Gravitational AdS to dS phase transition in five-dimensional Einstein-Maxwell-Gauss-Bonnet gravity

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In this work, we revisit a toy model proposed by Camanho *et al.* [J. High Energy Phys. 10 (2015) 179] and extensively study the possible existence of gravitational phase transition from AdS to dS geometries by adding the Maxwell field as an impurity substitution. We show that the phase transitions proceed via the bubble nucleation of spherical thin shells described by different branches of the solution which host a black hole in the interior. In order to demonstrate the existence of the phase transition, we examine how the free energy and temperature depend on the higher-order gravity coupling (λ) indicating the possibility of thermalon-mediated phase transition. We observe that the phase transitions of the charged case are possible in which the required (maximum) temperature is lower than that of the neutral case. Interestingly, we also discover that the critical temperature and the coupling λ of the phase transitions are modified when having the charge. Notably, our results agree with the claim that the generalized gravitational phase transition is a generic behavior of the higher-order gravity theories even when the matter field is added.

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I. INTRODUCTION

One of the greatest challenges in physics nowadays is to explain the positive value of the cosmological constant or, equivalently, the energy density of the vacuum. Regarding the positiveness of the cosmological constant, the phase transition in gravitational physics has posed one of the interesting subjects for decades. Indeed, phase transitions between two competing vacua of a given theory are quite common in physics. They occur when the free energy of the actual vacuum becomes greater than the other due to a variation of some parameter of the system. Phase transitions between two competing vacua with different cosmological constants have been so far discussed in the context of gravitational instantons [1,2]. In various proposed gauge and gravity dualities, another important example of the gravitational phase transitions is known as the Hawking-Page transition [3]. This is the first-order phase transition competing between thermal anti–de Sitter (AdS) space and the Schwarzschild-AdS black hole. In addition, a number of publications of the phase transition in context of the AdS and dS black hole thermodynamics have been actively studied in several aspects and various models of higher-order gravity [4–23]. It is worth mentioning here that pioneer works on the AdS and dS phase transition in higher derivative gravity have been theoretically proposed in Refs. [4,5,24].

More interestingly, in higher-order theories of gravity, a number of recent studies have focused on thermalon-mediated phase transitions [25–27] in many cases of Lovelock gravity with a vacuum solution. These types of phase transitions proceed through the nucleation of the spherical thin-shell bubbles, so-called thermalon which is the Euclidean sector of a static bubble. It is worth noting that the thermalon and the techniques associated to it were first introduced in Ref. [28]. This thin shell stays between two regions described by different branches of the solution which host the black hole in the interior. On the other hand, the thermalon is a finite-temperature instanton which is considered as a thermodynamic phase and described an intermediate state. In a finite time, when the thermalon forms, it is dynamically unstable and then expands to occupy the entire space. Hence, this effectively changes the asymptotic structure of the spacetime. Once the cosmological constant is fixed, it was shown in Refs. [24,26,29] that thermal AdS space underwent a thermalon-mediated phase transition to an asymptotically dS black hole geometry.

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It has been found that this type of gravitational phase transition is a generalized phase transition of the Hawking-Page mechanism in Lovelock gravity. In addition, the AdS to dS gravitational phase transition is claimed to be a generic behavior of the higher order of the gravitational theories [26]. However, the inclusion of matter in this toy model of the phase transition has not been studied yet. It is worth to investigate the phase transition profile of the model with the matter field.

Moreover, the Lovelock or higher-order gravity naturally arises in string theory. Therefore, a study of the phase transitions in this type of gravity might reveal some interesting features of the consequences in the string theory at low-energy regimes. In particular, it is expected that we can gain a better understanding of the phase transition phenomenon in the AdS/CFT correspondence paradigms. Although the AdS/CFT is extensively studied in various aspects and its nature is widely known, the dS/CFT counterpart is less studied and poorly understood. For this reason, the study of the AdS to dS phase transition in this work may be also useful for uncovering the nature of the dS/CFT. In the present work, we therefore revisit a toy model proposed by Ref. [26] and extensively study the possible existence of a gravitational phase transition from AdS to dS geometries by adding the Maxwell field. In addition, we also investigate the effects of the static charge on the critical temperature and the coupling of the higher-order gravity term, the Gauss-Bonnet term in this work, causing the phase transition.

The content of the paper is organized as follows. In Sec. II, we will review some basics of Lovelock gravity in the vierbein formalism [26] with the Maxwell field that are the starting point for the computations of the present work and construct a junction condition of Lovelock-Maxwell gravity. We then focus on a special case of Lovelock gravity in which the action is reduced to Einstein-Gauss-Bonnet-Maxwell (EGBM) gravity. In this section, we also derive the effective potential of the thermalon EGBM gravity and examine the thermalon solutions as well as the stability and dynamics of the thermalon. In Sec. III, we study the gravitational phase transition and the relevant thermodynamic quantities in five-dimensional EGBM gravity. Here we examine how the free energy and temperature depend on the coupling indicating the possibility of a thermalon-mediated phase transition. We conclude our findings in the last section.

II. FORMALISM

A. Lovelock-Maxwell gravity action

We start with the action of the Lovelock gravity in the vierbein formalism with inclusion of the Maxwell field in d dimensions, and it reads [23,30]

$$\mathcal{I} = \frac{1}{16\pi G_N(d-3)!}\left[\sum_{k=0}^{K}\frac{c_k}{d-2k}\left(\int_{\mathbb{M}}\mathcal{L}_k - \int_{\partial\mathbb{M}}\mathcal{B}_k\right) + \int_{\mathbb{M}}\mathcal{F}\wedge *\mathcal{F}\right], \tag{1}$$

where $\mathbb{M}$ and $\partial\mathbb{M}$ are the spacetime manifold and its boundary, respectively. Note that $\{c_k\}$ is a set of couplings with length dimensions $L^{2(k-1)}$ with L being a length scale related to the customary cosmological constant Λ such that $L^2 = -(d-1)(d-2)/2\Lambda$, and K is a positive integer given by $K \leq (d-1)/2$. In this work, all ingredients of the Lovelock gravity in vierbein formalisms are given by

$$\mathcal{L}_k = \epsilon_{a_1\ldots a_d}R^{a_1a_2}\wedge\cdots\wedge R^{a_{2k-1}a_{2k}}\wedge e^{a_{2k+1}}\wedge\cdots\wedge e^{a_d}, \tag{2}$$

$$\mathcal{B}_k = k\int_0^1 d\xi\epsilon_{a_1\ldots a_d}\theta^{a_1a_2}\wedge\mathfrak{F}^{a_3a_4}\wedge\cdots\wedge\mathfrak{F}^{a_{2k-1}a_{2k}}\wedge e^{a_{2k+1}}\wedge\cdots\wedge e^{a_d}, \tag{3}$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb}, \tag{4}$$

$$\mathfrak{F}^{ab} = R^{ab} + (\xi^2-1)\theta^a{}_c\wedge\theta^{cb}, \tag{5}$$

$$\theta^{ab} = (n^aK^b{}_c - n^bK^a{}_c)e^c, \tag{6}$$

where R^{ab} is the curvature two-form with ω^{ab}, the torsionless Levi-Civita spin connection. In this work, we use small Latin alphabets $a, b, c, \ldots$ for the bulk spacetime indices and focus on five dimensions in the latter. Moreover, $e^a = e^a{}_\mu dx^\mu$, n^a, and K^{ab} are the vierbein one-form, normal unit vector, and extrinsic curvature, respectively. The spherically symmetric solution of the theory in d dimensions is taken in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2_{(\sigma)d-2}, \tag{7}$$

where $d\Omega^2_{(\sigma)d-2}$ is the line element of the $(d-2)$-dimensional surface of the constant curvature σ with $\sigma = 1, 0, -1$ (spherical, planar, and hyperbolic geometries, respectively). More importantly, we will use the normalization of the gravitational constant such that $16\pi G_N(d-3)! = 1$ [26,27]. The equation of motion of the Maxwell field in the vacuum is given by

$$d*\mathcal{F} = 0 \quad \text{and} \quad d\mathcal{F} = 0, \quad \text{with} \quad \mathcal{F} = dA, \tag{8}$$

where A is the vector potential one-form. The field strength tensor $\mathcal{F}$ is given by the following ansatz:

$$\mathcal{F} = \frac{Q}{r^{d-1}} dt \wedge dr, \tag{9}$$

where Q is the electric charge. Having use all ingredients introduced, we can write the solution of the theory by introducing the following polynomial:

$$\Upsilon[g] = \sum_{k=0}^{K} c_k g^k = \frac{\mathcal{M}}{r^{d-1}} - \frac{\mathcal{Q}^2}{r^{2(d-2)}}, \tag{10}$$

$$g \equiv g(r) = \frac{\sigma - f(r)}{r^2}. \tag{11}$$

The parameters $\mathcal{M}$ and $\mathcal{Q}$ are related to the black hole Arnowitt-Deser-Misner mass (M) and the electric charge (Q) via

$$\mathcal{M} = \frac{\Gamma(\frac{d}{2})M}{(d-2)!\pi^{(d/2)-1}}, \qquad \mathcal{Q}^2 = \frac{Q^2}{(d-2)(d-3)}. \tag{12}$$

We refer the detail derivation of the Υ solution in Refs. [11–13,22].

B. Junction condition in Lovelock-Maxwell gravity

In this work, the main purpose is to study the dynamics of the unstable spherical thin shell (thermalon) of the Lovelock-Maxwell gravity. To do this, we first divide the manifold of the spacetime into two regions. We focus on the case of the timelike surface of the manifold. Then the manifold is decomposed as $\mathbb{M} = \mathbb{M}_- \cup (\Sigma \times \xi) \cup \mathbb{M}_+$, where Σ is the junction hypersurface of two regions of the manifolds and $\xi \in [0, 1]$ is interpolating for both regions. The $\mathbb{M}_+$ and $\mathbb{M}_-$ are outer and inner regions of the manifolds, respectively. The metric tensors $f_\pm(r)$ are also used to describe geometries of the outer and inner manifolds. One writes two different line elements of the spacetimes that is used to describe AdS outer ($+$) and dS inner ($-$) spacetime as

$$ds_\pm^2 = -f_\pm(r_\pm)dt_\pm^2 + \frac{dr_\pm^2}{f_\pm(r_\pm)} + r_\pm^2 d\Omega^2_{(\sigma),d-2}, \tag{13}$$

where again $\pm$ correspond to outer and inner spacetimes, respectively. In the latter, we will focus our study in five-dimensional spacetime. This gives $d\Omega^2_{(\sigma),d-2} \overset{d=5}{=} d\Omega^2_{(\sigma),3}$ and it is defined by

$$d\Omega^2_{(\sigma),d-2} = \begin{cases} d\theta^2 + \sin^2\theta d\chi^2 + \sin^2\theta \sin^2\chi d\phi^2 : \sigma = 1, \\ d\theta^2 + d\chi^2 + d\phi^2 : \sigma = 0, \\ d\theta^2 + \sinh^2\theta d\chi^2 + \sinh^2\theta \sinh^2\chi d\phi^2 : \sigma = -1. \end{cases} \tag{14}$$

Next, we construct a manifold $\mathbb{M}$ by matching $\mathbb{M}_\pm$ at their boundaries. We choose the boundary hypersurfaces $\partial\mathbb{M}_\pm$ as

$$\partial\mathbb{M}_\pm \coloneqq \{r_\pm = a | f_\pm > 0\} \tag{15}$$

with parameterizations of the coordinates

$$r_\pm = a(\tau), \qquad t_\pm = \tilde{t}_\pm(\tau), \tag{16}$$

where τ is comoving time of the induced line elements of the hypersurface (Σ) which takes the same form in both of two manifolds $\mathbb{M}_\pm$ at the boundaries; it reads

$$ds_\Sigma^2 = -d\tau^2 + a^2(\tau) d\Omega^2_{(\sigma),d-2}, \tag{17}$$

where $a(\tau)$ is the scale factor in the comoving frame and we will represent it as a thin-shell bubble radius in the latter. Applying the coordinate parameterizations to the line elements of the manifolds in Eq. (13), one finds

$$\begin{aligned} ds_\pm^2 &= -f_\pm(a) d\tilde{t}_\pm(\tau)^2 + \frac{da(\tau)^2}{f_\pm(a)} + r_\pm^2 d\Omega^2_{(\sigma),d-2} \\ &= -\left(f_\pm(a)\left(\frac{\partial \tilde{t}_\pm}{\partial\tau}\right)^2 - \frac{1}{f_\pm(a)}\left(\frac{\partial a}{\partial \tau}\right)^2\right) d\tau^2 \\ &\quad + a^2(\tau) d\Omega^2_{(\sigma),d-2}. \end{aligned} \tag{18}$$

As mentioned earlier, the line elements of the hypersurface must have the same form at the boundaries of the manifolds. Then we compare line elements in Eqs. (17) and (18), and we obtain the following constraint:

$$1 = f_\pm(a)\left(\frac{\partial \tilde{t}_\pm}{\partial\tau}\right)^2 - \frac{1}{f_\pm(a)}\left(\frac{\partial a}{\partial\tau}\right)^2. \tag{19}$$

It has been shown in detail in Refs. [27,30] that the continuity of the junction condition across the hypersurface in the electrovacuum case is written in terms of the canonical momenta $\pi^\pm_{AB}$ as

$$\pi^+_{AB} - \pi^-_{AB} = 0. \tag{20}$$

We note that the capital Latin alphabets $A, B, C, \ldots$ are the vierbein indices of the hypersurface Σ. The canonical momentum π_{AB} is derived by varying the gravitational action of the boundary with respect to the induced metric h_{AB} on the hypersurface Σ, i.e., [27,30],

$$\delta\mathcal{I}_{\partial\mathbb{M}} = -\int_{\partial\mathbb{M}} d^{(d-1)}x \pi_{AB} \delta h^{AB}, \tag{21}$$

where the canonical momentum $\pi^A{}_B$ is given by

$$\begin{aligned}\pi^A{}_B &= -\frac{1}{2}\frac{\delta \mathcal{I}_{\partial\mathbb{M}}}{\delta e^B}\wedge e^A \\ &= \sum_{k=1}^{K} k c_k \int_0^1 d\xi K^{A_1}\wedge \mathfrak{F}^{A_2A_3}(\xi) \\ &\quad \wedge\cdots \mathfrak{F}^{A_{2k-2}A_{2k-1}}(\xi) e^{A_{2k}\dots A_{d-2}A}\epsilon_{A_1\dots A_{d-2}B}, \end{aligned} \tag{22}$$

with $K^A = K^A{}_B e^B$.

It has been also demonstrated in Refs. [27,30] that, for our study case in the latter, the diagonal components of the $\pi^{\pm}_{AB}$ give the same relation between time and spatial parts via the following constraint in the five-dimensional case ($d = 5$) to yield

$$\frac{d}{d\tau}(a^3\pi^{\pm}_{\tau\tau}) = 3a^2\dot{a}\pi^{\pm}_{\varphi_i\varphi_i}, \qquad \varphi_i = \varphi_1,\varphi_2,\varphi_3 = \theta,\chi,\phi. \tag{23}$$

In addition, the (comoving) time component of the $\pi^{\pm}_{AB}$ is rewritten in the compact form as [26,27,30]

$$\begin{aligned}\Pi^{\pm} = \pi^{\pm}_{\tau\tau} &= \frac{\sqrt{\dot{a}^2 + f_{\pm}(r)}}{a} \\ &\quad\times \int_0^1 d\xi \Upsilon'\left[\frac{\sigma - \xi^2 f_{\pm}(a) + (1-\xi^2)\dot{a}^2}{a^2}\right], \end{aligned} \tag{24}$$

where $\Upsilon'[x] = d\Upsilon[x]/dx$. Furthermore, it is convenient to define new variables, $\tilde{\Pi} = \Pi^+ - \Pi^-$, and the junction conditions of the continuity across hypersurface are given by

$$\tilde{\Pi} = 0 = \frac{d\tilde{\Pi}}{d\tau}. \tag{25}$$

References [27,30] have derived the further compact form of $\tilde{\Pi}$ as

$$\tilde{\Pi} = \int_{\sqrt{H-g_-}}^{\sqrt{H-g_+}} dx \Upsilon'[H - x^2], \tag{26}$$

where $H = (\sigma + \dot{a}^2)/a^2$. We will specify the Lovelock-Maxwell gravity at $K = 2$, and this leads to a so-called Einstein-Gauss-Bonnet-Maxwell (EGBM) gravity in the following. From now on, we will work on the Euclidean signature, i.e., $t \to it$, for studying the thermalon which is the Euclidean sector of the spherical bubble thin shell. This gives $\dot{a}^2 \to -\dot{a}^2$ and $\ddot{a} \to -\ddot{a}$.

C. The Einstein-Gauss-Bonnet-Maxwell gravity

In this subsection, we focus on the EGBM theory to analyze the gravitational phase transition in the next section. The EGBM gravity is a reduction form of the Lovelock-Maxwell gravity at $K = 2$. One finds

$$\begin{aligned}\mathcal{I} &= \sum_{k=0}^{K=2}\frac{c_k}{d-2k}\left(\int_{\mathbb{M}}\mathcal{L}_k - \int_{\partial\mathbb{M}}\mathcal{B}_k\right) + \int_{\mathbb{M}}\mathcal{F}\wedge *\mathcal{F} \\ &= \int d^dx\left[-\epsilon_\Lambda\frac{(d-1)(d-2)}{L^2} + R + \frac{\lambda L^2}{(d-3)(d-4)}(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd})\right] \\ &\quad -\frac{1}{4}\int d^dx\mathcal{F}_{ab}\mathcal{F}^{ab} - \int_{\partial\mathbb{M}} d^{(d-1)}x\sqrt{-h}\left[K + \frac{2\lambda L^2}{(d-3)(d-4)}\left\{J - 2\left(\mathcal{R}^{AB} - \frac{1}{2}h^{AB}\mathcal{R}\right)K_{AB}\right\}\right], \end{aligned} \tag{27}$$

where $J \equiv h^{AB}J_{AB}$ is the trace of J_{AB} which is built up from K_{AB} as

$$\begin{aligned}J_{AB} = \frac{1}{3}(&2KK_{AC}K^C_B + K_{CD}K^{CD}K_{AB} \\ &- 2K_{AC}K^{CD}K_{DB} - K^2K_{AB})\end{aligned} \tag{28}$$

and $\mathcal{R}_{AB}$ is the Ricci tensor (intrinsic curvature) of the hypersurface Σ. More importantly, we note that the coefficients c_k of the Lovelock theory for the Gauss-Bonnet gravity case are given by

$$c_0 = \frac{1}{L^2}, \qquad c_1 = 1, \qquad c_2 = \lambda L^2. \tag{29}$$

Here we have identified the cosmological constant (Λ) of the theory as

$$\Lambda = \epsilon_\Lambda\frac{(d-1)(d-2)}{2L^2}, \tag{30}$$

where $\epsilon_\Lambda = \pm 1$ is the sign of the bare cosmological constant and we use the $\epsilon_\Lambda = +1$ (de Sitter) of the bare cosmological constant in this work. The solution of the polynomial, $\Upsilon[g]$ in the EGBM theory is given by

$$\Upsilon[g] = -\frac{1}{L^2} + g + \lambda L^2 g^2 = \frac{\mathcal{M}}{r^{d-1}} - \frac{\mathcal{Q}^2}{r^{2(d-2)}}. \tag{31}$$

One finds the solutions of g from the above equation as

$$g_\pm \equiv g_\pm(r) = -\frac{1}{2\lambda L^2}\left(1 \pm \sqrt{1+4\lambda\left[1+L^2\left(\frac{\mathcal{M}}{r^{d-1}} - \frac{\mathcal{Q}^2}{r^{2(d-2)}}\right)\right]}\right). \tag{32}$$

Therefore, the solutions of the line elements for inner and outer manifolds in Eq. (18) are given by [7,11]

$$f_\pm \equiv f_\pm(r) = \sigma + \frac{r^2}{2\lambda L^2}\left(1 \pm \sqrt{1+4\lambda\left[1+L^2\left(\frac{\mathcal{M}}{r^{d-1}} - \frac{\mathcal{Q}^2}{r^{2(d-2)}}\right)\right]}\right). \tag{33}$$

Superficially, one might encounter the naked singularity solutions from the f_+. However, we have provided the expression of the critical value of the charge $|\mathcal{Q}_c|$ in Eq. (76), and we have also restricted our study for $|\mathcal{Q}| < |\mathcal{Q}_c|$ that can avoid the naked singularity. Next, we turn to construct the junction condition of EGBM gravity. One recalls the compact form of the comoving time component of the canonical momenta, given by $\Pi = \pi_{\tau\tau}$ in Eq. (24) as [26,27,30]

$$\Pi_\pm = \frac{\sqrt{\dot{a}^2 + f_\pm(a)}}{a}\int_0^1 d\xi \Upsilon'\left[\frac{\sigma - \xi^2 f_\pm(a) + (1-\xi^2)\dot{a}^2}{a^2}\right] = \frac{2}{3}\lambda L^2 g_\pm \sqrt{H - g_\pm} + \frac{4}{3}H\lambda L^2\sqrt{H-g_\pm} + \sqrt{H-g_\pm}, \tag{34}$$

where the following definitions have been used to perform the above integration:

$$H(a,\dot{a}) = \frac{\sigma + \dot{a}^2}{a^2}, \tag{35}$$

$$g(a) = \frac{\sigma - f(a)}{a^2}, \tag{36}$$

$$\Upsilon'[x] = \frac{d\Upsilon[x]}{dx} = 1 + 2\lambda L^2 x. \tag{37}$$

The junction condition of the EGBM gravity in Eq. (20) implies

$$\tilde{\Pi} = \Pi_+ - \Pi_- = 0, \tag{38}$$

which is assumed that

$$\Pi_+^2 = \Pi_-^2. \tag{39}$$

Substituting the results of $\Pi_\pm$ in Eq. (34) in the junction condition (39), we find

$$-\dot{a}^2 = \frac{a^{d+1}}{12\lambda L^2}\frac{(g_+(2g_+\lambda L^2+3)^2 - g_-(2g_-\lambda L^2+3)^2)}{(\mathcal{M}_+ - \mathcal{M}_-) - (\mathcal{Q}_+^2 - \mathcal{Q}_-^2)/a^{d-3}} + \sigma, \tag{40}$$

where we have used the following identity in the denominator:

$$g_\pm + \lambda L^2 g_\pm^2 = \frac{1}{L^2} + \frac{\mathcal{M}_\pm}{a^{d-1}} - \frac{\mathcal{Q}_\pm^2}{a^{2(d-2)}}. \tag{41}$$

The result of the junction condition equation in Eq. (40) is rewritten in terms of kinetic and effective potential energies as

$$\frac{1}{2}\dot{a}^2 + V(a) = 0. \tag{42}$$

Then the effective potential $V(a)$ of the junction condition equation is given by

$$V(a) = \frac{a^{d+1}}{24\lambda L^2}\frac{(g_+(2g_+\lambda L^2+3)^2 - g_-(2g_-\lambda L^2+3)^2)}{(\mathcal{M}_+ - \mathcal{M}_-) - (\mathcal{Q}_+^2 - \mathcal{Q}_-^2)/a^{d-3}} + \frac{\sigma}{2}. \tag{43}$$

Moreover, one always can reduce the power of the $g_\pm$ functions via the following identities:

$$g_\pm^3 = \frac{g_\pm}{\lambda L^2}\left(\frac{\mathcal{M}_\pm}{a^{d-1}} - \frac{\mathcal{Q}_\pm^2}{a^{2(d-2)}} - g_\pm + \frac{1}{L^2}\right),$$
$$g_\pm^2 = \frac{1}{\lambda L^2}\left(\frac{\mathcal{M}_\pm}{a^{d-1}} - \frac{\mathcal{Q}_\pm^2}{a^{2(d-2)}} - g_\pm + \frac{1}{L^2}\right). \tag{44}$$

Using the power reductions of $g_\pm$, one rewrite the effective potential $V(a)$ in Eq. (43) at the first order of $g_\pm$ as

$$V(a) = \frac{a^{d+1}}{24\lambda L^2[(\mathcal{M}_+ - \mathcal{M}_-) - (\mathcal{Q}_+^2 - \mathcal{Q}_-^2)/a^{d-3}]} \times \left[(1+4\lambda)g + 4(2+g\lambda L^2)\left(\frac{\mathcal{M}}{a^{d-1}} - \frac{\mathcal{Q}^2}{a^{2(d-2)}}\right)\right]\Bigg|_-^+ + \frac{\sigma}{2}, \tag{45}$$

where the symbol $[\mathcal{O}]|_-^+$ is defined by

$$[\mathcal{O}]|_-^+ \equiv \mathcal{O}_+ - \mathcal{O}_-. \tag{46}$$

We continue to evaluate the derivative of the effective potential $V'(a)$, and it reads

$$V'(a) = \frac{a^d}{24\lambda L^2(\mathcal{M}_+ - \mathcal{M}_- - (\mathcal{Q}_+^2 - \mathcal{Q}_-^2)/a^{d-3})} \times \left[(1+d)(1+4\lambda)g - (d-17+2(d-5)\lambda L^2 g)\frac{\mathcal{M}}{a^{d-1}} + 2(5d-22+2(2d-7)\lambda L^2 g)\frac{\mathcal{Q}^2}{a^{2(d-2)}}\right]\Big|_-^+ + \frac{a^3(3-d)(\mathcal{Q}_+^2 - \mathcal{Q}_-^2)}{24\lambda L^2(\mathcal{M}_+ - \mathcal{M}_- - (\mathcal{Q}_+^2 - \mathcal{Q}_-^2)/a^{d-3})^2} \times \left[(1+4\lambda)g + 4(2+\lambda L^2 g)\left(\frac{\mathcal{M}}{a^{d-1}} - \frac{\mathcal{Q}^2}{a^{2(d-2)}}\right)\right]\Big|_-^+. \tag{47}$$

To eliminate g', we have used the following identities:

$$g' = \frac{1}{\Upsilon'[g]}\left((1-d)\frac{\mathcal{M}}{a^d} - 2(2-d)\frac{\mathcal{Q}^2}{a^{2d-3}}\right), \tag{48}$$

$$\Upsilon'[g] = 1 + 2\lambda L^2 g. \tag{49}$$

We note that if we drop the static charge, i.e., $\mathcal{Q}_\pm = 0$, the effective potential and its derivative convert to the same forms as the neutral case given in Refs. [27,31], i.e.,

$$V(a) \overset{\mathcal{Q}_\pm \to 0}{=} \frac{a^{d+1}}{24\lambda L^2[(\mathcal{M}_+ - \mathcal{M}_-)]} \times \left[(1+4\lambda)g + 4(2+g\lambda L^2)\frac{\mathcal{M}}{a^{d-1}}\right]\Big|_-^+ + \frac{\sigma}{2}, \tag{50}$$

$$V'(a) \overset{\mathcal{Q}_\pm \to 0}{=} \frac{a^d}{24\lambda L^2(\mathcal{M}_+ - \mathcal{M}_-)}\left[(1+d)(1+4\lambda)g - (d-17+2(d-5)\lambda L^2 g)\frac{\mathcal{M}}{a^{d-1}}\right]\Big|_-^+. \tag{51}$$

It is worth noting here that the wormhole solution in EGBM gravity has been studied in Ref. [32] considering $\mathcal{Q}_+ = -\mathcal{Q}_-$. This is so since the radial direction in one of the asymptotic regions is opposite to the other. However, in our case, we have made the assumption that there is no charge at the boundary (bubble) and the continuity of the (electric) static charge across the hypersurface is governed by

$$\mathcal{Q} = \mathcal{Q}_+ = \mathcal{Q}_-. \tag{52}$$

Having used the above continuity equation, the effective potential of the thermalon dynamics and its derivative become

$$V(a) = \frac{a^{d+1}}{24\lambda L^2(\mathcal{M}_+ - \mathcal{M}_-)}\left[(1+4\lambda)g + 4(2+\lambda L^2 g) \times \left(\frac{\mathcal{M}}{a^{d-1}} - \frac{\mathcal{Q}^2}{a^{2(d-2)}}\right)\right]\Big|_-^+ + \frac{\sigma}{2}, \tag{53}$$

$$V'(a) = \frac{a^d}{24\lambda L^2(\mathcal{M}_+ - \mathcal{M}_-)}\left[(1+d)(1+4\lambda)g - (d-17+2(d-5)\lambda L^2 g)\frac{\mathcal{M}}{a^{d-1}} + 4(2d-7)\lambda L^2 g\frac{\mathcal{Q}^2}{a^{2(d-2)}}\right]\Big|_-^+. \tag{54}$$

We close this subsection by discussing the effect of the Maxwell field (static charge) on the thermalon effective potential displaying in Fig. 1. One sees that the inclusion of the static charge does not change the shape of the effective potential except the existence of the potential. Increasing of the static charge makes the existence of the potential closer to the thermalon position as shown in Fig. 1. In addition, setting $\mathcal{Q} = 0$, we precisely reproduce the effective potential of the thermalon in the neutral case as done in Ref. [27].

D. Thermalon dynamics of EGBM gravity and its stability

We have derived the effective potential of the thermalon and its derivative in the previous section for the EGBM gravity. Now we are going to work out the thermalon solutions as well as investigating the stability and dynamics of the thermalon. We first consider the solutions of the thermalon configuration by imposing $V(a_\star) = 0 = V'(a_\star)$, where $a_\star$ is the location of the thermalon. Solving those two equations, one obtains the solutions of $\mathcal{M}_\pm$ in terms of $g_\pm^\star$, $a_\star$, λ, L, d, and $\mathcal{Q}$ as

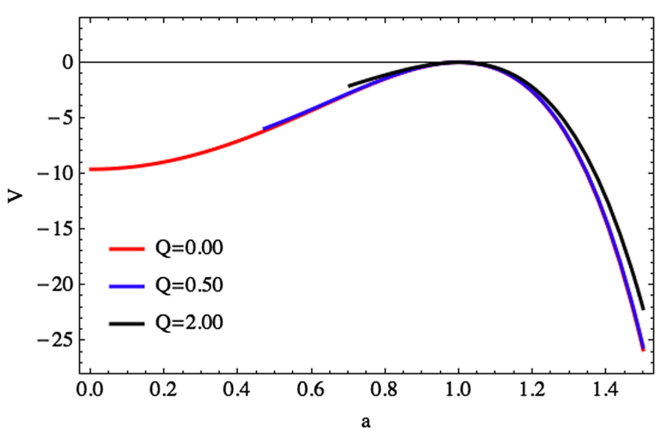

FIG. 1. The figure displays the shapes of the effective potential of the thermalon in various values of the charge $\mathcal{Q}$ with $\lambda = 0.015$, $a_\star = 1$, $L = 1$, $d = 5$, and $\sigma = 1$.

$$\mathcal{M}_{+}(g_{-}^{\star}, a_{\star}, \lambda, L^{2}, \mathcal{Q}^{2}) \equiv \mathcal{M}_{+}^{\star} = \frac{1}{4\lambda L^{2}(a_{\star}^{2}(d-1)+2(d-5)\lambda L^{2}\sigma)}[a_{\star}^{d+1}(d-1)(1+4\lambda)(3+2\lambda L^{2}g_{-}^{\star}) + 4a_{\star}^{d-1}(d+1)(1+4\lambda)\lambda L^{2}\sigma + 4a_{\star}^{5-d}(5d-13+2(d-3)\lambda L^{2}g_{-}^{\star})\lambda L^{2}\mathcal{Q}^{2} + 16a_{\star}^{3-d}(2d-7)\lambda^{2}L^{4}\mathcal{Q}^{2}\sigma], \tag{55}$$

$$\mathcal{M}_{-}(g_{+}^{\star}, a_{\star}, \lambda, L^{2}, \mathcal{Q}^{2}) \equiv \mathcal{M}_{-}^{\star} = \frac{1}{4\lambda L^{2}(a_{\star}^{2}(d-1)+2(d-5)\lambda L^{2}\sigma)}[a_{\star}^{d+1}(d-1)(1+4\lambda)(3+2\lambda L^{2}g_{+}^{\star}) + 4a_{\star}^{d-1}(d+1)(1+4\lambda)\lambda L^{2}\sigma + 4a_{\star}^{5-d}(5d-13+2(d-3)\lambda L^{2}g_{+}^{\star})\lambda L^{2}\mathcal{Q}^{2} + 16a_{\star}^{3-d}(2d-7)\lambda^{2}L^{4}\mathcal{Q}^{2}\sigma]. \tag{56}$$

Here we used $g_{\pm}^{\star} \equiv g_{\pm}(a_{\star})$. Then, we will solve for the solution of the functions $g_{\pm}^{\star} = g_{\pm}(a_{\star})$ in terms of $a_{\star}$, λ, L, d, and $\mathcal{Q}$ via the $\Upsilon[g_{\pm}]$ functions. One finds

$$-\frac{1}{L^{2}} + g_{+}^{\star} + \lambda L^{2}(g_{+}^{\star})^{2} = \mathcal{C}_{1}g_{-}^{\star} + \mathcal{C}_{2} \tag{57}$$

and

$$-\frac{1}{L^{2}} + g_{-}^{\star} + \lambda L^{2}(g_{-}^{\star})^{2} = \mathcal{C}_{1}g_{+}^{\star} + \mathcal{C}_{2}, \tag{58}$$

where the coefficients $\mathcal{C}_{1,2}$ are given by

$$\mathcal{C}_{1} = \frac{4a_{\star}^{2(3-d)}(d-3)\lambda L^{2}\mathcal{Q}^{2} + a_{\star}^{2}(d-1)(1+4\lambda)}{2a_{\star}^{2}(d-1)+4(d-5)\lambda L^{2}\sigma}, \tag{59}$$

$$\mathcal{C}_{2} = \frac{(1+4\lambda)(3a_{\star}^{2}(d-1)+4(d+1)\lambda L^{2}\sigma) + 8a_{\star}^{2(2-d)}(d-3)\lambda L^{2}\mathcal{Q}^{2}(2a_{\star}^{2}+3\lambda L^{2}\sigma)}{4\lambda L^{2}(a_{\star}^{2}(d-1)+2(d-5)\lambda L^{2}\sigma)}. \tag{60}$$

Solving the above two equations simultaneously, we obtain the solutions of $g_{\pm}^{\star}$ as

$$g_{+}^{\star} = \frac{-(1+\mathcal{C}_{1}) - \sqrt{1+4\lambda-2\mathcal{C}_{1}-3\mathcal{C}_{1}^{2}+4\mathcal{C}_{2}\lambda L^{2}}}{2\lambda L^{2}}, \tag{61}$$

$$g_{-}^{\star} = \frac{-(1+\mathcal{C}_{1}) + \sqrt{1+4\lambda-2\mathcal{C}_{1}-3\mathcal{C}_{1}^{2}+4\mathcal{C}_{2}\lambda L^{2}}}{2\lambda L^{4}}. \tag{62}$$

We note that $g_{-}^{\star}$ has a good behavior (stable) for $\lambda \to 0$, while $g_{+}^{\star}$ gives an infinite value (unstable) for $\lambda \to 0$. In addition, we need to study the phase transition between two manifolds of the spacetime, i.e., AdS (outer, $+$) to dS (inner, $-$), and then the condition $g_{+}^{\star} \neq g_{-}^{\star}$ is necessary.

To study the stability of the thermalon, we consider the bubble dynamics at the thermalon solutions at $a = a_{\star}$ and expand the junction condition in Eq. (38) up to the first order as

$$\tilde{\Pi} \approx \tilde{\Pi}_{\star} + \frac{\partial\tilde{\Pi}_{\star}}{\partial H}\frac{\dot{a}^{2}}{a_{\star}^{2}} + \frac{\partial\tilde{\Pi}_{\star}}{\partial a}(a-a_{\star}) + \frac{1}{2}\frac{\partial^{2}\tilde{\Pi}_{\star}}{\partial a^{2}}(a-a_{\star})^{2} + \cdots. \tag{63}$$

At $a = a_{\star}$, one finds

$$\tilde{\Pi}_{\star} = 0 = \frac{\partial\tilde{\Pi}_{\star}}{\partial a}. \tag{64}$$

The junction condition above can be rewritten as

$$\frac{1}{2}\dot{a}^{2} + V_{\text{eff}}(a) = 0 \quad \text{and} \quad \ddot{a} = -V'_{\text{eff}}(a), \tag{65}$$

where $\dot{a}$ is the derivative with respect to Euclidean time τ, i.e., $\dot{a} \equiv da(\tau)/d\tau$ and

$$V_{\text{eff}}(a) = \frac{1}{2}\tilde{k}(a-a_{\star})^{2}, \quad \tilde{k} = \frac{1}{2}a_{\star}^{2}\left(\frac{\partial\tilde{\Pi}_{\star}}{\partial H}\right)^{-1}\frac{\partial^{2}\tilde{\Pi}_{\star}}{\partial a^{2}}. \tag{66}$$

It has been shown in Refs. [30,31] that the sign of $\tilde{k}$ variable demonstrates the stability of the thermalon at $a = a_{\star}$. The thermalon configuration will be stable if $\tilde{k}$ is greater than zero. On the other hand, $\tilde{k} < 0$ gives the thermalon is unstable. Here we have expanded the junction condition and examined the stability of the effective potential around the thermalon location $a_{\star}$, in which this technique is analogous to the effective Hooke's constant. Having used the formula in

Eq. (66), we need the thermalon to expand and then give the phase transition of the bulk spacetime. This means that $\tilde{k} < 0$. With $|\mathcal{Q}| < |\mathcal{Q}_c|$ and $|\mathcal{Q}| = |\mathcal{Q}_c|$ limits, we have checked numerically, and we found $\tilde{k} < 0$ for all $\lambda > 0$ where $\mathcal{Q}_c$ is the critical charge [see the expression of $\mathcal{Q}_c$ in Eq. (76) and its implication in Sec. III A]. One may conclude, in this case, that the effective charge $\mathcal{Q}$ does not change the stability of the thermalon configuration. This can be depicted by the shapes of the potential in the various positive values of the coupling, $\lambda > 0$ in Fig. 2. We find that there are no local minima at the thermalon configuration at $a_\star = 1$. The thermalon position locates on the top of the potential and it is unstable. At this point, the thermalon expands, then reaches the asymptotic region in a finite time, and, therefore, changes the AdS to dS geometries of the whole spacetime.

Furthermore, we consider the expansion of the bubble thermalon escaping to infinity. Considering the matching condition $\tilde{\Pi}$ and keeping the first order of the $1/H$ expansion, we find (see [27] for a detailed derivation in the neutral case, $\mathcal{Q} = 0$)

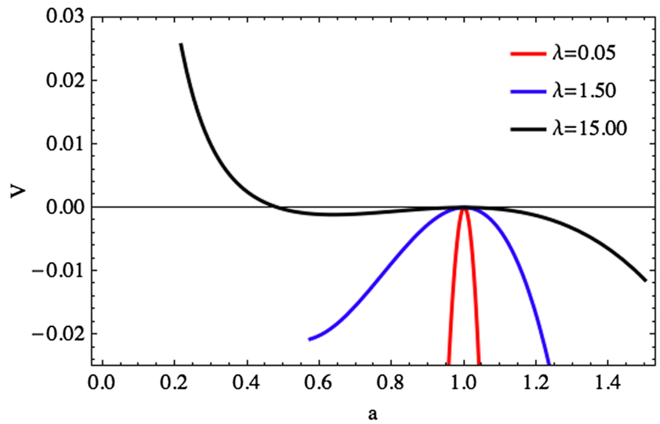

FIG. 2. The figure displays the shapes of the effective potential of the thermalon in various values of the coupling λ with $a_\star = 1$, $L = 1$, $d = 5$, $\sigma = 1$, and $\mathcal{Q} = 0.5$. We found that there are no local minima of the effective potential at the thermalon position ($a_\star = 1$) for any positive values of the λ coupling. The thermalon is always unstable and gives the phase transition from AdS to dS spacetimes.

$$H \approx \frac{1}{2(\mathcal{M}_+ - \mathcal{M}_-)}\left[a^{d-1}\int_{g_-}^{g_+}\Upsilon[x]dx - \left(g_+\left[\mathcal{M}_+ - \frac{\mathcal{Q}^2}{a^{d-3}}\right] - g_-\left[\mathcal{M}_- - \frac{\mathcal{Q}^2}{a^{d-3}}\right]\right)\right]. \tag{67}$$

At $a \to \infty$ limit, this gives $H \to \infty$. In addition, we observe that $H \to \infty$ is a bit slower than the neutral case.

III. GRAVITATIONAL PHASE TRANSITION

A. Thermalon configurations, horizons, and Nariai bound

We come to the crucial part of this work before moving forward to the gravitational phase transition with the relevant thermodynamics quantities. The thermalon (bubble) location $a_\star$ needs to ensure that it lies between the black hole radius (r_B) inside the bubble and the cosmological horizon (r_C). One can solve for $f(r_H) = 0$ in the function of $M_-^\star$ to obtain

$$f_-(r_H) = 0 \Rightarrow g_-(r_H) = \frac{\sigma}{r_{\rm BH}^2}, \tag{68}$$

where r_H is the radius of the existent horizons of the spacetime. The above equation gives

$$\Upsilon_-\left[\frac{\sigma}{r_H^2}\right] = \frac{\mathcal{M}_-^\star}{r_H^{d-1}} - \frac{\mathcal{Q}^2}{r_H^{2(d-2)}}, \tag{69}$$

where the expression of the $\mathcal{M}_-^\star \equiv \mathcal{M}_-^\star(g_\pm^\star, a_\star, \lambda, L^2, \mathcal{Q}^2)$ is given by Eq. (56). More importantly, we will focus our study of the AdS to dS gravitational transition in $d = 5$ and $\sigma = 1$ (spherical geometry). Setting $f_-(r_H) = 0$, the (de Sitter branch, inner spacetime) horizons r_H can be obtained from the following equation:

$$r_H^6 - L^2 r_H^4 + L^2(\mathcal{M}_-^\star - \lambda L^2) r_H^2 - L^2\mathcal{Q}^2 = 0. \tag{70}$$

Having rescaled $r_H^2 \to u$, we find that the above equation reduces to a cubic equation of a variable u. In order to obtain three real solutions of the cubic equation, it requires the discriminant of the cubic equation to be less than zero. Using the standard technique, the positive solutions of Eq. (70) for the horizons of the inner spacetime are given by

$$r_{H,1} = \sqrt{\frac{L^2}{3} + \frac{2\sqrt{\frac{L^4}{3} - (\mathcal{M}_-^* - \lambda L^2)}}{\sqrt{3}}\sin\left[\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}q}{2\left(\sqrt{\frac{L^4}{3} - (\mathcal{M}_-^* - \lambda L^2)}\right)^3}\right)\right]}, \tag{71}$$

$$r_{H,2} = \sqrt{\frac{L^2}{3} - \frac{2\sqrt{\frac{L^4}{3} - (\mathcal{M}^*_- - \lambda L^2)}}{\sqrt{3}} \sin\left[\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}q}{2\left(\sqrt{\frac{L^4}{3} - (\mathcal{M}^*_- - \lambda L^2)}\right)^3}\right) + \frac{\pi}{3}\right]}, \tag{72}$$

$$r_{H,3} = \sqrt{\frac{L^2}{3} + \frac{2\sqrt{\frac{L^4}{3} - (\mathcal{M}^*_- - \lambda L^2)}}{\sqrt{3}} \cos\left[\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}q}{2\left(\sqrt{\frac{L^4}{3} - (\mathcal{M}^*_- - \lambda L^2)}\right)^3}\right) + \frac{\pi}{6}\right]}, \tag{73}$$

where we have defined a new function

$$q = -\frac{2L^6}{3} + \frac{L^2(\mathcal{M}^*_- - \lambda L^2)}{3} - L^2\mathcal{Q}^2. \tag{74}$$

More importantly, the real positive values of r_H in Eqs. (71)–(73) must satisfy the following conditions:

$$\Delta = \frac{1}{4}\left(-\frac{2L^6}{3} + \frac{L^2(\mathcal{M}^*_- - \lambda L^2)}{3} - L^2\mathcal{Q}^2\right)^2 + \frac{1}{27}\left((\mathcal{M}^*_- - \lambda L^2) - \frac{L^4}{3}\right)^3 < 0. \tag{75}$$

In addition, a critical charge, $\mathcal{Q}_c$, is determined by setting $\Delta = 0$ and it reads

$$|\mathcal{Q}_c| = \frac{L^2}{3\sqrt{3}}\sqrt{\left(-2 + \frac{9}{L^2}(\mathcal{M}^\star_- - \lambda L^2) + 2\left[1 - \frac{3}{L^2}(\mathcal{M}^\star_- - \lambda L^2)\right]^{3/2}\right)}, \tag{76}$$

with

$$\lambda > \frac{3\mathcal{M}^\star_- - L^2}{3L^2}. \tag{77}$$

To confirm the condition of the horizons, we also did a numerical demonstration of the existence of the horizon in Fig. 3 for nonextremal and extremal cases. It is well known for the charged solutions of the static spherical symmetry ($\sigma = 1$) at the event horizon of the black hole (r_B) that if $|\mathcal{Q}_c| < \mathcal{Q}$, there are two horizons covering the singularity and the extremal black hole has a single event horizon for $|\mathcal{Q}_c| = \mathcal{Q}$. On the other hand, if $|\mathcal{Q}_c| > \mathcal{Q}$, the spacetime reveals the naked singularity. Next, we consider the smallest radius of the (inner) de Sitter EGBM black hole, r_S. According to Eq. (33), the smallest radius is the solution of the equation

$$(1 + 4\lambda)r_S^6 + 4\lambda L^2\mathcal{M}^\star_- r_S^4 - 4\lambda L^2\mathcal{Q}^2 = 0. \tag{78}$$

The solution of the r_S corresponds to the curvature singularity or Cauchy horizon. The appearance of charge $\mathcal{Q}$ in the solutions makes the study of horizons more complicated. However, it has been shown in Ref. [33] that for the cosmological horizon (r_C) exists in the de Sitter spacetime ($\Lambda \equiv 6/L^2 > 0$) and the Cauchy horizon is covered by the event horizon ($r_S < r_B$) with the following range of parameters:

$$\frac{\mathcal{M}^\star_-}{L^2} - \frac{1}{3} < \lambda < \frac{\mathcal{M}^\star_-}{L^2}. \tag{79}$$

More importantly, there is the existence of the horizons in the EGBM gravity with de Sitter spacetime. These have been proven in Ref. [34] that for $d = 5$, $\sigma = 1$, and $\Lambda \equiv 6/L^2 > 0$, there are three types of the horizons as

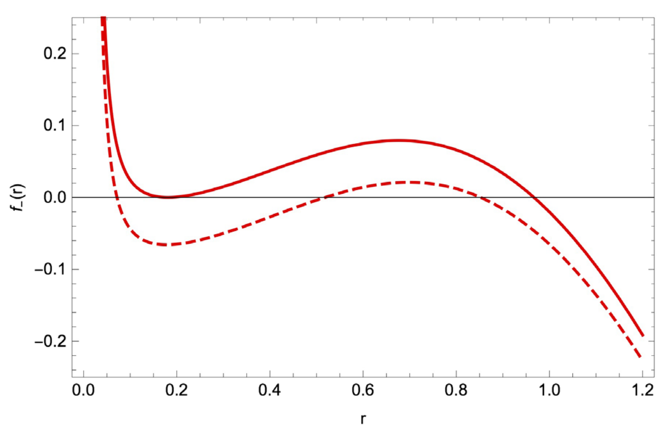

FIG. 3. The figure shows the existence of the event horizon and cosmological horizon for nonextremal ($|\mathcal{Q}| < |\mathcal{Q}_c|$) and extremal ($|\mathcal{Q}| = |\mathcal{Q}_c|$) cases. The dashed and solid lines represent three (inner, outer, and cosmological horizons) and two (degenerated and cosmological horizons) horizons for nonextremal and extremal metric in Eq. (33), respectively, with $\lambda = 1$, $M = 1.2$, $L = 1$, and $\mathcal{Q} = 0.001$.

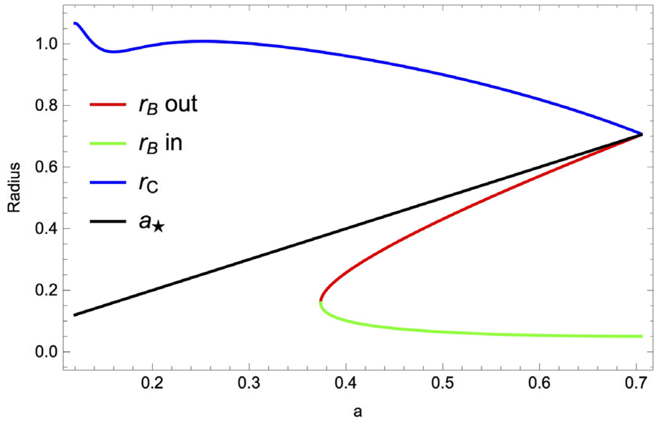

FIG. 4. A plot of an outer r_B (red), an inner r_B (green), r_C (blue), and $a_\star$ (black) as functions of $a_\star$ for $\lambda = 0.20$, $L = 1$, $d = 5$, $\sigma = 1$, and $\mathcal{Q} = 0.025$. We observe that the bubble location $a_\star$ is always found between the event horizons $r_{B,\mathrm{in}}$ and $r_{B,\mathrm{out}}$ and the cosmological horizon r_C, until the outer event and cosmological horizons meet at the point where $a^{(\mathcal{Q})}_{\mathrm{Nariai}} = 0.7062$ given by Eq. (81). This point is called the Nariai bound. For the neutral model, it is given by $a_{\mathrm{Nariai}} = \sqrt{3/\Lambda} = L/\sqrt{2} = 0.7074$ with $L = 1$ [10,16].

inner, black hole, and cosmological horizons. The types of horizons depend on the ranges of the mass and charge parameters [14,33,34]. In the present work, we limit our study to the nonextremal black hole case due to the complicated relation between the mass function $\mathcal{M}^\star_-$ and the charge $\mathcal{Q}$ in Eq. (56). Therefore, one can identify the radius of the outer, inner event, and cosmological horizons from the solutions in Eqs. (71)–(73) as

$$r_{B,\mathrm{out}} = r_{H,1}, \qquad r_{B,\mathrm{in}} = r_{H,2}, \qquad r_C = r_{H,3}. \tag{80}$$

The expressions of the above equations speculate that the thermalon position always lies between the event horizon and the cosmological horizon: $r_{B,\mathrm{in}} < r_{B,\mathrm{out}} < a_\star < r_C$. To demonstrate the above speculation, we therefore plot all horizons and the thermalon configuration shown in Fig. 4 as a function of the thermalon radius with $|\mathcal{Q}| < |\mathcal{Q}_c|$. One can see clearly that the outer (red line) and inner (green line) charged black hole event horizons are jointed smoothly and always covered by the thermalon radius (black line), while the cosmological horizon (blue line) is the largest radius and covers all horizons. The plot results in Fig. 4 are reproduced as found in Refs. [26,27,31] when $\mathcal{Q} = 0$ is taken into account. For EGBM gravity with the positive (bare) cosmological constant, in addition, our results are confirmed by Ref. [15], and it has been shown numerically that the event horizon is always covered by the cosmological horizon. Moreover, all ranges of the relevant parameters in the plots are numerically checked, and they are obeyed the condition in Eq. (79). It is interesting to see the case that the outer event horizon becomes larger until reaching the thermalon configuration and the cosmological horizon at some point. This point is called the Nariai bound, and it is given by $a_{\mathrm{Nariai}} = \sqrt{3/\Lambda} = L/\sqrt{2}$ [10,16] for the neutral case. Interestingly, the Nariai bound for the charge case is given by

$$a^{(\mathcal{Q})}_{\mathrm{Nariai}} = \sqrt{\frac{L^6 + L^2\mathcal{N} + \mathcal{N}^2}{6\mathcal{N}}},$$
$$\mathcal{N} = \left(L^6 - 54L^2\mathcal{Q}^2 + 6\sqrt{3L^4\mathcal{Q}^2(27\mathcal{Q}^2 - L^4)}\right)^{1/3}. \tag{81}$$

According to our results displayed in Fig. 4, we discovered that the Nariai bound of the charge case with a small charge is very close to the neutral one. We will see in the latter that the gravitational phase transition will take place, and it is satisfied by the Nariai bound.

B. Thermodynamics quantities and critical phenomena of AdS to dS phase transition

Next, we proceed further to quantify the relevant thermodynamics quantities for studying the phase transition. In order to investigate the thermal AdS to dS black hole phase transition as done in Refs. [25–27,31] for the neutral model, we shall take a short overview of the mechanisms of the gravitational phase transition in the literature. The initial thermal AdS (outer geometry) will decay and transit to the black hole inside the dS spacetime (inner geometry) via the thermalon mediation. Let us first clarify the precise meanings of the bubble and thermalon in order to avoid any confusion in the latter. On the one hand, the dynamics (expansion or contraction) in this scenario is always referred to as the (thin-shell) bubble. On the other hand, the thermodynamic phase of the bubble is meant by the word "thermalon." This means that the thermalon is the Euclidean sector of the static bubble.

Precisely, an initial thermal stage is AdS geometry which will decay to dS geometry mediated by the thermalon. In this sense, the bubble must expand. Such expansion is a purely dynamical process determined by the right-hand side of Eq. (42) in the Lorentzian sector. Note that the Euclidean thermalon potential has the opposite sign of that of the bubble effective potential [26,27]. Once the bubble has been formed, it will either expand or contract due to small metric fluctuations. When it expands, the dS phase in the interior will expand, eventually reaching the cosmological horizon. At that point, an observer in the interior of the cosmological horizon will measure the thermodynamics of a dS space. At the end, the boundary of a whole spacetime is changed from AdS to dS geometries; i.e., the cosmological constant changes from negative to positive values. Therefore, the observer inside the cosmological horizon can measure the thermodynamics quantities of the dS spacetime. One may conclude that the thermalon changes the solutions from one branch to another via the phase transition. More importantly, it has been shown that a reversible process for AdS to dS phase transition does not occur; see more detailed discussions in Refs. [26,27,31,35]. For example, a so-called reentrant phase transition process happening in the study of black hole thermodynamics [36,37] is not possible. The main purpose of this work is to investigate the phase transition profile of this scenario by including the static charge.

In general, the initial state of the gravitational phase transition in this work is the thermal AdS space with the static charge and requires the grand canonical ensemble to describe the thermodynamical system. In this particular case, the charge is not conserved during the process. Bear in mind that it is not necessarily incorrect, since the electrical potential Φ can be kept fixed at infinity [8,38]. However, this is not the case in the present study in which we consider keeping the charge fixed. By fixing the charge, the thermodynamical system can be basically reduced and described using the canonical ensemble [8,37,38]. It has been proven and demonstrated in Ref. [27] (see [30] for a detailed derivation) that, in the canonical ensemble including the bulk (both inner and outer manifolds) and the surface actions, the Euclidean action of the thermalon configuration ($\mathcal{I}_E$) is related to the inverse Hawking temperature (β_+), mass ($\mathcal{M}_+$) of the external observer in the asymptotic thermal AdS, and the entropy of the dS black hole. It reduces to a simple and compact form as

$$\mathcal{I}_E = \beta_+ \mathcal{M}_+ + S. \tag{82}$$

It is worth mentioning the decay channel in the present work. Here there are two vacua in our model. Precisely, the initial state is thermal AdS space, while the final one is black hole in dS spaces. The thermal AdS is initially in the false vacuum state or metastable state and then decays into a black hole inside dS space (true vacuum) via quantum tunneling or jumping over the wall of the quasiparticle state called the thermalon, also known as the Euclidean sector of the bubbles. In this work, the decay mechanism proceeds through nucleation of the bubbles or the thermalon of true vacuum (dS) inside the false vacuum (thermal AdS). The study of this process has been proposed in Refs. [25,27], and it was shown that the probability of the decay, P, of the thermalon effectively jumping from AdS to dS branch solutions is governed by $P \propto e^{-\mathcal{I}_E}$ with $\mathcal{I}_E$ being the Euclidean action difference between initial thermal AdS and the thermalon (bubble state). Therefore, the system will end up in the stable dS black hole after the thermalon expansion reaching the asymptotically dS region in a finite time.

Having used the on-shell regularization method by subtracting the thermal AdS space (outer branch solution) contribution as argued in Refs. [26,27,31] for the neutral model, this leads to the (Gibbs) free energy in the canonical ensemble with the fixed charge of the thermalon configuration. It reads [26,27,37]

$$F = \mathcal{M}_+ + T_+ S, \tag{83}$$

where $T_+ = 1/\beta_+$ is the Hawking temperature. In the latter, the free energy of the thermalon is compared to the thermal AdS space where the thermal AdS space is set to zero ($F_{\rm AdS} = 0$) because it was considered to be the background subtraction [26,27,31,35]. Before we go further to quantify the relevant thermodynamics variables, it is worth noting that there are former five free parameters in the theory of the neutral case, i.e., $\mathcal{M}_\pm$, $T_\pm$, and $a_\star$.

By using four conditions, there are two equations $V(a_\star) = 0 = V'(a_\star)$ from the configurations of the thermalon, Hawking temperature condition to avoid canonical singularity at the horizon, $T = f'(r_B)/4\pi$, and the matching temperature of the thermal circle at the thermalon configuration $\beta_+\sqrt{f_+(a_\star)} = \beta_-\sqrt{f_-(a_\star)}$. Notice that the subindex $\star$ means that it is the static value. We discovered that there are only free parameters and choose $T_+ = 1/\beta_+$. But the inclusion of the vacuum static charge in this work gives an additional free parameter $\mathcal{Q}$. The Hawking temperature T_+ is given by

$$T_+ = \sqrt{\frac{f_+(a_\star)}{f_-(a_\star)}}T_-, \tag{84}$$

where the T_- is the Hawking temperature of the inner dS black hole in EGBM gravity, and it is determined with d dimensions and general spatial curvature σ by [39]

$$T_- = \left[\sum_{k=0}^{2}(d-1-2k)\sigma c_k\left(\frac{\sigma}{r_B^2}\right)^{k-1} - (d-3)\frac{\mathcal{Q}_c^2}{r_B^{2d-6}}\right] \times\left[4\pi r_B\sum_{k=0}^{2}kc_k\left(\frac{\sigma}{r_B^2}\right)^{k-1}\right]^{-1}. \tag{85}$$

The entropy S is given by [39]

$$S = 4\pi\sum_{k=0}^{2}\frac{kc_k}{d-2k}\left(\frac{\sigma}{r_B^2}\right)^{k-1}. \tag{86}$$

We note that the entropy of the charged black hole has the same form as the neutral black hole [26,27,31]. In addition, the mass parameter $\mathcal{M}_-^\star$ is given by Eq. (55). Having used the outer event horizon in Eq. (71) and substituted into Eqs. (55), (84), and (86), we obtain all building blocks of the thermodynamics quantities as a function of the thermalon radius, and we are ready to study the thermalon properties and the gravitational phase transitions in the thermodynamics phase space.

The free energy in Eq. (83) plays the crucial role for investigating the phase transition. The behavior of the free energy is very interesting and is influenced by both the coupling λ and the charge $\mathcal{Q}$. Note that a cusp structure for the given values of the coupling λ and the charge $\mathcal{Q}$ indicates the lowest value of the free energy F which is lower than the thermal AdS space ($F_{\mathrm{AdS}} = 0$; in this work, it is zero as mentioned earlier) at the same temperature. Then the thermalon will jump to the dS branch solution and change the boundary from AdS to dS asymptotics resulting in the discontinuity (cusp) of the free energy F at the maximum temperature of the physical branch solutions. This leads to the zeroth-order phase transition; see [26,35] for a detailed discussion. To investigate the AdS to dS phase transition, we consider Fig. 5, displaying free energy F of the thermalon configuration as a function of the temperature $T = \beta_+^{-1}$ for several values of the coupling λ with the fixed value of the charge $\mathcal{Q}$. We have used $L = 1$, $\sigma = 1$, $d = 5$, and $\mathcal{Q} = 0.15$. From right to left: $\lambda = 0.05$ (cyan), $\lambda = 0.10$ (pink), $\lambda = 0.25$ (blue), $\lambda = 0.65$ (green), and $\lambda = 1.20$ (red). More importantly, for each value of the λ coupling of the T vs F phase diagram in Fig. 5, we point out that the upper branch beyond the cusp is unphysical branch solutions where it corresponds to $\Pi^+ = -\Pi^-$ solutions of the $V(a_\star) = 0 = V'(a_\star)$ conditions, while the lower branch is the physical solutions $\Pi^+ = \Pi^-$; see more discussions in Ref. [31]. We notice that for various ranges of temperatures the free energy at the maximum temperature of the (physical) branch is negative (i.e., less than the free energy of the thermal AdS, $F_{\mathrm{AdS}} = 0$), implying the possibility of a thermalon-mediated phase transition [26,31,35]. Note that interpolating the cusp structures of the free energy at the maximum temperature of the physical solution corresponds to the curve of the Nariai bound of the dS branch solution [31]. Additionally, we observe that the range of temperatures over which these transitions emerge increases as the coupling λ is given smaller with a small charge required. Moreover, thermalon-mediated phase transitions are possible over a wide range of temperatures for smaller values of the coupling λ, and the condition in Eq. (79) is still valid. However, for the given charge value $\mathcal{Q} = 0.15$ in Fig. 5, the phase transition is not possible for the coupling $\lambda \gtrsim 0.65$; see the green and red lines where the cusp structures of the free energy occur for $F \geq 0$. In contrast of the study of the AdS to dS phase transition in the neutral case, the phase transition takes place for the critical value of the coupling with $\lambda = 1.138$ [26]. Including the charge, however, we find that there is no phase transition (i.e., the free energy is greater than zero) for the critical value of the coupling $\lambda = 1.138$. We then extensively study by comparing the plot of the free energy F of the thermalon configuration vs the temperature $T = \beta_+^{-1}$ between the charge and the neutral models with $L = 1$, $\sigma = 1$, and $d = 5$. The red line shows F vs T of a charged case with $\lambda = 0.05$ and $\mathcal{Q} = 0.15$, while the

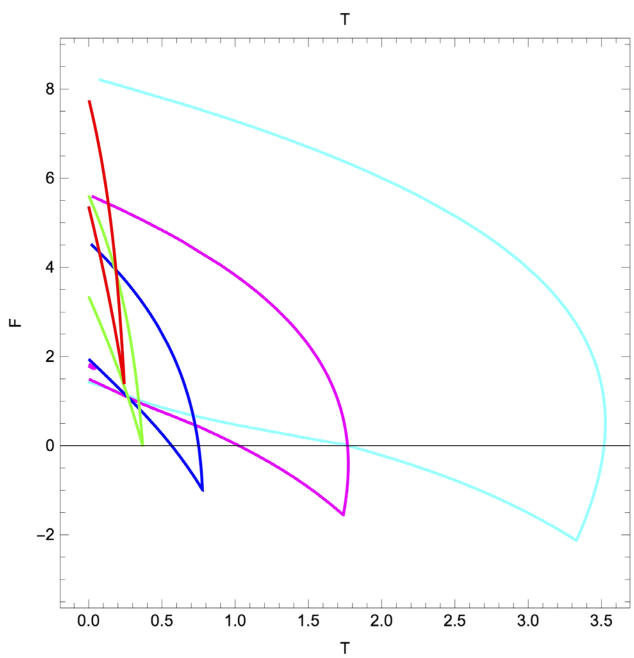

FIG. 5. The figure displays free energy F of the thermalon configuration as a function of the temperature $T = \beta_+^{-1}$ for several values of the coupling λ. We have used $L = 1$, $\sigma = 1$, $d = 5$, and $\mathcal{Q} = 0.175$. From right to left: $\lambda = 0.05$ (cyan), $\lambda = 0.10$ (pink), $\lambda = 0.25$ (blue), $\lambda = 0.65$ (green), and $\lambda = 1.20$ (red). For each value of the coupling λ, the upper branch beyond the cusp is unphysical where it corresponds to $\Pi^+ = -\Pi^-$ solutions, while the lower branch is the physical solutions of $\Pi^+ = \Pi^-$.

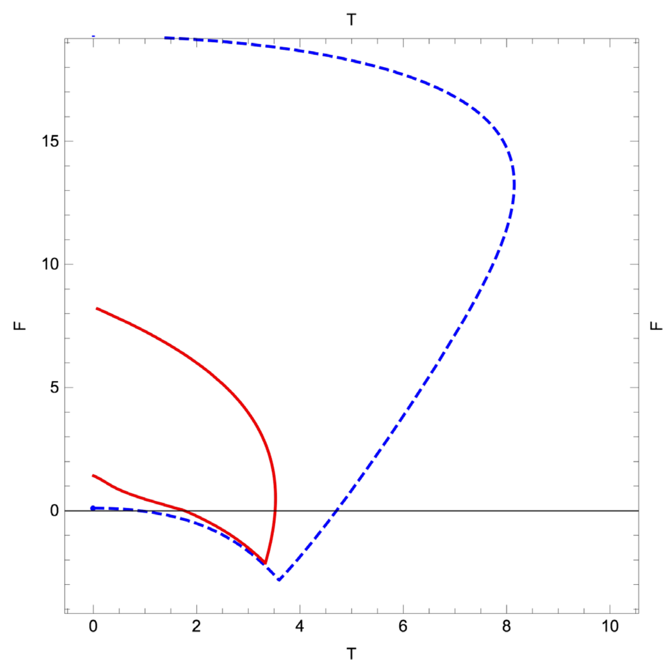

FIG. 6. The figure displays free energy F of the thermalon configuration as a function of the temperature $T = \beta_+^{-1}$. We have used $L = 1$, $\sigma = 1$, $d = 5$, and $\lambda = 0.05$. The red line shows F vs T of a charged case with $\mathcal{Q} = 0.15$, while the dashed blue line indicates F vs T of a neutral case with $\mathcal{Q} = 0$.

dashed blue line indicates F vs T of a neutral case with $\lambda = 0.05$ and $\mathcal{Q} = 0$. It is worth noting from Fig. 6 that the phase transitions of the charged case, $\mathcal{Q} \neq 0$, are possible in which the required maximum temperature of the physical branch is lower than that of the neutral case, $\mathcal{Q} = 0$. However, it is expected that in the planar $\sigma = 0$ and hyperbolic $\sigma = -1$ geometries the thermal properties and the phase transition profiles might be different. These interesting topics are worth further investigating, and we will leave these topics for our future works. Interestingly, we notice that the critical (maximum) temperature and coupling λ of the phase transitions are modified when adding the charge. We have also checked that at a fixed value of λ the critical temperature of the phase transition decreases when the charge gradually increases followed by a condition $|\mathcal{Q}| < |\mathcal{Q}_c|$. This phenomenon is similar to the physical situation in condensed matter physics, i.e., adding a charge as an impurity substitution. For instance, the conventional superconductivity is a single normal impurity with a small concentration. Increasing the size of the impurity in a fixed-size host superconductor gives a decreasing critical temperature of the host superconductor [40–42].

IV. CONCLUSION

In this work, we have revisited the toy model of the AdS to dS phase transition in higher-order gravity proposed by Ref. [26]. Notice that the gravitational phase transition for the neutral case in the vacuum solutions has been extensively studied in the literature. It was proposed that the thermalon, the Euclidean spherical thin shell, plays a crucial role of the phase transition as already mentioned in Sec. III. In other words, the thermalon changes the branches of the solutions from one branch to another via the thermal phase transition. This phenomenon is a generalization of the Hawking-Page phase transition, and it is expected to be a generic behavior of the phase transition in higher-order gravity. We then extend the study of the AdS to dS phase transition by adding the Maxwell field as an impurity substitution for investigating the profile of the phase transition in this framework. We therefore focus on five-dimensional EGBM gravity in this work. The junction condition in the EGBM theory is also constructed, and this leads to the effective potential of the thermalon in the nonextremal case ($|\mathcal{Q}| < |\mathcal{Q}_c|$). We found that the inclusion of the Maxwell field (the static charge) does not change the dynamics and stability of the thermalon as shown in Sec. II except the existence of the effective potential. As expected, we found that there are three horizons of the interior space existing in this scenario, i.e., outer event, inner event, and cosmological horizons. The thermalon radius is always located between outer event and cosmological horizons.

In addition to the study of phase transition in the thermodynamic phase space, the behaviors of the (Gibbs) free energy (F) vs temperature (T_+) exhibit a possibility of the phase transition with the presence of the static charge. The phase transition takes place when the free energy is lower than the thermal AdS space ($F_{\mathrm{AdS}} = 0$) at the maximum temperature of the (physical) branch solutions. This leads to the thermalon transition from the AdS to dS branch solutions. Comparing to the neutral case, we found that the inclusion of the static charge affects the critical higher-order coupling λ and the maximum value of the temperature of the phase transitions. For a fixed value of the charge $|\mathcal{Q}| < |\mathcal{Q}_c|$, the critical (maximum) temperature and the coupling λ of the thermalon transition are lower than the neutral case. When fixing a value of λ, the maximum temperature of the physical branch decreases if the static charge increases. According to the results presented in this work, we conclude that the inclusion of the Maxwell field (static charge) in the gravitational phase transition behaves in the same way as that of the impurity substitution in condensed matter physics as the zeroth-order phase transition. Moreover, adding a matter field in higher-order gravity does not change the profile of the phase transition. Our results agree with the claim that the gravitational AdS to dS phase transition is a generic transition mechanism of the theories of higher-order gravity.

Based on our analysis, inclusion of more complex fields, e.g., adding matter fields, might gain a deeper understanding of the dS/CFT structure. Some existing fields in string theory might reveal rich phenomena and new features of the gravitational phase transition; for instance, the three-form field is one of these interesting substitutions, and it is worth further investigating.

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