No-go result for covariance in models of loop quantum gravity

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(Received 3 July 2020; accepted 30 July 2020; published 10 August 2020)

Based on the observation that the exterior space-times of Schwarzschild-type solutions allow two symmetric slicings, a static spherically symmetric one and a timelike homogeneous one, modifications of gravitational dynamics suggested by symmetry-reduced models of quantum cosmology can be used to derive corresponding modified spherically symmetric equations. Generally covariant theories are much more restricted in spherical symmetry compared with homogeneous slicings, given by 1 + 1-dimensional dilaton models if they are local. As shown here, modifications used in loop quantum cosmology do not have a corresponding covariant spherically symmetric theory. Models of loop quantum cosmology therefore violate general covariance in the form of slicing independence. Only a generalized form of covariance with a non-Riemannian geometry could consistently describe space-time in models of loop quantum gravity.

DOI: 10.1103/PhysRevD.102.046006

I. INTRODUCTION

Models of black holes in quantum gravity are valuable not only because their strong-field effects draw considerable physical interest, but also because they are understood as a consequence of nontrivial dynamical properties of space-time. Given the incomplete status of all approaches to quantum gravity, the latter connection would, at present, seem even more important than the former. In this sense, black-hole models in quantum gravity have a clear advantage over models of quantum cosmology because basic cosmological solutions work with simpler space-times characterized by exact or perturbative spatial homogeneity with a preferred background time direction.

The connection between black holes and space-time structure is particularly relevant in canonical, background-independent approaches, such as loop quantum gravity. In such approaches, the structure of space-time is a derived concept and not presupposed. Explorations in a physically motivated context can therefore provide important insights into the viability of any specific proposal. Even if implied quantum-gravity effects in black holes may not be realistically observable, studying them in detail can often strengthen an analysis of purely mathematical consistency conditions on the theory.

An important step in this direction had recently been undertaken in Ref. [1]. Although the initial analysis was quickly found to be invalid—owing to an incorrect treatment of phase-space-dependent quantum corrections [2–6], a failure to recognize subtleties in the asymptotic structure [7], and unacceptable long-term effects in astrophysically relevant solutions [8]—it was based on an interesting suggestion that leads to a new and independent test of space-time structure in models of loop quantum gravity [9]. Here, we elaborate on this application and use it to demonstrate a no-go result that implies the noncovariance of any model of loop quantum gravity, if covariance is understood in the classical way related to slicing independence in Riemannian geometry.

II. SYMMETRIES IN SCHWARZSCHILD SPACE-TIME

The construction utilized in Ref. [1] is an application of minisuperspace results, originally derived for models of quantum cosmology, to a black-hole context. The most common example of this form is based on the well-known fact that the Schwarzschild solution,

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2m/r} + r^2(\mathrm{d}\vartheta^2+\sin^2\vartheta\,\mathrm{d}\varphi^2), \tag{1}$$

has a homogeneous spatial slicing in the interior, where $r < R_{\mathrm{S}} = 2m$ is less than the Schwarzschild radius. In this region, r can serve as a time coordinate because the restricted line element $\mathrm{d}s^2|_{r=\mathrm{const}} > 0$ is positive between any two distinct points at the same value of r. The r dependence of the coefficients in Eq. (1) therefore implies time dependence in this region, but not spatial inhomogeneity. The resulting homogeneous dynamics is described by the Kantowski-Sachs model [10].

A quantum scenario of the Schwarzschild interior can therefore be constructed by importing quantum effects found in anisotropic minisuperspace models from quantum cosmology. Of major interest to Ref. [1] was the possibility that a bounce in cosmological models, as sometimes claimed in models of loop quantum cosmology, might then be reinterpreted as a nonsingular transition through the black-hole interior. Note, however, that most bounce claims in loop quantum cosmology are based on oversimplified models that do not capture the correct physics near a spacelike singularity [11–13]. Moreover, such models are often in violation of general covariance [14], a conclusion that will be strengthened by our derivations below. Specific predictions made in this context, in particular of a quantitative nature as in the example of the ratio of masses before and after the bounce, therefore cannot be considered reliable.

Nevertheless, it is justified to assume the modified dynamics implied by a quantum version of the Kantowski-Sachs model as a possible substitute for the dynamics of general relativity in the Schwarzschild interior, and then to evaluate potential implications on qualitative features of the resulting model. An open question even in such less ambitious studies has been how to connect the Schwarzschild interior to a possible inhomogeneous exterior geometry. Such a connection has become possible by the useful suggestion of Ref. [1] to consider homogeneous *timelike* slicings in the exterior, given by constant r in Eq. (1) even if $r > R_S$, and apply modifications proposed in minisuperspace models. Using this method, the authors of Ref. [1] constructed a modified line element that could possibly describe the exterior geometry of a quantum-modified, nonsingular Schwarzschild black hole (or its Kruskal extension).

In order to do so, Ref. [1] *assumed* that the modified exterior is subject to the same space-time structure as the classical theory, given by Riemannian geometry and described by a line element such as Eq. (1) but with a modified r dependence in its coefficients. However, in background-independent models of quantum gravity, it is not guaranteed that the structure of space-time as seen in the classical theory remains intact. Space-time structure should rather be derived from the theory, which would then show whether a line element of the form $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$, restricted to spherical symmetry in the present context, can indeed describe the modified quantum dynamics. In canonical approaches to quantum gravity, such as loop quantum gravity used in Ref. [1], the task is to show that gauge transformations acting on components of g_{ab}, generated by modified constraints that generate the modified dynamics used to obtain any kind of nonsingular homogeneous model, are consistent with standard coordinate transformations of $\mathrm{d}x^a$ assumed in the definition of a line element. We will show that this is not the case for modifications suggested by loop quantum cosmology.

III. DYNAMICAL MODELS

For our demonstration, we will need the relevant equations that describe the classical and modified dynamics of the slicings involved in the construction of Ref. [1]. We present these equations and our new derivations in a form based on metric variables, which are more common and therefore more easily accessible than the triad variables used in models of loop quantum gravity. Our general result does not depend on the choice of variables because it is invariant under canonical transformations. For an explicit derivation in triad variables, see Ref. [9].

A. Spherical symmetry and interior geometry

We begin with the generic form of line elements subject to the symmetries of Kantowski-Sachs models:

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}x^2 + b(t)^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \quad (2)$$

with three free functions, N, a and b, depending on time. Such a line element can be used to describe the spatially homogeneous Schwarzschild interior. Because the line element is also spherically symmetric, it is of the general form

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + L(t,x)^2(\mathrm{d}x + M(t,x)\mathrm{d}t)^2 + S(t,x)^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2), \quad (3)$$

specialized to r-independent coefficients as well as vanishing shift, $M = 0$.

Since we will use spherically symmetric models later on, we quote the dynamical equations implied for the coefficients of Eq. (3) by a local, generally covariant theory in which a line element of this form would indeed correctly describe the symmetries of solutions. In the $1+1$-dimensional context in which spherically symmetric models are placed, it is well known that this set of theories is given by dilaton-gravity models [15–17] in which, up to field redefinitions, only a specific set of functions, including the dilaton potential $V(S)$, can be varied while all other terms in an action or Hamiltonian constraint are fully determined by covariance. The equivalence of the generalized dilaton models introduced in Ref. [17] with two-dimensional Horndeski theories [18], and therefore with the most general two-dimensional local scalar-tensor theory with second-order field equations, has recently been demonstrated in Ref. [19]. This general class of theories also includes Palatini-$f(R)$ models [20] through their equivalence with scalar-tensor theories with a nondynamical scalar field [21].

The action of any such theory can be written as

$$S[g,\phi] = \frac{1}{16\pi G}\int \mathrm{d}^2x\sqrt{-\det g}(\xi(\phi)R + k(\phi,X) + C(\phi,X)\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi) \quad (4)$$

with three free functions, $\xi(\phi)$, $k(\phi,X)$ and $C(\phi,X)$ of the scalar field ϕ and

$$X = -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi. \tag{5}$$

As a parametrization of the most general second-order theories in two dimensions, the three functions $\xi(\phi)$, $k(\phi, X)$ and $C(\phi, X)$ are not independent if field redefinitions of ϕ and g_{ab} are allowed. For instance, $\xi(\phi)$ can be mapped to one by a suitable ϕ-dependent conformal transformation of g_{ab}, adjusting also $k(\phi, X)$. This ambiguity will not concern us here. It is only important that for any local generally covariant theory for a two-dimensional metric and a scalar field with second-order field equations there is a choice of $\xi(\phi)$, $k(\phi, X)$ and $C(\phi, X)$ such that the action is of the form (4).

The canonical formulation of Eq. (4) in this general form (and without fixing the gauge) is rather involved because it requires inversions of some of the free functions or their derivatives. Since the models under consideration here are canonical, we will therefore begin with a restricted class of spherically symmetric theories in which, compared with Eq. (4), we have $\xi = 1$, $C = 0$ and k linear in X. That is, we will first consider minimally coupled scalar-tensor theories with quadratic kinetic terms. In a second step, we will then show that our result does not depend on field redefinitions that change $\xi(\phi)$, or on an introduction of nontrivial k and C.

The most general covariant theory under these conditions can be derived directly at the canonical level; see Refs. [22–24] for explicit derivations. This dynamics tells us that, up to canonical transformations, the momenta canonically conjugate to S and L, respectively, are given by

$$p_S = -\frac{1}{N}\left(\frac{\partial(SL)}{\partial t} - \frac{\partial(MSL)}{\partial x}\right), \tag{6}$$

$$p_L = -\frac{S}{N}\left(\frac{\partial S}{\partial t} - M\frac{\partial S}{\partial x}\right). \tag{7}$$

The Hamiltonian constraint

$$H_{\rm sph}[N] = \int {\rm d}x\, N\left(-\frac{p_S p_L}{S} + \frac{L p_L^2}{2S^2} + \frac{S}{L}\frac{\partial^2 S}{\partial x^2} - \frac{S}{L^2}\frac{\partial S}{\partial x}\frac{\partial L}{\partial x} + \frac{1}{2L}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{4} L S V(S)\right) \tag{8}$$

and diffeomorphism constraint

$$D_{\rm sph}[M] = \int {\rm d}x\, M\left(p_S\frac{\partial S}{\partial x} - L\frac{\partial p_L}{\partial x}\right) \tag{9}$$

then generate the equations of motion

$$\frac{1}{N}\frac{\partial p_S}{\partial t} = -\frac{p_S p_L}{S^2} + \frac{L p_L^2}{S^3} - \frac{1}{L}\frac{\partial^2 S}{\partial x^2} - \frac{1}{LN}\frac{\partial N}{\partial x}\frac{\partial S}{\partial x} + \frac{1}{L^2 N}\frac{\partial L}{\partial x}\frac{\partial (NS)}{\partial x} - \frac{S}{LN}\frac{\partial^2 N}{\partial x^2} - \frac{L}{4}\frac{{\rm d}(SV(S))}{{\rm d}S} + \frac{\partial(M p_S)}{\partial x} \tag{10}$$

and

$$\frac{1}{N}\frac{\partial p_L}{\partial t} = -\frac{p_L^2}{2S^2} - \frac{S}{NL^2}\frac{\partial S}{\partial x}\frac{\partial N}{\partial x} - \frac{1}{2L^2}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{1}{4} S V(S) + M\frac{\partial p_L}{\partial x} \tag{11}$$

while equations for $\partial S/\partial t$ and $\partial L/\partial t$ follow from the Eq. (6) for the momenta. For spherically symmetric general relativity, the dilaton potential is given by $V(S) = -2/S$.

Since Kantowski-Sachs models are spherically symmetric, we can derive the momenta, constraints, and equations of motion of Eq. (2) by specializing the equations of spherical symmetry, also using $M = 0$ and x independence. We obtain the momenta

$$p_a = -\frac{b}{N}\frac{\partial b}{\partial t}, \qquad p_b = -\frac{1}{N}\frac{\partial(ab)}{\partial t}, \tag{12}$$

with the Hamiltonian constraint

$$H_{\rm hom}[N] = N\left(-\frac{p_a p_b}{b} + \frac{a p_a^2}{2b^2} - \frac{a}{2}\right). \tag{13}$$

It implies the equations of motion

$$\frac{{\rm d}a}{{\rm d}t} = \frac{\partial H_{\rm hom}[N]}{\partial p_a} = N\left(-\frac{p_b}{b} + \frac{a p_a}{b^2}\right), \tag{14}$$

$$\frac{{\rm d}b}{{\rm d}t} = \frac{\partial H_{\rm hom}[N]}{\partial p_b} = -N\frac{p_a}{b}, \tag{15}$$

$$\frac{{\rm d}p_a}{{\rm d}t} = -\frac{\partial H_{\rm hom}[N]}{\partial a} = \frac{1}{2}N\left(1 - \frac{p_a^2}{b^2}\right), \tag{16}$$

$$\frac{{\rm d}p_b}{{\rm d}t} = -\frac{\partial H_{\rm hom}[N]}{\partial b} = -N\left(\frac{p_a p_b}{b^2} - \frac{a p_a^2}{b^3}\right). \tag{17}$$

B. Timelike homogeneity

In the Schwarzschild exterior, $r > R_{\rm S}$, slices of constant r are still homogeneous but timelike. The resulting canonical relationships can be derived from the Kantowski-Sachs equations by a complex canonical transformation from a, p_a and N to

$$A = ia, \qquad p_A = -ip_a, \qquad n = iN \tag{18}$$

while b and p_b remain unchanged. The line element (2) then takes the form

$$\mathrm{d}s^2 = n(t)^2\mathrm{d}t^2 - A(t)^2\mathrm{d}x^2 + b(t)^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \tag{19}$$

in which slices of constant t are timelike. This line element represents the symmetry of the exterior Schwarzschild solution, where x would be the Schwarzschild *time* coordinate and t the Schwarzschild *radial* coordinate.

We will derive the dynamics of Eq. (19) in a generalized form that takes into account possible modifications from loop quantum cosmology, applied to this homogeneous model. These modifications are subject to a large number of quantization ambiguities. Our result, however, will be insensitive to ambiguities because it holds for any Hamiltonian constraint

$$H_{\text{timelike}}[n] = n\left(-\frac{p_A p_b}{b} + \frac{A p_A^2}{2b^2} + \frac{A}{2} + \delta h(A, b, p_A, p_b)\right) \tag{20}$$

with a nonlinear function $h(A, b, p_A, p_b)$ of the canonical variables, multiplied by a parameter δ that vanishes in the classical limit. (In an explicit version, both δ and h would be obtained from so-called holonomy modifications of loop quantum cosmology, which always imply a nonlinear and even nonpolynomial h.) For $\delta = 0$, the classical terms in Eq. (20) are derived by applying the complex canonical transformation (18) to Eq. (13).

While A and b are still defined geometrically by their appearance in the line element (19), the new term δh in Eq. (20) modifies the relationship between momenta and time derivatives of A and b. The previous equations (14) and (15) are replaced by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial H_{\text{timelike}}[n]}{\partial p_A} = n\left(-\frac{p_b}{b} + \frac{A p_A}{b^2} + \delta\frac{\partial h}{\partial p_A}\right), \tag{21}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\partial H_{\text{timelike}}[n]}{\partial p_b} = n\left(-\frac{p_A}{b} + \delta\frac{\partial h}{\partial p_b}\right). \tag{22}$$

Deriving the momenta requires an inversion of these equations, which is now nontrivial unless h is a low-order polynomial in p_A and p_b. For our purposes, however, it is sufficient to invert these equations perturbatively in δ. Since our aim is to show that no modifications of this form are compatible with slicing independence, and since an effective theory parametrized by some δ is covariant if and only if it is covariant order by order in δ, a perturbative treatment to leading order in δ suffices to show that the theory violates covariance. To first order in δ, we then have the simple inversion

$$p_A = -\frac{b}{n}\frac{\mathrm{d}b}{\mathrm{d}t} + \delta b\frac{\partial h}{\partial p_b}, \tag{23}$$

$$p_b = -\frac{1}{n}\left(b\frac{\mathrm{d}A}{\mathrm{d}t} + A\frac{\mathrm{d}b}{\mathrm{d}t}\right) + \delta\left(A\frac{\partial h}{\partial p_b} + b\frac{\partial h}{\partial p_A}\right) \tag{24}$$

of the previous equations. We have not explicitly replaced the appearance of p_A and p_b in δ terms on the right-hand sides. To first order in δ, these appearances merely represent the classical form of the momenta.

In terms of time derivatives, the modified Hamiltonian therefore equals

$$H_{\text{timelike}}[n] = -n\left(\frac{b}{n^2}\frac{\mathrm{d}A}{\mathrm{d}t}\frac{\mathrm{d}b}{\mathrm{d}t} + \frac{1}{2}A\left(\frac{1}{N^2}\left(\frac{\mathrm{d}b}{\mathrm{d}t}\right)^2 - 1\right)\right) + n\delta\left(-p_A\frac{\partial h}{\partial p_A} - p_b\frac{\partial h}{\partial p_b} + h\right). \tag{25}$$

It is modified by a δ term if and only if h is nonlinear in momenta which, to repeat, is always the case in models of loop quantum gravity. Our main result will depend only on this general feature.

C. Testing slicing independence

If space-time is of classical Riemannian form, the condition of homogeneity in a timelike direction is equivalent to the existence of a static solution in a spacelike slicing. The direction in which the timelike slicing "evolves" then corresponds to a direction of inhomogeneity in the spacelike slicing. In the present context, both slicings share rotational symmetry implied by the angle-dependent spherical line element. For this statement, we do not need complete slices, and our derivations will therefore apply to local properties of space-time. They are insensitive to any renormalization procedures that may have to be applied to parameters in h or δ in an effective theory if fields are evolved over a wide range of scales.

The modification of Eq. (25) by δ terms does not change the symmetric nature in the geometrical interpretation of the solution as the dynamics of a timelike slicing of space-time. If it belongs to a generally covariant, slicing-independent theory, it must therefore allow an equivalent description as a static solution in a spherically symmetric spacelike slicing. The Hamiltonian (25) is based on a canonical formulation with the same phase space as the classical theory; therefore, it is a homogeneous model of a local gravitational theory in metric variables. Nonlocality would imply additional degrees of freedom through auxiliary fields that describe nonlocal terms, or higher time derivatives in a derivative expansion, but no such fields are implied by holonomy modifications in homogeneous models. Therefore, the corresponding spherically symmetric theory should be local if covariance is realized. Here, it is important that we are not just looking for an embedding of a single solution or a class of solutions in a covariant theory, but rather have to make sure that the complete canonical description, including the phase-space structure, can be realized in a generally covariant theory. Similarly, the number of phase-space degrees of freedom in holonomy-modified homogeneous models, with a single momentum per spatial metric or triad component, implies that we are looking for a theory with second-order field equations. No higher-derivative terms are therefore allowed, even if they are local.

Since all local, generally covariant theories with one inhomogeneous spatial dimension and second-order field equations are, up to field redefinitions, given by $1+1$-dimensional dilaton-gravity models of the form (4), where the scalar ϕ is now given by the function S which does transform like a scalar under two-dimensional coordinate transformations of t and x, there must be functions $\xi(S)$, $k(S,X)$ and $C(S,X)$ such that all solutions of the dynamics generated by Eq. (25) can be mapped to solutions of this generalized dilaton-gravity theory. This condition allows us to test whether models of loop quantum cosmology with nonlinear δ terms in Eq. (25) can be consistent with slicing independence in a generally covariant theory. As already mentioned, we will first evaluate this condition in the restricted setting in which a single dilaton potential $V(S)$ in Eq. (8) characterizes a given model.

In order to determine a possible mapping that could relate the two slicings, we compare the homogeneous line element (19) on timelike slices with a static spherically symmetric one,

$$\mathrm{d}s^2 = -K(X)^2\mathrm{d}T^2 + L(X)^2\mathrm{d}X^2 + S(X)^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{26}$$

Staticity restricts the dependence of the metric components to X, while it implies zero shift vector. This comparison uniquely determines the candidate mapping

$$X = t, \qquad T = x \tag{27}$$

for coordinates, combined with

$$A = K, \qquad b = S, \qquad n = L \tag{28}$$

for metric components.

Solutions in the homogeneous slicing must be such that the Hamiltonian constraint $H_{\mathrm{timelike}}[n] = 0$ is satisfied. In a covariant theory, the same equation must hold true after applying the mapping (28), but it need not correspond to the spherically symmetric Hamiltonian constraint. (In fact, it does not, as we will see soon.) For covariance, it would be sufficient if it were a combination of all the equations available in the spherically symmetric slicing, including the staticity condition in addition to the general spherically symmetric constraints and equations of motion.

An explicit transformation of $H_{\mathrm{timelike}}[n] = 0$ to a spherically symmetric model, using Eq. (28) together with a substitution of t derivatives by X derivatives, shows which spherically symmetric equations should be referred to. Transforming $H_{\mathrm{timelike}}[n]$, we obtain the expression

$$H_{\mathrm{timelike}}[L] = -\frac{S}{L}\frac{\mathrm{d}K}{\mathrm{d}X}\frac{\mathrm{d}S}{\mathrm{d}X} - \frac{1}{2}KL\left(\frac{1}{L^2}\left(\frac{\mathrm{d}S}{\mathrm{d}X}\right)^2 - 1\right) + L\delta\left(-p_A\frac{\partial h}{\partial p_A} - p_b\frac{\partial h}{\partial p_b} + h\right) \tag{29}$$

where

$$p_A = -\frac{S}{L}\frac{\mathrm{d}S}{\mathrm{d}X} + O(\delta), \tag{30}$$

$$p_b = -\frac{1}{L}\left(S\frac{\mathrm{d}K}{\mathrm{d}X} + K\frac{\mathrm{d}S}{\mathrm{d}X}\right) + O(\delta) \tag{31}$$

are implied by Eqs. (23) and (24). [We do not need to replace these expressions explicitly in the δ term of Eq. (29), allowing us to work with a more compact constraint.]

In the form (29), the constraint of the timelike slicing clearly cannot directly correspond to the spherically symmetric Hamiltonian constraint because it depends on the lapse function K of the spherically symmetric slicing not just through K itself but also through its derivative, $\mathrm{d}K/\mathrm{d}X$. An additional condition is therefore required if $H_{\mathrm{timelike}}[L]$ is to vanish for all solutions in a *static* spherically symmetric model. It turns out that staticity can be used to eliminate the derivative $\mathrm{d}K/\mathrm{d}X$ in favor of K itself and the remaining fields, L and S and their derivatives. In particular, evaluating Eq. (11) with $p_L = 0$ and $M = 0$, implied by staticity, (as well as $N = K$) leads to the differential equation

$$0 = -\frac{S}{KL^2}\frac{\mathrm{d}S}{\mathrm{d}X}\frac{\mathrm{d}K}{\mathrm{d}X} - \frac{1}{2L^2}\left(\frac{\mathrm{d}S}{\mathrm{d}X}\right)^2 - \frac{1}{4}SV(S). \tag{32}$$

[We need only one further condition, and therefore will not use a second independent equation (10) in the present context. This equation is more complicated but would give equivalent results.] Solving this equation *algebraically* for $\mathrm{d}K/\mathrm{d}X$, we can therefore eliminate this derivative from Eq. (29), such that

$$H_{\mathrm{timelike}}[L] = \frac{1}{2}KL\left(1 + \frac{1}{2}SV(S)\right) + O(\delta). \tag{33}$$

Disregarding δ terms, this expression vanishes in a spherically symmetric model provided the dilaton potential indeed belongs to classical spherically symmetric gravity, $V(S) = -2/S$. Classically, therefore, the Hamiltonian constraint equation in the timelike homogeneous slicing amounts to a combination of the staticity condition and a condition on the dilaton potential in the spherically symmetric slicing. It is independent of the spherically symmetric Hamiltonian constraint, which rather can be seen to correspond to one of the equations of motion in the timelike homogeneous slicing. (Recall that spatial derivatives in the spherically symmetric slicing correspond to normal derivatives in the timelike homogeneous slicing.)

The equivalence is no longer realized if we include δ terms of the general form as shown in Eq. (25) with nonlinear h. In particular, these terms depend on p_b which, following Eq. (31) depends on the lapse function K of a spherically symmetric slicing. If h is nonlinear in p_b, the δ terms are nonlinear in p_b, or nonlinear in K after the transformation to spherically symmetric variables. After factoring out a single factor of KL, as in Eq. (33), the remaining terms $H_{\rm timelike}[L]/(KL)$ therefore still depend on K. If the dynamics in this slicing corresponds to a dilaton model with Hamiltonian (8), $H_{\rm timelike}[L]/(KL)$ can depend only on S because the dilaton potential is restricted by the covariance condition to have only such a dependence.

However, if $H_{\rm timelike}[L]/(KL)$ depends on K when δ terms are included, it cannot just depend on S: while the staticity condition (32) can be used to solve for K in terms of S and L, it requires solving the differential equation; an algebraic solution for $\mathrm{d}K/\mathrm{d}X$ as in the classical case is not sufficient. A solution K of Eq. (32) depends on S and L nonlocally because integrations are required. No nonlocal S dependence, and no dependence on L at all, can be absorbed in a local dilaton potential $V(S)$. Therefore, there is no local generally covariant theory of the restricted form considered so far, that could describe a spherically symmetric slicing corresponding to the timelike homogeneous one which, by construction, is also local. Any δ term with nonlinear h therefore violates slicing independence and general covariance.

So far, we have shown that there is no minimally coupled generally covariant theory quadratic in momenta which could correspond to the modified dynamics of a timelike homogeneous slicing. It is not difficult to see that nonminimal coupling, leading to $\xi \neq 1$ in Eq. (4), does not change the result. Such a theory can always be obtained from a minimally coupled one by a field redefinition, using an S-dependent conformal transformation of the two-dimensional metric. Such a transformation, formulated canonically, would rescale K and L by S-dependent functions, such that there would be new terms in the equations of motion with spatial derivatives of S, but no new derivative terms of K or L. It is impossible for such terms to absorb a K dependence in a δ term as mentioned in the preceding paragraph, or a term nonlocal in L or with an entire derivative expansion of L if a solution of Eq. (32) for K is used. Our no-go result therefore extends to nonminimally coupled scalars. Similarly, allowing for terms with nonlinear k or nonzero C in Eq. (4) leads, in the static case which is relevant here, only to terms with additional spatial derivatives of S or at most first-order derivatives of L through the Christoffel symbol required in the C term of Eq. (4). Again, even with the freedom of choosing k or C it is impossible to absorb the dependence on K or nonlocally on L that results from a δ term.

Our no-go theorem is therefore complete: there is no generally covariant spherically symmetric theory that could have a solution space corresponding to the modified timelike homogeneous model. Slicing independence is therefore violated by holonomy modifications in symmetry-reduced models of loop quantum gravity.

IV. CONCLUSION

We have shown that the model proposed in Ref. [1] violates general covariance and therefore fails to describe space-time or black holes. This result has implications even if one is not interested in black holes but only in cosmological applications of models of loop quantum gravity. If such models are sufficiently general, it must be possible to apply any proposed modification to models with the symmetries of Kantowski-Sachs space-times, including those with a timelike homogeneous slicing. If they belong to a generally covariant theory, it must then be possible to find a consistent mapping to a static spherically symmetric slicing. Our results show that this is never the case for holonomy modifications proposed in models of loop quantum cosmology. This is our no-go result about general covariance in this setting. (Our result is consistent with the observation that all proposed analog actions that could describe holonomy modifications by higher-curvature terms in isotropic models [25,26], based on mimetic gravity [27,28], fail to describe related effects in anisotropic models [29] or for perturbative inhomogeneity [30].)

Such a general violation of covariance might be interpreted as ruling out not only a specific model but also the entire approach, based on loop quantization. Luckily, however, previous research had already independently shown a possible way out of this damning conclusion. It is possible to evade our no-go result if one takes into account the possibility of *generalized* space-time structures that may be considered covariant in the sense that the same number of gauge transformations is realized as in the classical theory, but in a way that no longer corresponds to slicing independence in Riemannian geometry [22,31]. In fact, the constructions of Ref. [1] implicitly assumed that space-time, even after modifying dynamical equations originally derived from general relativity, retains its Riemannian structure and can be described by line elements. Our no-go result rules out this implicit assumption, but it may be evaded if the assumption is relaxed.

It is difficult to describe non-Riemannian space-time structures in general terms because most of our intuition in general relativity is built on Riemannian properties. Nevertheless, in canonical theories, there are systematic methods that allow one to test whether gauge symmetries are respected by modifications or quantization, even while algebraic relations of the symmetries may be subject to modifications themselves. General covariance in canonical gravity is expressed as the condition of anomaly-freedom of the constraints that generate hypersurface deformations in space-time, given by the Hamiltonian and diffeomorphism constraints. If modified constraints still have closed Poisson brackets, the theory is anomaly-free and enjoys the same number of gauge transformations as classical general relativity, with hypersurface deformations realized in the classical limit. If this is the case, the theory may be considered covariant but in a generalized way. The modified constraints then may no longer generate hypersurface deformations in space-time, but they do provide a well-defined set of gauge transformations that allow one to remove the correct number of spurious degrees of freedom. Line elements might then be applicable in certain regions of space-time, after a field redefinition of metric components that in certain cases can be derived from the consistent generators of modified hypersurface deformations [32,33]. These constructions are now being investigated [34–36], but generalized covariance must be better understood before it is possible to derive complete and reliable models of black holes in loop quantum gravity.

ACKNOWLEDGMENTS

This work was supported in part by NSF Grant No. PHY-1912168.

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