Kerr analogue of Kinnersley's field of an arbitrarily accelerating point mass

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We construct the field of an arbitrarily accelerating and rotating point mass which specializes to the Kerr solution when the acceleration vanishes and specializes to Kinnersley's arbitrarily accelerating point mass when the rotation vanishes.

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I. INTRODUCTION

Although Kinnersley's [1] field of an arbitrarily accelerating point mass has been known for some time, and extensions of it to include charge have been given by Bonnor and Vaidya [2], there does not appear to be an extension that specializes to the Kerr solution [3] when the acceleration vanishes. There exist "radiating Kerr metrics" due to Vaidya and Patel [4] and, more generally, due to Herlt [5] (see [6] for a full description of what is available) but none appear to supply Kinnersley's field with spin included. The original solution by Kinnersley was also extensively studied in subsequent works on particles which accelerate by photon emission, so-called photon rockets, in the context of general relativity, see for example [7–13].

The geometrical construction of Kinnersley's field is described in detail in [6]. We require a modification of this construction to include spin. Since we are primarily interested in introducing acceleration of the source we will leave the mass and angular momentum per unit mass constant throughout. The construction here of the Kerr analogue of Kinnersley's model is a spin-off from the study of the equations of motion of a small, slowly rotating Kerr particle moving in an external gravitational field [14]. In particular the geometrical construction in the present work is motivated by a construction of the Kerr solution with three components of angular momentum in [15].

The structure of the paper is as follows: In Sec. II we give a description of the construction of Kinnersley's field of an arbitrarily accelerating mass point as important background for our construction and to establish our notations and sign conventions. In Sec. III we describe the geometrical construction of the Kerr analogue of Kinnersley's field, and properties of the new model are given in Sec. IV. Our results are concisely summarized in Sec. V.

II. KINNERSELY'S FIELD

We start with the Minkowskian line element in rectangular Cartesian coordinates and time $X^i = (X, Y, Z, T)$ for $i = 1, 2, 3, 4$ (and we will use units for which the speed of light in a vacuum $c = 1$ and the gravitational constant $G = 1$):

$$ds_0^2 = -(dX)^2 - (dY)^2 - (dZ)^2 + (dT)^2 = \eta_{ij} dX^i dX^j. \tag{1}$$

The Schwarzschild solution [16] of Einstein's vacuum field equations is given, in Kerr-Schild form [17], by the metric tensor

$$g_{ij} = \eta_{ij} - \frac{2m}{R} k_i k_j, \tag{2}$$

with $k_i dX^i = dT - dR$, and $R = (X^2 + Y^2 + Z^2)^{1/2}$. Here m (= constant) is the mass of the spherically symmetric source. With $k^i = \eta^{ij} k_j$ (and η^{ij} defined by $\eta^{ij}\eta_{jk} = \delta^i_k$) we have $k^i k_i = 0$. Also the inverse of the metric tensor (2) (denoted g^{ij} with $g^{ij} g_{jk} = \delta^i_k$) is given by

$$g^{ij} = \eta^{ij} + \frac{2m}{R} k^i k^j, \tag{3}$$

and thus $k^i = \eta^{ij} k_j = g^{ij} k_j$ so that k^i is a null vector field in the flat space-time with metric tensor η_{ij} and in the curved space-time with metric tensor g_{ij}. We see that in the flat space-time with line element (1) $R = 0 \Leftrightarrow X = Y = Z = 0$ is a timelike geodesic (the T axis). Effectively Kinnersley's [1] construction replaces this geodesic with an arbitrary

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timelike world line. To achieve this let $X^i = w^i(u)$ be an arbitrary timelike world line in Minkowskian space-time with u arc length or proper time along it. Then $v^i(u) = dw^i/du$ is the unit timelike tangent to this world line satisfying $\eta_{ij}v^iv^j = v_jv^j = +1$. Thus v^i is the 4-velocity of the particle with world line $X^i = w^i(u)$. The corresponding 4-acceleration is $a^i(u) = dv^i/du$ and satisfies $a_iv^i = 0$ since $v_iv^i = 1$. The position 4-vector of any point of Minkowskian space-time relative to the world line $X^i = w^i(u)$ may be written in the form

$$X^i = w^i(u) + rk^i, \tag{4}$$

for $0 \le r < +\infty$ with $k^ik_i = 0$ and $k^iv_i = +1$. Thus k^i is a normalized future pointing null vector field defined along the world line $X^i = w^i(u)$ and so k^i is tangent to the future null cones with vertices on $X^i = w^i(u)$. The *direction* of k^i is parametrized by two real parameters ξ, η (say) so that we can write

$$k^i = P_0^{-1}\left(-\xi, -\eta, -1 + \frac{1}{4}(\xi^2+\eta^2), 1 + \frac{1}{4}(\xi^2+\eta^2)\right), \tag{5}$$

for $-\infty < \xi,\ \eta < +\infty$ and with P_0 determined by the normalization $k^iv_i = 1$ of k^i to read

$$P_0 = v^1(u)\xi + v^2(u)\eta + v^3(u)\left(1 - \frac{1}{4}(\xi^2+\eta^2)\right) + v^4(u)\left(1 + \frac{1}{4}(\xi^2+\eta^2)\right). \tag{6}$$

We note that

$$\begin{aligned} h_0 \equiv a_ik^i &= P_0^{-1}\Big\{a^1(u)\xi + a^2(u)\eta \\ &\quad + a^3(u)\left(1 - \frac{1}{4}(\xi^2+\eta^2)\right) \\ &\quad + a^4(u)\left(1 + \frac{1}{4}(\xi^2+\eta^2)\right)\Big\} \\ &= \frac{\partial}{\partial u}(\log P_0), \end{aligned} \tag{7}$$

and thus the propagation law for k^i along $X^i = w^i(u)$ is

$$\frac{\partial k^i}{\partial u} = -h_0k^i. \tag{8}$$

From (4) we have

$$dX^i = (v^i - rh_0k^i)du + k^idr + r\frac{\partial k^i}{\partial \xi}d\xi + r\frac{\partial k^i}{\partial \eta}d\eta. \tag{9}$$

Hence, in particular,

$$k_idX^i = du \Leftrightarrow k_i = u_{,i}, \tag{10}$$

with the comma denoting partial differentiation with respect to X^i. Noting that

$$\frac{\partial k^i}{\partial \xi}\frac{\partial k_i}{\partial \xi} = -P_0^{-2} = \frac{\partial k^i}{\partial \eta}\frac{\partial k_i}{\partial \eta} \quad \text{and} \quad \frac{\partial k^i}{\partial \xi}\frac{\partial k_i}{\partial \eta} = 0, \tag{11}$$

we have

$$ds_0^2 = \eta_{ij}dX^idX^j = -r^2P_0^{-2}(d\xi^2 + d\eta^2) + 2dudr + (1 - 2h_0r)du^2. \tag{12}$$

If we now generalize the metric tensor (3) to provide the line element, in coordinates $x^i = (\xi, \eta, r, u)$,

$$ds^2 = g_{ij}dx^idx^j = -r^2P_0^{-2}(d\xi^2 + d\eta^2) + 2dudr + \left(1 - 2h_0r - \frac{2m}{r}\right)du^2, \tag{13}$$

we arrive at Kinnersley's line element which reduces to the Eddington-Finkelstein form of the Schwarzschild line element when $a^i = 0$ [i.e., when the world line $X^i = w^i(u)$ in Minkowskian space-time is a timelike geodesic]. For this special case we may take $v^i = \delta^i_4$ and so $P_0 = 1 + \frac{1}{4}(\xi^2+\eta^2) = p_0$ (say). Then (13) becomes the Schwarzschild line element

$$ds^2 = -r^2p_0^{-2}(d\xi^2 + d\eta^2) + 2dudr + \left(1 - \frac{2m}{r}\right)du^2. \tag{14}$$

Here ξ, η are stereographic coordinates on the unit 2-sphere, related to the polar angles θ, ϕ for $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ via $\eta/\xi = \tan\phi$ and $\cos\theta = (4 - \xi^2 - \eta^2)/(4 + \xi^2 + \eta^2)$. We can write (13) in terms of basis 1-forms $\vartheta^{(a)}$ with $a = 1, 2, 3, 4$ as

$$ds^2 = -(\vartheta^{(1)})^2 - (\vartheta^{(2)})^2 + 2\vartheta^{(3)}\vartheta^{(4)} = g_{(a)(b)}\vartheta^{(a)}\vartheta^{(b)}, \tag{15}$$

with

$$\begin{aligned} \vartheta^{(1)} &= rP_0^{-1}d\xi = -\vartheta_{(1)}, \\ \vartheta^{(2)} &= rP_0^{-1}d\eta = -\vartheta_{(2)}, \\ \vartheta^{(3)} &= du = \vartheta_{(4)}, \\ \vartheta^{(4)} &= dr + \left(\frac{1}{2} - h_0r - \frac{m}{r}\right)du = \vartheta_{(3)}, \end{aligned} \tag{16}$$

and $\vartheta_{(a)} = g_{(a)(b)}\vartheta^{(b)}$. The 1-forms define a half null tetrad and the components $R_{(a)(b)}$ of the Ricci tensor on this tetrad vanish with the exception of

$$R_{(3)(3)} = -\frac{6mh_0}{r^2}. \tag{17}$$

Hence if R_{ij} are the components of the Ricci tensor in coordinates $x^i = (\xi, \eta, r, u)$ then we can write

$$R_{ij} = -\frac{6mh_0}{r^2} n_i n_j \quad \text{with} \quad n_i dx^i = du. \tag{18}$$

We note that $g^{ij} n_i n_j = 0$.

In preparation for consideration of the axially symmetric Kerr case it is useful to specialize the Kinnersley model to the case for which the world line $X^i = w^i(u)$ is the history of a particle performing rectilinear motion. For this we take the 4-velocity v^i to be restricted by requiring $v^1(u) = 0 = v^2(u)$. Then writing $\lambda = v^4 + v^3$ and $\mu = v^4 - v^3$ we have $\lambda\mu = 1$ and, with a dot denoting differentiation with respect to u, we see that if

$$\frac{\dot{\lambda}}{\lambda} = A(u) \quad \text{then } A^2 = -a_i a^i = (a^3)^2 - (a^4)^2. \tag{19}$$

Now P_0 and h_0 in (6) and (7) read

$$P_0 = \lambda^{-1}\left(\lambda^2 + \frac{1}{4}(\xi^2 + \eta^2)\right), \tag{20}$$

and

$$h_0 = A(u)\left(\frac{\lambda^2 - \frac{1}{4}(\xi^2 + \eta^2)}{\lambda^2 + \frac{1}{4}(\xi^2 + \eta^2)}\right). \tag{21}$$

We introduce polar coordinates θ, ϕ by writing

$$\xi = 2\lambda\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)^{1/2} \cos\phi,$$
$$\eta = 2\lambda\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)^{1/2} \sin\phi, \tag{22}$$

and thus

$$\frac{\eta}{\xi} = \tan\phi \quad \text{and} \quad \frac{\lambda^2 - \frac{1}{4}(\xi^2 + \eta^2)}{\lambda^2 + \frac{1}{4}(\xi^2 + \eta^2)} = \cos\theta. \tag{23}$$

This results in $h_0 = A(u)\cos\theta$,

$$P_0^{-1} d\xi = -\sin\theta \sin\phi d\phi + (d\theta + A(u)\sin\theta du)\cos\phi, \tag{24}$$

$$P_0^{-1} d\eta = \sin\theta d\phi \cos\phi + (d\theta + A(u)\sin\theta du)\sin\phi, \tag{25}$$

and thus the line element (13) takes the form

$$\begin{aligned} ds^2 &= -r^2\{(d\theta + A(u)\sin\theta du)^2 + \sin^2\theta d\phi^2\} + 2dudr \\ &\quad + \left(1 - 2A(u) r\cos\theta - \frac{2m}{r}\right) du^2, \\ &= \left(\underset{(0)}{g_{ij}} - \frac{2m}{r} n_i n_j\right) dx^i dx^j, \\ &= g_{ij} dx^i dx^j. \end{aligned} \tag{26}$$

Here the Minkowskian metric tensor components in the curvilinear coordinates $x^i = (\theta, \phi, r, u)$ are denoted $\underset{(0)}{g_{ij}}$. The Minkowskian metric tensor components in the rectangular Cartesian coordinates and time $X^i = (X, Y, Z, T)$ are denoted η_{ij} and thus $\underset{(0)}{g_{ij}} dx^i dx^j = \eta_{ij} dX^i dX^j$. We note that $g_{ij} = \underset{(0)}{g_{ij}} - (2m/r) n_i n_j$ has Kerr-Schild form. Also $n^i = \underset{(0)}{g^{ij}} n_j = g^{ij} n_j = \delta^i_3$. The Kerr-Schild form of the metric tensor has the important algebraic property of the equality of determinants:

$$g = \det(g_{ij}) = \det(\underset{(0)}{g_{ij}}) = \underset{(0)}{g} = -r^4 \sin^2\theta. \tag{27}$$

Hence, if covariant differentiation with respect to the Riemannian connection calculated with the metric tensor g_{ij} is denoted by a stroke, we have

$$n^i{}_{|i} = \frac{1}{2g}\frac{\partial g}{\partial r} = \frac{1}{2\,\underset{(0)}{g}}\frac{\partial}{\partial r}(\underset{(0)}{g}) = \frac{2}{r}, \tag{28}$$

and so, for future reference, we see that the metric tensor above has the form

$$g_{ij} = \underset{(0)}{g_{ij}} - m n^k{}_{|k} n_i n_j. \tag{29}$$

We note that the optical scalar describing the expansion of the congruence of null geodesic integral curves of the vector field n^i is $(1/2) n^i{}_{|i}$.

III. KERR ANALOGUE OF KINNERSLEY'S FIELD

To introduce rotation or spin into the Kinnersley field we proceed by modifying (4) to read

$$X^i = w^i(u) + r k^i + U^i, \tag{30}$$

with

$$U^i(\xi, \eta, u) = P_0^2\left(\frac{\partial k^i}{\partial \xi}\frac{\partial F}{\partial \eta} - \frac{\partial k^i}{\partial \eta}\frac{\partial F}{\partial \xi}\right). \tag{31}$$

Here k^i and P_0 are given by (5) and (6) and

$$F = s^i(u)k_i \quad \text{with} \quad s^i v_i = 0 \quad \text{and} \quad \frac{ds^i}{du} = -(a_j s^j)v^i. \tag{32}$$

The spacelike vector $s^i(u)$ is here defined along the world line $X^i = w^i(u)$ by Fermi transport. We note the useful formulas:

$$\frac{\partial U^i}{\partial \xi} = (v^i - k^i)\frac{\partial F}{\partial \eta} + \frac{\partial k^i}{\partial \eta}F, \tag{33}$$

$$\frac{\partial U^i}{\partial \eta} = -(v^i - k^i)\frac{\partial F}{\partial \xi} - \frac{\partial k^i}{\partial \xi}F, \tag{34}$$

$$\frac{\partial U^i}{\partial u} = P_0^2\left(\frac{\partial h_0}{\partial \eta}\frac{\partial F}{\partial \xi} - \frac{\partial h_0}{\partial \xi}\frac{\partial F}{\partial \eta}\right)k^i + FP_0^2\left(\frac{\partial h_0}{\partial \xi}\frac{\partial k^i}{\partial \eta} - \frac{\partial h_0}{\partial \eta}\frac{\partial k^i}{\partial \xi}\right). \tag{35}$$

We now, for simplicity, specialize the world line $X^i = w^i(u)$ by requiring $v^1(u) = 0 = v^2(u)$ as we did at the end of the previous section. Then one can solve the propagation law (32) for s^i along $X^i = w^i(u)$ with

$$s^i(u) = \left(0, 0, \frac{1}{2}\underset{(0)}{S}(\lambda + \lambda^{-1}), \frac{1}{2}\underset{(0)}{S}(\lambda - \lambda^{-1})\right), \tag{36}$$

with $\lambda(u) = v^4 + v^3$ as before and $\underset{(0)}{S} = \text{constant}$. We then find that

$$F = s^i k_i = \underset{(0)}{S}\left(\frac{\lambda^2 - \frac{1}{4}(\xi^2 + \eta^2)}{\lambda^2 + \frac{1}{4}(\xi^2 + \eta^2)}\right), \tag{37}$$

and P_0, h_0 are given by (20). Now introducing the polar angles θ, ϕ via (22) we have $h_0 = A(u)\cos\theta$ as before and $F = \underset{(0)}{S}\cos\theta$ and we obtain from (31)

$$\begin{aligned} dX^i = {} & (du + \underset{(0)}{S}\sin^2\theta d\phi)v^i \\ & + (dr - rA\cos\theta du - \underset{(0)}{S}\sin^2\theta d\phi)k^i \\ & + (\lambda^{(1)}\cos\phi - \lambda^{(2)}\sin\phi)P_0\frac{\partial k^i}{\partial \xi} \\ & + (\lambda^{(1)}\sin\phi + \lambda^{(2)}\cos\phi)P_0\frac{\partial k^i}{\partial \eta}, \end{aligned} \tag{38}$$

with the 1-forms $\lambda^{(1)}$, $\lambda^{(2)}$ given by

$$\lambda^{(1)} = r(d\theta + A(u)\sin\theta du) - \underset{(0)}{S}\sin\theta\cos\theta d\phi, \tag{39}$$

$$\lambda^{(2)} = r\sin\theta d\phi + \underset{(0)}{S}\cos\theta d\theta. \tag{40}$$

We note that

$$k_i dX^i = du + \underset{(0)}{S}\sin^2\theta d\phi = n_i dx^i, \tag{41}$$

defining n_i in coordinates $x^i = (\theta, \phi, r, u)$ and, using the Minkowskian scalar products (11),

$$\begin{aligned} \eta_{ij}dX^i dX^j = {} & -(\lambda^{(1)})^2 - (\lambda^{(2)})^2 + 2(du + \underset{(0)}{S}\sin^2\theta) \\ & \times \left\{dr - rA\cos\theta du - \underset{(0)}{S}\sin^2\theta d\phi \right. \\ & \left. + \frac{1}{2}(du + \underset{(0)}{S}\sin^2\theta d\phi)\right\}, \\ = {} & \underset{(0)}{g}_{ij}dx^i dx^j. \end{aligned} \tag{42}$$

When $\underset{(0)}{S} = 0$ this line element coincides with (26) when $m = 0$. When $A = 0$ in (42) the line element is the Minkowskian *background* for the Kerr solution with angular momentum per unit mass $\underset{(0)}{S}$. The constant $\underset{(0)}{S}$ is thus playing the role of the constant a in the standard form of the Kerr solution. We are using $\underset{(0)}{S}$ to denote the angular momentum per unit mass or spin rather than the more familiar a to avoid any confusion with the acceleration. We now form the Kerr-Schild metric tensor (29) in this case with n_i given by (41). Once again we have $n^i = \delta^i_3$ and using (28) we find that

$$n^i{}_{|i} = \frac{2r + A\underset{(0)}{S}{}^2\sin^2\theta\cos\theta}{r^2 + \underset{(0)}{S}{}^2\cos^2\theta + rA\underset{(0)}{S}{}^2\sin^2\theta\cos\theta}. \tag{43}$$

Substituting (42) and (43) into (29) we arrive at the Kerr analogue of Kinnersley's model which can be written

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j &= -(\lambda^{(1)})^2 - (\lambda^{(2)})^2 + 2\lambda^{(3)}\lambda^{(4)} \\ &= g_{(a)(b)}\lambda^{(a)}\lambda^{(b)}, \end{aligned} \tag{44}$$

with $\lambda^{(1)}$, $\lambda^{(2)}$ given by (39) and (40) and

$$\lambda^{(3)} = du + \underset{(0)}{S}\sin^2\theta d\phi, \tag{45}$$

$$\lambda^{(4)} = dr - rA(u)\cos\theta du - \underset{(0)}{S}\sin^2\theta d\phi + \left(\frac{1}{2} - \frac{m\varphi'}{2\varphi}\right)\lambda^{(3)}, \tag{46}$$

with

$$\varphi = r^2 + \underset{(0)}{S}{}^2\cos^2\theta + rA\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta, \tag{47}$$

and $\varphi' = \partial\varphi/\partial r$. We emphasize that $A(u)$ is an arbitrary function of its argument while m and $\underset{(0)}{S}$ are constants.

Writing the basis 1-forms $\lambda^{(a)}$ given by (39), (40), (45) and (46) in the form

$$\lambda^{(1)} = f_i dx^i, \qquad \lambda^{(2)} = e_i dx^i,$$
$$\lambda^{(3)} = n_i dx^i, \qquad \lambda^{(4)} = l_i dx^i, \tag{48}$$

with $x^i = (\theta, \phi, r, u)$, all scalar products among the basis vectors f^i, e^i, n^i, l^i vanish except $f_i f^i = e_i e^i = -n_i l^i = -1$. An exact calculation of the Ricci tensor components R_{ij} in coordinates x^i results in

$$\begin{aligned} R_{ij} = {} & R_{(1)(1)} f_i f_j + R_{(2)(2)} e_i e_j + R_{(1)(2)}(f_i e_j + f_j e_i) \\ & + R_{(1)(3)}(f_i n_j + f_j n_i) + R_{(2)(3)}(e_i n_j + e_j n_i) \\ & + R_{(3)(4)}(n_i l_j + n_j l_i) + R_{(3)(3)} n_i n_j, \end{aligned} \tag{49}$$

with

$$\begin{aligned} R_{(1)(1)} = {} & mA^2\,\underset{(0)}{S}{}^4\sin^4\theta\cos^2\theta\varphi^{-3}\varphi' + mA\,\underset{(0)}{S}{}^2 \\ & \times \sin^2\theta\cos\theta\varphi^{-3}(r^2 - \underset{(0)}{S}{}^2\cos^2\theta), \end{aligned} \tag{50}$$

$$R_{(2)(2)} = -mA\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta\varphi^{-3}(r^2 - \underset{(0)}{S}{}^2\cos^2\theta), \tag{51}$$

$$R_{(1)(2)} = -mA\,\underset{(0)}{S}{}^3\sin^2\theta\cos^2\theta\varphi^{-3}\varphi', \tag{52}$$

$$R_{(3)(4)} = \frac{1}{2}mA^2\,\underset{(0)}{S}{}^4\sin^4\theta\cos^2\theta\varphi^{-3}\varphi', \tag{53}$$

and φ is given by (47). The remaining tetrad components of the Ricci tensor to be calculated are $R_{(1)(3)}$, $R_{(2)(3)}$ and $R_{(3)(3)}$. These latter components are considerably more complicated than (50)–(53) and are only required asymptotically (for large r) for our purposes and are given in (73)–(75) below. We note that

$$g^{ij}R_{ij} = -R_{(1)(1)} - R_{(2)(2)} + 2R_{(3)(4)} = 0. \tag{54}$$

The generalized Kerr congruence (i.e., the Kerr congruence generalized to include the influence of the acceleration) consists of the integral curves of the null vector field n which is given, in coordinates $x^i = (\theta, \phi, r, u)$ by

$$n = n^i\frac{\partial}{\partial x^i} = \frac{\partial}{\partial r} \quad \text{and} \quad n_i dx^i = \underset{(0)}{S}\sin^2\theta d\phi + du. \tag{55}$$

This is a geodesic congruence since

$$n^i{}_{|j}n^j = \frac{1}{2}g^{ij}(2g_{j3,3} - g_{33,j}) = 0, \tag{56}$$

and $g_{3i} = (0, \underset{(0)}{S}\sin^2\theta, 0, 1)$. Thus the integral curves of the vector field n are null geodesics with r an affine parameter along them. If we define the complex null vector m^i and its complex conjugate $\bar{m}^i$ by

$$m^i = \frac{1}{\sqrt{2}}(f^i + ie^i) \quad \text{and} \quad \bar{m}^i = \frac{1}{\sqrt{2}}(f^i - ie^i), \tag{57}$$

then the complex shear σ of this null geodesic congruence is given by

$$\sigma = n_{i|j}m^i m^j \Rightarrow |\sigma|^2 = \frac{1}{2}n_{(i|j)}n^{i|j} - \Theta^2, \tag{58}$$

where the round brackets denote symmetrization and

$$\Theta = \frac{1}{2}n^i{}_{|i}, \tag{59}$$

is the expansion scalar of the congruence. Also

$$\rho = n_{i|j}m^i\bar{m}^j = -\Theta + i\omega, \tag{60}$$

with

$$\omega^2 = \frac{1}{2}n_{[i|j]}n^{i|j}, \tag{61}$$

the twist of the congruence. The square brackets here denote antisymmetrization. For the particular case of the generalized Kerr congruence the complex shear σ is in fact real and is given by

$$\sigma = -\frac{1}{2}\varphi^{-1}A\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta, \tag{62}$$

while the complex scalar (60) takes the form

$$\rho = -\frac{1}{2}\varphi^{-1}\varphi' + i\varphi^{-1}\underset{(0)}{S}\cos\theta. \tag{63}$$

We already know that the tetrad component $R_{(4)(4)} = R_{ij}n^i n^j$ vanishes. This can now be verified using the propagation equation for ρ along the congruence:

$$\frac{\partial\rho}{\partial r} = \rho^2 + \sigma^2 + \frac{1}{2}R_{(4)(4)}. \tag{64}$$

We can write $R_{(1)(1)}$, $R_{(2)(2)}$, $R_{(1)(2)}$, $R_{(3)(4)}$ in terms of the geometrical variables σ, ω as follows: Using

$$\sigma = \underset{(0)}{\sigma}\varphi^{-1} \quad \text{with} \quad \underset{(0)}{\sigma} = -\frac{1}{2}A\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta, \tag{65}$$

and

$$\omega = \underset{(0)}{\omega}\varphi^{-1} \quad \text{with} \quad \underset{(0)}{\omega} = \underset{(0)}{S}\cos\theta, \tag{66}$$

we find from (50)–(53):

$$R_{(1)(1)} = -2m\,\underset{(0)}{\sigma}\varphi^{-2} + 4m\,\underset{(0)}{\sigma}{}^2 r\varphi^{-3} + 4m\,\underset{(0)}{\sigma}(\underset{(0)}{\omega}{}^2 - 2\,\underset{(0)}{\sigma}{}^2)\varphi^{-3}, \tag{67}$$

$$R_{(2)(2)} = 2m\,\underset{(0)}{\sigma}\varphi^{-2} + 4m\,\underset{(0)}{\sigma}{}^2 r\varphi^{-3} - 4m\,\underset{(0)(0)}{\sigma\,\omega^2}\varphi^{-3}, \tag{68}$$

$$R_{(1)(2)} = 4m\,\underset{(0)}{\sigma}\,\underset{(0)}{\omega}\,r\varphi^{-3} - 4m\,\underset{(0)}{\sigma}{}^2\,\underset{(0)}{\omega}\varphi^{-3}, \tag{69}$$

$$R_{(3)(4)} = 4m\,\underset{(0)}{\sigma}{}^2 r\varphi^{-3} - 4m\,\underset{(0)}{\sigma}{}^3\varphi^{-3}. \tag{70}$$

Here the dependence of the tetrad components of the Ricci tensor on the radial coordinate r is explicit, remembering that φ is given by (47) and since $\underset{(0)}{\sigma}$, $\underset{(0)}{\omega}$ are independent of r. The Ricci tensor components R_{ij}, in coordinates $x^i = (\theta, \phi, r, u)$, have the exact algebraic form given in (49). We have noted following (49) that the Ricci scalar $R = g^{ij}R_{ij}$ vanishes exactly and so the Ricci tensor R_{ij} and the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$ coincide. We now also note that the basis vector fields f^i, e^i, l^i, n^i are parallel transported along the future-directed null geodesic integral curves of the vector field n^i and thus

$$f^i{}_{|j}n^j = 0, \quad e^i{}_{|j}n^j = 0, \quad l^i{}_{|j}n^j = 0 \text{ and } n^i{}_{|j}n^j = 0. \tag{71}$$

For large positive values of the affine parameter r we see from (67)–(70) that

$$R_{(1)(1)} = O\left(\frac{1}{r^4}\right), \qquad R_{(2)(2)} = O\left(\frac{1}{r^4}\right),$$
$$R_{(1)(2)} = O\left(\frac{1}{r^5}\right), \qquad R_{(3)(4)} = O\left(\frac{1}{r^5}\right). \tag{72}$$

The remaining nonvanishing tetrad components of the Ricci tensor, namely $R_{(1)(3)}$, $R_{(2)(3)}$, and $R_{(3)(3)}$, have the following asymptotic forms for large r:

$$R_{(1)(3)} = O\left(\frac{1}{r^4}\right), \tag{73}$$

$$R_{(2)(3)} = -\frac{6mA\,\underset{(0)}{S}\sin\theta\cos\theta}{r^3} + O\left(\frac{1}{r^4}\right), \tag{74}$$

$$R_{(3)(3)} = -\frac{6mA\cos\theta}{r^2} + \frac{1}{r^3}\left\{-6m\dot{A}\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta - 6mA^2\,\underset{(0)}{S}{}^2\sin^2\theta + 48mA^2\,\underset{(0)}{S}{}^2\sin^2\theta\cos^2\theta\right\} + O\left(\frac{1}{r^4}\right). \tag{75}$$

Here, as always, $\dot{A} = dA/du$. Substituting (72)–(75) into (49) and given the energy-momentum-stress tensor of matter T_{ij} via Einstein's equations $R_{ij} = -8\pi T_{ij}$ we find that we can write

$$T_{ij} = \underset{(1)}{T_{ij}} + \underset{(2)}{T_{ij}} + O\left(\frac{1}{r^4}\right), \tag{76}$$

with

$$8\pi\underset{(1)}{T_{ij}} = \left(\frac{6mA\cos\theta}{r^2} - \frac{6mA^2\,\underset{(0)}{S}{}^2\sin^2\theta\cos^2\theta}{r^3}\right)n_i n_j, \tag{77}$$

and

$$8\pi\underset{(2)}{T_{ij}} = \frac{6mA\,\underset{(0)}{S}\sin\theta\cos\theta}{r^3}(e_i n_j + e_j n_i) + \frac{1}{r^3}(6m\dot{A}\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta - 42mA^2\,\underset{(0)}{S}{}^2\sin^2\theta\cos^2\theta + 6mA^2\,\underset{(0)}{S}{}^2\sin^2\theta)n_i n_j. \tag{78}$$

We have chosen the definitions of $\underset{(1)}{T_{ij}}$ and $\underset{(2)}{T_{ij}}$ here because now they each separately satisfy the approximate conservation equations

$$\underset{(1)}{T^{ij}}{}_{|j} = O\left(\frac{1}{r^5}\right) \quad \text{and} \quad \underset{(2)}{T^{ij}}{}_{|j} = O\left(\frac{1}{r^5}\right). \tag{79}$$

The first of these is easily verified using $n^i\frac{\partial}{\partial x^i} = \frac{\partial}{\partial r}$ and

$$n^i{}_{|i} = \frac{2}{r} - \frac{A\,\underset{(0)}{S}{}^2\sin^2\theta\cos\theta}{r^2} + O\left(\frac{1}{r^3}\right). \tag{80}$$

The leading r^{-2} term in the energy-momentum-stress tensor here does not involve the spin parameter of the Kerr solution and coincides with the Kinnersley energy-momentum-stress tensor. It is algebraically identical to Kinnersley's energy-momentum-stress tensor if, instead of using the affine parameter distance r, we use the parallax distance r_P defined by [18]

$$r_P = \Theta^{-1} = \frac{r^2 + rA\underset{(0)}{S}{}^2\sin^2\theta\cos\theta + \underset{(0)}{S}{}^2\cos^2\theta}{r + \frac{1}{2}A\underset{(0)}{S}{}^2\sin^2\theta\cos\theta}. \quad (81)$$

Now for large values of r we have

$$\frac{1}{r_P^2} = \frac{1}{r^2} - \frac{A\underset{(0)}{S}{}^2\sin^2\theta\cos\theta}{r^3} + O\left(\frac{1}{r^4}\right), \quad (82)$$

and so

$$8\pi\underset{(1)}{T}{}_{ij} = \frac{6mA\cos\theta}{r_P^2}n_i n_j. \quad (83)$$

With the shear σ given by (62) and the twist ω and expansion Θ of the integral curves of the vector field n^i given via (60) and (63) we have an exact formula relating the affine parameter distance r and the parallax distance r_P:

$$r = \frac{r_P(1+\sigma r_P)}{1 + (\omega r_P)^2 - (\sigma r_P)^2}. \quad (84)$$

This generalizes a formula derived by Sachs [Eq. (23) in [18]] in the shear-free case.

We note that $\underset{(1)}{T}{}^{ij}$ corresponds to the two leading terms in the expansion for large r of

$$8\pi\underset{(1)}{t}{}^{ij} = 6mA\varphi^{-1}\cos\theta n^i n^j, \quad (85)$$

with φ given by (47), which satisfies the exact conservation equation

$$\underset{(1)}{t}{}^{ij}{}_{|j} = 0, \quad (86)$$

on account of (43). Verifying the second of (79) requires

$$n^i{}_{|j}e^j = -\varphi^{-1}rA\underset{(0)}{S}\sin\theta\cos\theta n^i + \varphi^{-1}re^i + \varphi^{-1}\underset{(0)}{S}\cos\theta f^i, \quad (87)$$

$$e^i{}_{|i} = -\varphi^{-1}rA\underset{(0)}{S}\sin\theta\cos\theta - \varphi^{-1}\underset{(0)}{S}\cot\theta\cos\theta, \quad (88)$$

with φ expanded in inverse powers of r.

IV. PROPERTIES OF THE KERR ANALOG OF KINNERSLEY'S FIELD

To identify a curvature singularity in the Kerr analogue of Kinnersley's field, a convenient approach is to calculate the Kretschmann scalar. This is given in our case by

$$\begin{aligned} R_{ijkl}R^{ijkl} = \frac{2m^2}{\varphi^6}\{&24r^6 - 360r^4\underset{(0)}{S}{}^2\cos^2\theta + 360r^2\underset{(0)}{S}{}^4\cos^4\theta - 24\underset{(0)}{S}{}^6\cos^6\theta + 72r^5A\underset{(0)}{S}{}^2\sin^2\theta\cos\theta \\ &+ 104r^4A^2\underset{(0)}{S}{}^4\sin^4\theta\cos^2\theta - 720r^3A\underset{(0)}{S}{}^4\sin^2\theta\cos^3\theta + 88r^3A^3\underset{(0)}{S}{}^6\sin^6\theta\cos^3\theta + 50r^2A^4\underset{(0)}{S}{}^8\sin^8\theta\cos^4\theta \\ &- 608r^2A^2\underset{(0)}{S}{}^6\sin^4\theta\cos^4\theta + 360rA\underset{(0)}{S}{}^6\sin^2\theta\cos^5\theta - 248rA^3\underset{(0)}{S}{}^8\sin^6\theta\cos^5\theta + 18rA^5\underset{(0)}{S}{}^{10}\sin^{10}\theta\cos^5\theta \\ &+ 3A^6\underset{(0)}{S}{}^{12}\sin^{12}\theta\cos^6\theta - 42A^4\underset{(0)}{S}{}^{10}\sin^8\theta\cos^6\theta + 104A^2\underset{(0)}{S}{}^8\sin^4\theta\cos^6\theta\}, \end{aligned} \quad (89)$$

with φ given by (47). Thus the Kretschmann scalar is singular when $\varphi = 0$ and this occurs, in particular, at $r = 0$, $\theta = \pi/2$. When (31) is specialized to the axially symmetric case we have

$$X = rk^1 - \underset{(0)}{S}k^2, \quad (90)$$

$$Y = rk^2 + \underset{(0)}{S}k^1, \quad (91)$$

$$Z = w^3(u) + rk^3, \quad (92)$$

$$T = w^4(u) + rk^4. \quad (93)$$

Since $(k^1)^2 + (k^2)^2 = \sin^2\theta$ we see that when $r = 0$ and $\theta = \pi/2$ the singularity in the curvature occurs on the ring

$$X^2 + Y^2 = \underset{(0)}{S}{}^2, \quad (94)$$

which is accelerating in the positive Z direction when $A(u) \neq 0$. More generally, using $\underset{(0)}{\sigma}$ and $\underset{(0)}{\omega}$ given by (65) and (66), we can write φ in (47) in the form

$$\varphi = r^2 - 2r\underset{(0)}{\sigma} + \underset{(0)}{\omega}{}^2. \quad (95)$$

It therefore follows from (89) that curvature singularities occur when

$$\varphi = 0 \Leftrightarrow r = \underset{(0)}{\sigma} \pm \sqrt{\underset{(0)}{\sigma}{}^2 - \underset{(0)}{\omega}{}^2}, \quad (96)$$

and this requires

$$\underset{(0)}{\sigma}{}^2 \geq \underset{(0)}{\omega}{}^2. \tag{97}$$

Always assuming that $A(u) \neq 0$ we see from this inequality that *provided* $\theta \neq \frac{\pi}{2}$ *and* $\underset{(0)}{S} \neq 0$ *we must have*

$$1 \leq \frac{1}{4} A^2 \underset{(0)}{S}{}^2 \sin^4\theta < \frac{1}{4} A^2 \underset{(0)}{S}{}^2. \tag{98}$$

But this condition by itself excludes Kinnersley's model as a special case. Consequently we can say that the Kerr analogue of Kinnersely's field of an arbitrarily accelerating point mass has a curvature singularity on a ring of radius $|\underset{(0)}{S}|$ undergoing rectilinear motion with arbitrary acceleration.

From the foregoing we see that asymptotically the matter distribution created by the accelerating, spinning source is qualitatively similar to that of Kinnersley's accelerating point mass except that the propagation direction of the lightlike matter in the spinning case has shear on account of the acceleration/spin interaction. A further manifestation of the asymptotic similarity between the spinning case and Kinnersley's nonspinning case is displayed by comparing the formulas (4) and (30) for large values of r. We begin by writing (30) as

$$X^i = w^i(u) + rK^i \quad \text{with} \quad K^i = k^i + \frac{1}{r} U^i. \tag{99}$$

Written out explicitly

$$U^i = P_0^2 \left(\frac{\partial k^i}{\partial \xi} \frac{\partial k^j}{\partial \eta} - \frac{\partial k^j}{\partial \xi} \frac{\partial k^i}{\partial \eta} \right) s_j. \tag{100}$$

With k^i and P_0 given by (5) and (6) we can write

$$P_0^2 \left(\frac{\partial k^i}{\partial \xi} \frac{\partial k^j}{\partial \eta} - \frac{\partial k^j}{\partial \xi} \frac{\partial k^i}{\partial \eta} \right) = \epsilon_{ijkl} v^k k^l, \tag{101}$$

where ϵ_{ijkl} is the four-dimensional Levi-Civita permutation symbol (we take $\epsilon_{1234} = +1$). If we define the spin tensor

$$s_{ij} = \epsilon_{ijkl} s^k v^l = -s_{ji} \Leftrightarrow s_i = \frac{1}{2} \epsilon_{ijkl} s^{jk} v^l, \tag{102}$$

then

$$U^i = s^i{}_j k^j \Rightarrow K^i = \left(\delta^i_j + \frac{1}{r} s^i{}_j \right) k^j, \tag{103}$$

demonstrating that asymptotically K^i only differs from k^i by an infinitesimal Lorentz transformation. For the axially symmetric case

$$s_{ij} = \underset{(0)}{S} \epsilon_{ij34}, \tag{104}$$

and using (22)

$$\begin{aligned} K_i dX^i &= k_i dX^i + \frac{1}{r} \underset{(0)}{S} (k^2 dX - k^1 dY) \\ &= k_i dX^i + \frac{1}{r} \underset{(0)}{S} \sin\theta (\cos\phi dY - \sin\phi dX), \end{aligned} \tag{105}$$

with $k_i dX^i$ given by (41). Making use of (38) and (45) we find that

$$K_i dX^i = \lambda^{(3)} - \frac{1}{r} \underset{(0)}{S} \sin\theta \lambda^{(2)} = N_i dx^i, \tag{106}$$

defining N_i in coordinates $x^i = (\theta, \phi, r, u)$. In terms of the basis f^i, e^i, l^i, n^i we therefore have

$$N^i = n^i - \frac{1}{r} \underset{(0)}{S} \sin\theta e^i, \tag{107}$$

and thus it is immaterial whether we use n^i or N^i in the leading term in the energy-momentum-stress tensor components.

V. CONCLUSIONS

We have shown how to construct an exact solution for the gravitational field of an arbitrarily accelerating and rotating point mass in general relativity. This Kerr-like analogue of Kinnersely's field of an arbitrarily accelerating point mass has a curvature singularity on a ring of radius $|\underset{(0)}{S}|$ undergoing rectilinear motion with arbitrary acceleration. Asymptotically the matter distribution created by the accelerating, spinning source is qualitatively similar to that of Kinnersley's accelerating point mass except that the propagation direction of the lightlike matter in the spinning case has shear on account of the acceleration/spin interaction. The uniqueness of the solution given here is a topic for further study and is related to the uniqueness of the Kerr solution.

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