HQET vertex diagram: ε expansion

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Differential equations for the one-loop vertex diagram in heavy quark effective theory (HQET) with arbitrary self-energy insertions and arbitrary residual energies are reduced to the ε form and used to obtain the ε expansion in terms of Goncharov polylogarithms.

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We consider the one-loop vertex diagram (Fig. 1) with arbitrary degrees of all three denominators:

$$I_{n_1,n_2,n_3}(\vartheta;\omega_1,\omega_2) = \frac{1}{i\pi^{d/2}}\int\frac{d^dk}{D_1^{n_1}D_2^{n_2}D_3^{n_3}},$$
$$D_1 = -2(k+p_1)\cdot v_1,\qquad D_2 = -2(k+p_2)\cdot v_2,$$
$$D_3 = -k^2, \tag{1}$$

where $\omega_{1,2} = p_{1,2}\cdot v_{1,2}$, $\cosh\vartheta = v_1\cdot v_2$. It has obvious properties

$$I_{n_1,n_2,n_3}(\vartheta;\omega_1,\omega_2) = I_{n_1,n_2,n_3}(-\vartheta;\omega_1,\omega_2), \tag{2}$$

$$I_{n_1,n_2,n_3}(\vartheta;\omega_1,\omega_2) = I_{n_2,n_1,n_3}(\vartheta;\omega_2,\omega_1), \tag{3}$$

$$I_{n_1,0,n_3}(\vartheta;\omega_1,\omega_2) = I_{n_1,n_3}(-2\omega_1)^{d-n_1-2n_3}, \tag{4}$$

where

$$I_{n_1,n_2} = \frac{\Gamma(n_1+2n_2-d)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{5}$$

is the one-loop self-energy diagram in heavy quark effective theory (HQET).

Results exact in ε are known for $\omega_1 = \omega_2$ [1]

$$I_{n_1,n_2,n_3}(\vartheta;\omega,\omega) = I_{n_1+n_2,n_3}(-2\omega)^{d-n_1-n_2-2n_3} \times {}_3F_2\left(\left.\begin{matrix}n_1, n_2, \frac{d}{2}-n_3\\ \frac{n_1+n_2}{2}, \frac{n_1+n_2+1}{2}\end{matrix}\right|\frac{1-\cosh\vartheta}{2}\right) \tag{6}$$

and $\vartheta = 0$ [2]

$$I_{n_1,n_2,n_3}(0;\omega_1,\omega_2) = I_{n_1+n_2,n_3}(-2\omega_2)^{d-n_1-n_2-2n_3} \times {}_2F_1\left(\left.\begin{matrix}n_1, n_1+n_2+2n_3-d\\ n_1+n_2\end{matrix}\right|1-y\right) \tag{7}$$

(the symmetry (3) follows from a hypergeometric identity). Here and below we use $d = 4-2\varepsilon$,

$$x = e^{\vartheta},\qquad y = \frac{\omega_1}{\omega_2}. \tag{8}$$

We consider the one-loop vertex (Fig. 1) with any numbers of self-energy insertions into each of three lines, provided that all lines in these insertions are massless. If the full number of loops in all self-energy insertions into the line i is l_i, then $n_{1,2} = m_{1,2} + 2l_{1,2}\varepsilon$, $n_3 = m_3 + l_3\varepsilon$, where all m_i are integers. All integrals with a given set l_i can be reduced [1], using integration by parts (IBP), to three master integrals with $m_i = (0,1,1)$, $(1,0,1)$, and $(1,1,1)$. We choose the column of the basis integrals $(f_1, f_2, f_3)^T$, where

$$I_{2l_1\varepsilon,1+2l_2\varepsilon,1+l_3\varepsilon}(\vartheta;\omega_1,\omega_2) = I_{1+2(l_1+l_2)\varepsilon,1+l_3\varepsilon}(-2\omega_1)^{-l\varepsilon}(-2\omega_2)^{1-l\varepsilon}f_1(x,y),$$
$$I_{1+2l_1\varepsilon,2l_2\varepsilon,1+l_3\varepsilon}(\vartheta;\omega_1,\omega_2) = I_{1+2(l_1+l_2)\varepsilon,1+l_3\varepsilon}(-2\omega_1)^{1-l\varepsilon}(-2\omega_2)^{-l\varepsilon}f_2(x,y),$$
$$I_{1+2l_1\varepsilon,1+2l_2\varepsilon,1+l_3\varepsilon}(\vartheta;\omega_1,\omega_2) = I_{2+2(l_1+l_2)\varepsilon,1+l_3\varepsilon}(-2\omega_1)^{-l\varepsilon}(-2\omega_2)^{-l\varepsilon}f_3(x,y), \tag{9}$$

where $l = l_1+l_2+l_0$ is the total number of loops, $l_0 = l_3+1$. They have symmetry properties

$$f(x^{-1},y) = f(x,y), \tag{10}$$

$$f(x,y^{-1}) = S_y[f(x,y)]_{l_1\leftrightarrow l_2},\qquad S_y = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

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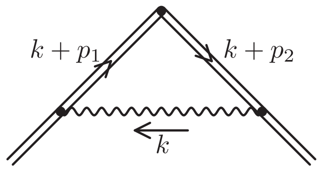

FIG. 1. The one-loop HQET vertex diagram.

The initial condition is $f(1,1) = (1,1,1)^T$. If $l_1 = 0$, f_1 is trivial (4); if $l_2 = 0$, f_2 is trivial; if $l_1 = l_2 = 0$, there is only one nontrivial master integral f_3. If $l_1 = l_2$, $f_2(x,y) = f_1(x,y^{-1})$ (10), and there are only two unknown functions f_1 and f_3.

We shall use the method of differential equations [3]. Using

$$\begin{aligned}
&\sinh\vartheta \frac{\partial}{\partial\vartheta} I_{n_1,n_2,n_3}\\
&\quad = n_1[\mathbf{1}^+\mathbf{2}^- - 2(\omega_1\cosh\vartheta - \omega_2)\mathbf{1}^+ - \cosh\vartheta] I_{n_1,n_2,n_3}\\
&\quad = n_2[\mathbf{2}^+\mathbf{1}^- - 2(\omega_2\cosh\vartheta - \omega_1)\mathbf{2}^+ - \cosh\vartheta] I_{n_1,n_2,n_3},\\
&\frac{\partial}{\partial\omega_1} I_{n_1,n_2,n_3} = 2n_1 \mathbf{1}^+ I_{n_1,n_2,n_3},\\
&\frac{\partial}{\partial\omega_2} I_{n_1,n_2,n_3} = 2n_2 \mathbf{2}^+ I_{n_1,n_2,n_3}
\end{aligned} \tag{11}$$

and the IBP reduction, we can derive the differential equations

$$\partial_x f = M_x f, \qquad \partial_y f = M_y f, \tag{12}$$

where the matrices $M_{x,y}$ (depending on x, y, and ε) satisfy

$$\partial_x M_y - \partial_y M_x - [M_x, M_y] = 0 \tag{13}$$

because $\partial_x\partial_y f = \partial_y\partial_x f$. The symmetries (10) lead to

$$\begin{aligned}
&M_x(x^{-1},y) + x^2 M_x(x,y) = 0, \quad M_y(x^{-1},y) = M_y(x,y);\\
&M_x(x,y^{-1}) = S_y[M_x]_{l_1\leftrightarrow l_2} S_y,\\
&M_y(x,y^{-1}) + y^2 S_y[M_y]_{l_1\leftrightarrow l_2} S_y = 0.
\end{aligned} \tag{14}$$

The differential equations (12) can be reduced to the canonical form [4] by a linear transformation $f = TF$ (the matrix T depends on x, y, ε),

$$dF = \varepsilon dMF, \qquad M(x,y) = \sum_i M_i \log p_i(x,y), \tag{15}$$

where $p_i(x,y)$ are polynomials in x and y, and M_i are constant matrices. We use the *Mathematica* package Libra [5] which implements the algorithm of [6], and obtain

$$\begin{aligned}
T &= \begin{pmatrix} 1 & 0 & l_1 \frac{1+x^2-2xy}{1-x^2}\\ 0 & 1 & l_2 \frac{1+x^2-2xy^{-1}}{1-x^2}\\ 0 & 0 & -\frac{1+2(l_1+l_2)\varepsilon}{\varepsilon}\frac{x}{1-x^2} \end{pmatrix},\\
T^{-1} &= \begin{pmatrix} 1 & 0 & \frac{l_1\varepsilon}{1+2(l_1+l_2)\varepsilon}\frac{1+x^2-2xy}{x}\\ 0 & 1 & \frac{l_2\varepsilon}{1+2(l_1+l_2)\varepsilon}\frac{1+x^2-2xy^{-1}}{x}\\ 0 & 0 & -\frac{\varepsilon}{1+2(l_1+l_2)\varepsilon}\frac{1-x^2}{x} \end{pmatrix}.
\end{aligned} \tag{16}$$

The symmetry properties of the canonical master integrals are

$$\begin{aligned}
&F(x^{-1},y) = S_x F(x,y), \qquad S_x = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{pmatrix},\\
&F(x,y^{-1}) = S_y[F(x,y)]_{l_1\leftrightarrow l_2}.
\end{aligned} \tag{17}$$

The initial conditions for the differential equations (15) are

$$F(1,1) = T^{-1}(1,1) f(1,1) = (1,1,0)^T. \tag{18}$$

The matrix $M(x,y)$ is

$$\begin{aligned}
M = {}& M_1 \log x + M_2[\log(1+x) + \log(1-x)] + M_3 \log y\\
&+ M_4 \log(x-y) + M_5 \log(1-xy),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&M_1 = \begin{pmatrix} l_1 & -l_1 & l_1(l_1-l_2+l_0)\\ -l_2 & l_2 & l_2(-l_1+l_2+l_0)\\ 1 & 1 & l_1+l_2-l_0 \end{pmatrix},\\
&M_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2l_0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} l & 0 & 0\\ -2l_2 & -l_1+l_2-l_0 & 0\\ 0 & 0 & l \end{pmatrix},\\
&M_4 = \begin{pmatrix} -l_1 & l_1 & l_1 l\\ l_2 & -l_2 & -l_2 l\\ 1 & -1 & -l \end{pmatrix}, \quad M_5 = \begin{pmatrix} -l_1 & l_1 & -l_1 l\\ l_2 & -l_2 & l_2 l\\ -1 & 1 & -l \end{pmatrix}
\end{aligned}$$

[only derivatives of M matter, and hence we may freely substitute $\log(y-x) \to \log(x-y)$, etc.]. This matrix has symmetry properties

$$\begin{aligned}
&M(x^{-1},y) = S_x M(x,y) S_x,\\
&M(x,y^{-1}) = S_y[M(x,y)]_{l_1\leftrightarrow l_2} S_y
\end{aligned} \tag{20}$$

(again, up to inessential additive constants).

If $l_1 = 0$ then $F_1(x,y) = y^{l\varepsilon}$. The first equation decouples, and this trivial function satisfies this equation. The two nontrivial master integrals $F_{2,3}$ are determined by coupled equations. The case $l_2 = 0$ is similar. If $l_1 = l_2 = 0$

then $F_{1,2}(x,y) = y^{\pm l\varepsilon}$; the only nontrivial master integral F_3 is determined by the third equation.

First we consider the single-scale case $y=1$. The differential equations for $x<1$ are

$$\frac{dF(x,1)}{dx} = \varepsilon\left[\frac{M_1}{x} + \frac{M_2}{x+1} + \frac{M_2+M_4+M_5}{x-1}\right]F(x,1),$$
$$M_2+M_4+M_5 = 2\begin{pmatrix} -l_1 & l_1 & 0 \\ l_2 & -l_2 & 0 \\ 0 & 0 & -l_1-l_2 \end{pmatrix}. \tag{21}$$

For $x>1$ we have $F(x,1) = S_x F(x^{-1},1)$; these functions satisfy the equations

$$\frac{dF(x,1)}{dx^{-1}} = \varepsilon\left[-\frac{M_1+2M_2+M_4+M_5}{x^{-1}} + \frac{M_2}{x^{-1}+1} + \frac{M_2+M_4+M_5}{x^{-1}-1}\right]F(x,1)$$

because $-S_x(M_1+2M_2+M_4+M_5)S_x = M_1$, $S_x M_2 S_x = M_2$, $S_x(M_2+M_4+M_5)S_x = M_2+M_4+M_5$ (this follows from (20)).

The solution of the differential equations (21) with the initial conditions (18) as a series in ε can be obtained using Libra. The coefficients are uniform-weight combinations of harmonic polylogarithms [7] (we use HPL [8,9] to reduce them to a minimal set):

$$\begin{aligned}
F_1(x,1) &= 1 + l_1(l_1-l_2+l_0)H_0^2(x)\varepsilon^2 \\
&+ 2l_1\Big\{(l_1-l_2+l_0)\Big[-4l_0H_{0,0,-1}(x) + 2l_0H_0(x)H_{0,-1}(x) + (2l-l_0)\frac{\pi^2}{6}H_0(x)\Big] \\
&- 2(l_1^2-l_2^2+(l_1+3l_2)l_0)H_{0,0,1}(x) + 4l_2l_0H_0(x)H_{0,1}(x) + (l_1-l_2)lH_0^2(x)H_1(x) \\
&+ \frac{1}{6}(2(l_1^2-l_2^2)+(l_1+l_2)l_0-l_0^2)H_0^3(x) + (2(l_1^2-l_2^2)+(5l_1+3l_2)l_0+3l_0^2)\zeta_3\Big\}\varepsilon^3 \\
&+ 2l_1\Big\{(l_1-l_2)l\Big[2l_0(4H_{0,0,1,-1}(x) + 4H_{0,0,-1,1}(x) + 2H_{0,1,0,-1}(x) - 3H_{0,0,0,-1}(x) \\
&- 4H_1(x)H_{0,0,-1}(x) + 2H_0(x)H_1(x)H_{0,-1}(x)) - (l_1+l_2)(2H_{0,1,0,1}(x) + 4H_1(x)H_{0,0,1}(x) - H_0^2(x)H_1^2(x)) \\
&- (2(l_1+l_2)-l_0)\Big(2H_{0,0,0,1}(x) - \frac{1}{3}H_0^3(x)H_1(x)\Big) + (2l-l_0)\frac{\pi^2}{3}H_0(x)H_1(x) + 2(2l+l_0)\zeta_3H_1(x)\Big] \\
&+ (l_1-l_2+l_0)\Big[4l_0^2H_0(x)H_{0,-1,-1}(x) - 2l_0^2H_{0,-1}^2(x) - 4(l_1+l_2)l_0H_{0,1}(x)H_{0,-1}(x) + (l_1+l_2-l_0)l_0H_0^2(x)H_{0,-1}(x) \\
&+ (2l-l_0)l_0\frac{\pi^2}{3}H_{0,-1}(x)\Big] + (l_1+l_2)[8l_2l_0H_0(x)H_{0,1,1}(x) + (l_1^2-l_2^2+(l_1-5l_2)l_0)H_{0,1}^2(x)] \\
&+ 8l_2l_0^2H_0(x)(H_{0,1,-1}(x) + H_{0,-1,1}(x)) - 2l_2(l_1+l_2-l_0)l_0H_0(x)(2H_{0,0,1}(x) - H_0(x)H_{0,1}(x)) \\
&+ 2(l_1-3l_2+l_0)l_0^2H_0(x)H_{0,0,-1}(x) + \frac{1}{24}(4(l_1^3-l_2^3)+2(l_1^2+l_2^2)l_0-(l_1+l_2)l_0^2+l_0^3)H_0^4(x) \\
&+ (2l-l_0)\frac{\pi^2}{12}\Big[8l_2l_0H_{0,1}(x) + (2(l_1^2-l_2^2)+(l_1+l_2)l_0-l_0^2)H_0^2(x) + (6(l_1^2-l_2^2)+(l_1-21l_2)l_0-5l_0^2)\frac{\pi^2}{30}\Big] \\
&- 2(2l_2l+l_1l_0+l_0^2)l_0\zeta_3H_0(x)\Big\}\varepsilon^4 + \mathcal{O}(\varepsilon^5), \\
F_2(x,1) &= [F_1(x,1)]_{l_1\leftrightarrow l_2}, \\
F_3(x,1) &= [F_3(x,1)]_{l_1\leftrightarrow l_2} = 2H_0(x)\varepsilon \\
&+ \Big[-4(l_1+l_2)(H_{0,1}(x) - H_0(x)H_1(x)) - 4l_0(H_{0,-1}(x) - H_0(x)H_{-1}(x)) + (l_1+l_2-l_0)H_0^2(x) + (2l-l_0)\frac{\pi^2}{3}\Big]\varepsilon^2 \\
&+ \Big\{4(l_1+l_2)^2(2H_{0,1,1}(x) - 2H_1(x)H_{0,1}(x) + H_0(x)H_1^2(x)) \\
&+ 2(l_1+l_2)\Big[4l_0(H_{0,1,-1}(x) + H_{0,-1,1}(x) - H_{-1}(x)H_{0,1}(x) - H_1(x)H_{0,-1}(x) + H_0(x)H_1(x)H_{-1}(x)) \\
&- (l_1+l_2-l_0)(2H_{0,0,1}(x) - H_0^2(x)H_1(x)) + (2l-l_0)\frac{\pi^2}{3}H_1(x)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+4l_0^2(2H_{0,-1,-1}(x)-2H_{-1}(x)H_{0,-1}(x)+H_0(x)H_{-1}^2(x))\\
&-(l_1+l_2-l_0)\left[4l_0H_{0,0,-1}(x)-2l_0H_0^2(x)H_{-1}(x)-(2l-l_0)\frac{\pi^2}{3}H_0(x)\right]\\
&+\frac{1}{3}(2(l_1^2+l_2^2)-(l_1+l_2)l_0+l_0^2)H_0^3(x)+2(2l-l_0)l_0\frac{\pi^2}{3}H_{-1}(x)-2(2(l_1+l_2)^2+3(l_1+l_2)l_0+2l_0^2)\zeta_3\Bigg\}\varepsilon^3\\
&+\Bigg\{8(l_1+l_2)^3\left(-2H_{0,1,1,1}(x)+2H_1(x)H_{0,1,1}(x)-H_1^2(x)H_{0,1}(x)+\frac{1}{3}H_0(x)H_1^3(x)\right)\\
&-2(l_1+l_2)^2\Big[(l_1+l_2-l_0)(2H_{0,1,0,1}(x)+4H_1(x)H_{0,0,1}(x)-H_{0,1}^2(x)-H_0^2(x)H_1^2(x))\\
&+4l_0(2H_{0,1,1,-1}(x)+2H_{0,1,-1,1}(x)+2H_{0,-1,1,1}(x)-2H_{-1}(x)H_{0,1,1}(x)-2H_1(x)H_{0,1,-1}(x)-2H_1(x)H_{0,-1,1}(x)\\
&+2H_1(x)H_{-1}(x)H_{0,1}(x)+H_1^2(x)H_{0,-1}(x)-H_0(x)H_{-1}(x)H_1^2(x))-(2l-l_0)\frac{\pi^2}{3}H_1^2(x)\Big]\\
&-8l_0^3\left(2H_{0,-1,-1,-1}(x)-2H_{-1}(x)H_{0,-1,-1}(x)+H_{-1}^2(x)H_{0,-1}(x)-\frac{1}{3}H_0(x)H_{-1}^3(x)\right)\\
&+2(l_1+l_2-l_0)\Big[(l_1+l_2)\Big(2l_0(2H_{0,0,1,-1}(x)+2H_{0,0,-1,1}(x)-2H_{-1}(x)H_{0,0,1}(x)-2H_1(x)H_{0,0,-1}(x)\\
&+H_0^2(x)H_1(x)H_{-1}(x))+(2l-l_0)\frac{\pi^2}{3}H_0(x)H_1(x)\Big)+l_0^2(4H_{0,0,-1,-1}(x)-4H_{-1}(x)H_{0,0,-1}(x)+H_0^2(x)H_{-1}^2(x))\\
&+(2l-l_0)l_0\frac{\pi^2}{3}H_0(x)H_{-1}(x)\Big]\\
&-2(l_1+l_2)\Big[4l_0^2(2H_{0,1,-1,-1}(x)+2H_{0,-1,1,-1}(x)+2H_{0,-1,-1,1}(x)-2H_{-1}(x)H_{0,1,-1}(x)-2H_{-1}(x)H_{0,-1,1}(x)\\
&-2H_1(x)H_{0,-1,-1}(x)+H_{-1}^2(x)H_{0,1}(x)+2H_1(x)H_{-1}(x)H_{0,-1}(x)-H_0(x)H_1(x)H_{-1}^2(x))\\
&-2(2l-l_0)l_0\frac{\pi^2}{3}H_1(x)H_{-1}(x)\Big]\\
&-4(2(l_1+l_2)(l_1^2+l_2^2)-(l_1^2+l_2^2-10l_1l_2)l_0+(l_1+l_2)l_0^2)H_{0,0,0,1}(x)\\
&-4(5(l_1^2+l_2^2)-6l_1l_2+2(l_1+l_2)l_0+l_0^2)l_0H_{0,0,0,-1}(x)\\
&+8l_1l_2l_0H_0(x)(4H_{0,0,1}(x)-H_0(x)H_{0,1}(x))+2((l_1-l_2)^2+(l_1+l_2)l_0)l_0H_0(x)(4H_{0,0,-1}(x)-H_0(x)H_{0,-1}(x))\\
&+\frac{1}{3}(2(l_1^2+l_2^2)-(l_1+l_2)l_0+l_0^2)H_0^2(x)\left[2(l_1+l_2)H_0(x)H_1(x)+2l_0H_0(x)H_{-1}(x)+(2l-l_0)\frac{\pi^2}{2}\right]\\
&+\frac{1}{12}(4(l_1^3+l_2^3)-2(l_1^2+l_2^2)l_0+(l_1+l_2)l_0^2-l_0^3)H_0^4(x)\\
&+(2l-l_0)\frac{\pi^2}{3}\left[2l_0^2H_{-1}^2(x)+(22(l_1^2+l_2^2)+28l_1l_2+13(l_1+l_2)l_0+9l_0^2)\frac{\pi^2}{60}\right]\\
&-4(2(l_1+l_2)^2+3(l_1+l_2)l_0+2l_0^2)\zeta_3[(l_1+l_2)H_1(x)+l_0H_{-1}(x)]\\
&-4(4l_1l_2(l_1+l_2)-2(l_1^2+l_2^2+l_1l_2)l_0-2(l_1+l_2)l_0^2-l_0^3)\zeta_3H_0(x)\Bigg\}\varepsilon^4+\mathcal{O}(\varepsilon^5).
\end{aligned}
\tag{22}
$$

This expansion can be straightforwardly extended to any order in ε. We have also expanded the exact hypergeometric representations of $F_{1,3}(x,1)$ which follow from (6) up to ε^3 using HypExp [10,11]. The results can be expressed via ordinary polylogarithms up to Li_3, and agree with (22). They also agree with the expansions up to ε^3 obtained in [1] [also using (6) and HypExp]. When $l_1=l_2=0$, the only non-trivial master integral F_3 is expressed in Eq. (6) via the ${}_2F_1$ function whose ε expansion is known to all orders [12]. The expansion in euclidean case is given there (there is a typo in the journal version corrected in the version 4 in arXiv); the Minkowski case is given by the formula (41) in [13]. Our result (22) at $l_1 = l_2 = 0$ agrees with the formula (B.10) in [13] (it contains 3 further expansion terms).

Any finite number of terms in the expansion of $F(x,1)$ in $\bar{x} = 1 - x$ can be straightforwardly obtained from (6):

$$\begin{aligned} F_1(x,1) &= 1 + \frac{\varepsilon^2 l_1 \bar{x}^2}{(1+(l_1+l_2)\varepsilon)(1+2(l_1+l_2)\varepsilon)} \\ &\quad \times \{(l_1 - l_2 + l_0 + 2l_2 l_0 \varepsilon)(1+\bar{x}) + \mathcal{O}(\bar{x}^2)\}, \\ F_2(x,1) &= [F_1(x,1)]_{l_1 \leftrightarrow l_2}, \\ F_3(x,1) &= -\frac{\varepsilon \bar{x}}{1+2(l_1+l_2)\varepsilon}[2 + \bar{x} + \mathcal{O}(\bar{x}^2)] \end{aligned} \tag{23}$$

(we have obtained them up to $\bar{x}^{20}$). The coefficients are exact functions of ε. This expansion satisfies the differential equation (21) with the initial condition (18). Expanding each coefficient of (23) in ε, and each coefficient of (22) in $\bar{x}$, we obtain two identical double expansions up to ε^4 and $\bar{x}^{20}$; this is a strong check of our result (22).

Next we consider the straight-line case $x = 1$. From the form of the matrix T^{-1} (16) at $x = 1$ we see that $F_3(1,y) = 0$. The differential equations for $y < 1$ are

$$\begin{aligned} \frac{dF(1,y)}{dy} &= \varepsilon\left[\frac{M_3}{y} + \frac{M_4 + M_5}{y-1}\right]F(1,y), \\ M_4 + M_5 &= 2\begin{pmatrix} -l_1 & l_1 & 0 \\ l_2 & -l_2 & 0 \\ 0 & 0 & -l \end{pmatrix} \end{aligned} \tag{24}$$

(they are, of course, consistent with $F_3 = 0$). For $y > 1$ we have $F(1,y) = S_y[F(1,y^{-1})]_{l_1 \leftrightarrow l_2}$; these functions satisfy the equations

$$\frac{dF(1,y)}{dy^{-1}} = \varepsilon\left[-\frac{M_3 + M_4 + M_5}{y^{-1}} + \frac{M_4 + M_5}{y^{-1} - 1}\right]F(1,y)$$

because $-S_y[M_3 + M_4 + M_5]_{l_1 \leftrightarrow l_2} S_y = M_3$, $S_y[M_4 + M_5]_{l_1 \leftrightarrow l_2} S_y = M_4 + M_5$ [this follows from (20)].

Solving the differential equations (24) with the initial conditions (18) we obtain

$$\begin{aligned} y^{-l\varepsilon} F_1(1,y) = 1 &- 4 l_1 l \left(H_{0,1}(y) - H_0(y) H_1(y) - \frac{\pi^2}{6}\right)\varepsilon^2 \\ &+ 4 l_1 l \Bigg[(l_1 + l_2)\left(2H_{0,1,1}(y) - 2H_1(y)H_{0,1}(y) + H_0(y)H_1^2(y) + \frac{\pi^2}{3}H_1(y)\right) \\ &- (l_1 + l_0)(2H_{0,0,1}(y) - 2H_0(y)H_{0,1}(y) + H_0^2(y)H_1(y)) - 2(l_2 - l_0)\zeta_3\Bigg]\varepsilon^3 \\ &- 4 l_1 l \Bigg[2(l_1 + l_2)^2\left(2H_{0,1,1,1}(y) - 2H_1(y)H_{0,1,1}(y) + H_1^2(y)H_{0,1}(y) - \frac{1}{3}H_0(y)H_1^3(y) - \frac{\pi^2}{6}H_1^2(y)\right) \\ &+ 2(l_1 + l_0)^2\left(2H_{0,0,0,1}(y) - 2H_0(y)H_{0,0,1}(y) + H_0^2(y)H_{0,1}(y) - \frac{1}{3}H_0^3(y)H_1(y)\right) \\ &+ 2(l_1 l - l_2 l_0)\left(H_{0,1,0,1}(y) + 2H_0(y)H_{0,1,1}(y) + 2H_1(y)H_{0,0,1}(y)\right) \\ &- (l_1 l - 3 l_2 l_0)H_{0,1}^2(y) - 4 l_1 l H_0(y) H_1(y) H_{0,1}(y) + (l_1 + l_2)(l_1 + l_0)H_0^2(y)H_1^2(y) \\ &- 2 l_2 l_0 \frac{\pi^2}{3}(H_{0,1}(y) - H_0(y)H_1(y)) + 4(l_2(l_1 + l_2) - (l_1 - l_2)l_0)\zeta_3 H_1(y) \\ &- (7 l_1 l + 4(l_2^2 - l_2 l_0 + l_0^2))\frac{\pi^4}{90}\Bigg]\varepsilon^4 + \mathcal{O}(\varepsilon^5), \end{aligned}$$

$$
\begin{aligned}
y^{l\varepsilon}F_2(1,y) = {} & 1 + 2l_2 l\left(2H_{0,1}(y) - 2H_0(y)H_1(y) - H_0^2(y) - \frac{\pi^2}{3}\right)\varepsilon^2 \\
& - 4l_2 l\Bigg[(l_1+l_2)\left(2H_{0,1,1}(y) - 2H_1(y)H_{0,1}(y) + H_0(y)H_1^2(y) + \frac{\pi^2}{3}(H_0(y)+H_1(y))\right) \\
& - 2(l_1-l_0)H_{0,0,1}(y) - 2(l_2+l_0)(H_0(y)H_{0,1}(y)+\zeta_3) + (l_2+l)H_0^2(y)\left(H_1(y)+\frac{1}{3}H_0(y)\right)\Bigg]\varepsilon^3 \\
& + 4l_2 l\Bigg[2(l_1+l_2)^2\left(2H_{0,1,1,1}(y) - 2H_1(y)H_{0,1,1}(y) + H_1^2(y)H_{0,1}(y) - \frac{1}{3}H_0(y)H_1^3(y) - \frac{\pi^2}{6}H_1^2(y)\right) \\
& + 2(l_1 l - l_2 l_0)(H_{0,1,0,1}(y) + 2H_1(y)H_{0,0,1}(y)) + 4(l_1+l_0)^2 H_{0,0,0,1}(y) \\
& - 4(l_2(l_1+l_2) - (l_1-l_2)l_0)(H_0(y)H_{0,1,1}(y) - \zeta_3(H_0(y)+H_1(y))) + 4(l_1 l_2 - l l_0)H_0(y)H_{0,0,1}(y) \\
& - (l_1(l_1+l_2+3l_0) - l_2 l_0)H_{0,1}^2(y) + 2((l_2+l_0)^2 + l_1 l_0)H_0^2(y)H_{0,1}(y) \\
& + 4l_2 l H_0(y)H_1(y)H_{0,1}(y) - (l_1+l_2)(l_2+l)H_0^2(y)H_1^2(y) \\
& - \frac{1}{6}((l_1+l_0)^2 + 3l_2 l)H_0^3(y)(4H_1(y)+H_0(y)) \\
& + \frac{\pi^2}{3}(2l_1 l_0 H_{0,1}(y) - ((l_1+l_2)^2 + l_1 l_0)H_0(y)(2H_1(y)+H_0(y))) \\
& - (7l_1(l_1+l_2) + 4l_2^2 + (12l_1+l_2)l_0 + 4l_0^2)\frac{\pi^4}{90}\Bigg]\varepsilon^4 + \mathcal{O}(\varepsilon^5).
\end{aligned}
\tag{25}
$$

This expansion can be straightforwardly extended to any order in ε. We have also expanded the exact hypergeometric representations of $F_{1,2}(1,y)$ which follow from (7) up to ε^3 using HypExp [10,11]. The results can be expressed via ordinary polylogarithms up to Li_3, and agree with (25).

Any finite number of terms in the expansion of $F(1,y)$ in $\bar{y} = 1-y$ can be straightforwardly obtained from (7):

$$
\begin{aligned}
F_1(1,y) &= 1 - \frac{\varepsilon l\bar{y}}{1+2(l_1+l_2)\varepsilon}[1 - 2(l_1-l_2)\varepsilon + \mathcal{O}(\bar{y})], \\
F_2(1,y) &= 1 + \frac{\varepsilon l\bar{y}}{1+2(l_1+l_2)\varepsilon}[1 + 2(l_1-l_2)\varepsilon + \mathcal{O}(\bar{y})]
\end{aligned}
\tag{26}
$$

(we have obtained them up to $\bar{y}^{20}$). This expansion satisfies the differential equations (24) with the initial conditions (18). Expanding each coefficient of (26) in ε, and each coefficient of (25) in $\bar{y}$, we obtain two identical double expansions up to ε^4 and $\bar{y}^{20}$; this is a strong check of our result (25).

Finally, we discuss the general case. Due to the symmetry relations (17) it is sufficient to consider the region $x \le 1$, $y \le 1$. We can solve the differential equations (15) along one of the two paths in Fig. 2. The result is a combination of products of Goncharov polylogarithms [14]

$$
G_{\underbrace{0,\ldots,0}_{n}}(x) = \frac{1}{n!}\log^n x, \qquad G_{a,\ldots}(x) = \int_0^x \frac{dt}{t-a} G_{\ldots}(t)
$$

of $\bar{x} = 1-x$ and $\bar{y} = 1-y$. Numerical evaluation of Goncharov polylogarithms is available [15] in GiNaC [16]. We make no efforts to express some of them via harmonic polylogarithms of x and y because some Goncharov polylogarithms are bound to remain. Using Libra we obtain

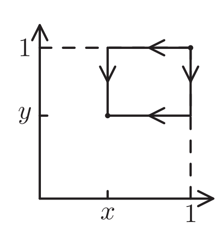

FIG. 2. Paths from (1,1) to (x,y).

$$
\begin{aligned}
y^{-l\varepsilon}F_1(x,y) &= 1 + 2l_1\{l[G_{\bar{y},1}(\bar{x}) - G_{\hat{y},1}(\bar{x}) - 2G_{0,1}(\bar{y}) + G_1(\bar{y})(G_1(\bar{x}) - G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x}))] + (l_1 - l_2 + l_0)G_{1,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2l_1\{l[l_0(2(G_{\hat{y},0,1}(\bar{x}) - G_{\bar{y},0,1}(\bar{x}) + G_{\hat{y},2,1}(\bar{x}) - G_{\bar{y},2,1}(\bar{x})) + G_{\bar{y},\hat{y},1}(\bar{x}) - G_{\hat{y},\bar{y},1}(\bar{x}) \\
&\quad + G_1(\bar{y})(G_{\bar{y},\hat{y}}(\bar{x}) + G_{\hat{y},\bar{y}}(\bar{x}) - G_{\hat{y},1}(\bar{x}))) \\
&\quad + (2l - l_0)(G_{\bar{y},\bar{y},1}(\bar{x}) - G_{\hat{y},\hat{y},1}(\bar{x}) - G_1(\bar{y})(G_{\bar{y},\bar{y}}(\bar{x}) + G_{\hat{y},\hat{y}}(\bar{x}) - G_{\bar{y},1}(\bar{x}))) \\
&\quad + 2(l_1 + l_2)(G_{0,1}(\bar{y})(G_1(\bar{x}) - G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x})) - 2G_{0,0,1}(\bar{y})) + (2l_1 + l_0)(G_{1,\hat{y},1}(\bar{x}) + G_1(\bar{y})G_{1,\hat{y}}(\bar{x})) \\
&\quad + (2l_1 - l_0)G_{\hat{y},1,1}(\bar{x}) - (2l_2 - l_0)(G_{\bar{y},1,1}(\bar{x}) + G_{1,\bar{y},1}(\bar{x}) - G_1(\bar{y})(G_{1,\bar{y}}(\bar{x}) - G_{1,1}(\bar{x}))) \\
&\quad + 2(l_1 + l_0)(G_{1,1}(\bar{y})(G_1(\bar{x}) - G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x})) - 2G_{0,1,1}(\bar{y}))] \\
&\quad - 2l_0(l_1 - l_2 + l_0)(G_{1,0,1}(\bar{x}) + G_{1,2,1}(\bar{x})) - (2(l_1^2 - l_2^2) + (l_1 + l_2)l_0 - l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \\
&= 1 + 2l_1\{l[G_1(\bar{x})(G_{\bar{x}}(\bar{y}) - G_{\hat{x}}(\bar{y})) - G_{\bar{x},1}(\bar{y}) - G_{\hat{x},1}(\bar{y})] + (l_1 - l_2 + l_0)G_{1,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2l_1\{l[l_0(2G_{2,1}(\bar{x})(G_{\hat{x}}(\bar{y}) - G_{\bar{x}}(\bar{y})) + G_1(\bar{x})(G_{\bar{x},\hat{x}}(\bar{y}) - G_{\hat{x},\bar{x}}(\bar{y})) + G_{\bar{x},\hat{x},1}(\bar{y}) + G_{\hat{x},\bar{x},1}(\bar{y})) \\
&\quad + (2l - l_0)(G_1(\bar{x})(G_{\bar{x},\bar{x}}(\bar{y}) - G_{\hat{x},\hat{x}}(\bar{y})) - G_{\bar{x},\bar{x},1}(\bar{y}) - G_{\hat{x},\hat{x},1}(\bar{y})) \\
&\quad + 2(l_1 + l_2)G_{0,1}(\bar{x})(G_{\bar{x}}(\bar{y}) - G_{\hat{x}}(\bar{y})) + 2(l_1 - l_2)G_{0,1,1}(\bar{x}) \\
&\quad + (2l_1 - l_0)G_{1,1}(\bar{x})G_{\hat{x}}(\bar{y}) - (2l_2 - l_0)G_{1,1}(\bar{x})G_{\bar{x}}(\bar{y}) - 2(l_1 + l_0)(G_{\bar{x},1,1}(\bar{y}) + G_{\hat{x},1,1}(\bar{y}))] \\
&\quad - 2(l_1 - l_2 + l_0)[l_0G_{1,2,1}(\bar{x}) - (l_1 + l_2)G_{1,0,1}(\bar{x})] - (2(l_1^2 - l_2^2) + (l_1 + l_2)l_0 - l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4), \\
y^{l\varepsilon}F_2(x,y) &= 1 + 2l_2\{l[2(G_{0,1}(\bar{y}) - G_{1,1}(\bar{y})) - G_{\bar{y},1}(\bar{x}) + G_{\hat{y},1}(\bar{x}) - G_1(\bar{y})(G_1(\bar{x}) - G_{\hat{y}}(\bar{x}) - G_{\bar{y}}(\bar{x}))] \\
&\quad - (l_1 - l_2 - l_0)G_{1,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2l_2\{l[l_0(G_1(\bar{y})(G_{\bar{y},1}(\bar{x}) - G_{\bar{y},\hat{y}}(\bar{x}) - G_{\hat{y},\bar{y}}(\bar{x})) + 2(G_{\bar{y},0,1}(\bar{x}) - G_{\hat{y},0,1}(\bar{x}) + G_{\bar{y},2,1}(\bar{x}) - G_{\hat{y},2,1}(\bar{x})) \\
&\quad - G_{\bar{y},\hat{y},1}(\bar{x}) + G_{\hat{y},\bar{y},1}(\bar{x})) + (2l - l_0)(G_1(\bar{y})(G_{\bar{y},\bar{y}}(\bar{x}) + G_{\hat{y},\hat{y}}(\bar{x}) - G_{\hat{y},1}(\bar{x})) - G_{\bar{y},\bar{y},1}(\bar{x}) + G_{\hat{y},\hat{y},1}(\bar{x})) \\
&\quad + (l_1 + l_2)(4(G_{0,0,1}(\bar{y}) - G_{1,0,1}(\bar{y})) - 2G_{0,1}(\bar{y})(G_1(\bar{x}) - G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x}))) \\
&\quad - (2l_1 - l_0)(G_1(\bar{y})(G_{1,\hat{y}}(\bar{x}) - G_{1,1}(\bar{x})) + G_{\hat{y},1,1}(\bar{x}) + G_{1,\hat{y},1}(\bar{x})) - (2l_2 + l_0)(G_1(\bar{y})G_{1,\bar{y}}(\bar{x}) - G_{1,\bar{y},1}(\bar{x})) \\
&\quad + (2l_2 - l_0)G_{\bar{y},1,1}(\bar{x}) + 2(l_2 + l)(2(G_{1,1,1}(\bar{y}) - G_{0,1,1}(\bar{y})) + G_{1,1}(\bar{y})(G_1(\bar{x}) - G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x})))] \\
&\quad + 2l_0(l_1 - l_2 - l_0)(G_{1,0,1}(\bar{x}) + G_{1,2,1}(\bar{x})) + (2(l_1^2 - l_2^2) - (l_1 + l_2)l_0 + l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \\
&= 1 + 2l_2\{l[G_1(\bar{x})(G_{\hat{x}}(\bar{y}) - G_{\bar{x}}(\bar{y})) - 2G_{1,1}(\bar{y}) + G_{\bar{x},1}(\bar{y}) + G_{\hat{x},1}(\bar{y})] - (l_1 - l_2 - l_0)G_{1,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2l_2\{2l^2G_1(\bar{x})(G_{\bar{x},1}(\bar{y}) - G_{\hat{x},1}(\bar{y})) + l[l_0(2G_{2,1}(\bar{x})(G_{\bar{x}}(\bar{y}) - G_{\hat{x}}(\bar{y})) + G_1(\bar{x})(G_{\hat{x},\bar{x}}(\bar{y}) - G_{\bar{x},\hat{x}}(\bar{y})) \\
&\quad - G_{\bar{x},\hat{x},1}(\bar{y}) - G_{\hat{x},\bar{x},1}(\bar{y})) + (2l - l_0)(G_1(\bar{x})(G_{\hat{x},\hat{x}}(\bar{y}) - G_{\bar{x},\bar{x}}(\bar{y})) + G_{\bar{x},\bar{x},1}(\bar{y}) + G_{\hat{x},\hat{x},1}(\bar{y})) \\
&\quad + 2(l_1 + l_2)(G_{0,1}(\bar{x})(G_{\hat{x}}(\bar{y}) - G_{\bar{x}}(\bar{y})) + G_1(\bar{x})(G_{1,\bar{x}}(\bar{y}) - G_{1,\hat{x}}(\bar{y})) - G_{1,\bar{x},1}(\bar{y}) - G_{1,\hat{x},1}(\bar{y})) \\
&\quad + 2(l_1 - l_2)(G_{1,1}(\bar{x})G_1(\bar{y}) - G_{0,1,1}(\bar{x})) - (2l_1 - l_0)G_{1,1}(\bar{x})G_{\hat{x}}(\bar{y}) + (2l_2 - l_0)G_{\bar{x}}(\bar{y})G_{1,1}(\bar{x}) \\
&\quad + 2(l_2 + l)(2G_{1,1,1}(\bar{y}) - G_{\bar{x},1,1}(\bar{y}) - G_{\hat{x},1,1}(\bar{y}))] \\
&\quad + 2(l_1 - l_2 - l_0)[l_0G_{1,2,1}(\bar{x}) - (l_1 + l_2)G_{1,0,1}(\bar{x})] + (2(l_1^2 - l_2^2) - (l_1 + l_2)l_0 + l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4),
\end{aligned}
$$

$$\begin{aligned}
F_3(x,y) &= 2G_1(\bar{x})\varepsilon + 2\{l[G_1(\bar{y})(G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x})) - G_{\bar{y},1}(\bar{x}) - G_{\hat{y},1}(\bar{x})] + 2l_0(G_{0,1}(\bar{x}) + G_{2,1}(\bar{x})) + (l_1 + l_2 - l_0)G_{1,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2\{l[l_0(G_1(\bar{y})(2(G_{0,\hat{y}}(\bar{x}) - G_{0,\bar{y}}(\bar{x}) + G_{2,\hat{y}}(\bar{x}) - G_{2,\bar{y}}(\bar{x})) - G_{\bar{y},\hat{y}}(\bar{x}) + G_{\hat{y},\bar{y}}(\bar{x})) \\
&\quad + 2(G_{0,\bar{y},1}(\bar{x}) + G_{0,\hat{y},1}(\bar{x}) + G_{2,\bar{y},1}(\bar{x}) + G_{2,\hat{y},1}(\bar{x}) + G_{\bar{y},0,1}(\bar{x}) + G_{\hat{y},0,1}(\bar{x}) + G_{\bar{y},2,1}(\bar{x}) + G_{\hat{y},2,1}(\bar{x})) \\
&\quad - G_{\bar{y},\hat{y},1}(\bar{x}) - G_{\hat{y},\bar{y},1}(\bar{x})) \\
&\quad + (2l - l_0)(G_1(\bar{y})(G_{\bar{y},\bar{y}}(\bar{x}) - G_{\hat{y},\hat{y}}(\bar{x})) - G_{\bar{y},\bar{y},1}(\bar{x}) - G_{\hat{y},\hat{y},1}(\bar{x})) \\
&\quad + (l_1 + l_2)(2G_{0,1}(\bar{y})(G_{\bar{y}}(\bar{x}) - G_{\hat{y}}(\bar{x})) - G_1(\bar{y})(G_{\bar{y},1}(\bar{x}) - G_{\hat{y},1}(\bar{x}))) + (l_1 - l_2)(2G_{0,1}(\bar{y})G_1(\bar{x}) - G_1(\bar{y})G_{1,1}(\bar{x})) \\
&\quad + (2l_1 - l_0)(G_1(\bar{y})G_{1,\hat{y}}(\bar{x}) + G_{\hat{y},1,1}(\bar{x}) + G_{1,\hat{y},1}(\bar{x})) - (2l_2 - l_0)(G_1(\bar{y})G_{1,\bar{y}}(\bar{x}) - G_{1,\bar{y},1}(\bar{x}) - G_{\bar{y},1,1}(\bar{x})) \\
&\quad - (l_1 - l_2 + l_0)G_1(\bar{x})G_{1,1}(\bar{y}) + 2l_2G_{1,1}(\bar{y})(G_{\hat{y}}(\bar{x}) - G_{\bar{y}}(\bar{x}))] \\
&\quad - 4l_0^2(G_{0,0,1}(\bar{x}) + G_{0,2,1}(\bar{x}) + G_{2,0,1}(\bar{x}) + G_{2,2,1}(\bar{x})) - 2l_0(l_1 + l_2 - l_0)(G_{0,1,1}(\bar{x}) + G_{1,0,1}(\bar{x}) \\
&\quad + G_{1,2,1}(\bar{x}) + G_{2,1,1}(\bar{x})) - (2(l_1^2 + l_2^2) - (l_1 + l_2)l_0 + l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \\
&= 2G_1(\bar{x})\varepsilon + 2\{l[G_1(\bar{x})(G_1(\bar{y}) - G_{\bar{x}}(\bar{y}) - G_{\hat{x}}(\bar{y})) + G_{\bar{x},1}(\bar{y}) - G_{\hat{x},1}(\bar{y})] \\
&\quad + (l_1 + l_2 - l_0)G_{1,1}(\bar{x}) - 2(l_1 + l_2)G_{0,1}(\bar{x}) + 2l_0G_{2,1}(\bar{x})\}\varepsilon^2 \\
&\quad - 2\{l^2[G_1(\bar{x})(G_{1,\bar{x}}(\bar{y}) + G_{1,\hat{x}}(\bar{y}) + G_{\bar{x},1}(\bar{y}) + G_{\hat{x},1}(\bar{y}) - G_{1,1}(\bar{y})) - G_{1,\bar{x},1}(\bar{y}) + G_{1,\hat{x},1}(\bar{y})] \\
&\quad + l[l_0(2G_{2,1}(\bar{x})(G_{\bar{x}}(\bar{y}) + G_{\hat{x}}(\bar{y}) - G_1(\bar{y})) - G_1(\bar{x})(G_{\bar{x},\hat{x}}(\bar{y}) + G_{\hat{x},\bar{x}}(\bar{y})) - G_{\bar{x},\hat{x},1}(\bar{y}) + G_{\hat{x},\bar{x},1}(\bar{y})) \\
&\quad - (2l - l_0)[G_1(\bar{x})(G_{\bar{x},\bar{x}}(\bar{y}) + G_{\hat{x},\hat{x}}(\bar{y})) - G_{\bar{x},\bar{x},1}(\bar{y}) + G_{\hat{x},\hat{x},1}(\bar{y})] \\
&\quad + 2(l_1 + l_2)[G_{0,1}(\bar{x})(G_1(\bar{y}) - G_{\hat{x}}(\bar{y})) - G_{0,1}(\bar{x})G_{\bar{x}}(\bar{y})] \\
&\quad + G_{1,1}(\bar{x})[(2l_1 - l_0)G_{\hat{x}}(\bar{y}) + (2l_2 - l_0)G_{\bar{x}}(\bar{y}) - (l_1 + l_2 - l_0)G_1(\bar{y})] - 2l_2(G_{\bar{x},1,1}(\bar{y}) - G_{\hat{x},1,1}(\bar{y}))] \\
&\quad - 4(l_1 + l_2)^2G_{0,0,1}(\bar{x}) - 2(l_1 + l_2 - l_0)[l_0(G_{2,1,1}(\bar{x}) + G_{1,2,1}(\bar{x})) - (l_1 + l_2)(G_{1,0,1}(\bar{x}) + G_{0,1,1}(\bar{x}))] \\
&\quad + 4l_0(l_1 + l_2)(G_{2,0,1}(\bar{x}) + G_{0,2,1}(\bar{x})) - 4l_0^2G_{2,2,1}(\bar{x}) \\
&\quad - (2(l_1^2 + l_2^2) - (l_1 + l_2)l_0 + l_0^2)G_{1,1,1}(\bar{x})\}\varepsilon^3 + \mathcal{O}(\varepsilon^4),
\end{aligned} \tag{27}$$

where $\hat{x} = 1 - x^{-1}$, $\hat{y} = 1 - y^{-1}$. All Goncharov polylogarithms up to weight 2 can be expressed via Li_2 and logarithms. This expansion can be straightforwardly extended to any order in ε.

The results (22), (25), (27) are available in the Supplemental Material, *Mathematica* file [17].

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