
- [21] A. A. Grib and Y. V. Pavlov, arXiv:1007.3222.
- [22] O. B. Zaslavskii, *Pis'ma Zh. Eksp. Teor. Fiz.* **92**, 635 (2010) [*JETP Lett.* **92**, 571 (2010)].
- [23] T. Harada and M. Kimura, *Phys. Rev. D* **83**, 024002 (2011).
- [24] A. A. Grib and Yu. V. Pavlov, *Gravitation Cosmol.* **17**, 42 (2011).
- [25] M. Banados, B. Hassanain, J. Silk, and S. M. West, *Phys. Rev. D* **83**, 023004 (2011).
- [26] A. J. Williams, *Phys. Rev. D* **83**, 123004 (2011).
- [27] T. Harada and M. Kimura, *Phys. Rev. D* **83**, 084041 (2011).
- [28] M. Patil and P. S. Joshi, *Phys. Rev. D* **84**, 104001 (2011).
- [29] Y. Zhu, S.-F. Wu, Y.-X. Liu, and Y. Jiang, *Phys. Rev. D* **84**, 043006 (2011).
- [30] C. Liu, S. Chen, and J. Jing, *Chin. Phys. Lett.* **30**, 100401 (2013).
- [31] O. B. Zaslavskii, *Phys. Rev. D* **84**, 024007 (2011).
- [32] O. B. Zaslavskii, arXiv:1105.0303.
- [33] A. A. Grib, Yu. V. Pavlov, and O. F. Piattella, *Int. J. Mod. Phys. A* **26**, 3856 (2011); **18**, 70 (2012).
- [34] W.-P. Yao, S. Chen, C. Liu, and J. Jing, *Eur. Phys. J. C* **72**, 1898 (2012).
- [35] S. Gao and C. Zhong, *Phys. Rev. D* **84**, 044006 (2011).
- [36] O. B. Zaslavsky, *Gravitation Cosmol.* **18**, 139 (2012).
- [37] M. Patil, P. S. Joshi, M. Kimura, and K.-i. Nakao, *Phys. Rev. D* **86**, 084023 (2012).
- [38] Y. Zhu, S.-F. Wu, Y. Jiang, and G.-H. Yang, *Phys. Rev. D* **84**, 123002 (2011).
- [39] T. Harada and M. Kimura, *Phys. Rev. D* **84**, 124032 (2011).
- [40] O. B. Zaslavsky, *Int. J. Mod. Phys. A* **26**, 3845 (2011).
- [41] O. B. Zaslavskii, *Phys. Rev. D* **85**, 024029 (2012).
- [42] V. P. Frolov, *Phys. Rev. D* **85**, 024020 (2012).
- [43] O. B. Zaslavskii, *Classical Quantum Gravity* **29**, 205004 (2012).
- [44] A. A. Grib, Yu. V. Pavlov, and O. F. Piattella, *Gravitation Cosmol.* **18**, 70 (2012).
- [45] I. Hussain, *Mod. Phys. Lett. A* **27**, 1250017 (2012).
- [46] T. Harada, H. Nemoto, and U. Miyamoto, *Phys. Rev. D* **86**, 024027 (2012); **86**, 069902(E) (2012).
- [47] I. V. Tanatarov and O. B. Zaslavskii, *Phys. Rev. D* **86**, 044019 (2012).
- [48] H. Nemoto, U. Miyamoto, T. Harada, and T. Kokubu, *Phys. Rev. D* **87**, 127502 (2013).
- [49] A. A. Grib and Yu. V. Pavlov, *Europhys. Lett.* **101**, 20004 (2013).
- [50] A. Galajinsky, *Phys. Rev. D* **88**, 027505 (2013).
- [51] O. B. Zaslavsky, *Mod. Phys. Lett. A* **28**, 1350037 (2013).
- [52] Z. Stuchlik and J. Schee, *Classical Quantum Gravity* **30**, 075012 (2013).
- [53] A. Abdujabbarov, N. Dadhich, B. Ahmedov, and H. Eshkuvatov, *Phys. Rev. D* **88**, 084036 (2013).
- [54] N. Tsukamoto, M. Kimura, and T. Harada, *Phys. Rev. D* **89**, 024020 (2014).
- [55] O. B. Zaslavskii, *Mod. Phys. Lett. A* **29**, 1450112 (2014).
- [56] H. Yumisaki, *Prog. Theor. Exp. Phys.* (2017), 063B04.
- [57] J. Sadeghi, B. Pourhassan, and H. Farahani, *Commun. Theor. Phys.* **62**, 358 (2014).
- [58] P. Pradhan, arXiv:1402.2748.
- [59] P. Pradhan, *Astropart. Phys.* **62**, 217 (2015).
- [60] S. G. Ghosh, P. Sheoran, and M. Amir, *Phys. Rev. D* **90**, 103006 (2014).
- [61] I. Hussain, M. Jamil, and B. Majeed, *Int. J. Theor. Phys.* **54**, 1567 (2015).
- [62] A. Zakria and M. Jamil, *J. High Energy Phys.* 05 (2015) 147.
- [63] M. Amir and S. G. Ghosh, *J. High Energy Phys.* 07 (2015) 015.
- [64] M. Halilsoy and A. Ovgun, *Adv. High Energy Phys.* **2017**, 1 (2017).
- [65] B. Pourhassan and U. Debnath, *Gravitation Cosmol.* **25**, 196 (2019).
- [66] S. G. Ghosh and M. Amir, *Eur. Phys. J. C* **75**, 553 (2015).
- [67] H. Saadat, *Can. J. Phys.* **92**, 1562 (2014).
- [68] A. Abdujabbarov, B. Ahmedov, and B. Ahmedov, *Phys. Rev. D* **84**, 044044 (2011).
- [69] J. Sadeghi and B. Pourhassan, *Eur. Phys. J. C* **72**, 1984 (2012).
- [70] S.-W. Wei, Y.-X. Liu, H.-T. Li, and F.-W. Chen, *J. High Energy Phys.* 12 (2010) 066.
- [71] P.-J. Mao, R. Li, L.-Y. Jia, and J.-R. Ren, *Chin. Phys. C* **41**, 065101 (2017).
- [72] Y. Li, J. Yang, Y.-L. Li, S.-W. Wei, and Y.-X. Liu, *Classical Quantum Gravity* **28**, 225006 (2011).
- [73] C. Liu, S. Chen, C. Ding, and J. Jing, *Phys. Lett. B* **701**, 285 (2011).
- [74] M. Halilsoy and A. Ovgun, *Can. J. Phys.* **95**, 1037 (2017).
- [75] A. Abdujabbarov, F. Atamurotov, N. Dadhich, B. Ahmedov, and Z. Stuchlík, *Eur. Phys. J. C* **75**, 399 (2015).
- [76] U. Debnath, *Mod. Phys. Lett. A* **35**, 2050033 (2020).
- [77] B. Toshmatov, A. Abdujabbarov, B. Ahmedov, and Z. Stuchlík, *Astrophys. Space Sci.* **360**, 19 (2015).
- [78] J. Sultana and B. Bose, *Phys. Rev. D* **92**, 104022 (2015).

- [79] O. B. Zaslavskii, *Int. J. Mod. Phys. D* **26**, 1750108 (2017).
- [80] M. Amir, F. Ahmed, and S. G. Ghosh, *Eur. Phys. J. C* **76**, 532 (2016).
- [81] T. Oteev, A. Abdujabbarov, Z. Stuchlík, and B. Ahmedov, *Astrophys. Space Sci.* **361**, 269 (2016).
- [82] A. Jawad, F. Ali, M. Jamil, and U. Debnath, *Commun. Theor. Phys.* **66**, 509 (2016).
- [83] S. Fernando, *Mod. Phys. Lett. A* **32**, 1750074 (2017).
- [84] S. Fernando, *Mod. Phys. Lett. A* **32**, 1750088 (2017).
- [85] B. Majeed and M. Jamil, *Int. J. Mod. Phys. D* **26**, 1741017 (2017).
- [86] M. Sharif and M. Shahzadi, *Eur. Phys. J. C* **77**, 363 (2017).
- [87] N. Tsukamoto, K. Ogasawara, and Y. Gong, *Phys. Rev. D* **96**, 024042 (2017).
- [88] J. An and S. Gao, [arXiv:1708.09576](https://arxiv.org/abs/1708.09576).
- [89] J. An, J. Peng, Y. Liu, and X.-H. Feng, *Phys. Rev. D* **97**, 024003 (2018).
- [90] R. Bécar, P. A. González, and Y. Vásquez, *Eur. Phys. J. C* **78**, 335 (2018).
- [91] P. A. González, M. Olivares, E. Papantonopoulos, and Y. Vásquez, *Phys. Rev. D* **97**, 064034 (2018).
- [92] F. Ahmed, M. Amir, and S. G. Ghosh, *Astrophys. Space Sci.* **364**, 10 (2019).
- [93] S. Shaymatov, B. Ahmedov, Z. Stuchlík, and A. Abdujabbarov, *Int. J. Mod. Phys. D* **27**, 1850088 (2018).
- [94] K. Ogasawara and N. Tsukamoto, *Phys. Rev. D* **99**, 024016 (2019).
- [95] P. A. González, M. Olivares, Y. Vásquez, J. Saavedra, and A. Övgün, *Eur. Phys. J. C* **79**, 528 (2019).
- [96] P. Saha and U. Debnath, *Mod. Phys. Lett. A* **34**, 1950127 (2019).
- [97] A. Rudra, H. Nandan, R. Gannouji, S. Chakraborty, and A. K. Chatterjee, [arXiv:1906.03566](https://arxiv.org/abs/1906.03566).
- [98] R. Rahim and K. Saifullah, [arXiv:1906.05632](https://arxiv.org/abs/1906.05632).
- [99] T. Harada and M. Kimura, *Classical Quantum Gravity* **31**, 243001 (2014).
- [100] M. Mathisson, *Acta Phys. Pol.* **6**, 163 (1937).
- [101] A. Papapetrou, *Proc. R. Soc. A* **209**, 248 (1951).
- [102] W. G. Dixon, *Nuovo Cimento* (1955–1965) **34**, 317 (1964).
- [103] C. Armaza, M. Bañados, and B. Koch, *Classical Quantum Gravity* **33**, 105014 (2016).
- [104] Y.-P. Zhang, B.-M. Gu, S.-W. Wei, J. Yang, and Y.-X. Liu, *Phys. Rev. D* **94**, 124017 (2016).
- [105] J. Jiang and S. Gao, *Eur. Phys. J. C* **79**, 378 (2019).
- [106] R. Hojman and S. Hojman, *Phys. Rev. D* **15**, 2724 (1977).
- [107] N. Zalaquett, S. A. Hojman, and F. A. Asenjo, *Classical Quantum Gravity* **31**, 085011 (2014).
- [108] M. Saijo, K.-i. Maeda, M. Shibata, and Y. Mino, *Phys. Rev. D* **58**, 064005 (1998).
- [109] O. B. Zaslavskii, *Euophys. Lett.* **114**, 30003 (2016).
- [110] C. Møller, *Communications of the Dublin Institute for Advanced Studies, Series A* **5**, 1 (1949).
- [111] R. Wald, *Phys. Rev. D* **6**, 406 (1972).
- [112] M. Zhang, J. Jiang, Y. Liu, and W.-B. Liu, *Phys. Rev. D* **98**, 044006 (2018).
- [113] S. A. Hojman and F. A. Asenjo, *Phys. Rev. D* **93**, 028501 (2016).
- [114] S. Suzuki and K.-i. Maeda, *Phys. Rev. D* **58**, 023005 (1998).
- [115] S. A. Hojman and F. A. Asenjo, *Classical Quantum Gravity* **30**, 025008 (2013).
- [116] U. Nucamendi and D. Sudarsky, *Classical Quantum Gravity* **14**, 1309 (1997).
- [117] U. Nucamendi and D. Sudarsky, *Classical Quantum Gravity* **17**, 4051 (2000).
- [118] E. Hackmann, C. Lämmerzahl, Y. N. Obukhov, D. Puetzfeld, and I. Schaffer, *Phys. Rev. D* **90**, 064035 (2014).
- [119] Z. Stuchlík, *Bull. Astron. Inst. Czech.* **34**, 129 (1983).
- [120] B. Toshmatov, Z. Stuchlík, and B. Ahmedov, *Eur. Phys. J. Plus* **132**, 98 (2017).
- [121] S. A. Hojman (unpublished) (1975).
- [122] A. A. Deriglazov and W. G. Ramírez, *Adv. High Energy Phys.* **2016**, 1 (2016).
- [123] A. A. Deriglazov and W. G. Ramírez, *Int. J. Mod. Phys. D* **26**, 1750047 (2017).
- [124] E. Hackmann, H. Nandan, and P. Sheoran, [arXiv:2006.05045](https://arxiv.org/abs/2006.05045).
- [125] O. B. Zaslavskii, *Phys. Rev. D* **102**, 044051 (2020).