Fermi coordinates and a static observer in Schwarzschild spacetime

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In this paper we construct the Fermi coordinates along any arbitrary line in a simple analytical way without use of orthogonal frames and their transport. In this manner we extend the Eddington approach to the construction of the Fermi metric in terms of the Riemann tensor. In the second part of the present article we show how the proposed approach works practically by applying it for deriving the Fermi coordinates for the static observer in the Schwarzschild spacetime.

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I. INTRODUCTION

It is known that for any metric and any line exists a set of Fermi coordinates [1] in which all Christoffel symbols are zero at points of this line and *this is the definition of Fermi coordinates*. However, the elimination of the Christoffel symbols on a line does not fix completely the corresponding coordinate transformations which means that there is an infinity of the Fermi coordinates associated to a given line. To make a concrete choice it is reasonable to search for some additional coordinate restrictions (not violating the vanishing of Christoffel symbols on line) appropriate from a physical point of view. The natural physical support has been proposed by Arthur Eddington [2] who also developed the way for the corresponding analytical calculations. Eddington did this for the case of the Riemann coordinates in the neighborhood of a point in four-dimensional spacetime(by definition in Riemann coordinates all Christoffel symbols are zero at some point of spacetime and not along a line). His idea was to specify the coordinate transformations so as to represent the quadratic terms of the expansion of the metric near such point by the components of the Riemann tensor. It turns out that the generalization of the Eddington approach to the case of Fermi coordinates in the neighborhood of an arbitrary line is straightforward. Such extension is the target of the first part of the present paper. It should be stressed that it is done for any original metric and any given curve, no matter what its geometric character (geodesic or not, timelike, spacelike or null) and in a pure analytical way without necessity to use orthogonal frames and their Fermi-Walker transport. Such a simplified universal method has some value, because the majority of papers in the literature have been dedicated only to some specific type of the line and have been essentially based on the use of transported frames (for example, Manasse and Misner [3] did this for timelike geodesics and Blau, Frank and Weiss [4] extended their results for a null geodesic curve).

In the second part of the present article we show the proposed approach in practical action by applying it for construction of the Fermi coordinates for the static observer in the Schwarzschild spacetime. This result is new since the known analogous constructions (for example, see [5] and references therein) have been restricted to a quasi-Fermi system defined by Synge [6] when not all Christoffel symbols on the world line of interest disappear.

It is worth remarking that since the work of Synge some terminological muddle has been widely spread in the literature. Synge introduced coordinates which he named “Fermi coordinates” in spite of the fact that in general this contradicts the generally accepted understanding of what Fermi coordinates are (which disparity was noted by Synge himself in his publication). The Synge and Fermi coordinates coincide only for geodesic world lines but for nongeodesics no Fermi coordinates can be constructed by the Synge prescription. In general this prescription lead to the nonzero values of Christoffel symbols $\Gamma^0_{0\alpha}$ and Γ^α_{00} at points of a line and this is the reason to attribute to the Synge approach the aforementioned “quasi-Fermi” appellation [7].

The problem is that for a nongeodesic line the Synge coordinate system is essentially different from the Fermi coordinates and no article is known where the Fermi coordinates would be constructed along the nongeodesic line. The present paper set aside to remove this shortage of traditional activity in this field.

II. CONSTRUCTION OF FERMI COORDINATES IN GENERAL

It is known that for any metric $g_{ik}(x)$ (by symbol x we denote the set of four coordinates x^0, x^1, x^2, x^3) in four-dimensional spacetime [9] and any line

$$x^\alpha = f^\alpha(x^0) \tag{1}$$

exists a set of Fermi coordinates $\acute{x}$ (that is $\acute{x}^0, \acute{x}^1, \acute{x}^2, \acute{x}^3$) in which all Christoffel $\acute{\Gamma}$ symbols are zero at points of this line. For the corresponding coordinates transformation $\acute{x}^i = \acute{x}^i(x^0, x^1, x^2, x^3)$ we denote the Jacobi matrix by A^i_k:

$$A^i_k(x) = \frac{\partial \acute{x}^i}{\partial x^k}. \tag{2}$$

The transformation of Γ symbols can be written as

$$\Gamma^i_{kl} A^q_i = \acute{\Gamma}^q_{nm} A^n_k A^m_l + A^q_{k,l}. \tag{3}$$

From the last formula follows that $\acute{\Gamma}^q_{nm}$ in Fermi coordinates vanish on the line (1) if matrix A^i_k satisfies the differential equation:

$$[A^i_{k,l}]_{\mathcal{L}} = [\Gamma^m_{kl} A^i_m]_{\mathcal{L}}, \tag{4}$$

where $[F]_{\mathcal{L}}$ means the value of any function F on the line (1), that is

$$[F(x^0, x^1, x^2, x^3)]_{\mathcal{L}} = F[x^0, f^1(x^0), f^2(x^0), f^3(x^0)]). \tag{5}$$

It is easy to see that Eq. (4) represents the set of ordinary differential equations with respect to the variable x^0. Indeed, in the vicinity of the line (1) the transformation between Fermi and original coordinates can be represented in the form of an expansion with respect to the three small deviations $x^\alpha - f^\alpha(x^0)$ from the line,

$$\begin{aligned} \acute{x}^m = {} & X^m(x^0) + Y^m_\alpha(x^0)[x^\alpha - f^\alpha(x^0)] \\ & + Z^m_{\alpha\beta}(x^0)[x^\alpha - f^\alpha(x^0)][x^\beta - f^\beta(x^0)] + O(3), \end{aligned} \tag{6}$$

where $O(n)$ means collection of terms of the order n and higher with respect to the small functional parameters $x^\alpha - f^\alpha(x^0)$. From (6) and definition (2) follows expansion for the components of matrix A^m_k:

$$\begin{aligned} A^m_0 = {} & \frac{dX^m}{dx^0} - Y^m_\alpha \frac{df^\alpha}{dx^0} + \left(\frac{dY^m_\beta}{dx^0} - 2Z^m_{\alpha\beta}\frac{df^\alpha}{dx^0}\right)(x^\beta - f^\beta) \\ & + O(2), \end{aligned} \tag{7}$$

$$A^m_\alpha = Y^m_\alpha + 2Z^m_{\alpha\beta}(x^\beta - f^\beta) + O(2). \tag{8}$$

Consequently on the line the components A^m_k are

$$[A^m_0]_{\mathcal{L}} = \frac{dX^m}{dx^0} - Y^m_\alpha \frac{df^\alpha}{dx^0}, \tag{9}$$

$$[A^m_\beta]_{\mathcal{L}} = Y^m_\beta. \tag{10}$$

From (7) and (8) follows values of the partial derivatives $A^m_{k,l}$ of matrix A^m_k on the line:

$$[A^m_{0,0}]_{\mathcal{L}} = \frac{d}{dx^0}\left(\frac{dX^m}{dx^0} - Y^m_\alpha \frac{df^\alpha}{dx^0}\right) - \left(\frac{dY^m_\beta}{dx^0} - 2Z^m_{\alpha\beta}\frac{df^\alpha}{dx^0}\right)\frac{df^\beta}{dx^0}, \tag{11}$$

$$[A^m_{0,\beta}]_{\mathcal{L}} = \frac{dY^m_\beta}{dx^0} - 2Z^m_{\alpha\beta}\frac{df^\alpha}{dx^0}, \tag{12}$$

$$[A^m_{\beta,0}]_{\mathcal{L}} = \frac{dY^m_\beta}{dx^0} - 2Z^m_{\alpha\beta}\frac{df^\alpha}{dx^0}, \tag{13}$$

$$[A^m_{\alpha,\beta}]_{\mathcal{L}} = 2Z^m_{\alpha\beta}. \tag{14}$$

It is convenient to use for the quantity $[A^m_0]_{\mathcal{L}}$ from (9) the special notation Λ^m:

$$\Lambda^m = \frac{dX^m}{dx^0} - Y^m_\alpha \frac{df^\alpha}{dx^0}. \tag{15}$$

After substituting expressions (9)–(14) into Eq. (4) we find that this equation is equivalent to the following system:

$$\begin{aligned} \frac{d\Lambda^m}{dx^0} = {} & \left\{[\Gamma^0_{\beta 0}]_{\mathcal{L}}\frac{df^\beta}{dx^0} + [\Gamma^0_{00}]_{\mathcal{L}}\right\}\Lambda^m \\ & + \left\{[\Gamma^\alpha_{\beta 0}]_{\mathcal{L}}\frac{df^\beta}{dx^0} + [\Gamma^\alpha_{00}]_{\mathcal{L}}\right\}Y^m_\alpha, \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{dY^m_\beta}{dx^0} = {} & \left\{[\Gamma^0_{\alpha\beta}]_{\mathcal{L}}\frac{df^\alpha}{dx^0} + [\Gamma^0_{\beta 0}]_{\mathcal{L}}\right\}\Lambda^m \\ & + \left\{[\Gamma^\gamma_{\alpha\beta}]_{\mathcal{L}}\frac{df^\alpha}{dx^0} + [\Gamma^\gamma_{\beta 0}]_{\mathcal{L}}\right\}Y^m_\gamma, \end{aligned} \tag{17}$$

$$Z_{\alpha\beta} = \frac{1}{2}[\Gamma^0_{\alpha\beta}]_{\mathcal{L}}\Lambda^m + \frac{1}{2}[\Gamma^\gamma_{\alpha\beta}]_{\mathcal{L}}Y^m_\gamma, \tag{18}$$

$$\frac{dX^m}{dx^0} = \Lambda^m + Y^m_\alpha \frac{df^\alpha}{dx^0}. \tag{19}$$

Because all Γ symbols of the original metric and functions f^α are given Eqs. (16) and (17) represent the closed linear system of the ordinary differential equations of first order with respect to the variable x^0 for coefficients $\Lambda^m(x^0)$ and $Y^m_\alpha(x^0)$ in expansion (6). These solutions should be substituted to the right-hand side of Eq. (18) which gives coefficients $Z_{\alpha\beta}(x^0)$. After that we need to substitute $\Lambda^m(x^0)$ and $Y^m_\alpha(x^0)$ into Eq. (19) where we obtain the last coefficients $X^m(x^0)$ by quadrature.

This is the general procedure how to construct the Fermi coordinates for any metric in the vicinity of any given curve. There is also a possibility to specialize the Fermi coordinates in such a way that the metric in these coordinates in the first two approximations will be Minkowskian:

$$\acute{g}_{ik}(\acute{x}) = \eta_{ik} + O(2), \tag{20}$$

where η_{ik} is the Minkowski metric tensor. This can be done by choosing in a special way the arbitrary constants of integration which contain the general solution of the Eqs. (16)–(17) and (19) [there are 20 such constants, ten of which should be fixed in order to obtain the form (20) and another ten will remain arbitrary reflecting the Poincarè symmetry of the Minkowskian spacetime].

III. METRIC IN FERMI COORDINATES

The same line (1) in Fermi coordinates $\acute{x}$ has an equation of similar form:

$$\acute{x}^\alpha = F^\alpha(\acute{x}^0). \tag{21}$$

The functions $F^\alpha(\acute{x}^0)$ follow from transformation (6). This transformation tells that on the line $\acute{x}^0 = X^0(x^0)$ and $\acute{x}^\alpha = X^\alpha(x^0)$. Then

$$F^\alpha(\acute{x}^0) = [X^\alpha(\zeta)]_{\zeta=(arcX^0)(\acute{x}^0)}, \tag{22}$$

where $arcX^0$ is a function inverse to X^0.

Because in Fermi coordinates

$$[\acute{g}_{ik}(\acute{x})]_{\mathcal{L}} = c_{ik}, \qquad \left[\frac{\partial \acute{g}_{ik}(\acute{x})}{\partial \acute{x}^l}\right]_{\mathcal{L}} = 0, \qquad c_{ik} = \text{const} \tag{23}$$

the expansion for metric near the line has the form

$$\acute{g}_{ik}(\acute{x}) = c_{ik} + \frac{1}{2}\left[\frac{\partial^2 \acute{g}_{ik}(\acute{x})}{\partial \acute{x}^\alpha \partial \acute{x}^\beta}\right]_{\mathcal{L}} [\acute{x}^\alpha - F^\alpha(\acute{x}^0)] \times [\acute{x}^\beta - F^\beta(\acute{x}^0)] + O(3). \tag{24}$$

Then to obtain this metric we need the second derivatives of the metric tensor with respect to the space coordinates $\acute{x}^\alpha$ on the line. However, these second derivatives depend on the cubic terms $O(3)$ in expansion (6) and up to now remain completely arbitrary. To make a choice for this cubic addend it is necessary to accept some additional coordinate restrictions which will not violate conditions (23). We already mentioned in the Introduction that the natural physical arguments for such a choice have been proposed by Eddington and here we will follow his proposal; that is we will specify the cubic addends in coordinate transformation to the Fermi coordinates so as to represent the second derivatives in metric (24) in terms of the Riemann tensor. Eddington showed that Riemann coordinates can be further specified in such a way that cyclic combination $\acute{\Gamma}^i_{kl,m} + \acute{\Gamma}^i_{mk,l} + \acute{\Gamma}^i_{lm,k}$ of derivatives of $\acute{\Gamma}$ symbols at a point where $\acute{\Gamma}^i_{kl}$ are zero also vanish. Under this condition it is a simple matter to express second derivatives of the metric at this point in terms of the components of the Riemann tensor. In case of Fermi coordinates we described in the preceding section the full four-dimensional Eddington condition cannot be accepted because it contradicts Eqs. (16)–(19). However, it is possible to restrict the choice of Fermi coordinates by the following reduced version of the same condition:

$$\left[\frac{\partial \acute{\Gamma}^i_{\nu\lambda}(\acute{x})}{\partial \acute{x}^\mu} + \frac{\partial \acute{\Gamma}^i_{\mu\nu}(\acute{x})}{\partial \acute{x}^\lambda} + \frac{\partial \acute{\Gamma}^i_{\lambda\mu}(\acute{x})}{\partial \acute{x}^\nu}\right]_{\mathcal{L}} = 0, \tag{25}$$

where the upper index remains four dimensional and all three lower indices are three dimensional. The proof of the possibility of this restriction we placed in Appendix B.

Under the restriction $[\acute{\Gamma}^i_{kl}(\acute{x})]_{\mathcal{L}} = 0$ from the general expression for the Riemann tensor we have

$$[\acute{R}^i_{klm}(\acute{x})]_{\mathcal{L}} = \left[\frac{\partial \acute{\Gamma}^i_{km}(\acute{x})}{\partial \acute{x}^l} - \frac{\partial \acute{\Gamma}^i_{kl}(\acute{x})}{\partial \acute{x}^m}\right]_{\mathcal{L}}. \tag{26}$$

Let us apply this formula for the three-dimensional indices $(k, l, m) = (\nu, \lambda, \mu)$, that is

$$[\acute{R}^i_{\nu\lambda\mu}(\acute{x})]_{\mathcal{L}} = \left[\frac{\partial \acute{\Gamma}^i_{\nu\mu}(\acute{x})}{\partial \acute{x}^\lambda} - \frac{\partial \acute{\Gamma}^i_{\nu\lambda}(\acute{x})}{\partial \acute{x}^\mu}\right]_{\mathcal{L}}. \tag{27}$$

By simple manipulation with indices it is easy to show that the last expression with the help of condition (25) can be inverted:

$$\left[\frac{\partial \acute{\Gamma}^i_{\nu\lambda}(\acute{x})}{\partial \acute{x}^\mu}\right]_{\mathcal{L}} = -\frac{1}{3}[\acute{R}^i_{\nu\lambda\mu}(\acute{x}) + \acute{R}^i_{\lambda\nu\mu}(\acute{x})]_{\mathcal{L}}. \tag{28}$$

Now from the identity $[\acute{g}_{ik}(\acute{x})]_{;l;m} = 0$, taking into account the restriction $[\acute{\Gamma}^i_{kl}(\acute{x})]_{\mathcal{L}} = 0$, one can express the second derivatives of the metric tensor on the line $\mathcal{L}$ in Fermi coordinates in the form

$$\left[\frac{\partial \acute{g}_{ik}(\acute{x})}{\partial \acute{x}^\lambda \partial \acute{x}^\mu}\right]_{\mathcal{L}} = \left[\frac{\partial \acute{\Gamma}^l_{i\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{lk}(\acute{x}) + \frac{\partial \acute{\Gamma}^l_{k\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{li}(\acute{x})\right]_{\mathcal{L}}. \tag{29}$$

From this formula we have

$$\left[\frac{\partial \acute{g}_{00}(\acute{x})}{\partial \acute{x}^\lambda \partial \acute{x}^\mu}\right]_{\mathcal{L}} = 2\left[\frac{\partial \acute{\Gamma}^l_{0\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{l0}(\acute{x})\right]_{\mathcal{L}}, \tag{30}$$

$$\left[\frac{\partial \acute{g}_{0\alpha}(\acute{x})}{\partial \acute{x}^\lambda \partial \acute{x}^\mu}\right]_{\mathcal{L}} = \left[\frac{\partial \acute{\Gamma}^l_{0\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{l\alpha}(\acute{x}) + \frac{\partial \acute{\Gamma}^l_{\alpha\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{l0}(\acute{x})\right]_{\mathcal{L}}, \tag{31}$$

$$\left[\frac{\partial \acute{g}_{\alpha\beta}(\acute{x})}{\partial \acute{x}^\lambda \partial \acute{x}^\mu}\right]_{\mathcal{L}} = \left[\frac{\partial \acute{\Gamma}^l_{\alpha\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{l\beta}(\acute{x}) + \frac{\partial \acute{\Gamma}^l_{\beta\lambda}(\acute{x})}{\partial \acute{x}^\mu}\acute{g}_{l\alpha}(\acute{x})\right]_{\mathcal{L}}. \tag{32}$$

The first two of these formulas show that in order to express all second derivatives of the metric in terms of the Riemann tensor the relation (28) is not enough. It is necessary to find analogous expression also for the quantity $\partial\acute{\Gamma}^{l}_{0\lambda}(\acute{x})/\partial\acute{x}^{\mu}$ on the line. To do this let us take the general four-dimensional relation (26) for indices $k=\nu$, $l=\lambda$, $m=0$ and sum it with Eq. (27) being multiplied by the derivative $dF^{\mu}(\acute{x}^{0})/d\acute{x}^{0}$. In the right-hand side of this sum will appear the quantity

$$\left[\frac{\partial\acute{\Gamma}^{i}_{\nu\lambda}(\acute{x})}{\partial\acute{x}^{0}}\right]_{\mathcal{L}}+\left[\frac{\partial\acute{\Gamma}^{i}_{\nu\lambda}(\acute{x})}{\partial\acute{x}^{\mu}}\right]_{\mathcal{L}}\frac{dF^{\mu}(\acute{x}^{0})}{d\acute{x}^{0}}, \tag{33}$$

which is zero because for any function $\acute{\Psi}(\acute{x})$ which is zero along line $\mathcal{L}$, that is which satisfy the restriction $\acute{\Psi}[\acute{x}^{0},F^{1}(\acute{x}^{0}),F^{2}(\acute{x}^{0}),F^{3}(\acute{x}^{0})]=0$, the ordinary derivative of its value on the line with respect to $\acute{x}^{0}$ is also zero and due to this evident fact we deduce

$$\frac{d}{d\acute{x}^{0}}\acute{\Psi}[\acute{x}^{0},F^{1}(\acute{x}^{0}),F^{2}(\acute{x}^{0}),F^{3}(\acute{x}^{0})] =\left[\frac{\partial\acute{\Psi}(\acute{x})}{\partial\acute{x}^{0}}\right]_{\mathcal{L}}+\left[\frac{\partial\acute{\Psi}(\acute{x})}{\partial\acute{x}^{\mu}}\right]_{\mathcal{L}}\frac{dF^{\mu}(\acute{x}^{0})}{d\acute{x}^{0}}=0. \tag{34}$$

Then the resulting sum gives the following equation:

$$[\acute{R}^{i}_{\nu\lambda0}(\acute{x})]_{\mathcal{L}}+[\acute{R}^{i}_{\nu\lambda\mu}(\acute{x})]_{\mathcal{L}}\frac{dF^{\mu}(\acute{x}^{0})}{d\acute{x}^{0}} =\left[\frac{\partial\acute{\Gamma}^{i}_{\nu0}(\acute{x})}{\partial\acute{x}^{\lambda}}\right]_{\mathcal{L}}+\left[\frac{\partial\acute{\Gamma}^{i}_{\nu\mu}(\acute{x})}{\partial\acute{x}^{\lambda}}\right]_{\mathcal{L}}\frac{dF^{\mu}(\acute{x}^{0})}{d\acute{x}^{0}}, \tag{35}$$

where from the quantity $[\partial\acute{\Gamma}^{i}_{\nu0}(\acute{x})/\partial\acute{x}^{\lambda}]_{\mathcal{L}}$ can be represented in terms of the Riemann tensor since for the derivatives $[\partial\acute{\Gamma}^{i}_{\nu\mu}(\acute{x})/\partial\acute{x}^{\lambda}]_{\mathcal{L}}$ we already have such representation, see formula (28). The result is

$$\left[\frac{\partial\acute{\Gamma}^{i}_{\nu0}(\acute{x})}{\partial\acute{x}^{\lambda}}\right]_{\mathcal{L}}=[\acute{R}^{i}_{\nu\lambda0}(\acute{x})]_{\mathcal{L}} +\frac{1}{3}[\acute{R}^{i}_{\mu\nu\lambda}(\acute{x})-2\acute{R}^{i}_{\nu\mu\lambda}]_{\mathcal{L}}\frac{dF^{\mu}(\acute{x}^{0})}{d\acute{x}^{0}}. \tag{36}$$

Now from (24) and (30)–(32) (using definition $R_{iklm}=g_{in}R^{n}_{klm}$) we obtain the final general [10] result for the canonical (Eddington's terminology) metric in Fermi coordinates:

$$\acute{g}_{00}(\acute{x})=c_{00}+\left[\acute{R}_{0\lambda\mu0}(\acute{x})-\frac{2}{3}\acute{R}_{0\lambda\nu\mu}(\acute{x})\frac{dF^{\nu}(\acute{x}^{0})}{d\acute{x}^{0}}\right]_{\mathcal{L}} \times[\acute{x}^{\lambda}-F^{\lambda}(\acute{x}^{0})][\acute{x}^{\mu}-F^{\mu}(\acute{x}^{0})]+O(3), \tag{37}$$

$$\acute{g}_{0\alpha}(\acute{x})=c_{0\alpha}+\left[\frac{2}{3}\acute{R}_{\alpha\lambda\mu0}(\acute{x})-\frac{1}{3}\acute{R}_{\alpha\lambda\nu\mu}(\acute{x})\frac{dF^{\nu}(\acute{x}^{0})}{d\acute{x}^{0}}\right]_{\mathcal{L}} \times[\acute{x}^{\lambda}-F^{\lambda}(\acute{x}^{0})][\acute{x}^{\mu}-F^{\mu}(\acute{x}^{0})]+O(3), \tag{38}$$

$$\acute{g}_{\alpha\beta}(\acute{x})=c_{\alpha\beta}+\frac{1}{3}[\acute{R}_{\alpha\lambda\mu\beta}(\acute{x})]_{\mathcal{L}}[\acute{x}^{\lambda}-F^{\lambda}(\acute{x}^{0})] \times[\acute{x}^{\mu}-F^{\mu}(\acute{x}^{0})]+O(3). \tag{39}$$

IV. FERMI COORDINATES FOR STATIC OBSERVER IN SCHWARZSCHILD SPACETIME

Let us take the Schwarzschild metric in its standard form:

$$-ds^{2}=-\left(1-\frac{2m}{r}\right)dt^{2}+\left(1-\frac{2m}{r}\right)^{-1}dr^{2} +r^{2}(d\theta^{2}+\sin^{2}\theta d\varphi^{2}), \tag{40}$$

with the following designation for coordinates:

$$t,r,\theta,\varphi=x^{0},x^{1},x^{2},x^{3}. \tag{41}$$

The world line of a static observer is

$$x^{\alpha}=x^{\alpha}_{*}, \tag{42}$$

where $x^{\alpha}_{*}=(x^{1}_{*},x^{2}_{*},x^{3}_{*})=(r_{*},\theta_{*},\varphi_{*})$ are arbitrary constants. The transformation to Fermi coordinates $\acute{x}$ along this line is given by the formula (6), that is

$$\acute{x}^{m}=X^{m}(t)+Y^{m}_{\alpha}(t)(x^{\alpha}-x^{\alpha}_{*}) +Z^{m}_{\alpha\beta}(t)(x^{\alpha}-x^{\alpha}_{*})(x^{\beta}-x^{\beta}_{*})+O(3). \tag{43}$$

In Eqs. (16), (17) and (19) all terms containing df^{α}/dx^{0} disappear and among those Γ symbols which are present in these equations there are only two non-zero, namely

$$[\Gamma^{1}_{00}]_{\mathcal{L}}=\frac{m}{r_{*}^{2}}\left(1-\frac{2m}{r_{*}}\right),\qquad [\Gamma^{0}_{10}]_{\mathcal{L}}=\frac{m}{r_{*}^{2}}\left(1-\frac{2m}{r_{*}}\right)^{-1}. \tag{44}$$

Under these conditions equations (16)–(17) and (19) become very simple and can be integrated easily. The solution for the functions $\Lambda^{m}(t)$ is $\Lambda^{m}=C^{m}_{1}e^{\omega t}+C^{m}_{2}e^{-\omega t}$ and for coefficients $X^{m}(t)$ and $Y^{m}_{\alpha}(t)$ we have

$$X^{m}=\omega^{-1}(C^{m}_{1}e^{\omega t}-C^{m}_{2}e^{-\omega t})+C^{m}_{3}, \tag{45}$$

$$Y^{m}_{1}=\left(1-\frac{2m}{r_{*}}\right)^{-1}(C^{m}_{1}e^{\omega t}-C^{m}_{2}e^{-\omega t}), \tag{46}$$

$$Y^{m}_{2}=C^{m}_{4},\qquad Y^{m}_{3}=C^{m}_{5}, \tag{47}$$

where $C_1^m, \ldots, C_5^m$ are arbitrary constants of integration and

$$\omega = \frac{m}{r_*^2}. \tag{48}$$

Without loss of generality we can chose constants $C_1^m, C_2^m, C_4^m, C_5^m$ in the following way:

$$C_1^m = (C_1^0, C_1^1, C_1^2, C_1^3) = (\lambda, \lambda, 0, 0), \tag{49}$$

$$C_2^m = (C_2^0, C_2^1, C_2^2, C_2^3) = (\lambda, -\lambda, 0, 0), \tag{50}$$

$$C_4^m = (C_4^0, C_4^1, C_4^2, C_4^3) = (0, 0, r_*, 0), \tag{51}$$

$$C_5^m = (C_5^0, C_5^1, C_5^2, C_5^3) = (0, 0, 0, r_* \sin\theta_*), \tag{52}$$

where quantity λ is defined by the relation

$$\lambda^2 = \frac{1}{4}\left(1 - \frac{2m}{r_*}\right). \tag{53}$$

This choice for free parameters fixes the arbitrary constants c_{ik} in the metric (37)–(39) as

$$c_{00} = -1, \qquad c_{0\alpha} = 0, \qquad c_{\alpha\beta} = \delta_{\alpha\beta}, \tag{54}$$

that is in the first two approximations the metric is Minkowskian in Fermi coordinates.

Now we substitute the constants (49)–(52) into expressions (45)–(47) to obtain the final form for coefficients $X^m(t)$, $Y_\alpha^m(t)$ and after that insert them together with Schwarzschild Γ symbols $[\Gamma^0_{\alpha\beta}]_{\mathcal{L}}$ and $[\Gamma^\gamma_{\alpha\beta}]_{\mathcal{L}}$ into the right-hand side of the equation (18). This gives the coefficients $Z_{\alpha\beta}(t)$ after which we can write the final form of transformation to Fermi coordinates along the world line of static Schwarzschild observer:

$$\begin{aligned} \acute{x}^0 &= C_3^0 + \frac{2\lambda}{\omega}\sinh\omega t + \frac{1}{2\lambda}(r - r_*)\sinh\omega t \\ &\quad - \left[\frac{\omega}{16\lambda^3}(r - r_*)^2 + r_*\lambda(\theta - \theta_*)^2 + r_*\lambda\sin^2\theta_*(\varphi - \varphi_*)^2\right] \\ &\quad \times \sinh\omega t + O(3), \end{aligned} \tag{55}$$

$$\begin{aligned} \acute{x}^1 &= C_3^1 + \frac{2\lambda}{\omega}\cosh\omega t + \frac{1}{2\lambda}(r - r_*)\cosh\omega t \\ &\quad - \left[\frac{\omega}{16\lambda^3}(r - r_*)^2 + r_*\lambda(\theta - \theta_*)^2 + r_*\lambda\sin^2\theta_*(\varphi - \varphi_*)^2\right] \\ &\quad \times \cosh\omega t + O(3), \end{aligned} \tag{56}$$

$$\begin{aligned} \acute{x}^2 &= C_3^2 + r_*(\theta - \theta_*) + (r - r_*)(\theta - \theta_*) \\ &\quad - \frac{1}{2} r_* \sin\theta_* \cos\theta_* (\varphi - \varphi_*)^2 + O(3), \end{aligned} \tag{57}$$

$$\begin{aligned} \acute{x}^3 &= C_3^3 + r_* \sin\theta_*(\varphi - \varphi_*) + \sin\theta_*(r - r_*)(\varphi - \varphi_*) \\ &\quad + r_* \cos\theta_*(\theta - \theta_*)(\varphi - \varphi_*) + O(3). \end{aligned} \tag{58}$$

The metric for the static Schwarzschild observer in canonical Fermi coordinates follows from formulas (37)–(39). The arbitrary constants c_{ik} we already specified, see (54). Now we need to find the functions $F^\alpha(\acute{x}^0)$ and components of the Riemann tensor $\acute{R}_{iklm}(\acute{x})$. The equation of the Schwarzschild static world line in the Fermi coordinates can be extracted from transformation (55)–(58). On the line we have

$$\begin{aligned} &\acute{x}^0 = C_3^0 + \frac{2\lambda}{\omega}\sinh\omega t, \qquad \acute{x}^1 = C_3^1 + \frac{2\lambda}{\omega}\cosh\omega t, \\ &\acute{x}^2 = C_3^2, \qquad \acute{x}^3 = C_3^3. \end{aligned} \tag{59}$$

Then functions $F^\alpha(\acute{x}^0)$ are

$$\begin{aligned} &F^1(\acute{x}^0) = C_3^1 + \sqrt{a + (\acute{x}^0 - C_3^0)^2}, \\ &F^2 = C_3^2, \qquad F^3 = C_3^3, \end{aligned} \tag{60}$$

where

$$a = \frac{r_*^4}{m^2}\left(1 - \frac{2m}{r_*}\right). \tag{61}$$

The arbitrary constants C_3^i are not important, they can be eliminated by the shift of the origin of the Fermi coordinates.

The Riemann tensor $\acute{R}_{iklm}(\acute{x})$ can be found by transformation (55)–(58) from its known counterpart $R_{iklm}(x)$ for the Schwarzschild metric (40) which has the following nonzero components:

$$R_{0101} = R_{trtr} = -\frac{2m}{r^3}, \tag{62}$$

$$R_{0202} = R_{t\theta t\theta} = \frac{m}{r}\left(1 - \frac{2m}{r}\right), \tag{63}$$

$$R_{0303} = R_{t\varphi t\varphi} = \frac{m}{r}\left(1 - \frac{2m}{r}\right)\sin^2\theta, \tag{64}$$

$$R_{1212} = R_{r\theta r\theta} = -\frac{m}{r}\left(1 - \frac{2m}{r}\right)^{-1}, \tag{65}$$

$$R_{1313} = R_{r\varphi r\varphi} = -\frac{m}{r}\left(1 - \frac{2m}{r}\right)^{-1}\sin^2\theta, \tag{66}$$

$$R_{2323} = R_{\theta\varphi\theta\varphi} = 2mr\sin^2\theta. \tag{67}$$

We do not include in this list those nonzero components of $R_{iklm}(x)$ which can be obtained from (62)–(67) by application of all symmetries of the Riemann tensor. These components transform to the components of $\acute{R}_{iklm}(\acute{x})$ by the usual tensor law and on the line this transformation takes the form

$$[\acute{R}_{psqn}]_{\mathcal{L}} = [R_{iklm} Q^i_p Q^k_s Q^l_q Q^m_n]_{\mathcal{L}}, \tag{68}$$

where matrix Q^i_k is inverse to the Jacobian matrix A^i_k introduced in (2), see also (B3). For the transformation (55)–(58) these matrices calculated on the line $\mathcal{L}$ (the upper index numerates the matrix lines and the lower index corresponds to the columns) are

$$[A^i_k]_{\mathcal{L}} = \begin{pmatrix} 2\lambda\cosh\omega t & (2\lambda)^{-1}\sinh\omega t & 0 & 0 \\ 2\lambda\sinh\omega t & (2\lambda)^{-1}\cosh\omega t & 0 & 0 \\ 0 & 0 & r_* & 0 \\ 0 & 0 & 0 & r_*\sin\theta_* \end{pmatrix}, \tag{69}$$

$$[Q^i_k]_{\mathcal{L}} = \begin{pmatrix} (2\lambda)^{-1}\cosh\omega t & -(2\lambda)^{-1}\sinh\omega t & 0 & 0 \\ -2\lambda\sinh\omega t & 2\lambda\cosh\omega t & 0 & 0 \\ 0 & 0 & (r_*)^{-1} & 0 \\ 0 & 0 & 0 & (r_*\sin\theta_*)^{-1} \end{pmatrix}. \tag{70}$$

Calculations of $[\acute{R}_{psqn}]_{\mathcal{L}}$ from (68) using $[Q^i_k]_{\mathcal{L}}$ from (70) and $[R_{iklm}]_{\mathcal{L}} = R_{iklm}(r_*, \theta_*)$ from (62)–(67) give

$$[\acute{R}_{0101}]_{\mathcal{L}} = -\frac{2m}{r_*^3}, \qquad [\acute{R}_{0202}]_{\mathcal{L}} = \frac{m}{r_*^3}, \qquad [\acute{R}_{0303}]_{\mathcal{L}} = \frac{m}{r_*^3}, \tag{71}$$

$$[\acute{R}_{1212}]_{\mathcal{L}} = -\frac{m}{r_*^3}, \qquad [\acute{R}_{1313}]_{\mathcal{L}} = -\frac{m}{r_*^3}, \qquad [\acute{R}_{2323}]_{\mathcal{L}} = \frac{2m}{r_*^3}. \tag{72}$$

We see that on line $\mathcal{L}$ the Riemann tensor in the Fermi coordinates contains the same set of nonzero components as in Schwarzschild coordinates but their values are simpler. We again do not include in formulas (71) and (72) those nonzero components of $[\acute{R}_{iklm}]_{\mathcal{L}}$ which can be obtained by application of the symmetries of the Riemann tensor.

To write down the final form of the metric it is convenient to introduce shifting Fermi coordinates τ, u, v, w:

$$\tau = \acute{x}^0 - C^0_3, \qquad u = \acute{x}^1 - C^1_3,$$
$$v = \acute{x}^2 - C^2_3, \qquad w = \acute{x}^3 - C^3_3. \tag{73}$$

Collecting all information on the constants c_{ik} (54), functions $F^\alpha(\acute{x}^0)$ (60), and components of the Riemann tensor $[\acute{R}_{iklm}(\acute{x})]_{\mathcal{L}}$ (71) and (72), we obtain from (37)–(39) the final form of the metric for the static Schwarzschild observer in Fermi coordinates τ, u, v, w (73):

$$\begin{aligned} -ds^2 &= \acute{g}_{ik}(\acute{x}) d\acute{x}^i d\acute{x}^k \\ &= \acute{g}_{\tau\tau} d\tau^2 + 2\acute{g}_{\tau u} d\tau du + 2\acute{g}_{\tau v} d\tau dv + 2\acute{g}_{\tau w} d\tau dw \\ &\quad + \acute{g}_{uu} du^2 + \acute{g}_{vv} dv^2 + \acute{g}_{ww} dw^2 + 2\acute{g}_{uv} du dv \\ &\quad + 2\acute{g}_{uw} du dw + 2\acute{g}_{vw} dv dw, \end{aligned} \tag{74}$$

where components of the metric tensor (up to the quadratic terms with respect to the three small deviations $u - \sqrt{\tau^2 + a}$, v, w from the line) are

$$\acute{g}_{\tau\tau} = -1 + \frac{m}{r_*^3}\left[2\left(u - \sqrt{\tau^2 + a}\right)^2 - v^2 - w^2\right], \tag{75}$$

$$\acute{g}_{uu} = 1 + \frac{m}{3r_*^3}(v^2 + w^2), \tag{76}$$

$$\acute{g}_{vv} = 1 + \frac{m}{3r_*^3}\left[\left(u - \sqrt{\tau^2 + a}\right)^2 - 2w^2\right], \tag{77}$$

$$\acute{g}_{ww} = 1 + \frac{m}{3r_*^3}\left[\left(u - \sqrt{\tau^2 + a}\right)^2 - 2v^2\right], \tag{78}$$

$$\acute{g}_{\tau u} = \frac{m\tau(v^2 + w^2)}{3r_*^3\sqrt{\tau^2 + a}}, \qquad \acute{g}_{\tau v} = \frac{m\tau(\sqrt{\tau^2 + a} - u)v}{3r_*^3\sqrt{\tau^2 + a}},$$
$$\acute{g}_{\tau w} = \frac{m\tau(\sqrt{\tau^2 + a} - u)w}{3r_*^3\sqrt{\tau^2 + a}}, \tag{79}$$

$$\acute{g}_{uv} = \frac{m}{3r_*^3}\left(\sqrt{\tau^2 + a} - u\right)v, \qquad \acute{g}_{uw} = \frac{m}{3r_*^3}\left(\sqrt{\tau^2 + a} - u\right)w,$$
$$\acute{g}_{vw} = \frac{2m}{3r_*^3}vw. \tag{80}$$

To understand better the relation between the Synge and Fermi approach it would be instructive to take a timelike nongeodesic line and construct along it two different coordinate systems: (1) Fermi coordinates and (2) Singe's quasi-Fermi coordinates (that is coordinates associated with the observer's proper reference frame along this line) and work out the transformation between these two coordinate systems. In general this is not a simple enterprise, however, in the particular case of the static observer in Schwarzschild spacetime considered in this section the task can be resolved easily. In this case Fermi coordinates along the static world line in terms of Schwarzschild coordinates we already found [see (55)–(58)]. For simplicity let us take in these formulas $\theta_* = \pi/2$ in which case the Fermi coordinates τ, u, v, w (73) are

$$\tau = \rho \sinh \omega t + O(3), \tag{81}$$

$$u = \rho \cosh \omega t + O(3), \tag{82}$$

$$v = r_*(\theta - \pi/2) + (r - r_*)(\theta - \pi/2) + O(3), \tag{83}$$

$$w = r_*(\varphi - \varphi_*) + (r - r_*)(\varphi - \varphi_*) + O(3), \tag{84}$$

where we introduced the notation

$$\rho = \frac{2\lambda}{\omega} + \frac{1}{2\lambda}(r - r_*) - \frac{\omega}{16\lambda^3}(r - r_*)^2 - r_*\lambda(\theta - \pi/2)^2 - r_*\lambda(\varphi - \varphi_*)^2. \tag{85}$$

The Synge's quasi-Fermi coordinates T, X, Y, Z along the same world line in terms of the same Schwarzschild coordinates have been constructed in [5] and they are [11]

$$T = 2\lambda t, \tag{86}$$

$$X = \rho - \frac{2\lambda}{\omega} + O(3), \tag{87}$$

$$Y = r_*(\theta - \pi/2) + (r - r_*)(\theta - \pi/2) + O(3), \tag{88}$$

$$Z = r_*(\varphi - \varphi_*) + (r - r_*)(\varphi - \varphi_*) + O(3). \tag{89}$$

Then from (81)–(89) follows transformation between these two coordinate systems:

$$\tau = \left(\frac{2\lambda}{\omega} + X\right) \sinh \frac{\omega T}{2\lambda} + O(3), \tag{90}$$

$$u = \left(\frac{2\lambda}{\omega} + X\right) \cosh \frac{\omega T}{2\lambda} + O(3), \tag{91}$$

$$v = Y + O(3), \qquad w = Z + O(3). \tag{92}$$

Because of spherical symmetry the angular coordinatization in both systems coincides as it should. The transformation in the radial-time sector is of Rindler-Minkowski type as it also should be because (up to the second order with respect to the deviation from the line) the proper frame of a static observer in Schwarzschild metric (with Synge's coordinates T, X) is equivalent to one-dimensional accelerated motion in flat space (with Fermi coordinates τ, u).

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APPENDIX A: STANDARD FORMULAS

We use notations of the book [12]. In any spacetime with coordinates x^i and metric tensor g_{ik} the Γ symbols and Riemann tensor are

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}(g_{mk,l} + g_{lm,k} - g_{kl,m}), \tag{A1}$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}, \tag{A2}$$

$$R_{iklm} = g_{\text{in}} R^n_{klm}. \tag{A3}$$

There are four symmetry identities for the Riemann tensor:

$$R_{iklm} = -R_{kilm}, \quad R_{iklm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}, \tag{A4}$$

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{A5}$$

From definitions (A1)–(A3) follows another representation for R_{iklm}:

$$R_{iklm} = \frac{1}{2}(g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}). \tag{A6}$$

APPENDIX B: ON THE REDUCED EDDINGTON COORDINATES RESTRICTION

The transformation (6) with cubic terms is

$$\acute{x}^m = X^m(x^0) + Y^m_\alpha(x^0)[x^\alpha - f^\alpha(x^0)] + Z^m_{\alpha\beta}(x^0)[x^\alpha - f^\alpha(x^0)][x^\beta - f^\beta(x^0)] + W^m_{\alpha\beta\gamma}(x^0)[x^\alpha - f^\alpha(x^0)][x^\beta - f^\beta(x^0)] \times [x^\gamma - f^\gamma(x^0)] + O(4), \tag{B1}$$

where coefficients $W^m_{\alpha\beta\gamma}$ are symmetric with respect to the transposition of any two of the lower indices. Then we have 40 (ten for each four-dimensional index m) independent coefficients $W^m_{\alpha\beta\gamma}$. Now we apply the four-dimensional partial derivative $\partial/\partial x^s$ to the general transformation of Γ symbols (3) and restrict the result to the line $\mathcal{L}$ (taking into account that all $\acute{\Gamma}^q_{nm}$ are zero on this line). This operation gives

$$\left[\frac{\partial}{\partial x^s}(\Gamma^i_{kl}A^q_i) = \left(\frac{\partial}{\partial \acute{x}^p}\acute{\Gamma}^q_{nm}\right)A^p_s A^n_k A^m_l + \frac{\partial^3 \acute{x}^q}{\partial x^k \partial x^l \partial x^s}\right]_{\mathcal{L}}. \tag{B2}$$

Let us denote the four-dimensional matrix inverse to A^i_k by Q^i_k, that is

$$Q^k_i A^l_k = \delta^l_i, \tag{B3}$$

and multiply relation (B2) by $(Q^s_\alpha Q^k_\beta Q^l_\gamma)_{\mathcal{L}}$ with all three lower indices three dimensional. We obtain

$$\left[Q^s_\alpha Q^k_\beta Q^l_\gamma \frac{\partial}{\partial x^s}(\Gamma^i_{kl}A^q_i) = \frac{\partial}{\partial \acute{x}^\alpha}\acute{\Gamma}^q_{\beta\gamma} + \frac{\partial^3 \acute{x}^q}{\partial x^k \partial x^l \partial x^s}Q^s_\alpha Q^k_\beta Q^l_\gamma\right]_{\mathcal{L}}. \tag{B4}$$

Then we repeat this relation 2 times more with cyclic permutation of the three-dimensional indices $\beta,\gamma,\alpha \to \alpha,\beta,\gamma \to \gamma,\alpha,\beta$ and sum all three expressions. In result we have

$$\left[\frac{\partial}{\partial \acute{x}^\alpha}\acute{\Gamma}^q_{\beta\gamma} + \frac{\partial}{\partial \acute{x}^\gamma}\acute{\Gamma}^q_{\alpha\beta} + \frac{\partial}{\partial \acute{x}^\beta}\acute{\Gamma}^q_{\gamma\alpha}\right]_{\mathcal{L}}$$
$$= -\left\{3 Q^s_\alpha Q^k_\beta Q^l_\gamma \frac{\partial^3 \acute{x}^q}{\partial x^k \partial x^l \partial x^s}\right\}_{\mathcal{L}}$$
$$+ \left\{Q^s_\alpha Q^k_\beta Q^l_\gamma \left[\frac{\partial}{\partial x^s}(\Gamma^i_{kl}A^q_i) + \frac{\partial}{\partial x^l}(\Gamma^i_{sk}A^q_i) + \frac{\partial}{\partial x^k}(\Gamma^i_{ls}A^q_i)\right]\right\}_{\mathcal{L}}. \tag{B5}$$

Consequently the three-dimensional Eddington condition (25) will be satisfied if we choose the cubic addend in transformation (B1) to satisfy the requirement:

$$\left\{Q^s_\alpha Q^k_\beta Q^l_\gamma \frac{\partial^3 \acute{x}^q}{\partial x^k \partial x^l \partial x^s}\right\}_{\mathcal{L}}$$
$$= \frac{1}{3}\left\{Q^s_\alpha Q^k_\beta Q^l_\gamma \left[\frac{\partial}{\partial x^s}(\Gamma^i_{kl}A^q_i) + \frac{\partial}{\partial x^l}(\Gamma^i_{sk}A^q_i) + \frac{\partial}{\partial x^k}(\Gamma^i_{ls}A^q_i)\right]\right\}_{\mathcal{L}}. \tag{B6}$$

The left and right sides in relation (B6) are symmetric with respect to the transposition of any two of the indices α, β, γ, consequently this relation represents 40 independent equations for 40 unknown coefficients $W^m_{\alpha\beta\gamma}$ which enter the third derivatives of $\acute{x}^q$. No other quantity in (B6) contains these $W^m_{\alpha\beta\gamma}$. It is important that terms $(\partial^3 \acute{x}^q/\partial x^k \partial x^l \partial x^s)_{\mathcal{L}}$ are linear with respect to $W^m_{\alpha\beta\gamma}(x^0)$ and do not contain x^0 derivatives of these functions. Then the system (B6) is the set of the linear algebraic equations with respect to the unknowns $W^m_{\alpha\beta\gamma}$. Indeed, only the last term in expansion (B1) for Fermi coordinates $\acute{x}^q$ contains quantities $W^m_{\alpha\beta\gamma}$ and it is easy to show that the left-hand side of equation (B6) has the structure

$$\left[Q^s_\alpha Q^k_\beta Q^l_\gamma \frac{\partial^3 \acute{x}^q}{\partial x^k \partial x^l \partial x^s}\right]_{\mathcal{L}} = 6 W^q_{\mu\lambda\nu}[N^\mu_\alpha N^\lambda_\beta N^\nu_\gamma]_{\mathcal{L}} + \cdots, \tag{B7}$$

where dots mean all terms which do not contain coefficients $W^q_{\mu\lambda\nu}$ and 3×3 matrix $(N^\alpha_\beta)_{\mathcal{L}}$ is

$$[N^\alpha_\beta]_{\mathcal{L}} = \left[Q^\alpha_\beta - Q^0_\beta \frac{df^\alpha}{dx^0}\right]_{\mathcal{L}}. \tag{B8}$$

Then using the matrix inverse to $(N^\alpha_\beta)_{\mathcal{L}}$, the system (B6) can be uniquely resolved with respect to the unknown coefficients $W^q_{\mu\lambda\nu}$. This is the proof of the possibility to specialize the Fermi coordinate in the way to achieve the three-dimensional analog of the Eddington coordinates condition (25).

[1] E. Fermi, Sopra i fenomeni che avvengono in vicinanza di una linea oraria, in *Collected Papers (Note e Memorie), Accademia Nazionale dei Lincei*, edited by E. Fermi (University of Chicago Press, Chicago, 1962), Serie 5, Vol. 31, pp. 51–52.

[2] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1924).

[3] F. K. Manasse and C. W. Misner, Fermi normal coordinates and some basic concepts in differential geometry, J. Math. Phys. (N.Y.) **4**, 735 (1963).

[4] M. Blau, D. Frank, and S. Weiss, Fermi coordinates and Penrose limit, Classical Quantum Gravity **23**, 3993 (2006).

[5] D. Bini, A. Geralico, and R. T. Jantzen, Fermi coordinates in Schwarzschild spacetime: Closed form expressions, Gen. Relativ. Gravit. **43**, 1837 (2011).

[6] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), see Sec. X, Chap. II.

[7] The Synge's quasi-Fermi approach has been applied by Misner, Thorne and Wheeler (MTW) [8] for construction of the so-called "proper reference frame" of an accelerating and rotating observer. For an accelerating observer without rotation the MTW construction corresponds directly to the Synge approach with its Fermi-Walker transport but for the accelerating and rotating frame MTW generalized the transport law by adding to the evolution equation of the frame the term proportional to its angular velocity. In any case there are no Fermi coordinates in these constructions and this was clear for the authors because they do not claim any relation of their frame of an accelerating observer to Fermi coordinates. They refer shortly to Fermi coordinates only at the end of their analysis in connection to the particular case of the nonaccelerating and nonrotating geodesic observer when the Synge prescription indeed lead to coordinates of Fermi type.

[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), see Sec. 13.6.

[9] The Latin indices run through the four values 0,1,2,3 and Greek indices take three values 1,2,3. The simple partial derivatives we denote by comma and covariant derivatives by semicolon. We stress that in general x^0 has no obligation to be timelike. One can choose any distribution of signs in the signature of the metric tensor.

[10] Of course, the particular case for geodesic lines is also included. If the line (1) is geodesics then functions $f^\alpha(x^0)$ cannot be arbitrary but should follow from geodesic equations in metric $g_{ik}(x)$. Without loss of generality we can consider the metric in Fermi coordinates (37)–(39) having the form $\acute{g}_{ik} = \eta_{ik} + O(2)$ where η_{ik} is the Minkowski tensor. Then near the line the spacetime is flat and geodesics will be just a straight line. We can use the Lorentz rotation and shift of the origin of coordinates (which group do not changes η_{ik} and zero value of $\acute{\Gamma}$ symbols on line) in order to make this geodesics coincide with the coordinate line of the variable $\acute{x}^0$ starting from the origin of the Fermi coordinate system. This means that equations of line will be $\acute{x}^\alpha = 0$ [which means that transformation coefficients $X^\alpha(x^0)$ in (6) vanish] and all functions $F^\alpha(\acute{x}^0)$ in the metric (37)–(39) disappear. In this case some part of the equations (16), (17) and (19) will be equivalent to the geodesic equations for the line $x^\alpha = f^\alpha(x^0)$ in metric $g_{ik}(x)$ and all other relations will define the rest of the transformation coefficients in (6). The results will be the same as in paper [3] (with the difference that the Riemann tensor in our paper is defined with opposite sign).

[11] Expressions ((86)–(89)) follow from the first of the formulas (14) and from the formulas (74) in the paper [5]. With approximation up to the second order $O(2)$ the expansion (74) can be easily inverted, that is Singe's coordinates X, Y, Z can be written in the form of expansion in terms of the Schwarzschild coordinates. This results in expansion (86)–(89) in which we used our notations m, r_*, φ_* for the same constant parameters M, r_0, φ_0 adopted in paper [5].

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