

## Tracing the cosmic history by Gauss-Bonnet gravity

Ivan de Martino<sup>1,2,3</sup>, Mariafelicia De Laurentis<sup>4,5,6</sup> and Salvatore Capozziello<sup>4,7,8,6</sup>

<sup>1</sup>*Dipartimento di Fisica, Università di Torino, Via P. Giuria 1, I-10125 Torino, Italy*

<sup>2</sup>*Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy*

<sup>3</sup>*Donostia International Physics Center (DIPC), 20018 Donostia-San Sebastian (Gipuzkoa) Spain*

<sup>4</sup>*Dipartimento di Fisica, Università di Napoli “Federico II”, Complesso Universitario di Monte Sant’Angelo, Edificio G, Via Cinthia, I-80126 Napoli, Italy*

<sup>5</sup>*INFN Sezione di Napoli, Complesso Universitario di Monte Sant’Angelo, Edificio G, Via Cinthia, I-80126 Napoli, Italy*

<sup>6</sup>*Laboratory of Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia*

<sup>7</sup>*INFN Sezione di Napoli, Compl. Universitario di Monte Sant’Angelo, Edificio G, Via Cinthia, I-80126 Napoli, Italy*

<sup>8</sup>*Gran Sasso Science Institute (INFN), Via F. Crispi 7, I-67100 L’Aquila, Italy*



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Cosmic history can be traced considering further curvature contributions inside the gravitational action. Assuming that standard general relativity can be extended by other curvature invariants, we discuss the possibility that an action containing higher-order curvature terms can fit, in principle, the whole universe evolution. In particular, a theory like  $F(R, \mathcal{G})$ , with  $R$  the Ricci scalar and  $\mathcal{G}$  the Gauss-Bonnet topological term, contains all the curvature invariants that, depending on the energy regime, can address inflation, and matter dominated and dark energy regimes. In this paper, we investigate this possibility considering how  $F(R, \mathcal{G})$  models can lead gravity from ultraviolet to infrared scales. Specifically, we will take into account a cosmographic approach for this purpose.

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### I. INTRODUCTION

According to our current perspective, the Universe seems dominated by two ingredients we find frustratingly difficult to understand at fundamental level. One of these is the *dark matter* which drives the formation of self-gravitating systems. Despite extensive searches for new candidates beyond the Standard Model of particle physics that could account for the discrepancy between luminous and non-luminous matter, no final evidence emerged so far [1].

Furthermore, the recent issue of the accelerated expansion of the Hubble flow gave rise to the possibility of another elusive component of cosmic pie, the so-called *dark energy*. Also in this case, no fundamental particle, capable of addressing the cosmological dynamics, has been detected so far (for a comprehensive review see, for instance, Refs. [2,3]). On the one hand, we need a mechanism able to cluster structures and, on the other hand, we need a mechanism able to speed up the cosmic fluid. Despite the existence of macroscopic evidences, no fundamental ingredient emerged until now to account for the 95% of cosmic matter-energy content (for a comprehensive review we refer to [4]).

This state of art triggered the development of alternative theories of gravity formulated with the aim to replace,

correct, or extend general relativity (GR) in view of accounting for the discrepancies in apparent mass and cosmic acceleration. Nevertheless GR explains a broad range of phenomena and it works very well to describe the Universe as a whole, provided dark matter and dark energy exist as separate entities. According to this fact, any alternative theory has to reproduce GR results trying to account for additional effects as those detected at infrared scales [5–9]. Furthermore, the lack of a self-consistent quantum gravity drives the search for effective theories [10] as alternatives at ultraviolet scales. These approaches led to a *semiclassical* picture where geometry is described by a space-time continuum and matter-energy side is given by some scalar fields or geometric corrections [11,12]. However, the picture is consistent if we are quite far from the Planck scales and matter energy can be averaged to give classical counterparts.

One of the common features of these effective models includes invariant terms and scalar fields in the gravitational action [13]. In particular, curvature invariants, constructed by Ricci, Riemann, and Weyl tensors, emerge as soon as renormalization and regularization of quantum fields on curved spaces are considered [11,12]. As it is well known, these terms have a cosmological impact and can trigger inflation at early epochs [14] and dark energy at late epochs [15].

The basic idea is that gravity agrees with GR at certain scales and epochs but may have quite different behaviors at very short (ultraviolet) and very large (infrared) scales. In principle, considering corrections related only with scalar curvature, one can expect effective actions like [16]

$$F(R) = .. + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} + \alpha_0 + \frac{R}{\beta_1} + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + \dots, \quad (1)$$

where negative powers work at infrared while positive powers work at ultraviolet scales [17]. Here  $\alpha_0$  has the role of a cosmological constant, and  $\alpha_i$  and  $\beta_i$  are dimensional couplings. However, this is a phenomenological approach giving just a coarse-grained picture. In a more refined theory, other curvature invariants should be considered.

According to the previous considerations, issues such as the trace anomaly or the renormalization at one-loop level require higher-order curvature invariants which are mostly related to the recovery of effective gravitational actions in curved space-time [11,12]. Specifically, these corrections to the Hilbert-Einstein action are second-order curvature invariants giving rise to fourth-order field equations in metric derivatives. Considering a generic theory containing curvature invariants means to take into account Lagrangians like  $F(R, R_{\mu\nu}, R_{\mu\nu\delta\gamma})$  and then improving the number of degrees of freedom related to the gravitational field. However, as discussed in detail in [18–20], surface terms coming from combinations of curvature invariants can reduce the complexity of these theories.

A specific role is played by the Gauss-Bonnet topological term  $\mathcal{G}$  which is a second-order combination of curvature invariants defined as

$$\mathcal{G} \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}, \quad (2)$$

where  $R$  is the curvature scalar,  $R_{\alpha\beta}$  is the Ricci tensor, and  $R_{\alpha\beta\rho\sigma}$  is the Riemann tensor. In differential geometry, it is

$$\int_{\mathcal{M}} \mathcal{G} d^n x = \chi(\mathcal{M}), \quad (3)$$

where  $\chi(\mathcal{M})$  is the Euler characteristic of a manifold  $\mathcal{M}$  in  $n$  dimensions. For  $n = 4$ ,  $\chi(\mathcal{M}) = 0$  so it can be considered a surface term not affecting dynamics. Furthermore,  $\mathcal{G}$  emerges in the trace anomaly as soon as one wants to regularize and renormalize gravity at one-loop level (see [12,13] for details). However, for any nonlinear function of  $\mathcal{G}$ , this property does not hold and then contributions to the gravitational action are nontrivial [21].

According to these considerations, a generic function  $F(R, \mathcal{G})$  can contain the whole information related to

fourth-order dynamics. In other words, a two-scalar field theory with combinations of  $R$  and  $\mathcal{G}$  well represents gravity with second-order curvature invariants.

This class of models is capable of describing several phenomena at different scales, such as the current acceleration of the Universe at late epochs and double inflation at early epochs [22,23]. Strong field phenomena, like extreme neutron star configurations, can be also framed in the context of Gauss-Bonnet gravity [24]. Moreover, this kind of theories can satisfy the Solar System tests in the weak field limit [25].

In general, Gauss-Bonnet gravity can be relevant in cosmology as shown in [21,26–46].

In this perspective, some important issues can be the following: being such a theory well motivated at UV and IR scales, is it possible to select Gauss-Bonnet gravity models capable of tracing the Universe history at any epoch? In some sense, is it possible to connect inflation and dark energy epochs, passing through matter epoch, without choosing *ad hoc* models [47,48]?

These questions may find an answer using the cosmographic approach [49,50] that can be extended also to higher redshift regimes [51–53] and, in principle, is able to constrain the value of the Hubble constant and the other cosmographic parameters. Once these parameters are determined, they can be used to fix reliable constraints on theoretical models. This method is especially suited to the study of higher-order gravity theories [54,55] like the one we shall study in the next sections.

Obviously, considering higher-order theories with more than one field, such as  $F(R, \mathcal{G})$ , involves some mathematical difficulties in handling field equations and therefore also in finding analytical expressions for the cosmographic parameters. The choice of the function  $F(R, \mathcal{G})$  is crucial in this type of approach. Therefore, assuming very general hypotheses like analyticity and derivability, one is able to obtain useful relations among cosmographic parameters and the  $n$ th derivatives  $F^{(n)}(R, \mathcal{G}) = d^n F / d(R^n \mathcal{G}^n)$  for any choice of the function  $F(R, \mathcal{G})$ .

The paper is organized as follows. In Sec. II, we sketch  $F(R, \mathcal{G})$  gravity deriving its field equations. Section III is devoted to Gauss-Bonnet cosmology. In particular, we derive the Friedman equations that will be used in the following analysis. In Sec. IV, we discuss how  $\mathcal{G}$ -terms are particularly useful in early epochs and, together with  $R^2$  corrections, give rise to inflation. The cosmographic approach is developed in Sec. V giving all the technical details of the cosmographic series. Observational data and the methodology adopted for the analysis are described in Sec. VI. In Sec. VII, the  $F(R, \mathcal{G})$  cosmography is derived together with the observational constraints on the models. Discussion and conclusions are drawn in Sec. VIII. In the Appendix, calculations adopted for the cosmographic analysis are reported.

## II. GAUSS-BONNET GRAVITY

Let us start by writing a general action for Gauss-Bonnet gravity<sup>1</sup> [22,25,56]

$$\mathcal{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} F(R, \mathcal{G}) + \mathcal{A}_M, \quad (4)$$

where  $\kappa = 8\pi G$ ,  $\mathcal{A}_M$  is the standard matter action, and  $F(R, \mathcal{G})$  is a function of the Ricci scalar and the Gauss-Bonnet topological invariant. Variation of the action (4) with respect to the metric provides the following gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}^{(\text{mat})} + T_{\mu\nu}^{(\mathcal{G}B)}, \quad (5)$$

where  $T_{\mu\nu}^{(\mathcal{G}B)}$  is defined as

$$\begin{aligned} T_{\mu\nu}^{(\mathcal{G}B)} = & \nabla_\mu \nabla_\nu F_R - g_{\mu\nu} \square F_R + 2R \nabla_\mu \nabla_\nu F_{\mathcal{G}} \\ & - 2g_{\mu\nu} R \square F_{\mathcal{G}} - 4R_\mu{}^\lambda \nabla_\lambda \nabla_\nu F_{\mathcal{G}} - 4R_\nu{}^\lambda \nabla_\lambda \nabla_\mu F_{\mathcal{G}} \\ & + 4R_{\mu\nu} \square F_{\mathcal{G}} + 4g_{\mu\nu} R^{\alpha\beta} \nabla_\alpha \nabla_\beta F_{\mathcal{G}} \\ & + 4R_{\mu\alpha\beta\nu} \nabla^\alpha \nabla^\beta F_{\mathcal{G}} - \frac{1}{2} g_{\mu\nu} (R F_R + \mathcal{G} F_{\mathcal{G}} - F) \\ & + (1 - F_R) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right), \end{aligned} \quad (6)$$

involving all extra terms with respect to GR. As shown in [57],  $T_{\mu\nu}^{(\mathcal{G}B)}$  can be recast as a perfect fluid of geometric origin.

The trace equation is

$$\begin{aligned} -2F + F_R R + 3\nabla^2 F_R + 2F_{\mathcal{G}} \mathcal{G} + 2R \nabla^2 F_{\mathcal{G}} \\ - 4R_{\rho\sigma} \nabla^\rho \nabla^\sigma F_{\mathcal{G}} = 2\kappa^2 T, \end{aligned} \quad (7)$$

and derivatives are denoted as

$$F_R \equiv \frac{\partial F(R, \mathcal{G})}{\partial R}, \quad F_{\mathcal{G}} \equiv \frac{\partial F(R, \mathcal{G})}{\partial \mathcal{G}}. \quad (8)$$

GR is recovered as soon as  $F(R, \mathcal{G}) \rightarrow R$ . If  $\mathcal{G}$  is not taken into account, we recover immediately  $F(R)$  gravity. In this sense, Gauss-Bonnet cosmology is a straightforward two-field extension of  $F(R)$  gravity.

## III. GAUSS-BONNET COSMOLOGY

Starting from the above theory, it is possible to derive the related cosmology. We consider a spatially flat Friedman-Robertson-Walker (FRW) metric like

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (9)$$

where  $a(t)$  is the scale factor of the Universe. In this background, the cosmological equations [for details see Eqs. (A3) and (A4)], without standard matter, can be written in a simplified way as [23,56]

$$\dot{H} = \Psi [H \dot{F}_R - \ddot{F}_R + 4H^3 \dot{F}_{\mathcal{G}} - 4H^2 \ddot{F}_{\mathcal{G}}], \quad (10)$$

$$H^2 = \frac{\Psi}{3} [F_R R - F(R, \mathcal{G}) - 6H \dot{F}_R + \mathcal{G} F_{\mathcal{G}}], \quad (11)$$

where we have defined

$$\Psi \equiv \frac{1}{2F_R + 8H \dot{F}_{\mathcal{G}}}. \quad (12)$$

Furthermore, two Lagrange multipliers, defining  $R$  and  $\mathcal{G}$  as functions of  $a$  and its derivative  $\dot{a}$  and  $\ddot{a}$ , have to be considered to complete the dynamical system. See (A5) reported in the Appendix.

This system of equations will be the starting point to develop our considerations at the early epoch. As said above, our aim is to track back the cosmic history investigating the curvature regimes related to  $F(R, \mathcal{G})$  gravity. As we shall see, combining information coming from the behavior of  $\mathcal{G}$  and  $R$  functions can, in principle, give a self-consistent picture of cosmic evolution at any epoch.

## IV. EARLY EPOCH COSMOLOGY

A first consideration is related to the fact that, as pointed out by Starobinsky [14], adding higher-order curvature invariants in the effective action gives rise to inflationary episodes that naturally emerge without introducing *ad hoc* inflaton fields. In other words, improving geometry by curvature invariants gives the possibility to obtain accelerated expansions.

In the present case, the Ricci and Gauss-Bonnet invariants play the role of scalar fields, whose dynamics is determined by a Klein-Gordon-like equation coming from the trace equation (7). In fact, such an equation can be recast as [22]

$$3[\square F_R + V_R] + R[\square F_{\mathcal{G}} + W_{\mathcal{G}}] = 0, \quad (13)$$

according to (8) and the definition

$$V_R = \frac{\partial V}{\partial R} = \frac{1}{3} [R F_R - 2F(R, \mathcal{G})], \quad W_{\mathcal{G}} = \frac{\partial W}{\partial \mathcal{G}} = 2 \frac{\mathcal{G}}{R} F_{\mathcal{G}}. \quad (14)$$

Here  $\square$  is the d'Alembert operator in curved space-time. Clearly, as demonstrated in [22], the dominance of one of the two terms in square brackets determines the evolution related to  $R$  or  $\mathcal{G}$ . Consequently, we expect a double inflation in which both geometric fields play a role at different scales. In other words, we can have an  $R$ -dominated inflation and a further  $\mathcal{G}$ -dominated inflation working at different scales.

<sup>1</sup>We are using physical units.

However, before starting our discussion, the possible presence of ghosts is an issue to be considered in the framework of Gauss-Bonnet gravity. It is well known that, in general, higher-order theories of gravity are not ghost-free so that viable ranges of parameters have to be selected in order to obtain self-consistent models. This is particularly relevant in cosmology to obtain reliable cosmic histories to be matched with data. Specifically, higher-derivative gravity contains ghosts due to the Ostrogradsky instability [58,59].

These ghost terms may occur at fundamental and cosmological level for  $F(R, \mathcal{G})$  gravity and can be parametrized by superluminal modes  $k^4$  where  $k$  is the wave number of the specific mode. The reason why one gets  $k^4$  is due to the fact that  $R$  evolves as  $\ell^{-2}$  and  $\mathcal{G}$  as  $\ell^{-4}$ .

To obtain viable models, procedures to eliminate ghosts have been developed. In [60], a method based on the introduction of an auxiliary scalar field  $\chi = \chi(\mathcal{G})$  into the  $F(R, \mathcal{G})$  action is proposed. Such a field quantifies the propagation of scalar modes. Then, in order to make the scalar mode not a ghost, a canonical kinetic term of  $\chi$  can be introduced in the  $F(R, \mathcal{G})$  action. See also [61]. According to this procedure, it is possible to obtain second-order field equations (and then eliminate higher than second-order derivatives) and impose suitable initial conditions determining a regular and unique evolution without ghost fields. In particular, the new dynamical degree of freedom can be eliminated adopting a ‘‘mimetic gravity’’ procedure by introducing a Lagrange multiplier [62–65]. Such a multiplier gives a natural mass constraint by which the kinetic term becomes a constant. This constraint determines the range of parameters where the model is ghost-free. Clearly, being the Gauss-Bonnet terms of fourth order, the mass parameter has to be  $\mu^4$ . See [60] for details.

With the above considerations in mind, we can take into account the simplest natural extension of the Starobinsky model [14] adding a nontrivial Gauss-Bonnet contribution, that is [22],

$$F(R, \mathcal{G}) = R + \alpha R^2 + \beta \mathcal{G}^2, \quad (15)$$

where  $\alpha$  and  $\beta$  are coupling constants with dimensions  $\ell^2$  and  $\ell^4$ , respectively. This choice means that the Gauss-Bonnet invariant gives a further scale where curvature can play a relevant role. In a scenario where the Universe is homogeneous and isotropic, it is  $\mathcal{G}^2 \sim R^4$  and this allows us to rewrite the above function as

$$F(R) \simeq R + \alpha R^2 + \beta R^4, \quad (16)$$

which is an improved  $F(R)$  model with respect to the quadratic Starobinsky one. Obviously, considering anisotropies and inhomogeneities,  $\mathcal{G}^2 \neq R^4$  due to the fact that we cannot neglect the extra diagonal components of the Ricci and Riemann tensors.

In general, an inflationary behavior is achieved if the following conditions on the Hubble parameter and its derivatives are satisfied:

$$\left| \frac{\dot{H}}{H^2} \right| \ll 1, \quad \left| \frac{\ddot{H}}{H\dot{H}} \right| \ll 1. \quad (17)$$

From the energy condition, given by Eq. (11), we have

$$12\alpha H\ddot{H} + H^2 + 36\alpha H^2\dot{H} + 288\beta H^4\dot{H}^2 + 192\beta H^5\ddot{H} + 576\beta H^6\ddot{H} - 96\beta H^8 - 6\alpha\dot{H}^2 = 0, \quad (18)$$

and from (10), we obtain

$$576\beta H^2\dot{H}^3 + 768\beta H^3\dot{H}\ddot{H} + \beta H^4(1728\ddot{H}^2 + 96\ddot{H}) + 288\beta H^5\ddot{H} - 384\beta H^6\dot{H}^2 + 18\alpha H\ddot{H} + 24\alpha\dot{H}^2 + 6\alpha\ddot{H} + \dot{H} = 0. \quad (19)$$

Imposing that  $H$  is slowly varying, which means  $\dot{H} \ll H^2$  and  $\ddot{H} \ll H\dot{H}$ , Eq. (18) takes the form

$$H^2 + 6\alpha(2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2) + 96\beta H^4(3\dot{H}^2 + 2H\ddot{H} + 6H^2\dot{H} - H^4) = 0. \quad (20)$$

To study the evolution of the model, we need to find approximate solutions of Eq. (20) in different regimes. Let us suppose that

$$6\alpha \gg 96\beta H^4. \quad (21)$$

Then Eq. (20) becomes

$$H^2 + 6\alpha(2H\ddot{H} + 6H^2\dot{H} - \dot{H}^2) \cong 0 \quad (22)$$

obtaining the well-known Starobinsky scalaron mass [14,66]

$$m_R^2 = \frac{1}{6\alpha} \quad (23)$$

and the scale factor

$$a(t) \sim \exp\left[\frac{t}{\sqrt{6\alpha}}\right]. \quad (24)$$

Considering now the regime

$$96\beta H^4 \gg 6\alpha, \quad (25)$$

we have

$$H^2 + 96\beta H^4(3\dot{H}^2 + 2H\ddot{H} + 6H^2\dot{H} - H^4) \cong 0. \quad (26)$$

From the above relation, we get an additional mass term due to the presence of higher-order correction [22]

$$m_{\mathcal{G}}^2 = \frac{1}{2\sqrt[3]{12\beta}}, \quad (27)$$

with the scale factor

$$a(t) \sim \exp\left[\frac{t}{\sqrt[6]{96\beta}}\right]. \quad (28)$$

This result shows that we have two inflationary regimes. In general, the two epochs are determined by the two terms in Eq. (13) and potentials (14) that are valid for any  $F(R, \mathcal{G})$  model.

We can conclude that considering the entire budget of curvature in the effective action, two effective masses, leading dynamics, are naturally introduced. These correspond to two different regimes ruling large scale and very large-scale structures. A part the above toy model, any  $F(R, R_{\mu\nu}, R_{\mu\nu\alpha\beta})$  can be recast as  $F(R, \mathcal{G})$  thanks to the constraint (2). This means that  $F(R, \mathcal{G})$  models are good candidates to figure out the cosmic evolution at early epochs.

However, we discussed only a toy model that could become paradigmatic for tracing any cosmic history considering the evolution of curvature contributions. Clearly the more the model is independent from the fine-tuning of parameters, the more its naturalness is recovered. In the present model, the coupling constants  $\alpha$  and  $\beta$  in Eq. (15) should be  $\mathcal{O}(1)$  in natural units where  $M_{\text{Planck}} = 1$ . In this regard, one of the main drawbacks of the Starobinsky model is that it requires the unnatural hierarchy  $\alpha \gg \beta$  to give rise to a satisfactory inflationary behavior. In our case, we have a similar fine-tuning issue so that both  $\alpha$  and  $\beta$  have to be adjusted as reported above. The problem can be partially alleviated considering that the double-inflationary regime allows us to clearly distinguish between the  $R^2$ -driven phase with respect to the  $R^4 \sim \mathcal{G}^2$ -driven phase. In other words, when the energy scales are very different, the required fine-tuning is less severe.

## V. THE COSMOGRAPHIC APPROACH

The above considerations can be extended towards late cosmic epochs adopting cosmography, which is a model independent approach to constrain cosmological evolution by observational data. It relies on the hypothesis of large-scale homogeneity and isotropy, and combines kinematic parameters via the Taylor expansion of the scale factor. The starting point is the definition of the cosmographic parameters [50]:

$$H(t) = \frac{1}{a} \frac{da}{dt}, \quad (29)$$

$$q(t) = -\frac{1}{a} \frac{d^2 a}{dt^2} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-2}, \quad (30)$$

$$j(t) = \frac{1}{a} \frac{d^3 a}{dt^3} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-3}, \quad (31)$$

$$s(t) = \frac{1}{a} \frac{d^4 a}{dt^4} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-4}, \quad (32)$$

$$l(t) = \frac{1}{a} \frac{d^5 a}{dt^5} \left[ \frac{1}{a} \frac{da}{dt} \right]^{-5}, \quad (33)$$

where, as above,  $H(t)$  is the Hubble parameter,  $q(t)$  is the deceleration parameter which accounts for the decelerating or accelerating expansion of the Universe, and  $j(t)$  and  $s(t)$  are the jerk and snap parameters, respectively. The latter may serve as a geometrical diagnostic of dark energy models [67,68]. Finally,  $l(t)$  is the lerk parameter also related to higher-order corrections of the cosmic expansion. It is then useful to relate the Hubble parameter derivative with respect to the cosmic time to the other cosmographic parameters, that is,

$$\dot{H} = -H^2(1 + q), \quad (34)$$

$$\ddot{H} = H^3(j + 3q + 2), \quad (35)$$

$$\dddot{H} = H^4[s - 4j - 3q(q + 4) - 6], \quad (36)$$

$$H^{(4)} = H^5[l - 5s + 10(q + 2)j + 30(q + 2)q + 24], \quad (37)$$

Using these definitions, it is straightforward to rewrite the Hubble parameter in terms of the cosmographic ones by expanding it in Taylor series and evaluating cosmographic parameters at the present epoch, that is, at redshift  $z \simeq 0$  [69–71]:

$$\begin{aligned} H(z) &= H_0 + \left. \frac{dH}{dz} \right|_{z=0} z + \left. \frac{1}{2!} \frac{d^2 H}{dz^2} \right|_{z=0} z^2 + \left. \frac{1}{3!} \frac{d^3 H}{dz^3} \right|_{z=0} z^3 + \dots \\ &= H_0 \left[ 1 + (1 + q_0)z + \frac{1}{2}(-q_0^2 + j_0)z^2 \right. \\ &\quad + \frac{1}{6}(3q_0^2 + 3q_0^3 - 4q_0 j_0 - 3j_0 - s_0)z^3 \\ &\quad + \frac{1}{24}(-12q_0^2 - 24q_0^3 - 15q_0^4 + 32q_0 j_0 + 25q_0^2 j_0 \\ &\quad \left. + 7q_0 s_0 + 12j_0 - 4j_0^2 + 8s_0 + l_0)z^4 \right] + \dots \end{aligned} \quad (38)$$

Starting from the Hubble parameter, it is possible to express the luminosity distance as a redshift polynomial in terms of cosmographic quantities as [51,70–74]

$$d_L(z) = cH_0^{-1}(z + \mathcal{D}_{L,1}z^2 + \mathcal{D}_{L,2}z^3 + \mathcal{D}_{L,3}z^4 + \mathcal{D}_{L,4}z^5), \quad (39)$$

where the  $\mathcal{D}_{L,i}$  quantities are defined as follows:

$$\mathcal{D}_{L,1} = \frac{1}{2}(1 - q_0), \quad (40)$$

$$\mathcal{D}_{L,2} = -\frac{1}{6}(1 - q_0 - 3q_0^2 + j_0), \quad (41)$$

$$\mathcal{D}_{L,3} = \frac{1}{24}(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0j_0 + s_0), \quad (42)$$

$$\begin{aligned} \mathcal{D}_{L,4} = & \frac{1}{120}(-6 + 6q_0 + 81q_0^2 + 165q_0^3 + 105q_0^4 \\ & + 10j_0^2 - 27j_0 - 110q_0j_0 - 105q_0^2j_0 - 15q_0s_0 \\ & - 11s_0 - l_0), \end{aligned} \quad (43)$$

whereas, the angular diameter distance can be recast as

$$d_A(z) = cH_0^{-1}(z + \mathcal{D}_{A,1}z^2 + \mathcal{D}_{A,2}z^3 + \mathcal{D}_{A,3}z^4 + \mathcal{D}_{A,4}z^5), \quad (44)$$

defining

$$\mathcal{D}_{A,1} = -\frac{1}{2}(3 + q_0), \quad (45)$$

$$\mathcal{D}_{A,2} = \frac{1}{6}(11 + 7q_0 + 3q_0^2 - j_0), \quad (46)$$

$$\begin{aligned} \mathcal{D}_{A,3} = & -\frac{1}{24}(50 + 46q_0 + 39q_0^2 + 15q_0^3 - 13j_0 \\ & - 10q_0j_0 - s_0), \end{aligned} \quad (47)$$

$$\begin{aligned} \mathcal{D}_{A,4} = & \frac{1}{120}(274 + 326q_0 + 411q_0^2 + 315q_0^3 \\ & + 105q_0^4 - 210q_0j_0 - 105q_0^2j_0 - 15q_0s_0 \\ & + -137j_0 + 10j_0^2 - 21s_0 - l_0). \end{aligned} \quad (48)$$

Cosmological observables may be used to obtain constraints on the cosmographic parameters by comparing the theoretical predictions with data [75].

### A. The $y$ -redshift cosmography

Although cosmography has been successfully used to constrain the evolution of the late Universe, it is worth noticing that, at high redshift, i.e.,  $z > 1$ , the Taylor expansion does not converge. Therefore cosmography, as introduced above, can be consistently applied only to low redshift datasets. Nevertheless, in the last decade, a lot of high redshift observations with unprecedented accuracy have been acquired. To be able to use such datasets, one has to solve the convergence problem. The latter may be

overcome reparametrizing the Taylor expansion using the expansion in  $y$  redshift [51], that is

$$y = \frac{z}{1+z} \quad (49)$$

or  $y = 1 - a(t)$ . By definition, the new parameter spans the range  $[0, 1]$ , which corresponds to a redshift range  $[0, \infty]$ , allowing us to perform high redshift cosmography.

In the  $y$ -redshift parametrization, the Hubble function can be rewritten as

$$\begin{aligned} H(y) = H_0 \left[ & 1 + (q_0 + 1)y + \frac{1}{2}(j_0 - q_0^2 + 2q_0 + 2)y^2 \right. \\ & - \frac{1}{6}(4j_0q_0 - 3j_0 - 3q_0^3 + 3q_0^2 - 6q_0 + s_0 - 6)y^3 \\ & + \frac{1}{24}(-4j_0^2 + 25j_0q_0^2 - 16j_0q_0 + 12j_0 + l_0 - 15q_0^4 \\ & \left. + 12q_0^3 - 12q_0^2 + 7q_0s_0 + 24q_0 - 4s_0 + 24)y^4 \right]. \end{aligned} \quad (50)$$

While luminosity distance can be recast as

$$d_L(y) = cH_0^{-1}(y + \mathcal{D}_{L,1}^y y^2 + \mathcal{D}_{L,2}^y y^3 + \mathcal{D}_{L,3}^y y^4 + \mathcal{D}_{L,4}^y y^5), \quad (51)$$

with

$$\mathcal{D}_{L,1}^y = \frac{1}{2}(3 - q_0)$$

$$\mathcal{D}_{L,2}^y = \frac{1}{6}(11 - 5q_0 + 3q_0^2 - j_0)$$

$$\mathcal{D}_{L,3}^y = \frac{1}{24}(50 - 26q_0 + 21q_0^2 - 15q_0^3 - 7j_0 + 10q_0j_0 + s_0)$$

$$\begin{aligned} \mathcal{D}_{L,4}^y = & \frac{1}{120}(274 - 154q_0 + 141q_0^2 - 135q_0^3 + 105q_0^4 \\ & + 10j_0^2 - 47j_0 + 90q_0j_0 - 105q_0^2j_0 - 15q_0s_0 \\ & + 9s_0 - l_0). \end{aligned} \quad (52)$$

Finally, the angular diameter distance is rewritten as

$$d_A(z) = cH_0^{-1}(z + \mathcal{D}_{A,1}^y z^2 + \mathcal{D}_{A,2}^y z^3 + \mathcal{D}_{A,3}^y z^4 + \mathcal{D}_{A,4}^y z^5), \quad (53)$$

where we have defined

$$\mathcal{D}_{A,1}^y = -\frac{1}{2}(1 + q_0), \quad (54)$$

$$\mathcal{D}_{A,2}^y = \frac{1}{6}(-j_0 + q_0(3q_0 - 2) - 10), \quad (55)$$

$$\mathcal{D}_{A,3}^y = \frac{1}{24} [j_0(10q_0 - 7) + q_0(3q_0(7 - 5q_0) + 10) + s_0 - 58], \quad (56)$$

$$\mathcal{D}_{A,4}^y = \frac{1}{120} [10j_0^2 + j_0(35(4 - 3q_0)q_0 - 22) - l_0 + q_0(3q_0(35(q_0 - 2)q_0 + 22) - 15s_0 + 196) + 14s_0 - 316]. \quad (57)$$

In the following sections, we will use the  $y$ -redshift cosmography to obtain constraints on cosmographic parameters, and on generic analytic  $F(R, \mathcal{G})$  models.

## VI. OBSERVATIONAL DATA AND METHODOLOGY

In order to constrain the cosmographic parameters, we use measurements of luminosity distances coming from supernovae type Ia (SNeIa) and gamma ray bursts (GRBs), of  $H(z)$ , and of baryonic acoustic oscillation (BAO). This will allow us to set up a large sample of data and then consistently constrain the model.

### A. Luminosity distance

We employ a catalogue of 557 SNeIa in the redshift range  $z = [0, 1.4]$  (UnionII catalogue [76]), and a list of 109 GRBs given in [77] (the catalogue was compiled using the Amati relation [78–80]), of which 50 GRBs at  $z < 1.4$  and 59 GRBs distributed in the range of redshift  $[0.1, 8.1]$ . The observable is the distance modulus  $\mu_{\text{obs}}$ , and its theoretical counterpart is given by

$$\mu_{\text{th}}(z) = 5 \log_{10} \hat{d}_L(z) + \mu_0, \quad (58)$$

where  $\hat{d}_L(z) \equiv d_L(z)/(cH_0^{-1})$  and  $\mu_0 = 42.38 - 5 \log_{10} h$ , with  $h \equiv H_0/100$ . It worth noticing that the above equation holds for both cosmographic parametrizations discussed above. Finally,  $\chi^2$  can be computed as

$$\begin{aligned} -2 \log \mathcal{L}_k(\mathbf{p}) &= \chi_k^2(\mathbf{p}) \\ &= \sum_{i=1}^{N_{\text{obj}}} \left( \frac{\mu_{\text{th}}(z_i, \mathbf{p}) - \mu_{\text{obs}}(z_i)}{\sigma_\mu(z_i)} \right)^2, \end{aligned} \quad (59)$$

where  $k = [\text{SN}, \text{GRB}]$ ,  $N_{\text{obj}} = [557; 109]$  for SNeIa and GRBs, respectively. Here,  $\sigma_\mu(z)$  is the error on  $\mu_{\text{obs}}(z)$ . Nevertheless, as is well known, the parameter  $\mu_0$  encodes  $H_0$  and it must be marginalized over. Therefore, the  $\chi^2$  function can be defined as [81–84]

$$\tilde{\chi}^2(\mathbf{p}) = \tilde{A} - \frac{\tilde{B}^2}{\tilde{C}}, \quad (60)$$

where

$$\tilde{A} = \sum_{i=1}^{N_{\text{obj}}} \left( \frac{\mu_{\text{th}}(z_i, \mathbf{p}, \mu_0 = 0) - \mu_{\text{obs}}(z_i)}{\sigma_\mu(z_i)} \right)^2, \quad (61)$$

$$\tilde{B} = \sum_{i=1}^{N_{\text{obj}}} \frac{\mu_{\text{th}}(z_i, \mathbf{p}, \mu_0 = 0) - \mu_{\text{obs}}(z_i)}{\sigma_\mu^2(z_i)}, \quad (62)$$

$$\tilde{C} = \sum_{i=1}^{N_{\text{obj}}} \frac{1}{\sigma_\mu^2(z_i)}. \quad (63)$$

### B. Expansion rate

An additional dataset is composed by 30 uncorrelated measurements of the expansion rate,  $H(z)$ , [85–92]. Thus, following the prescription in [93], the corresponding  $\chi^2$  can be defined as

$$\begin{aligned} -2 \log \mathcal{L}_H(\mathbf{p}) &= \chi_H^2(\mathbf{p}) \\ &= \sum_{i=1}^{30} \left( \frac{H(z_i, \mathbf{p}) - H_{\text{obs}}(z_i)}{\sigma_H(z_i)} \right)^2, \end{aligned} \quad (64)$$

where  $\sigma_H(z)$  is the error on  $H_{\text{obs}}(z)$ .

### C. Baryonic acoustic oscillations

Finally, another additional dataset we are going to use is given by the data from the 6dFGS [94], the SDSS DR7 [95], and the BOSS DR11 [96–98], which are also reported in Table I of [93]. As usual, we define the BAOs observable as  $\hat{\Xi} \equiv r_d/D_V(z)$ , where  $r_d$  is the sound horizon at the drag epoch and  $D_V$  the spherically averaged distance measure [99]

$$D_V(z) \equiv \left[ (1+z)^2 d_A^2(z) \frac{cz}{H(z)} \right]^{1/3}. \quad (65)$$

For the sake of simplicity, since we are adopting a model independent approach, we will set  $r_d = 144.57$  Mpc [100]. Finally, the  $\chi^2$  can be straightforwardly computed as

$$\begin{aligned} -2 \log \mathcal{L}_{\text{BAO}}(\mathbf{p}) &= \chi_{\text{BAO}}^2(\mathbf{p}) = \sum_{i=1}^6 \left( \frac{\hat{\Xi}(\mathbf{p}, z_i) - \Xi_{\text{obs}}(z_i)}{\sigma_\Xi(z_i)} \right)^2, \\ & \quad (66) \end{aligned}$$

where  $\sigma_\Xi(z)$  is the error on  $\Xi(z)$ .

### D. The Monte Carlo Markov chain

The theoretical counterparts are predicted using Eqs. (50), (51), and (53), and fit to the aforementioned datasets in order to compute the log-likelihood,  $-2 \log \mathcal{L} = \chi^2(\mathbf{p})$  where  $\mathbf{p} = [H_0, q_0, j_0, s_0, l_0]$  are the parameters of the model.

TABLE I. Parameter space explored by the MCMC algorithm. The first column lists the parameters, the second column lists the prior used in our MCMC pipeline, and, finally, the third column lists the best fit values.

Parameter	Priors	Best Fit
$H_0$	[50, 100]	$69.84^{+0.83}_{-0.82}$
$q_0$	[-1, 1]	$-0.16^{+0.04}_{-0.05}$
$j_0$	[-20, 20]	$-13.25^{+0.83}_{-0.62}$
$s_0$	[-200, 200]	$-138.47^{+13.52}_{-10.21}$
$l_0$	[-200, 200]	$36.41^{+18.64}_{-16.80}$

We explore the parameter space with a Monte Carlo Markov chain (MCMC) employing a Metropolis-Hastings [101,102] sampling algorithm. The step size is adapted to guarantee an acceptance rate between 20% and 50% [103,104], and the convergence is ensured by the Gelman-Rubin criteria [105]. Finally, the different chains are merged to constrain the model parameters. Our priors are listed in Table I.

Once our MCMC reaches the convergence, we join together all likelihoods related to different datasets,  $\mathcal{L}(\mathbf{p}) = \prod_i \mathcal{L}_i$  where  $i$  indicates the given dataset, to obtain the 2D joint contours represented in Fig. 1, and the best fit parameters summarized in Table I. On the one side, our purpose is not just related to find the best fit cosmographic parameters; thus, we only show the joint contours at 68% and 95% of confidence levels, and then report the corresponding best fit parameters, without analyzing the constraining power of each dataset. On the other side, having the posterior distribution of the cosmographic parameters, it will allow us to predict, adopting a Monte Carlo sampling of those distributions, the corresponding values of the theoretical parameters of the  $F(R, \mathcal{G})$  gravity models, once we relate them to the cosmographic ones. Hence, in the next sections we will introduce the  $F(R, \mathcal{G})$  cosmography.

## VII. $F(R, \mathcal{G})$ COSMOGRAPHY

Assuming again a flat FRW metric, we will compute the equations to describe the cosmological evolution in terms of cosmographic parameters. Here, to make the text

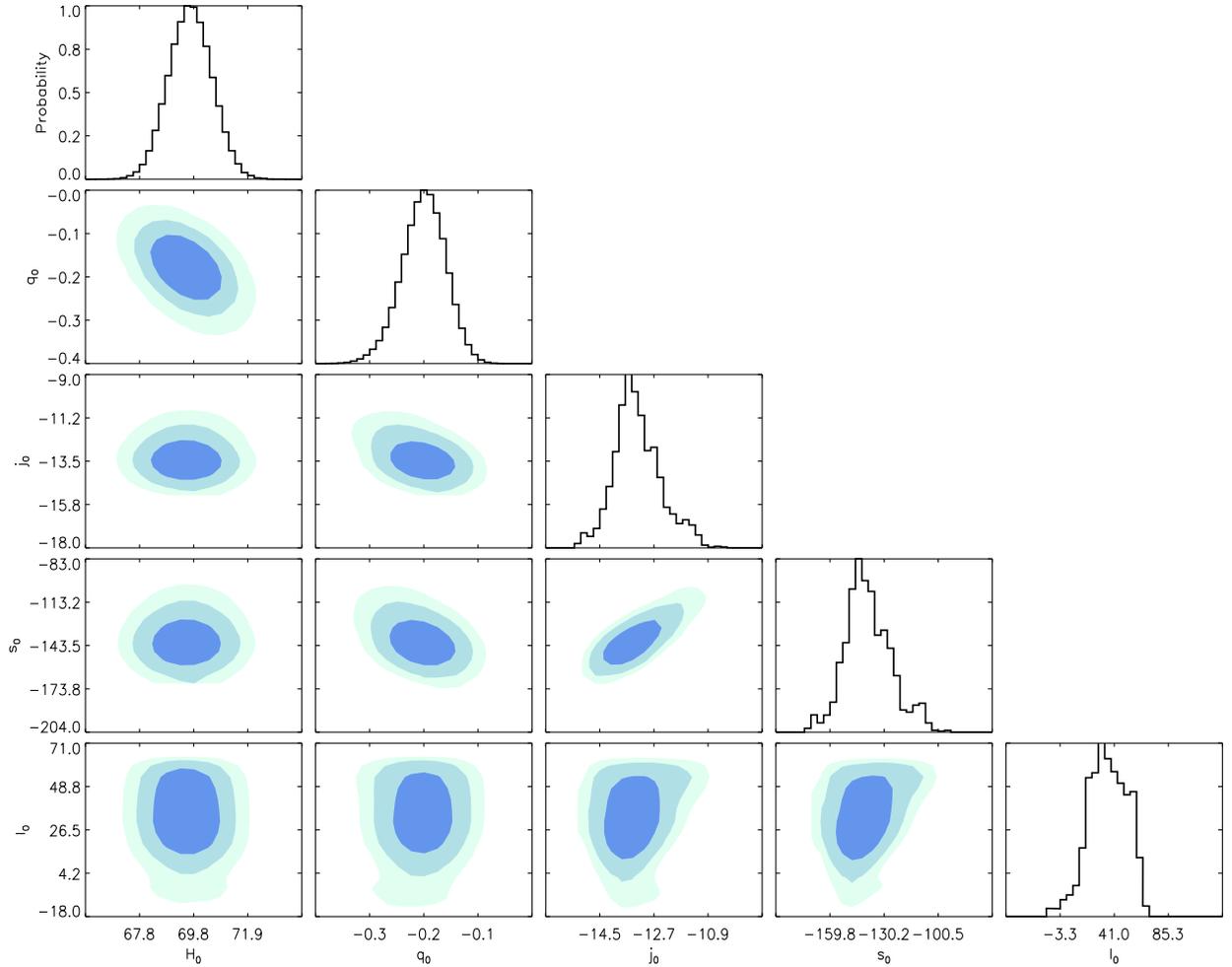


FIG. 1. Two dimensional joint contours of the cosmographic parameters obtained using SNeIa + BAO + H(z) + GRB.

readable, we summarize the whole procedure and report only the final equations needed to translate the constraints from the cosmographic to the theoretical parameters of  $F(R, \mathcal{G})$  models. For details in calculations, we refer the reader to the Appendix.

Starting from Eqs. (A1) and (A2), we assume that  $F(R, \mathcal{G})$  is expandable in the Taylor series, that is,

$$\begin{aligned} F(R, \mathcal{G}) \approx & F(R_0, \mathcal{G}_0) + F_{\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} + \frac{1}{2}F_{\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2 \\ & + R \left( F_R(R_0, \mathcal{G}_0) + F_{R\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} \right. \\ & \left. + \frac{1}{2}F_{R\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2 \right) \\ & + R^2 \left( \frac{F_{RR}(R_0, \mathcal{G}_0)}{2} + \frac{1}{2}F_{RR\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} \right. \\ & \left. + \frac{1}{4}F_{RR\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2 \right). \end{aligned} \quad (67)$$

Supposing that the models may be well approximated by the second-order Taylor expansion in  $(R - R_0)$  and  $(\mathcal{G} - \mathcal{G}_0)$ , one can evaluate the main equations describing the cosmological evolution, that is, (A3) and (A4) at the present day, using the definition of the Ricci and Gauss-Bonnet scalars in term of the cosmographic parameters. To this end, we need to consider the time derivative of (A4) and time derivatives of  $R$  and  $\mathcal{G}$ . The result is the system of equations in (A6)–(A16). Then, we have to solve the system of equations with respect to the present day values of  $F(R, \mathcal{G})$  and its derivatives up to the second order of the expansion. A reasonable approximation is to neglect all terms beyond  $R^3$  in the Taylor expansion, which means we set the following conditions:

$$F_{\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = F_{R\mathcal{G}}(R_0, \mathcal{G}_0) = F_{R\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = 0, \quad (68)$$

and

$$F_{RR\mathcal{G}}(R_0, \mathcal{G}_0) = F_{RR\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = 0. \quad (69)$$

Finally, since we want to recover GR at lower order, we set the prior:

$$F_R(R_0, \mathcal{G}_0) = 1. \quad (70)$$

In such a way, one obtains

$$\frac{F(R_0, \mathcal{G}_0)}{H_0^2} = \frac{\mathcal{P}_1(q_0, j_0, s_0, l_0) + \mathcal{P}_2(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (71)$$

$$\frac{F_{RR}(R_0, \mathcal{G}_0)}{(6H_0^2)^{-1}} = \frac{\mathcal{P}_3(q_0, j_0, s_0, l_0) + \mathcal{P}_4(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (72)$$

$$\frac{F_{R\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)}{(48H_0^4)^{-1}} = \frac{\mathcal{P}_5(q_0, j_0, s_0, l_0) + \mathcal{P}_6(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \quad (73)$$

where we have defined the auxiliary functions:  $\mathcal{P}_1(q_0, j_0, s_0, l_0)$ ,  $\mathcal{P}_2(q_0, j_0, s_0, l_0, \Omega_M)$ ,  $\mathcal{P}_3(q_0, j_0, s_0, l_0)$ ,  $\mathcal{P}_4(q_0, j_0, s_0, l_0, \Omega_M)$ ,  $\mathcal{P}_5(q_0, j_0, s_0, l_0)$ ,  $\mathcal{P}_6(q_0, j_0, s_0, l_0, \Omega_M)$ , and  $\mathcal{R}(q_0, j_0, s_0, l_0)$ , which are reported in the Appendix; see Eqs. (A28)–(A34).

Assumptions in Eqs. (68)–(70) are made in order to (i) analyze the  $F(R, \mathcal{G})$  model without any terms contributing as a cosmological constant; thus, we force the zero order of the Taylor expansion to be zero and (ii) have only the Ricci scalar as first term of the Taylor expansion to recover the GR. In such a way we are focusing on the contributions given by higher-order terms in  $R$  and  $\mathcal{G}$  to the cosmological evolution and, also, we are avoiding to consider extra equations in our systems which would lead to the introduction of extra cosmographic parameters that would decrease the constraining power of the dataset.

Equations (71)–(73) allow us to use constraints on the cosmographic parameters, given in Table I, to bound the derivatives of the Taylor expansion of the  $F(R, \mathcal{G})$  model. Thus, we carried out 1000 Monte Carlo simulations randomly choosing the values of the cosmographic parameters from their posterior distribution, and deriving the 2D contours and their best values. Final results are summarized in Table II, and in Fig. 2.

Results deserve some comments. As it is possible to see from the bounded values, higher-order Gauss-Bonnet terms do not affect the late-time cosmological evolution. It is fully driven by the  $R^2$ -term of the Taylor expansion. In other words, the  $F(R, \mathcal{G})$  theory reduces to  $F(R)$  models when late time evolution is considered.

This output is rather expected because the Gauss-Bonnet invariant, scaling as quadratic gravity, works very well in high curvature regimes as those of primordial epochs. As shown above, a term like  $\mathcal{G}^2$  scales as  $R^4$  and then it is effective at very high energies naturally producing inflationary behaviors. Clearly, it must be negligible at late time.

Furthermore, bounds on the second derivative with respect to the Ricci scalar are consistent with constraints obtained in  $F(R)$  gravity as shown, for example, in [54,106]. This may be considered as a self-consistent test for the procedure. In a different perspective, it seems that

TABLE II. Best fit parameters of the  $F(R, \mathcal{G})$  model corresponding to the one dimensional posterior distribution in Fig. 2.

Parameter	Best Fit
$F_{00}(10^3)$	$31.73^{+0.58}_{-0.57}$
$F_{RR}(10^{-6})$	$-0.95^{+0.05}_{-0.08}$
$F_{RG}(10^{-11})$	$1.97^{+0.17}_{-0.12}$

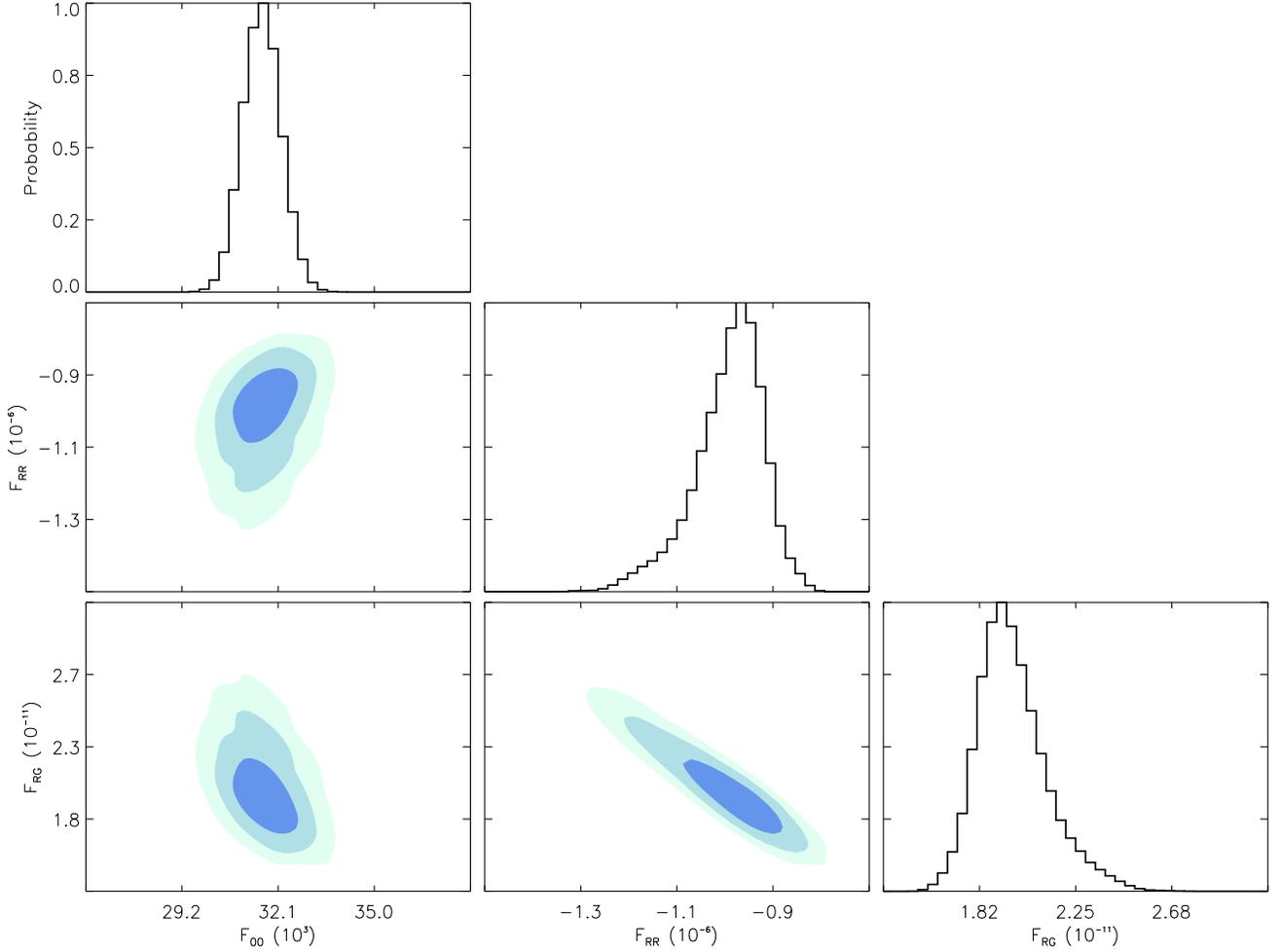


FIG. 2. Two dimensional joint contours of the  $F(R, \mathcal{G})$  model's parameters corresponding to a Monte Carlo sampling of the posterior distribution in Fig. 1.

cosmological evolution is efficiently described by two fields at early epoch while one field is sufficient to describe late epochs. In other words, the Klein-Gordon dynamics, given by (13), turns on or turns off the scalar fields (related to  $R$  and  $\mathcal{G}$ ) according to the scale.

## VIII. DISCUSSION AND CONCLUSIONS

One of the main goals of modern cosmology is to construct self-consistent models capable of tracking cosmic history from early to late epochs. According to this program, inflation, dark energy, and dark matter issues have to be considered in order to describe evolution at any era.

As, up to now, the dark side is so elusive because no final matter candidates have been detected at the fundamental level, improving the geometric sector seems a reliable approach. Adding further curvature and torsion invariants is a paradigm supported by effective gravitational theories formulated in curved space-time: these terms emerge in the action as soon as one faces the problem to regularize and renormalize the theory. Specifically, terms containing

$R^2$ ,  $R_{\mu\nu}R^{\mu\nu}$ , and  $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ , eventually constrained by the Gauss-Bonnet invariant  $\mathcal{G}$ , have to be considered in any approach aimed at obtaining a theory of gravity renormalized at one-loop level [12].

In this perspective, any cosmological model that wants to take advantage from these extended theories with respect to GR has to consider such invariants.

Models like  $F(R, \mathcal{G})$ , in principle, take into account second-order curvature invariants that give rise to fourth-order field equations by varying with respect to the metric. Considering  $\mathcal{G}$  as a constraint for the other terms means that such an action is capable of representing all the effective degrees of freedom related to second-order curvature invariants and, furthermore, one is dealing with an effective theory with two-scalar fields.

In this paper, we discussed  $F(R, \mathcal{G})$  models at early and late time epochs, which means at ultraviolet and infrared regimes.

The main result at early epochs is that a double inflation, depending on  $\mathcal{G}$  and  $R$ , can be naturally achieved. This feature is very important to produce large and very large

structures and to give a gateway mechanism to regularize the inflation (see for example [107]).

A cosmographic approach has been adopted at late epochs for redshift  $z \rightarrow 0$ . Without choosing specific  $F(R, \mathcal{G})$  models, we investigated if GR extensions, due to  $R$  and  $\mathcal{G}$ , affect dynamics at recent time. Among the priors, we imposed that GR has to be recovered and cosmographic parameters, in the observed ranges, are restored. In the approximation, we excluded adding further cosmographic parameters due to the order of approximation. The result is that  $F(R)$  corrections have to be included while  $\mathcal{G}$  and its functions are negligible at late time. The interpretation of this fact is quite straightforward: the dark energy regime can be restored just considering an effective field [i.e., the field related to  $F(R)$ ] and two-scalar fields are not necessary at late epochs.

This result is coherent with the following fact. It is well known that terms like  $\mathcal{G}^2$  can give rise to phantom solutions in the limit  $t \rightarrow \infty$ . In this case, the de Sitter phase is asymptotically unstable as reported in literature (see [26–46] and references therein). In the present study, however, this problem is overcome because terms like  $\mathcal{G}^2$  dominate at  $t \rightarrow 0$ , leading the first inflationary phase while they decay at late epochs: here  $R$ -terms are dominating and then the de Sitter solution results are stable and phantom-free. From an observational viewpoint, this statement is supported by the above contour plots based on the reported datasets: cosmographic results exclude the contributions of  $\mathcal{G}$ -terms into late time dynamics and, as a consequence, phantom instability is also excluded from the ranges of parameters. In other words, the best fit values in Table I for the cosmographic parameters and in Table II for  $F(R, \mathcal{G})$  parameters exclude phantom behaviors.

Clearly, what we discussed here is a coarse-grained approach that deserves further investigation and more refined studies. This will be the topic of a next paper.

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## APPENDIX: DETAILED CALCULATIONS FOR $F(R, \mathcal{G})$ COSMOGRAPHY

In the FRW background in Eq. (9) with a perfect fluid equation of state for ordinary matter, the field equations for  $F(R, \mathcal{G})$  gravity are given by

$$\begin{aligned} \left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right)F_R &= -\frac{\kappa}{2}(p^{(m)} + \rho^{(m)}) + \frac{1}{2} \left[ \left(\frac{\dot{a}}{a}\right)\dot{F}_R - \ddot{F}_R \right. \\ &\quad + 4\left(\frac{\dot{a}}{a}\right)^3\dot{F}_G - 8\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right)\dot{F}_G \\ &\quad \left. - 4\left(\frac{\dot{a}}{a}\right)^2\ddot{F}_G \right], \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} F_R\left(\frac{\dot{a}}{a}\right)^2 &= \frac{\kappa}{3}\rho^{(m)} + \frac{1}{6} \left[ RF_R - F(R, \mathcal{G}) - 6\left(\frac{\dot{a}}{a}\right)\dot{F}_R \right. \\ &\quad \left. + \mathcal{G}F_G - 24\left(\frac{\dot{a}}{a}\right)^3\dot{F}_G \right], \end{aligned} \quad (\text{A2})$$

where  $\rho^{(m)}$  and  $p^{(m)}$  are the energy density and pressure of ordinary matter, respectively, and the overdot denotes a derivative with respect to the time coordinate,  $t$ .

Assuming that  $\rho^{(m)} = 3H_0^2\Omega_m a^{-3}(t)$  and  $p^{(m)} = 0$ , and expanding the time derivatives, the above equations (A1) and (A2) become

$$\begin{aligned} F_R H^2 &= \frac{1}{6}(-6H(4H^2(\dot{\mathcal{G}}F_{\mathcal{G}\mathcal{G}} + \dot{R}F_{R\mathcal{G}}) + \dot{\mathcal{G}}F_{R\mathcal{G}} + \dot{R}F_{RR})) \\ &\quad + \mathcal{G}F_G + RF_R - F + \frac{H_0^2\kappa\Omega_m}{a^3}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} 2F_R\dot{H} &= -\ddot{\mathcal{G}}F_{R\mathcal{G}} + H(\dot{\mathcal{G}}(F_{R\mathcal{G}} - 8\dot{H}F_{\mathcal{G}\mathcal{G}}) + \dot{R}(F_{RR} - 8\dot{H}F_{R\mathcal{G}})) \\ &\quad + 4H^3(\dot{\mathcal{G}}F_{\mathcal{G}\mathcal{G}} + \dot{R}F_{R\mathcal{G}}) \\ &\quad - \dot{\mathcal{G}}(\dot{\mathcal{G}}F_{R\mathcal{G}\mathcal{G}} + 2\dot{R}F_{RR\mathcal{G}}) - 4H^2(\ddot{\mathcal{G}}F_{\mathcal{G}\mathcal{G}} + 2\dot{\mathcal{G}}\dot{R}F_{R\mathcal{G}\mathcal{G}} \\ &\quad + \dot{\mathcal{G}}^2F_{\mathcal{G}\mathcal{G}\mathcal{G}} + \dot{R}\dot{F}_{R\mathcal{G}} + \dot{R}^2F_{RR\mathcal{G}}) \\ &\quad - \dot{R}F_{RR} - \dot{R}^2F_{RRR} - \frac{3H_0^2\kappa\Omega_m}{a^3}. \end{aligned} \quad (\text{A4})$$

For the sake of simplicity, we have used the following abbreviations:  $F \equiv F(R, \mathcal{G})$ ,  $F_x \equiv F_x(R, \mathcal{G})$  where  $x$  can be the Ricci or the Gauss-Bonnet scalar, as well as any combination of them indicating derivatives with respect to these variables and, finally, we have also defined  $a \equiv a(t)$ ,  $H \equiv H(t)$ ,  $R \equiv R(t)$ , and  $\mathcal{G}(t) \equiv \mathcal{G}$ . To complete the dynamical system, we have to consider constraints coming from the definitions of  $R$  and  $\mathcal{G}$ , that is,

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right], \quad \text{and} \quad \mathcal{G} = 24\left(\frac{\ddot{a}\dot{a}^2}{a^3}\right). \quad (\text{A5})$$

These last equations are Lagrange multipliers that constrain dynamics. See [56] for details. In order to get a close system of equations useful for cosmographic analysis, let us differentiate Eq. (A4) with respect to  $t$ . We obtain

$$\begin{aligned}
2F_R^2\ddot{H} = & 4H^3[(F_{GGG}F_R - F_{GG}F_{RG})\dot{\mathcal{G}}^2 - \dot{R}(F_{RG}^2 - 2F_RF_{RGG} + F_{GG}F_{RR})\dot{\mathcal{G}} + \ddot{\mathcal{G}}F_{GG}F_R \\
& + F_{RG}(\ddot{R}F_R - \dot{R}^2F_{RR}) + \dot{R}^2F_RF_{RRG}] + 4H^2[(F_{GGG}F_{RG} - F_{GGGG}F_R)\dot{\mathcal{G}}^3 \\
& + \dot{R}(2F_{RG}F_{RGG} - 3F_RF_{RGGG} + F_{GGG}F_{RR})\dot{\mathcal{G}}^2 + (2F_{RGG}F_{RR}\dot{R}^2 + F_{RG}F_{RRG}\dot{R}^2 - 3F_RF_{RRGG}\dot{R}^2 \\
& + \ddot{R}F_{RG}^2 + 3\dot{H}F_{GG}F_R + \ddot{\mathcal{G}}(F_{GG}F_{RG} - 3F_{GGG}F_R) - 3\dot{R}F_RF_{RGG})\dot{\mathcal{G}} - G^{(3)}F_{GG}F_R \\
& + 3\dot{H}\dot{R}F_RF_{RG} - R^{(3)}F_RF_{RG} - 3\dot{R}\ddot{\mathcal{G}}F_RF_{RGG} + \dot{R}\ddot{\mathcal{G}}F_{GG}F_{RR} + \dot{R}\ddot{R}F_{RG}F_{RR} - 3\dot{R}\ddot{R}F_RF_{RRG} \\
& + \dot{R}^3F_{RR}F_{RRG} - \dot{R}^3F_RF_{RRRG}] + [(F_{RG}F_{RGG} - F_RF_{RGGG})\dot{\mathcal{G}}^3 + \dot{R}\dot{\mathcal{G}}^2(F_{RGG}F_{RR} + 2F_{RG}F_{RRG} \\
& - 3F_RF_{RRGG}) + (-8F_{GG}F_R\dot{H}^2 + F_RF_{RG}\dot{H} + \ddot{\mathcal{G}}(F_{RG}^2 - 3F_RF_{RGG}) + \ddot{R}F_{RG}F_{RR} - 3\dot{R}F_RF_{RRG} \\
& + 2\dot{R}^2F_{RR}F_{RRG} + \dot{R}^2F_{RG}F_{RRR} - 3\dot{R}^2F_RF_{RRRG})\dot{\mathcal{G}} + \dot{R}\ddot{R}F_{RR}^2 - 8\dot{H}^2\dot{R}F_RF_{RG} - G^{(3)}F_RF_{RG} \\
& + \dot{H}\dot{R}F_RF_{RR} - R^{(3)}F_RF_{RR} + \dot{R}\ddot{\mathcal{G}}F_{RG}F_{RR} - 3\dot{R}\ddot{\mathcal{G}}F_RF_{RRG} - 3\dot{R}\ddot{R}F_RF_{RRR} + \dot{R}^3F_{RR}F_{RRR} \\
& - \dot{R}^3F_RF_{RRRR}] + H\left\{[\dot{\mathcal{G}}(8\dot{H}F_{GG} - F_{RG}) + \dot{R}(8\dot{H}F_{RG} - F_{RR})](\dot{\mathcal{G}}F_{RG} + \dot{R}F_{RR})\right. \\
& + F_R\left[(F_{RGG}\dot{\mathcal{G}}^2 + (2\dot{R}F_{RRG} - 8\dot{H}F_{GG})\dot{\mathcal{G}} + (\ddot{\mathcal{G}} - 8\dot{R}\ddot{H})F_{RG} + \ddot{R}F_{RR}\right. \\
& \left. - 16\dot{H}(F_{GGG}\dot{\mathcal{G}}^2 + 2\dot{R}F_{RGG}\dot{\mathcal{G}} + \ddot{\mathcal{G}}F_{GG} + \ddot{R}F_{RG} + \dot{R}^2F_{RRG}) + \dot{R}^2F_{RRR}\right] + \frac{9H_0^2\kappa\Omega_m}{a^3}\left. \right\} \\
& + \frac{3H_0^2\kappa\Omega_m}{a^3}(\dot{\mathcal{G}}F_{RG} + \dot{R}F_{RR}). \tag{A6}
\end{aligned}$$

In addition to the previous equations, we may define the Ricci and Gauss-Bonnet scalars and their derivatives with respect to the cosmic time  $t$  as a function of the Hubble parameter  $H$  and its derivatives. Thus, we have

$$R = 6(2H^2 + \dot{H}), \tag{A7}$$

$$\dot{R} = 6(\ddot{H} + 4H\dot{H}), \tag{A8}$$

$$\ddot{R} = 6(H^{(3)} + 4H\ddot{H} + 4\dot{H}^2), \tag{A9}$$

$$R^{(3)} = 6(H^{(4)} + 4HH^{(3)} + 12\dot{H}\ddot{H}), \tag{A10}$$

and

$$\mathcal{G} = 24H^2(H^2 + \dot{H}), \tag{A11}$$

$$\dot{\mathcal{G}} = 48H\dot{H}(\dot{H} + H^2) + 24H^2(\ddot{H} + 2H\dot{H}), \tag{A12}$$

$$\ddot{\mathcal{G}} = 48\dot{H}^2(\dot{H} + H^2) + 48H(\dot{H} + H^2)\ddot{H} + 96H\dot{H}(\ddot{H} + 2H\dot{H}) + 24H^2(H^{(3)} + 2H\ddot{H} + 2\dot{H}^2), \tag{A13}$$

$$\begin{aligned}
\mathcal{G}^{(3)} = & 48H^{(3)}H(\dot{H} + H^2) + 144H\ddot{H}(\ddot{H} + 2H\dot{H}) + 144\dot{H}(\dot{H} + H^2)\ddot{H} + 144\dot{H}^2(\ddot{H} + 2H\dot{H}) \\
& + 144H\dot{H}(H^{(3)} + 2H\ddot{H} + 2\dot{H}^2) + 24H^2(H^{(4)} + 2HH^{(3)} + 6\dot{H}\ddot{H}), \tag{A14}
\end{aligned}$$

Let us now suppose that the  $F(R, \mathcal{G})$  Lagrangian may be well approximated by its second-order Taylor expansion in  $(R - R_0)$  and  $(\mathcal{G} - \mathcal{G}_0)$ . We set

$$\begin{aligned}
F(R, \mathcal{G}) \approx & F(R_0, \mathcal{G}_0) + F_{\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} + \frac{1}{2}F_{\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2 + R\left(F_R(R_0, \mathcal{G}_0) + F_{R\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} + \frac{1}{2}F_{R\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2\right) \\
& + R^2\left(\frac{F_{RR}(R_0, \mathcal{G}_0)}{2} + \frac{1}{2}F_{RR\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G} + \frac{1}{4}F_{RR\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0)\mathcal{G}^2\right).
\end{aligned}$$

We can make further assumptions to reduce the complexity of the problem. We want our model to be an extension of GR then we retain corrections related to the Gauss-Bonnet invariant up to  $R^3$ . The reason for this choice is to simplify the problem in view of obtaining analytic solutions, and to avoid the introduction of new cosmographic parameters beyond the parameter  $l$ , which would be needed if higher-order corrections to GR are taken into account. All those assumptions can be translated in the following bounds:

$$F_{\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = F_{R\mathcal{G}}(R_0, \mathcal{G}_0) = F_{R\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = F_{RR\mathcal{G}}(R_0, \mathcal{G}_0) = F_{RR\mathcal{G}\mathcal{G}}(R_0, \mathcal{G}_0) = 0, \quad (\text{A15})$$

$$F_R(R_0, \mathcal{G}_0) = 1. \quad (\text{A16})$$

Inserting Eqs. (A7)–(A16) in Eqs. (A3)–(A6), one gets

$$(12\mathcal{D})^{-1} \left[ \frac{H_0^2 k \Omega_m}{a^3} - (2F_{00} + 12H(F_{R\mathcal{G}}\dot{\mathcal{G}} + F_{RR}\dot{R}) + 12H^2(F_{R\mathcal{G}}G + F_{RR}R + 1) - R(2F_{R\mathcal{G}}G + F_{RR}R) + 48F_{R\mathcal{G}}H^3\dot{R}) \right] = 0, \quad (\text{A17})$$

$$\dot{H} - (2\mathcal{D})^{-1} \left( H(\dot{R}(F_{RR} - 8F_{R\mathcal{G}}H') + F_{R\mathcal{G}}\dot{\mathcal{G}}) - F_{R\mathcal{G}}\ddot{\mathcal{G}} - 4F_{R\mathcal{G}}H^2\ddot{R} + 4F_{R\mathcal{G}}H^3\dot{R} - F_{RR}\ddot{R} - \frac{3H_0^2 k \Omega_m}{a^3} \right) = 0, \quad (\text{A18})$$

$$\begin{aligned} \ddot{H} - (2\mathcal{D}^2)^{-1} \left\{ [F_{R\mathcal{G}}^2\dot{\mathcal{G}}\ddot{\mathcal{G}} + F_{R\mathcal{G}}(\mathcal{D}(H'(\dot{\mathcal{G}} - 8H'\dot{R}) - G^{(3)}) + F_{RR}\ddot{\mathcal{G}}\dot{R} \right. \\ + F_{RR}\dot{\mathcal{G}}\ddot{R}) + F_{RR}(H'\dot{R} - R^{(3)})\mathcal{D} + F_{RR}^2\dot{R}\ddot{R}] + 4F_{R\mathcal{G}}H^2(\ddot{R}(F_{R\mathcal{G}}\dot{\mathcal{G}} \\ + F_{RR}\dot{R}) + 3H'\dot{R}\mathcal{D} - R^{(3)}\mathcal{D}) + 4F_{R\mathcal{G}}H^3(\ddot{R}\mathcal{D} - \dot{R}(F_{R\mathcal{G}}\dot{\mathcal{G}} + F_{RR}\dot{R})) \\ + H(\mathcal{D}((\ddot{R}(F_{RR} - 16F_{R\mathcal{G}}H') + F_{R\mathcal{G}}\ddot{\mathcal{G}} - 8F_{R\mathcal{G}}H''\dot{R}) + 9H_0^2 k \Omega_m) \\ \left. + (F_{R\mathcal{G}}\dot{\mathcal{G}} + F_{RR}\dot{R})(-\dot{R}(F_{RR} - 8F_{R\mathcal{G}}H') - F_{R\mathcal{G}}\dot{\mathcal{G}})) + \frac{3H_0^2 k \Omega_m}{a^3} (F_{R\mathcal{G}}\dot{\mathcal{G}} + F_{RR}\dot{R}) \right\} = 0, \quad (\text{A19}) \end{aligned}$$

where we have defined

$$\mathcal{D} = F_{R\mathcal{G}}G + F_{RR}R + 1. \quad (\text{A20})$$

The next step is to evaluate Eqs. (34)–(37) at redshift zero, and to use them in the previous equations (also evaluated at redshift zero) in order to find a relation between the derivatives of the  $F(R, \mathcal{G})$  model and the cosmographic parameters. Thus, Eqs. (A17)–(A20), evaluated at  $z = 0$ , are

$$\frac{F_{00} + 6H_0^2(24F_{R\mathcal{G}}H_0^4(2j_0 + q_0(q_0 + 2) - 2) + 3F_{RR}H_0^2(2j_0 - q_0(q_0 + 2) - 3) - k\Omega_m + 1)}{6\mathcal{K}} = 0, \quad (\text{A21})$$

$$\begin{aligned} \frac{H_0^2}{2\mathcal{K}} [48F_{R\mathcal{G}}H_0^4(-j_0(4q_0 + 5) - (q_0 + 6)q_0^2 + q_0 + s_0 + 6) \\ + 6F_{RR}H_0^2(-j_0 + 3q_0(q_0 + 3) + s_0 + 6) + 3k\Omega_m - 2q_0 - 2] = 0, \quad (\text{A22}) \end{aligned}$$

$$\begin{aligned}
& \frac{H_0^3}{2\mathcal{K}^2} [-1152F_{RG}^2H_0^8(5j_0^2 + j_0(2q_0^3 - 11q_0^2 - 5q_0 - s_0 - 6) + q_0(-l_0 + 2q_0^4 - 4q_0^2 + 4q_0s_0 + 26q_0 + 5s_0 + 18)) \\
& - 6F_{RR}H_0^2(24F_{RG}H_0^4(19j_0^2 - j_0(12q_0^3 + 56q_0^2 - 13q_0 + 3s_0) - 3l_0q_0 + 2l_0 - 38q_0^4 - 29q_0^3 + 10q_0^2s_0 + 112q_0^2 + 5q_0s_0 \\
& + 12q_0 - 12s_0 - 48) - j_0(3k\Omega_m + 5q_0 + 3) + 12kq_0\Omega_m - 3k\Omega_m + l_0 - 32q_0^2 - 55q_0 - 2s_0 - 24) - 24F_{RG}H_0^4(8j_0^2 \\
& + j_0(-3k\Omega_m + 14q_0^2 + 78q_0 + 40) + 2(q_0^2(22 - 3k\Omega_m) + l_0 + 17q_0^3 - 2q_0(3s_0 + 20) - 4(2s_0 + 9))) \\
& - 36F_{RR}^2H_0^4(j_0^2 + j_0(2q_0^2 - 8q_0 - s_0 - 15) - l_0q_0 + l_0 + 3(9q_0^3 + 14q_0^2 + q_0(s_0 - 1) - 4)) - 2j_0 + 9k\Omega_m - 6q_0 - 4] = 0,
\end{aligned} \tag{A23}$$

where we have defined

$$\mathcal{K} = 24F_{RG}H_0^4q_0 + 6F_{RR}H_0^2(q_0 - 1) - 1. \tag{A24}$$

The final step is to solve the previous equations with respect to the derivatives of  $F(R, \mathcal{G})$  to get

$$\frac{F(R_0, \mathcal{G}_0)}{H_0^2} = \frac{\mathcal{P}_1(q_0, j_0, s_0, l_0) + \mathcal{P}_2(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \tag{A25}$$

$$\frac{F_{RR}(R_0, \mathcal{G}_0)}{(6H_0^2)^{-1}} = \frac{\mathcal{P}_3(q_0, j_0, s_0, l_0) + \mathcal{P}_4(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \tag{A26}$$

$$\frac{F_{RG\mathcal{G}}(R_0, \mathcal{G}_0)}{(48H_0^4)^{-1}} = \frac{\mathcal{P}_5(q_0, j_0, s_0, l_0) + \mathcal{P}_6(q_0, j_0, s_0, l_0, \Omega_M)}{\mathcal{R}(q_0, j_0, s_0, l_0)}, \tag{A27}$$

where we have defined

$$\begin{aligned}
\mathcal{P}_1(q_0, j_0, s_0, l_0) = & -3(8j_0^3 - 24j_0^2q_0^3 - 94j_0^2q_0^2 - 64j_0^2q_0 - 8j_0^2s_0 - 14j_0^2 - 8j_0l_0(q_0 + 1) + 12j_0q_0^5 - 4j_0q_0^4 + 16j_0q_0^3 \\
& + 14j_0q_0^2s_0 + 134j_0q_0^2 - 28j_0q_0s_0 + 96j_0q_0 - 22j_0s_0 + 6j_0 + 2l_0q_0^3 - 6l_0q_0^2 - 6l_0q_0 + 2l_0 + 24q_0^6 + 6q_0^5 \\
& - 12q_0^4s_0 - 72q_0^4 - 34q_0^3s_0 - 12q_0^3 + 4q_0^2s_0 + 144q_0^2 + 12q_0s_0^2 + 98q_0s_0 + 150q_0 + 12s_0^2 + 60s_0 + 48),
\end{aligned} \tag{A28}$$

$$\begin{aligned}
\mathcal{P}_2(q_0, j_0, s_0, l_0, \Omega_M) = & 3\Omega_mk(16j_0^3 + 56j_0^2q_0 + 8j_0^2s_0 - 12j_0^2 + 8j_0l_0(q_0 + 1) + 11j_0q_0^4 + 28j_0q_0^3 + 14j_0q_0^2s_0 + 63j_0q_0^2 \\
& + 48j_0q_0s_0 - 57j_0q_0 + 12j_0s_0 - 24j_0 + 2l_0q_0^3 + 12l_0q_0^2 + 4l_0q_0 - 3l_0 + 6q_0^5 - 33q_0^4 + 8q_0^3s_0 \\
& - 162q_0^3 - 44q_0^2s_0 - 276q_0^2 - 12q_0s_0^2 - 114q_0s_0 - 138q_0 - 12s_0^2 - 45s_0 - 18, )
\end{aligned} \tag{A29}$$

$$\begin{aligned}
\mathcal{P}_3(q_0, j_0, s_0, l_0) = & 2j_0^2 - 12j_0q_0^3 - 56j_0q_0^2 - 72j_0q_0 - 2j_0s_0 - 30j_0 - 2l_0q_0 - 2l_0 - 24q_0^4 - 30q_0^3 + 12q_0^2s_0 + 60q_0^2 \\
& + 22q_0s_0 + 114q_0 + 12s_0 + 48,
\end{aligned} \tag{A30}$$

$$\mathcal{P}_4(q_0, j_0, s_0, l_0, \Omega_m) = \Omega_mk(12j_0^2 + 21j_0q_0^2 + 81j_0q_0 + 12j_0 + 3l_0 + 36q_0^3 + 6q_0^2 - 18q_0s_0 - 114q_0 - 15s_0 - 54), \tag{A31}$$

$$\mathcal{P}_5(q_0, j_0, s_0, l_0) = -24 - 6j_0 - 2j_0^2 + 2l_0 - 78q_0 - 12j_0q_0 + 2l_0q_0 - 84q_0^2 - 4j_0q_0^2 - 30q_0^3 + 2j_0s_0 + 2q_0s_0, \tag{A32}$$

$$\mathcal{P}_6(q_0, j_0, s_0, l_0, \Omega_m) = \Omega_mk(15j_0q_0 + 30j_0 - 3l_0 + 45q_0^2 + 72q_0 - 3s_0 + 18) \tag{A33}$$

$$\begin{aligned}
\mathcal{R}(q_0, j_0, s_0, l_0) = & [4j_0^3 + j_0^2(15q_0^2 + 56q_0 - 4s_0 + 30) - j_0(4l_0(q_0 + 1) + 16q_0^4 + 32q_0^3 + q_0^2(7s_0 + 69) + 6q_0(7s_0 + 22) \\
& + 24(s_0 + 3)) + q_0^3(-l_0 + 5s_0 + 159) + q_0^2(-9l_0 + 46s_0 + 306) + q_0(6(s_0^2 + 16s_0 + 40) - 8l_0) - 21q_0^5 \\
& + 6(s_0^2 + 8s_0 + 12)].
\end{aligned} \tag{A34}$$

These are all the ingredients used to construct our Gauss-Bonnet cosmography. See also [54] for the case of  $F(R)$  gravity.

- [1] M. Tanabashi *et al.* (Particle Data Group), *Phys. Rev. D* **98**, 030001 (2018).
- [2] P. Brax, *Rep. Prog. Phys.* **81**, 016902 (2018).
- [3] D. Huterer and D. L. Shafer, *Rep. Prog. Phys.* **81**, 016901 (2018).
- [4] R. T. Génova-Santos, The establishment of the standard cosmological model through observations, in *Reviews in Frontiers of Modern Astrophysics*, edited by P. Kabáth, D. Jones, and M. Skarka (Springer, Cham, 2020).
- [5] I. de Martino, M. De Laurentis, and S. Capozziello, *Universe* **1**, 123 (2015).
- [6] S. Capozziello and M. De Laurentis, *Ann. Phys. (Amsterdam)* **524**, 545 (2012).
- [7] I. De Martino, R. Lazkoz, and M. De Laurentis, *Phys. Rev. D* **97**, 104067 (2018).
- [8] M. De Laurentis, I. De Martino, and R. Lazkoz, *Phys. Rev. D* **97**, 104068 (2018).
- [9] M. De Laurentis, I. De Martino, and R. Lazkoz, *Eur. Phys. J. C* **78**, 916 (2018).
- [10] C. J. Hogan, *Rev. Mod. Phys.* **72**, 1149 (2000).
- [11] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP, Bristol, UK, 1992).
- [12] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
- [13] S. Capozziello and M. De Laurentis, *Phys. Rep.* **509**, 167 (2011).
- [14] A. A. Starobinsky, *Phys. Lett.* **91B**, 99 (1980).
- [15] S. Capozziello, *Int. J. Mod. Phys. D* **11**, 483 (2002).
- [16] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
- [17] S. Capozziello, M. De Laurentis, and O. Luongo, *Int. J. Mod. Phys. D* **24**, 1541002 (2015).
- [18] C. Bogdanos, S. Capozziello, M. De Laurentis, and S. Nesseris, *Astropart. Phys.* **34**, 236 (2010).
- [19] F. Bajardi and S. Capozziello, *Eur. Phys. J. C* **80**, 704 (2020).
- [20] S. Capozziello, M. De Laurentis, and K. F. Dialektopoulos, *Eur. Phys. J. C* **76**, 629 (2016).
- [21] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **631**, 1 (2005).
- [22] M. De Laurentis, M. Paoletta, and S. Capozziello, *Phys. Rev. D* **91**, 083531 (2015).
- [23] M. De Laurentis, *Mod. Phys. Lett. A* **30**, 1550069 (2015).
- [24] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, *J. Cosmol. Astropart. Phys.* **01** (2015) 001.
- [25] M. De Laurentis and A. J. Lopez-Revelles, *Int. J. Geom. Methods Mod. Phys.* **11**, 1450082 (2014).
- [26] S. Nojiri and S. D. Odintsov, *J. Phys. Conf. Ser.* **66**, 012005 (2007).
- [27] S. Nojiri, S. D. Odintsov, and M. Sami, *Phys. Rev. D* **74**, 046004 (2006).
- [28] D. Bazeia, B. C. da Cunha, R. Menezes, and A. Y. Petrov, *Phys. Lett. B* **649**, 445 (2007).
- [29] G. Cognola, E. Elizalde, S. Nojiri, S. Odintsov, and S. Zerbini, *Phys. Rev. D* **75**, 086002 (2007).
- [30] B. Li, J. D. Barrow, and D. F. Mota, *Phys. Rev. D* **76**, 044027 (2007).
- [31] D. Bazeia, R. Menezes, and A. Y. Petrov, *Eur. Phys. J. C* **58**, 171 (2008).
- [32] N. Goheer, R. Goswami, P. K. S. Dunsby, and K. Ananda, *Phys. Rev. D* **79**, 121301 (2009).
- [33] N. Goheer, R. Goswami, P. K. S. Dunsby, and K. Ananda, *Phys. Lett. B* **679**, 302 (2009).
- [34] M. Mohseni, *Phys. Lett. B* **682**, 89 (2009).
- [35] H. M. Sadjadi, *Europhys. Lett.* **92**, 50014 (2010).
- [36] M. Alimohammadi and A. Ghalee, *Phys. Rev. D* **79**, 063006 (2009).
- [37] C. G. Boehmer and F. S. N. Lobo, *Phys. Rev. D* **79**, 067504 (2009).
- [38] K. Uddin, J. E. Lidsey, and R. Tavakol, *Gen. Relativ. Gravit.* **41**, 2725 (2009).
- [39] S. Y. Zhou, E. J. Copeland, and P. M. Saffin, *J. Cosmol. Astropart. Phys.* **07** (2009) 009.
- [40] M. M. Ivanov and A. V. Toporensky, *Gravitation Cosmol.* **18**, 43 (2012).
- [41] S. Nojiri, S. D. Odintsov, A. Toporensky, and P. Tretyakov, *Gen. Relativ. Gravit.* **42**, 1997 (2010).
- [42] A. Kumar Sanyal, C. Rubano, and E. Piedipalumbo, *Gen. Relativ. Gravit.* **43**, 2807 (2011).
- [43] L. N. Granda, *Mod. Phys. Lett. A* **27**, 1250018 (2012).
- [44] L. N. Granda, *Mod. Phys. Lett. A* **28**, 1350117 (2013).
- [45] M. Benetti, S. Santos da Costa, S. Capozziello, J. S. Alcaniz, and M. De Laurentis, *Int. J. Mod. Phys. D* **27**, 1850084 (2018).
- [46] S. Santos Da Costa, F. V. Roig, J. S. Alcaniz, S. Capozziello, M. De Laurentis, and M. Benetti, *Classical Quantum Gravity* **35**, 075013 (2018).
- [47] S. Nojiri and S. D. Odintsov, *Phys. Rep.* **505**, 59 (2011).
- [48] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, *Phys. Rep.* **692**, 1 (2017).
- [49] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972), pp. 407–463.
- [50] M. Visser *Gen. Relativ. Gravit.* **37**, 1541 (2005).
- [51] C. Cattoën and M. Visser, *Classical Quantum Gravity* **24**, 5985 (2007).
- [52] S. Capozziello, R. D’Agostino, and O. Luongo, *Mon. Not. R. Astron. Soc.* **494**, 2576 (2020).
- [53] M. Benetti and S. Capozziello, *J. Cosmol. Astropart. Phys.* **12** (2019) 008.
- [54] S. Capozziello, V. F. Cardone, and V. Salzano, *Phys. Rev. D* **78**, 063504 (2008).
- [55] S. Capozziello, R. D’Agostino, and O. Luongo, *Int. J. Mod. Phys. D* **28**, 1930016 (2019).
- [56] S. Capozziello, M. De Laurentis, and S. D. Odintsov, *Mod. Phys. Lett. A* **29**, 1450164 (2014).
- [57] S. Capozziello, C. A. Mantica, and L. G. Molinari, *Int. J. Geom. Methods Mod. Phys.* **16**, 1950133 (2019).
- [58] R. P. Woodard, *Scholarpedia* **10**, 32243 (2015).
- [59] A. De Felice and T. Suyama, *J. Cosmol. Astropart. Phys.* **06** (2009) 034.
- [60] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, *Phys. Rev. D* **99**, 044050 (2019).
- [61] S. Nojiri, S. D. Odintsov, and M. Sasaki, *Phys. Rev. D* **71**, 123509 (2005).
- [62] A. H. Chamseddine and V. Mukhanov, *J. High Energy Phys.* **11** (2013) 135.
- [63] S. Nojiri and S. D. Odintsov, *Mod. Phys. Lett. A* **29**, 1450211 (2014).

- [64] J. Dutta, W. Khylllep, E. N. Saridakis, N. Tamanini, and S. Vagnozzi, *J. Cosmol. Astropart. Phys.* **02** (2018) 041.
- [65] L. Sebastiani, S. Vagnozzi, and R. Myrzakulov, *Adv. High Energy Phys.* **2017**, 3156915 (2017).
- [66] A. Vilenkin, *Phys. Rev. D* **32**, 2511 (1985).
- [67] V. Sahni, T. D. Saini, A. A. Starobinsky, and U. Alam, *J. Exp. Theor. Phys. Lett.* **77**, 201 (2003).
- [68] U. Alam, V. Sahni, T. D. Saini, and A. Starobinsky, *Mon. Not. R. Astron. Soc.* **344**, 1057 (2003).
- [69] T. Chiba and T. Nakamura, *Prog. Theor. Phys.* **100**, 1077 (1998).
- [70] S. Capozziello, R. Lazkoz, and V. Salzano, *Phys. Rev. D* **84**, 124061 (2011).
- [71] M. Demianski, E. Piedipalumbo, C. Rubano, and P. Scudellaro, *Mon. Not. R. Astron. Soc.* **426**, 1396 (2012).
- [72] M. Demianski, E. Piedipalumbo, D. Sawant, and L. Amati, *Astron. Astrophys.* **598**, A113 (2017).
- [73] E. Piedipalumbo, E. Della Moglie, and R. Cianci, *Int. J. Mod. Phys. D* **24**, 1550100 (2015).
- [74] E. Piedipalumbo, E. Della Moglie, M. De Laurentis, and P. Scudellaro, *Mon. Not. R. Astron. Soc.* **441**, 3643 (2014).
- [75] A. Aviles, C. Gruber, O. Luongo, and H. Quevedo, *Phys. Rev. D* **86**, 123516 (2012).
- [76] R. Amanullah *et al.*, *Astron. Astrophys.* **486**, 375 (2008).
- [77] H. Wei, *J. Cosmol. Astropart. Phys.* **08** (2010) 020.
- [78] L. Amati, F. Frontera, and C. Guidorzi, *Astron. Astrophys.* **508**, 173 (2009).
- [79] L. Amati, F. Frontera, M. Tavani *et al.*, *Astron. Astrophys.* **390**, 81 (2002).
- [80] L. Amati, C. Guidorzi, F. Frontera, M. Della Valle, F. Finelli, R. Landi, and E. Montanari, *Mon. Not. R. Astron. Soc.* **391**, 577 (2008).
- [81] E. Di Pietro and J. F. Claeskens, *Mon. Not. R. Astron. Soc.* **341**, 1299 (2003).
- [82] S. Nesseris and L. Perivolaropoulos, *Phys. Rev. D* **72**, 123519 (2005).
- [83] L. Perivolaropoulos, *Phys. Rev. D* **71**, 063503 (2005).
- [84] H. Wei, *Phys. Lett. B* **687**, 286 (2010).
- [85] R. Jimenez and A. Loeb, *Astrophys. J.* **573**, 37 (2002).
- [86] J. Simon, L. Verde, and R. Jimenez, *Phys. Rev. D* **71**, 123001 (2005).
- [87] D. Stern, R. Jimenez, L. Verde, S. A. Stanford, and M. Kamionkowski, *Astrophys. J. Suppl. Ser.* **188**, 280 (2010).
- [88] C. Zhang, H. Zhang, S. Yuan, S. Liu, T. J. Zhang, and Y.-C. Sun, *Res. Astron. Astrophys.* **14**, 1221 (2014).
- [89] M. Moresco, *Mon. Not. R. Astron. Soc.* **450**, L16 (2015).
- [90] M. Moresco, L. Verde, L. Pozzetti, R. Jimenez, and A. Cimatti, *J. Cosmol. Astropart. Phys.* **07** (2012) 053.
- [91] M. Moresco, A. Cimatti, R. Jimenez *et al.*, *J. Cosmol. Astropart. Phys.* **08** (2012) 006.
- [92] M. Moresco, L. Pozzetti, A. Cimatti, R. Jimenez, C. Maraston, L. Verde, D. Thomas, A. Citro, R. Tojeiro, and D. Wilkinson, *J. Cosmol. Astropart. Phys.* **05** (2016) 014.
- [93] V. V. Luković, R. D'Agostino, and N. Vittorio, *Astron. Astrophys.* **595**, A109 (2016).
- [94] F. Beutler, C. Blake, M. Colless, D. H. Jones, L. Staveley-Smith, L. Campbell, Q. Parker, W. Saunders, and F. Watson, *Mon. Not. R. Astron. Soc.* **416**, 3017 (2011).
- [95] A. J. Ross, L. Samushia, C. Howlett, W. J. Percival, A. Burden, and M. Manera *Mon. Not. R. Astron. Soc.* **449**, 835 (2015).
- [96] L. Anderson, E. Aubourg, S. Bailey *et al.*, *Mon. Not. R. Astron. Soc.* **441**, 24 (2014).
- [97] T. Delubac, J. E. Bautista, N. G. Busca *et al.*, *Astron. Astrophys.* **574**, A59 (2015).
- [98] A. Font-Ribera, D. Kirkby, N. Busca *et al.*, *J. Cosmol. Astropart. Phys.* **05** (2014) 027.
- [99] D. J. Eisenstein, I. Zehavi, D. W. Hogg *et al.*, *Astrophys. J.* **633**, 560 (2005).
- [100] Planck Collaboration *et al.*, arXiv:1807.06209.
- [101] W. K. Hastings, *Biometrika* **57**, 97 (1970).
- [102] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, *J. Chem. Phys.* **21**, 1087 (1953).
- [103] A. Gelman, G. O. Roberts, and W. R. Gilks, *Bayesian Stat.* **5**, 599 (1996), <https://ci.nii.ac.jp/naid/10010345459/>.
- [104] G. O. Roberts, A. Gelman, and W. R. Gilks, *Ann. Appl. Probab.* **7**, 110 (1997).
- [105] A. Gelman and D. B. Rubin, *Stat. Sci.* **7**, 457 (1992).
- [106] A. Aviles, A. Bravetti, S. Capozziello, and O. Luongo, *Phys. Rev. D* **87**, 044012 (2013).
- [107] L. Amendola, S. Capozziello, M. Litterio, and F. Occhionero, *Phys. Rev. D* **45**, 417 (1992).