Equivalence of inflationary models between the metric and Palatini formulation of scalar-tensor theories

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With a scalar field nonminimally coupled to curvature, the underlying geometry and variational principle of gravity—metric or Palatini—becomes important and makes a difference, as the field dynamics and observational predictions generally depend on this choice. In the present paper, we describe a classification principle which encompasses both metric and Palatini models of inflation, employing the fact that inflationary observables can be neatly expressed in terms of certain quantities which remain invariant under conformal transformations and scalar field redefinitions. This allows us to elucidate the specific conditions when a model yields equivalent phenomenology in the metric and Palatini formalisms and also to outline a method how to systematically construct different models in both formulations that produce the same observables.

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I. INTRODUCTION

Recent observations of the cosmic microwave background radiation indicate that at large scales the Universe is flat and homogeneous. These features can be explained by postulating a quasi–de Sitter expansion during the very early moments of the Universe. Furthermore, this inflationary era is able to generate and preserve the primordial inhomogeneities which became the seeds for the subsequent large-scale structure that we observe. Inflation is usually formulated by supplementing the Einstein-Hilbert action with one or more real scalar fields whose energy density drives the near-exponential expansion.

Recently, the Planck satellite mission [1] has constrained the available parameter space and essentially excluded many inflationary models. Two of the most popular models, namely, Starobinsky [2] and nonminimal Higgs inflation [3–6], still lie in the allowed region. Incidentally, these theories, even though seemingly very different, belong to the same equivalence class which is why they give the same predictions for the observables. They also belong to the class of scalar-tensor theories where the inflaton is generally nonminimally coupled to gravity but minimally coupled to matter (Jordan frame). Of course, one can always perform a rescaling of the metric and a scalar field reparametrization and move to the Einstein frame where the scalar field is minimally coupled to gravity. One can work in either frame, while there is an ongoing debate as to which one is physical [7–28]. To circumvent the issue, a frame-invariant approach was developed in [29–31], then fruitfully applied to slow-roll inflation [32–34], and extended to related theories and formulations [35–39]. The advantage of this method is that, starting from any scalar-tensor theory, one can define quantities that remain invariant under the conformal Weyl rescaling of the metric and scalar field reparametrization and then express the inflationary observables in terms of these invariants.

Another issue that arises when one is interested in nonminimally coupled theories is that of the employed variational principle. In the metric formalism, the metric is the only dynamical degree of freedom and the connection is the Levi-Civita. However, in the Palatini or first order formalism [40,41], the metric and the connection are assumed to be independent variables and one has to vary the action with respect to both of them. Both approaches lead to the same field equation for an action whose Lagrangian is linear in R and is minimally coupled, but this is no longer true for more general actions. Regarding inflation, the difference in the predictions between the two variational principles has been recently studied in [37,38,42–77]. In most of the previous studies, it was shown that the metric and Palatini formulations generally

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give different results when inflation is concerned (see, however, [49,50]). In this paper, we focus on the cases when the two formalisms can produce similar results and extend the classification scheme of [33] to include Palatini models.

Future space missions (LITEBIRD [78], PIXIE [79], PICO [80]) promise to determine the inflationary observables at high precision that will considerably narrow the range of viable models. However, even when the invariant potential can be effectively pinned down, there will remain a degeneracy, as many fundamental actions in different formulations and parametrizations can lead to the same invariant potential and hence to the same values for the observables. The aim of the current paper is to clarify the situation and to outline a method of how to explore and reconstruct such equivalent actions in a systematic way. In the end, some actions in a given equivalence class would be better motivated from the theoretical point of view, while the degeneracy could be also broken by some observations of noninflationary physics.

The paper is organized as follows. In the next section, we adopt the approach of invariants to study general scalar-tensor theories in both metric and Palatini formalisms. In Sec. III, we focus on inflation and express the slow-roll parameters and inflationary observables in terms of the invariant potential and its derivatives. Then, in Sec. IV, we determine under which conditions the metric and Palatini formalisms can generate the same slow-roll parameters when one starts from the same action and study some examples. Conversely, starting from the same invariant potential in Sec. V, we explore the reconstruction of the corresponding metric and Palatini actions. We summarize our results and conclude in Sec. VI. Finally, we include an Appendix where we illustrate how an additional independent (conformal) transformation of the connection enlarges the general Palatini action, but a suitable choice neutralizes the effect, a point that has not received much attention in the literature so far.

II. ACTION AND INVARIANT QUANTITIES

Regardless of the gravity formulation, the action for general scalar-tensor theory can be written as[1] [81]

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2}\mathcal{A}(\Phi)R - \frac{1}{2}\mathcal{B}(\Phi)(\nabla\Phi)^2 - \mathcal{V}(\Phi)\right\} + \mathcal{S}_m[\mathrm{e}^{2\sigma(\Phi)}g_{\mu\nu}, \chi_m], \tag{1}$$

where we used Planck units $M_{\mathrm{Pl}} = 1$ and metric signature $(-,+,+,+)$. The Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}[\Gamma, \partial\Gamma]$ is a function of the metric tensor $g_{\mu\nu}$ and the connection Γ. The choice of the gravity formulation is reflected on the expression of Γ in Eq. (1) [43],

$$\Gamma^{\lambda}_{\alpha\beta} = \left\{ {\lambda \atop \alpha\beta} \right\} + (1-\delta_{j\Gamma})[\delta^{\lambda}_{\alpha}\partial_{\beta}\omega(\Phi) + \delta^{\lambda}_{\beta}\partial_{\alpha}\omega(\Phi) - g_{\alpha\beta}\partial^{\lambda}\omega(\Phi)], \tag{2}$$

where

$$\omega(\Phi) = \ln\sqrt{\mathcal{A}(\Phi)}, \tag{3}$$

$\left\{ {\lambda \atop \alpha\beta} \right\}$ is the Levi-Civita connection, δ_{jk} is the Kronecker delta, and $j = g$ stands for the metric case while $j = \Gamma$ for the Palatini one.

We refer to the set of $\{\mathcal{A}, \mathcal{B}, \mathcal{V}, \sigma\}$ as the model functions. By considering a Weyl rescaling of metric (referred later as a change of frame) and scalar field redefinition (referred later as a reparametrization)

$$g_{\mu\nu} = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\bar{g}_{\mu\nu}, \tag{4a}$$

$$\Phi = \bar{f}(\bar{\Phi}), \tag{4b}$$

the action functional (1) preserves its structure (up to the boundary term) if the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{V}$, and σ transform as [81]

$$\bar{\mathcal{A}}(\bar{\Phi}) = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}\mathcal{A}(\bar{f}(\bar{\Phi})), \tag{5a}$$

$$\bar{\mathcal{B}}(\bar{\Phi}) = \mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}(\bar{f}')^2\mathcal{B}(\bar{f}(\bar{\Phi})) - 6\delta_{j\Gamma}\mathrm{e}^{2\bar{\gamma}(\bar{\Phi})}[(\bar{\gamma}')^2\mathcal{A}(\bar{f}(\bar{\Phi})) - \bar{\gamma}'\bar{f}'\mathcal{A}'], \tag{5b}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = \mathrm{e}^{4\bar{\gamma}(\bar{\Phi})}\mathcal{V}(\bar{f}(\bar{\Phi})), \tag{5c}$$

$$\bar{\sigma}(\bar{\Phi}) = \sigma(\bar{f}(\bar{\Phi})) + \bar{\gamma}(\bar{\Phi}), \tag{5d}$$

where prime denotes a derivative with respect to the scalar field. The Jordan frame is defined by the condition $\sigma(\Phi) = 0$. For what follows, we omit the matter part of the action and take $\mathcal{S}_m = 0$, since our interest is now on the scalar nonminimally coupled to gravity which will be identified with the inflaton field.

By a straightforward calculation, it is possible to make sure that in every spacetime point the numerical value of the quantities [29]

$$\mathcal{I}_{\mathrm{m}}(\Phi) = \frac{\mathrm{e}^{2\sigma(\Phi)}}{\mathcal{A}(\Phi)}, \tag{6}$$

[1]The most general Palatini action contains also additional terms due to the nonmetricity of the theory [37]. However, it is possible to show that the action can always be cast in the form of Eq. (1). For the interested reader, the concerning details are given in the Appendix.

$$\mathcal{I}_{\mathcal{V}}(\Phi) = \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^2}, \tag{7}$$

$$\mathcal{I}_{\Phi}(\Phi) = \int \mathrm{d}\Phi \sqrt{\frac{\mathcal{B}(\Phi)}{\mathcal{A}(\Phi)} + \frac{3}{2}\delta_{j\Gamma}\left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2} \tag{8}$$

remain invariant, i.e., $\bar{\mathcal{I}}_i(\bar{\Phi}) = \mathcal{I}_i(\Phi)$. In a similar vein, we may introduce an invariant metric $\hat{g}_{\mu\nu} = \mathcal{A} g_{\mu\nu}$, which is unaffected by the conformal transformation (4a). One can see that the invariant field $\mathcal{I}_\Phi$ has a different dependence on the model functions when one considers the metric (we use the notation $\mathcal{I}_\Phi^g$) or Palatini formalism (denoted as $\mathcal{I}_\Phi^\Gamma$). Still, in both formalisms, we may take the quantity $\mathcal{I}_\Phi$ as an invariant description of the scalar degree of freedom in the theory [29,37]. Negative values for the expression under the square root in Eq. (8) suggest that the scalar field is a ghost, while identically constant $\mathcal{I}_\Phi$ indicates that the scalar is not dynamical. In the metric formulation, this occurs only when $\mathcal{B}(\Phi) = -\frac{3}{2}\frac{(\mathcal{A}'(\Phi))^2}{\mathcal{A}(\Phi)}$, while in the Palatini for $\mathcal{B}(\Phi) = 0$. In both cases, the theory is equivalent to general relativity plus a cosmological constant. A multiscalar generalization of the integrand in Eq. (8) plays the role of the invariant volume form on the space of scalar fields; hence, here $\mathcal{I}_\Phi$ has a natural interpretation as an invariant "distance" in the one-dimensional space of the scalar field [32,35].

By inverting Eq. (8), we may switch to use $\mathcal{I}_\Phi$ as the basic variable instead of Φ, and employing the invariant metric $\hat{g}_{\mu\nu}$, we can rewrite the action (1) in terms of invariant quantities only [29],

$$\hat{\mathcal{S}} = \int \mathrm{d}^4x\sqrt{-\hat{g}}\left\{\frac{1}{2}\hat{g}^{\mu\nu}R_{\mu\nu}[\Gamma,\hat{g}_{\mu\nu}] - \frac{1}{2}(\hat{\nabla}\mathcal{I}_\Phi)^2 - \mathcal{I}_{\mathcal{V}}\right\} + \mathcal{S}_m[\mathcal{I}_m\hat{g}_{\mu\nu},\psi]. \tag{9}$$

An arbitrary scalar-tensor theory with four free functions (1) can therefore be cast by the two transformations of frame change and reparametrization (4) into the action (9) endowed by two functions that carry invariant meaning. The quantity $\mathcal{I}_\mathrm{m}(\mathcal{I}_\Phi)$ characterizes the coupling of gravity to matter fields. For constant $\mathcal{I}_\mathrm{m}$, the theory is equivalent to general relativity with a minimally coupled scalar field; otherwise, the scalar field participates in mediating the gravitational interaction and the effective gravitational "constant" starts to vary according to the scalar field value. The quantity $\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\Phi)$ is the invariant scalar potential. In the case of inflation where the matter fields can be neglected, the physics of the model is encoded by the invariant potential alone [33]. The form of the invariant action (9) coincides with the usual Einstein frame action, a circumstance which will help us to write down the inflationary parameters in terms of the invariants in the next section.

III. SLOW-ROLL PARAMETERS AND COMPUTATIONAL ALGORITHM

The action functional (9) can be identified as the Einstein frame regarding the $\hat{g}_{\mu\nu}$ metric. Then the equations of motion coincide in both formulations of gravity, although in the Palatini formalism the Levi-Civita connection is derived on shell from its constraint equation $\delta_{(\Gamma)}\hat{\mathcal{S}} = 0$. The invariant quantity $\mathcal{I}_\Phi$ assumes the role of the inflaton field driving inflation, governed by its potential $\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\Phi)$. Assuming then the usual slow-roll conditions, we can rewrite the potential slow-roll parameters (PSRPs) as [32–34]

$$\epsilon = \frac{1}{2}\left(\frac{\mathrm{d}\ln\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi}\right)^2, \tag{10}$$

$$\eta = \frac{1}{\mathcal{I}_{\mathcal{V}}}\frac{\mathrm{d}^2\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi^2}. \tag{11}$$

At this point, we assumed that the integral in Eq. (8) is solvable and the relation of $\mathcal{I}_\Phi(\Phi)$ invertible,[2] so that we can obtain a relation of $\Phi(\mathcal{I}_\Phi)$. Then, after a direct substitution into $\mathcal{I}_{\mathcal{V}}$, we express the PSRPs in terms of $\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\Phi)$.

The tensor-to-scalar ratio r, the scalar spectral index n_s, and the amplitude of the scalar power spectrum A_s are some of the inflationary observable quantities posing strict constraints on the parameter space of the inflationary models. These are usually computed in the slow-roll approximation and, up to first order in PSRPs, they read as follows [33,34]:

$$r = 8\left(\frac{\mathrm{d}\ln\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi}\right)^2, \tag{12}$$

$$n_s = 1 - 3\left(\frac{\mathrm{d}\ln\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi}\right)^2 + 2\frac{1}{\mathcal{I}_{\mathcal{V}}}\frac{\mathrm{d}^2\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi^2}, \tag{13}$$

$$A_s = \frac{\mathcal{I}_{\mathcal{V}}}{12\pi^2}\left(\frac{\mathrm{d}\ln\mathcal{I}_{\mathcal{V}}}{\mathrm{d}\mathcal{I}_\Phi}\right)^{-2}. \tag{14}$$

Note that all of the above observables are calculated at horizon exit, $\mathcal{I}_\Phi = \mathcal{I}_\Phi^*$. The number of e-foldings, characterizing the duration of inflation, is given by

$$N = \int_{\mathcal{I}_\Phi^{\mathrm{end}}}^{\mathcal{I}_\Phi^*} \mathcal{I}_{\mathcal{V}}(\mathcal{I}_\Phi)\left(\frac{\mathrm{d}\mathcal{I}_{\mathcal{V}}(\mathcal{I}_\Phi)}{\mathrm{d}\mathcal{I}_\Phi}\right)^{-1}\mathrm{d}\mathcal{I}_\Phi, \tag{15}$$

where $\mathcal{I}_\Phi^{\mathrm{end}}$ and $\mathcal{I}_\Phi^*$ are the field values at the end and start of inflation, respectively.

The invariant formalism can be applied in a straightforward way to any model that can be recast in the form of

[2]The problem is still solvable also when $\mathcal{I}_\Phi(\Phi)$ is not invertible. In that case, Φ is used as a *new* variable and the chain rule is applied in the computation of the derivatives.

Eq. (9), by first identifying the model functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$, and $\sigma(\Phi)$. This is under the implicit assumption that the model under consideration includes only one dynamical scalar field Φ. As we explained previously, we may use (8) to compute the invariant quantity $\mathcal{I}_\Phi(\Phi)$ and invert that relation to obtain $\Phi(\mathcal{I}_\Phi)$. By using $\Phi(\mathcal{I}_\Phi)$, we can calculate the invariant potential $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$ and then solve $\epsilon(\mathcal{I}_\Phi^{\mathrm{end}}) = 1$ to obtain the field value at the end of inflation. The field value $\mathcal{I}_\Phi^*$ is obtained by integrating (15) and assuming that the number of e-folds lies somewhere in the allowed region of $N \simeq (50\text{–}60)$ e-folds. Finally, the inflationary observables are readily obtained from Eqs. (12)–(15) using the field value $\mathcal{I}_\Phi^*$.

In the following sections, we apply this procedure in the study of the inflationary predictions for scalar-tensor theories in the metric and Palatini formulations.

IV. WHEN DO IDENTICAL METRIC AND PALATINI ACTIONS YIELD (ALMOST) THE SAME OSERVABLES?

Comparing Eqs. (7) and (8), we can see that the difference between the metric and Palatini formulation arises from a different definition of the invariant field value. Therefore, given the action in Eq. (1) (i.e., a set of functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{V}$), the metric and Palatini formulations usually generate different invariant actions and therefore different predictions. However, it might happen that the two formulations produce the same slow-roll parameters when the invariant potential and the invariant field value possess certain properties. The slow-roll parameters are independent of the overall normalization of the invariant potential; therefore, it is enough to assume that invariant potential in the two formulations, as functions of the corresponding invariant field values, are proportional to each other,

$$\mathcal{I}_\mathcal{V}^{\mathrm{g}} \propto \mathcal{I}_\mathcal{V}^{\Gamma}. \tag{16}$$

Unfortunately, we cannot provide a general criterion that is more explicit than Eq. (16), because Eq. (8) contains an integral over Φ and the corresponding solving technique is strongly dependent on the actual definition of $\mathcal{A}$ and $\mathcal{B}$. On the other hand, we can provide a couple of explicit examples: one relatively simple (A) and one more complicated (B).

A. Example: Power law invariant potential

Given Eqs. (7) and (8), the simplest way to satisfy Eq. (16) is by requiring

$$\mathcal{I}_\mathcal{V} \propto \mathcal{I}_\Phi^n, \tag{17}$$

$$\mathcal{I}_\Phi^{\mathrm{g}} \propto \mathcal{I}_\Phi^{\Gamma}, \tag{18}$$

where n is some nonzero power. The class of models (17) is well known (e.g., [1]) and goes under the name of monomial inflation. Using Eqs. (10), (11), and (17), we see that the corresponding slow-roll parameters are

$$\epsilon = \frac{n^2}{2\mathcal{I}_\Phi^2}, \tag{19}$$

$$\eta = \frac{n(n-1)}{\mathcal{I}_\Phi^2}. \tag{20}$$

We can appreciate that the case $n = 2$ is even more special, since it accidentally implies also that $\epsilon = \eta$. With a couple of straightforward computations, we can easily verify that the tensor-to-scalar ratio and the scalar spectral index are

$$r = \frac{16n}{n + 4N}, \tag{21}$$

$$n_s = 1 - \frac{2(n+2)}{n + 4N}. \tag{22}$$

Furthermore, the combination of Eqs. (8) and (18) implies

$$\frac{\mathcal{B}(\Phi)}{\mathcal{A}(\Phi)} \propto \left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2, \tag{23}$$

therefore

$$\mathcal{I}_\Phi^{\Gamma,\mathrm{g}}(\Phi) \propto \int d\Phi \left|\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right| = \ln\frac{\mathcal{A}(\Phi)}{\mathcal{A}_0}, \tag{24}$$

where $\mathcal{A}_0$ is a constant of integration that does not carry any physical meaning and can be used to conveniently set the zero value of the invariant field according to the problem at hand. Imposing Eq. (17), we obtain

$$\left(\ln\frac{\mathcal{A}(\Phi)}{\mathcal{A}_0}\right)^n \propto \frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}. \tag{25}$$

Therefore, the metric and Palatini formulations produce the same slow-roll parameters when

$$\mathcal{A}(\Phi)\mathcal{B}(\Phi) \propto (\mathcal{A}'(\Phi))^2, \tag{26}$$

$$\mathcal{V}(\Phi) \propto \mathcal{A}(\Phi)^2\left(\ln\frac{\mathcal{A}(\Phi)}{\mathcal{A}_0}\right)^n. \tag{27}$$

From the two equations, we can immediately see that the following class of nonminimal Coleman-Weinberg models where:[3]

[3]Without loss of generality, we assume $\Phi > 0$ in Secs. IV and V.

$$\mathcal{A}(\Phi) = \xi\Phi^2, \tag{28}$$

$$\mathcal{B}(\Phi) = 1, \tag{29}$$

$$\mathcal{V}(\Phi) = \beta\left(\ln\frac{\Phi}{\Phi_0}\right)^n \Phi^4 \tag{30}$$

satisfy the conditions (26) and (27), and therefore generate slow-roll parameters that cannot discriminate between metric and Palatini gravity. These results are in agreement with the findings of [49] and the strong coupling limit of [50].

Moreover, Eqs. (26) and (27) can also be used to back-engineer models. For instance, choosing $n = 1$ and a natural inflation potential

$$\mathcal{V}(\Phi) = M^4\left(1 - \cos\left(\frac{\Phi}{\Phi_0}\right)\right), \tag{31}$$

the addition of the following nontrivial nonminimal coupling to gravity and noncanonical kinetic function:

$$\mathcal{A}(\Phi) = \sqrt{\frac{z}{W(z)}}, \tag{32}$$

$$\mathcal{B}(\Phi) = \frac{\sin^2(\frac{\Phi}{\Phi_0})W(z)^{3/2}}{4z^{3/2}(W(z)+1)^2}, \tag{33}$$

where $W(z)$ is the Lambert W-function and $z = 1 - \cos(\frac{\Phi}{\Phi_0})$, would generate

$$\mathcal{I}_\mathcal{V} \propto \mathcal{I}_\Phi, \tag{34}$$

$$\epsilon = \frac{1}{2\mathcal{I}_\Phi^2}, \tag{35}$$

$$\eta = 0, \tag{36}$$

regardless of the adopted gravity formulation. Therefore, for $n = 1$, the scalar spectral index and the tensor-to-scalar ratio have the following values:

$$\begin{aligned} n_s &= 0.9701, \quad r = 0.0796, \quad \text{for } N = 50,\\ n_s &= 0.9751, \quad r = 0.0664, \quad \text{for } N = 60, \end{aligned} \tag{37}$$

which are out of the 2σ Planck boundaries [1], but still allowed at 3σ. Therefore, if the Universe happened to be described by a nonminimal scalar field with model functions (31)–(33) in action (1), the slow-roll parameters would not be able to distinguish whether the underlying theory is metric or Palatini in character.

B. Example: Logarithmic invariant potential

A more complicated way to satisfy Eq. (16) is the following choice:

$$\mathcal{I}_\mathcal{V} \propto (\ln(\mathcal{I}_\Phi^m))^n, \tag{38}$$

$$\mathcal{I}_\Phi^{\mathrm{g}} \propto (\mathcal{I}_\Phi^\Gamma)^l, \tag{39}$$

where Φ is a subscript while l, m, n are some powers. The class of models (38) can be interpreted as a cosmological constant subject to quantum corrections. Being somehow new (only the case $n = 1$ is well known (e.g., [1])), such a class deserves a deeper investigation than the previous example. Despite the nonlinear relation between the invariant fields in the two formalisms in Eq. (39), the expressions of the slow-roll parameters coming from Eq. (38) are

$$\epsilon = \frac{n^2}{2\mathcal{I}_\Phi^2(\ln\mathcal{I}_\Phi)^2}, \tag{40}$$

$$\eta = \frac{n(n-1-\ln\mathcal{I}_\Phi)}{\mathcal{I}_\Phi^2(\ln\mathcal{I}_\Phi)^2}, \tag{41}$$

where the power m canceled out because of properties of the logarithm. Inflation ends when

$$\mathcal{I}_\Phi^{\mathrm{end}} = \frac{n}{\sqrt{2}}W\left(\frac{n}{\sqrt{2}}\right)^{-1}, \tag{42}$$

while the number of e-folds turns out to be

$$N = \frac{1}{n}\left[-\frac{\mathcal{I}_\Phi^2}{4} + \frac{\mathcal{I}_\Phi^2}{2}\ln\mathcal{I}_\Phi\right]_{\mathcal{I}_\Phi^{\mathrm{end}}}^{\mathcal{I}_\Phi^*}. \tag{43}$$

Therefore, the value of the invariant field at the horizon crossing is

$$\mathcal{I}_\Phi^* = \sqrt{4N'W\left(\frac{4N'}{e}\right)^{-1}}, \tag{44}$$

where we have defined

$$N' = nN - \frac{n^2}{8}W\left(\frac{n}{\sqrt{2}}\right)^{-2} + \frac{n^2}{4}W\left(\frac{n}{\sqrt{2}}\right)^{-2}\ln\left(\frac{n}{\sqrt{2}}W\left(\frac{n}{\sqrt{2}}\right)^{-1}\right). \tag{45}$$

Using (44), the scalar spectral index and the tensor-to-scalar ratio can be expressed by

$$n_s = 1 - \frac{nW(\frac{4N'}{e})(1+n+\ln(4N'W(\frac{4N'}{e})^{-1}))}{N'(\ln(4N'W(\frac{4N'}{e})^{-1}))^2}, \tag{46}$$

$$r = \frac{n^2 W(\frac{4N'}{e})}{2N'(\ln(4N'W(\frac{4N'}{e})^{-1}))^2}. \tag{47}$$

An example to illustrate this possibility can be realized by the model functions

$$\mathcal{A}(\Phi) = \exp\left(\frac{1}{2\sqrt{6}}\mathrm{acosh}(2\Phi) - \frac{1}{\sqrt{6}}\Phi\sqrt{4\Phi^2-1}\right), \tag{48}$$

$$\mathcal{B}(\Phi) = \exp\left(\frac{1}{2\sqrt{6}}\mathrm{acosh}(2\Phi) - \frac{1}{\sqrt{6}}\Phi\sqrt{4\Phi^2-1}\right), \tag{49}$$

$$\mathcal{V}(\Phi) = \mathcal{A}(\Phi)^2(\ln\Phi^2)^2. \tag{50}$$

In this case, by integrating Eq. (8), one obtains the invariant field in the two formalisms as

$$\mathcal{I}_\Phi^\Gamma = \Phi, \tag{51}$$

$$\mathcal{I}_\Phi^g = \Phi^2. \tag{52}$$

While the invariant potentials differ by a constant factor,

$$\mathcal{I}_\mathcal{V}^\Gamma = 4(\ln\mathcal{I}_\Phi^\Gamma)^2, \tag{53}$$

$$\mathcal{I}_\mathcal{V}^g = (\ln\mathcal{I}_\Phi^g)^2, \tag{54}$$

as expected, the slow-roll parameters coincide,

$$\epsilon = \frac{4}{2\mathcal{I}_\Phi^2(\ln\mathcal{I}_\Phi)^2}, \tag{55}$$

$$\eta = \frac{2(2-\ln\mathcal{I}_\Phi)}{\mathcal{I}_\Phi^2(\ln\mathcal{I}_\Phi)^2} \tag{56}$$

and therefore yield the same n_s and r (as functions of $\mathcal{I}_\Phi$) in both metric and Palatini formulations [cf. Eqs. (38), (55), and (56) for $n=2$]. In this case, the scalar spectral index and the tensor-to-scalar ratio take the following values:

$$n_s = 0.9699, \quad r = 0.0556, \quad \text{for } N = 50,$$
$$n_s = 0.9752, \quad r = 0.0450, \quad \text{for } N = 60, \tag{57}$$

which are within the 2σ Planck boundaries [1].

The model (48)–(50) looks rather contrived, but it employs a parametrization where the calculational logic is easy to see. However, hidden somewhere in the infinite possibilities of reparametrizations, there might exist a physically better motivated form of the same model, but where the calculations become harder to deal with. It is not easy to guess what a nicer parametrization could be, but as an extra illustration let us just perform a simple scalar field redefinition

$$\Phi = \frac{1+\bar{\Phi}^2}{4\bar{\Phi}}. \tag{58}$$

The model functions (48)–(50) transform under Eq. (58) into

$$\bar{\mathcal{A}}(\bar{\Phi}) = \bar{\Phi}^{\frac{1}{2\sqrt{6}}}\mathrm{e}^{-\frac{\sqrt{6}(\bar{\Phi}^4-1)}{48\bar{\Phi}^2}}, \tag{59}$$

$$\bar{\mathcal{B}}(\bar{\Phi}) = \bar{\Phi}^{\frac{1}{2\sqrt{6}}}\frac{(1-\bar{\Phi}^2)^2}{16\bar{\Phi}^4}\mathrm{e}^{-\frac{\sqrt{6}(\bar{\Phi}^4-1)}{48\bar{\Phi}^2}}, \tag{60}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = \bar{\Phi}^{\frac{1}{\sqrt{6}}}\mathrm{e}^{-\frac{\sqrt{6}(\bar{\Phi}^4-1)}{24\bar{\Phi}^2}}\left(\ln\frac{(1+\bar{\Phi}^2)^2}{16\bar{\Phi}^2}\right)^2, \tag{61}$$

and contrary to the previous form the functions, $\bar{\mathcal{A}}(\bar{\Phi})$ and $\bar{\mathcal{B}}(\bar{\Phi})$ do not coincide any more. A direct integration of Eq. (8) yields the expressions of the invariant field now as

$$\mathcal{I}_{\bar{\Phi}}^\Gamma = \frac{1+\bar{\Phi}^2}{4\bar{\Phi}}, \tag{62}$$

$$\mathcal{I}_{\bar{\Phi}}^g = \frac{1+\bar{\Phi}^4}{16\bar{\Phi}^2} + \frac{1}{8}, \tag{63}$$

where the last term had to be added as an integration constant to maintain an explicit equivalence. By construction, we get the same invariant potentials (53), (54), and PSRPs (55), (56). Note that if we had omitted the constant of integration in (63), the proportionality of invariant fields (39) would still hold, but the proportionality of invariant potentials (38) would not be completely obvious at first sight. Nevertheless, a direct calculation of the derivatives in the inflationary parameters (12), (13) would yield the same result with or without the integration constant.

As a final comment, let us stress that in all the examples of this section the invariant PSRPs, and therefore the predictions for r, n_s, and N coincide in the metric and Palatini cases, since the invariant potentials are proportional to each other and the overall factor cancels out. However, the amplitude of the scalar power spectrum (14) depends explicitly on the invariant potential, and thus this observable will be sensitive to the difference in the actual normalization of the invariant potentials. The normalization can be crucial in satisfying the observational constraints, currently $A_s \simeq 2.1\times10^{-9}$ [1]. The metric and Palatini models will yield the same phenomenology also in this respect if a strict equivalence between the invariant potentials holds, not just a proportionality (16). Starting from exactly the same invariant actions, this is never the case. However, for the examples considered before, a change in the normalization of the model functions of the metric and Palatini action will have the final effect of generating exactly the same invariant potentials. For instance, for what concerns Example A, the invariant potential under metric and Palatini are the same when the nonminimal couplings in Eq. (28) satisfy the following condition:

$$\xi_\Gamma = \frac{\xi_g}{1+6\xi_g}, \tag{64}$$

where $\xi_{g,\Gamma}$ are, respectively, the nonminimal coupling under the metric and the Palatini formulation [49,50]. A more general discussion about the generation of exactly equivalent invariant potentials is presented in the next section.

V. WHEN DO DIFFERENT METRIC AND PALATINI ACTIONS YIELD THE SAME OBSERVABLES?

As described in [33], equivalent inflationary theories are described by one invariant function[4]: $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$. However, inflationary models can be produced by using three generating functions: $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, and $\mathcal{V}(\Phi)$. Therefore, *a priori* knowledge of $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$ allows us to derive only one constraint that $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, and $\mathcal{V}(\Phi)$ have to satisfy, leaving two functions out of the three completely undetermined. Generally, we can express the invariant field $\mathcal{I}_\Phi$ as the inverse function of the invariant potential[5] in Eq. (7),

$$\mathcal{I}_\Phi = \left(\frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}\right)^{-1} \equiv \mathcal{I}_\mathcal{V}^{-1}(\Phi), \tag{65}$$

where the superscript "−1" stands for inverse function. Using Eq. (8), we can write

$$\mathcal{I}_\mathcal{V}^{-1}(\Phi) = \int \mathrm{d}\Phi \sqrt{\frac{\mathcal{B}(\Phi)}{\mathcal{A}(\Phi)} + \frac{3}{2}\delta_{j\Gamma}\left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2}, \tag{66}$$

where the parameter $\delta_{j\Gamma}$ indicates the adopted gravity formulation. Therefore, given the invariant function $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$, Eq. (66) is the constraint that $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, and $\mathcal{V}(\Phi)$ must satisfy in order to create equivalent inflationary theories also among different gravity formulations. This means that (apart from pathological cases) we can randomly choose two functions among $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, and $\mathcal{V}(\Phi)$. If the third one satisfies Eq. (66), then the correct $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$ is always generated. However, the solution of the constraint (66) is strongly dependent on the initial choice of model functions, invariant potential, and gravity formulation (metric or Palatini). Nevertheless, until the constraint (66) is satisfied, the same invariant potential $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$ and therefore the same inflationary observables [Eqs. (12)–(15)] are generated, regardless of initial model functions and gravity formulation.

When $\mathcal{A}(\Phi)$ and $\mathcal{V}(\Phi)$ are given, it is always possible to solve Eq. (66) and obtain the corresponding value for the noncanonical kinetic function

$$\mathcal{B}(\Phi) = \mathcal{A}(\Phi)\left\{\left[\frac{d\mathcal{I}_\mathcal{V}^{-1}(\Phi)}{d\Phi}\right]^2 - \frac{3}{2}\delta_{j\Gamma}\left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2\right\}, \tag{67}$$

where $\delta_{j\Gamma}$ reflects the adopted gravity formulation [see Eq. (5)].

Instead, if $\mathcal{A}(\Phi)$ and $\mathcal{B}(\Phi)$ are fixed, the constraint can be formally solved as

$$\mathcal{V}(\Phi) = \mathcal{A}(\Phi)^2 \mathcal{I}_\mathcal{V}(\mathcal{I}_\mathcal{V}^{-1}(\Phi)), \tag{68}$$

where $\mathcal{I}_\mathcal{V}^{-1}(\Phi)$ is given in Eq. (66). However, in this case, since the integral of an elementary function is not automatically elementary, choosing $\mathcal{A}(\Phi)$ and $\mathcal{B}(\Phi)$ as elementary functions of Φ does not always ensure that $\mathcal{V}(\Phi)$ is elementary as well.

Finally, when $\mathcal{B}(\Phi)$ and $\mathcal{V}(\Phi)$ are chosen, the constraint equation (66) becomes the following differential equation:

$$\mathcal{A}(\Phi)\left\{\frac{3}{2}\delta_{j\Gamma}\left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2 - \left[\frac{d\mathcal{I}_\mathcal{V}^{-1}(\Phi)}{d\Phi}\right]^2\right\} + \mathcal{B}(\Phi) = 0 \tag{69}$$

to be solved in order to determine $\mathcal{A}(\Phi)$.

Next, we present an example in order to better illustrate the different issues arising in each configuration.

A. Example: Generalized Starobinsky invariant potential

Let us consider the following invariant potential:

$$\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi) = M^4\left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\mathcal{I}_\Phi}\right)^2, \tag{70}$$

which generalizes the Starobinsky potential [87–89]. The model is well known too; therefore, we just summarize briefly the main features. At the leading order in the invariant field value, the slow-roll parameters are

[4]As already discussed in [33], the full gravitation equivalence needs to take into account also the invariant $\mathcal{I}_m$, that describes the couplings to matter. Therefore, ensuring only the same invariant potential, the (p)reheating mechanism might still affect the value of the observables and break the equivalence [49,82]. On the other hand, we can see from (6) that, by adjusting accordingly the function σ, we can easily obtain theories where $\mathcal{I}_m$ is the same under both gravity formulations, restoring the equivalence of observables also when (p)reheating is considered.

[5]The computation of $\mathcal{I}_\mathcal{V}^{-1}$ is quite delicate. In many cases, $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$ is not a bijective (i.e., invertible) function; therefore, $\mathcal{I}_\mathcal{V}^{-1}$ can be consistently identified only after a proper definition of the domain of $\mathcal{I}_\mathcal{V}(\mathcal{I}_\Phi)$. For more details about this topic, see [83] and also [30,84]. This is related to the possible occurrence of a singularity or exceptional point in the theory which could have implications for cosmology or in the presence of black holes (see, e.g., [85,86]). Nevertheless, here we restrict our attention to slow-roll inflation and smooth functions, expecting no singularities to arise.

$$\epsilon \approx \frac{4}{3\alpha e^{2\sqrt{\frac{2}{3\alpha}}\mathcal{I}_\Phi}}, \tag{71}$$

$$\eta \approx \frac{-4}{3\alpha e^{\sqrt{\frac{2}{3\alpha}}\mathcal{I}_\Phi}}, \tag{72}$$

while the number of e-folds is

$$N \approx \frac{3\alpha}{4} e^{\sqrt{\frac{2}{3\alpha}}\mathcal{I}_\Phi^*}. \tag{73}$$

Therefore, at the leading order in N, we get that

$$n_s \approx 1 - \frac{2}{N}, \tag{74}$$

$$r \approx \frac{12\alpha}{N^2}. \tag{75}$$

Planck data sets $\log_{10}\alpha < 1.3$ at 95% CL [1].

We can invert Eq. (70) to obtain

$$\mathcal{I}_\mathcal{V}^{-1}(\Phi) = -\sqrt{\frac{3\alpha}{2}} \ln\left(1 - \frac{1}{M^2}\sqrt{\frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}}\right). \tag{76}$$

Therefore, the constraint in Eq. (66) becomes

$$\int \mathrm{d}\Phi \sqrt{\frac{\mathcal{B}(\Phi)}{\mathcal{A}(\Phi)} + \frac{3}{2}\delta_{j\Gamma}\left(\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\right)^2} = -\sqrt{\frac{3\alpha}{2}} \ln\left(1 - \frac{1}{M^2}\sqrt{\frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)^2}}\right). \tag{77}$$

Let us consider now some case by case examples and see how the initial choice of the model functions affects the solving strategy of Eq. (77).

1. $\mathcal{A}$ and $\mathcal{V}$ are fixed

Taking, for instance, the following natural inflation potential and nonminimal coupling to gravity:

$$\mathcal{A}(\Phi) = 1 + \xi\Phi^2, \tag{78}$$

$$\mathcal{V}(\Phi) = M^4\left(1 - \cos\left(\frac{\Phi}{\Phi_0}\right)\right), \tag{79}$$

we obtain the invariant potential in Eq. (70) if

$$\mathcal{B}(\Phi)_\Gamma = \frac{3\alpha}{4\mathcal{A}(\Phi)\Phi_0^2}\left(\frac{\mathcal{A}(\Phi)\cos(\frac{\Phi}{2\Phi_0}) - 4\xi\Phi_0\Phi\sin(\frac{\Phi}{2\Phi_0})}{\mathcal{A}(\Phi) - \sqrt{2}\sin(\frac{\Phi}{2\Phi_0})}\right)^2 \tag{80}$$

in the Palatini case, with $\mathcal{A}(\Phi)$ given in Eq. (78), and

$$\mathcal{B}(\Phi)_g = \mathcal{B}(\Phi)_\Gamma - 6\frac{\xi^2\Phi^2}{\mathcal{A}(\Phi)} \tag{81}$$

in the metric case.

2. $\mathcal{A}$ and $\mathcal{B}$ are fixed

We consider now a nonminimally coupled scalar field with a canonical kinetic term

$$\mathcal{A}(\Phi) = \frac{2}{3\alpha}\Phi^2, \tag{82}$$

$$\mathcal{B}(\Phi) = 1. \tag{83}$$

Solving Eq. (66), we can see that we reproduce the invariant potential in Eq. (70) if the potential is

$$\mathcal{V}(\Phi)_\Gamma = \frac{4M^4}{9\alpha^2}\left(1 - \frac{\Phi_0}{\Phi}\right)^2\Phi^4 \tag{84}$$

in the Palatini case and

$$\mathcal{V}(\Phi)_g = \frac{4M^4}{9\alpha^2}\left(1 - \left(\frac{\Phi_0}{\Phi}\right)^{\sqrt{\frac{4+\alpha}{\alpha}}}\right)^2\Phi^4 \tag{85}$$

in the metric case, where Φ_0 is an integration constant.

3. $\mathcal{B}$ and $\mathcal{V}$ are fixed

In this last example, we consider a noncanonically normalized scalar and a quartic potential

$$\mathcal{B}(\Phi) = \frac{6(\alpha - 1)\Phi^2}{\mathcal{A}(\Phi)}, \tag{86}$$

$$\mathcal{V}(\Phi) = M^4\Phi^4, \tag{87}$$

with $\alpha > 1$. We need to determine $\mathcal{A}(\Phi)$ by solving the differential equation in Eq. (69), where $\mathcal{I}_\mathcal{V}^{-1}$, $\mathcal{B}(\Phi)$ and $\mathcal{V}(\Phi)$ are given, respectively, in Eqs. (76), (86), and (87). The specific choice in Eq. (86) allows us to solve such differential equation in both the metric and the Palatini cases. With a convenient choice of the integration constants, the corresponding solution is

$$\mathcal{A}(\Phi)_g = 1 + \Phi^2 \tag{88}$$

in the metric case and

$$\mathcal{A}(\Phi)_\Gamma = \Phi^2\left(1 + \Phi^{-2\sqrt{\frac{\alpha-1}{\alpha}}}\right) \tag{89}$$

in the Palatini case. The special value of $\alpha = 1$ requires an additional comment. In this case, the potential (70) becomes exactly the Starobinsky one and the noncanonical kinetic term (86) becomes identically zero in both the metric and the Palatini formulations. For the first case, this not a problem because it coincides with the formulation of the Starobinsky model via the auxiliary field in the Jordan frame. On the other hand, as discussed in Sec. II, in the Palatini formulation, the invariant field $\mathcal{I}_\Phi$ is not dynamical and the problem does not have a solution. However, it is still possible to reproduce the potential (70) from (87) in the Palatini formulation by relaxing the condition (86). For instance, choosing

$$\mathcal{B}(\Phi)_\Gamma = \frac{\alpha\Phi^2}{\mathcal{A}(\Phi)}, \tag{90}$$

we would get

$$\mathcal{A}(\Phi)_\Gamma = \Phi^2(1 + \Phi^{-\sqrt{\frac{2}{3}}}). \tag{91}$$

VI. SUMMARY AND CONCLUSIONS

In the present paper, we studied the slow-roll parameters and inflationary observables in the framework of scalar-tensor theories of gravity in the metric and Palatini formulations. The model functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ allow us to construct quantities, which are invariant under a conformal transformation of the metric and behave as scalar functions under the scalar field redefinition. Using this frame invariant approach, we expressed the slow-roll parameters ϵ, η, as well as the inflationary observable quantities n_s, r, A_s, and explained in detail how to compute them in the case of different model functions.

Next, in the main part of the paper, we clarified what conditions must be met for the metric and Palatini formalisms to give the same observable quantities. Due to the fact that most of the observable quantities are independent of the overall normalization factor, we concluded that it is sufficient for the invariant potentials in both formulations to be proportional to each other in order to obtain equal predictions for r, n_s, and N (but not A_s) in both formulations. We illustrated this general statement by two specific examples. After that, starting from the same invariant potential, we showed how by fixing two out of the three model functions we can straightforwardly obtain the third. We demonstrated the different possibilities by considering as an example an invariant potential of the Starobinsky form. One then sees how seemingly different models of inflation can give the same values of the observed parameters.

A deeper case-by-case study may unveil other configurations where the same model functions, but with different values of the free parameters, share the same invariant potential and therefore give the same values for observables. The framework described here provides a tool that enables to easily check different models against observations, as well as to reconstruct variations of models with a given phenomenology.

If the next generation satellites (LITEBIRD [78], PIXIE [79], PICO [80]) will be launched and after data will be collected, the available parameter space will be even more constrained, leaving us with a reduced set of allowed invariant potentials and more indications about which gravity formulation satisfies additional criteria like *elegance, simplicity*, or *minimality*.

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APPENDIX: MORE DETAILS ABOUT THE PALATINI ACTION

The most general action for a class of Palatini scalar-tensor theories of gravity featuring nonmetricity vectors entering the action functional in a linear way can be written as follows [37,38]:

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\bigg\{&\frac{1}{2}\mathcal{A}(\Phi)R(g,\Gamma) - \frac{1}{2}\mathcal{B}(\Phi)(\nabla\Phi)^2 \\ &- \mathcal{V}(\Phi) - \mathcal{C}_1(\Phi)Q^\mu\nabla_\mu\Phi - \mathcal{C}_2(\Phi)\bar{Q}^\mu\nabla_\mu\Phi\bigg\} \\ &+ \mathcal{S}_m[\mathrm{e}^{2\sigma(\Phi)}g_{\mu\nu}, \chi_m].\end{aligned} \tag{A1}$$

The action contains three independent variables: metric tensor, affine connection, and scalar field. It also features six arbitrary functions of the scalar field: $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{V}, \sigma\}$, providing, together with the dynamical variables, the so-called “frame” for the action (A1). The vectors Q^μ and $\bar{Q}^\mu$ are defined as

$$Q^\mu = g^{\mu\nu}g^{\alpha\beta}\nabla^\Gamma_\nu g_{\alpha\beta} = g^{\mu\nu}g^{\alpha\beta}Q_{\nu\alpha\beta}, \tag{A2a}$$

$$\bar{Q}^\mu = -g^{\mu\nu}g^{\alpha\beta}\nabla^\Gamma_\alpha g_{\nu\beta} = -g^{\mu\nu}g^{\alpha\beta}Q_{\alpha\nu\beta}. \tag{A2b}$$

The ∇^Γ is defined with respect to the independent connection; therefore, the covariant derivative of the metric will not vanish in general.

In the Palatini approach, the metric tensor is fundamentally independent of the connection. When we use the Weyl (or conformal) transformation of the metric, the connection remains unchanged. We might use this freedom and postulate additional transformations of the connection preserving the light cones. We introduce the following transformation formulas for the dynamical variables entering the action functional:

$$g_{\mu\nu} = e^{2\bar{\gamma}_1(\bar{\Phi})}\bar{g}_{\mu\nu}, \tag{A3a}$$

$$\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + 2\delta^{\alpha}{}_{(\mu}\partial_{\nu)}\bar{\gamma}_2(\bar{\Phi}) - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\partial_\beta\bar{\gamma}_3(\bar{\Phi}), \tag{A3b}$$

$$\Phi = \bar{f}(\bar{\Phi}). \tag{A3c}$$

The transformations are governed by three smooth functions of the scalar field $(\gamma_1, \gamma_2, \gamma_3)$ and are accompanied by a redefinition of the scalar field. The transformations (A3a)–(A3c) are invertible,

$$\bar{g}_{\mu\nu} = e^{2\gamma_1(\Phi)}g_{\mu\nu}, \tag{A4a}$$

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + 2\delta^{\alpha}{}_{(\mu}\partial_{\nu)}\gamma_2(\Phi) - g_{\mu\nu}g^{\alpha\beta}\partial_\beta\gamma_3(\Phi), \tag{A4b}$$

$$\bar{\Phi} = f(\Phi), \tag{A4c}$$

and the relations between the gamma functions and the diffeomorphism of the scalar field are given by

$$\bar{\gamma}_i = -\gamma_i \circ f, \tag{A5a}$$

$$\bar{f} = f^{-1}. \tag{A5b}$$

The action (A1) turns out to be form invariant under the action of transformations (A3a)–(A3c), which means that solutions to the field equations obtained in one frame are mapped into corresponding solutions in another frame, assuming that the six functions of the scalar field $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{V}, \alpha\}$ change in the following way:

$$\bar{\mathcal{A}}(\bar{\Phi}) = e^{2\bar{\gamma}_1(\bar{\Phi})}\mathcal{A}(\bar{f}(\bar{\Phi})), \tag{A6a}$$

$$\begin{aligned}\bar{\mathcal{B}}(\bar{\Phi}) = e^{2\bar{\gamma}_1(\bar{\Phi})}\Big[&\mathcal{B}(\bar{f}(\bar{\Phi}))(\bar{f}'(\bar{\Phi}))^2 + \bar{f}'(\bar{\Phi})(\mathcal{C}_1(\bar{f}(\bar{\Phi}))(8\bar{\gamma}_1'(\bar{\Phi}) - 10\bar{\gamma}_2'(\bar{\Phi}) + 2\bar{\gamma}_3'(\bar{\Phi}))\\ &- \mathcal{C}_2(\bar{f}(\bar{\Phi}))(2\bar{\gamma}_1'(\bar{\Phi}) - 7\bar{\gamma}_2'(\bar{\Phi}) + 5\bar{\gamma}_3'(\bar{\Phi}))) + 3\Big(4\mathcal{A}(\bar{f}(\bar{\Phi}))\bar{\gamma}_2'(\bar{\Phi})\bar{\gamma}_3'(\bar{\Phi}) - \mathcal{A}(\bar{f}(\bar{\Phi}))(\bar{\gamma}_2'(\bar{\Phi}))^2 - \mathcal{A}(\bar{f}(\bar{\Phi}))(\bar{\gamma}_3'(\bar{\Phi}))^2\\ &+ \frac{d\mathcal{A}(\bar{f}(\bar{\Phi}))}{f\bar{\Phi}}(\bar{\gamma}_2'(\bar{\Phi}) + \bar{\gamma}_3'(\bar{\Phi})) - 2\mathcal{A}(\bar{f}(\bar{\Phi}))\bar{\gamma}_1'(\bar{\Phi})(\bar{\gamma}_2'(\bar{\Phi}) + \bar{\gamma}_3'(\bar{\Phi}))\Big)\Big],\end{aligned} \tag{A6b}$$

$$\bar{\mathcal{C}}_1(\bar{\Phi}) = e^{2\bar{\gamma}_1(\bar{\Phi})}\left[\bar{f}'(\bar{\Phi})\mathcal{C}_1(\bar{f}(\bar{\Phi})) - \mathcal{A}(\bar{f}(\bar{\Phi}))\left(\frac{3}{2}\bar{\gamma}_2'(\bar{\Phi}) + \frac{1}{2}\bar{\gamma}_3'(\bar{\Phi})\right)\right], \tag{A6c}$$

$$\bar{\mathcal{C}}_2(\bar{\Phi}) = e^{2\bar{\gamma}_1(\bar{\Phi})}[\bar{f}'(\bar{\Phi})\mathcal{C}_2(\bar{f}(\bar{\Phi})) - \mathcal{A}(\bar{f}(\bar{\Phi}))(3\bar{\gamma}_2'(\bar{\Phi}) - \bar{\gamma}_3'(\bar{\Phi}))], \tag{A6d}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = e^{4\bar{\gamma}_1(\bar{\Phi})}\mathcal{V}(\bar{f}(\bar{\Phi})), \tag{A6e}$$

$$\bar{\sigma}(\bar{\Phi}) = \sigma(\bar{f}(\bar{\Phi})) + \bar{\gamma}_1(\bar{\Phi}). \tag{A6f}$$

It is always possible to choose the functions (γ_2, γ_3) in such a way that the functions $\mathcal{C}_1$ and $\mathcal{C}_2$ vanish. Indeed, one must take

$$\bar{\gamma}_2'(\Phi) = \frac{-2\mathcal{C}_1(\Phi) - \mathcal{C}_2(\Phi)}{6\mathcal{A}(\Phi)}, \tag{A7a}$$

$$\bar{\gamma}_3'(\Phi) = \frac{-2\mathcal{C}_1(\Phi) + \mathcal{C}_2(\Phi)}{2\mathcal{A}(\Phi)}. \tag{A7b}$$

Such a choice will transform the action (A1) to the following one:

$$\begin{aligned}\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\mathcal{A}(\Phi)R(g,\bar{\Gamma}) - \frac{1}{2}\bar{\mathcal{B}}(\Phi)(\nabla\Phi)^2 - \mathcal{V}(\Phi)\right\}\\ + \mathcal{S}_m[e^{2\sigma(\Phi)}g_{\mu\nu}, \chi_m],\end{aligned} \tag{A8}$$

where

$$\begin{aligned}\bar{\mathcal{B}}(\Phi) = \mathcal{B}(\Phi) + \frac{\mathcal{A}'(\Phi)(\mathcal{C}_2(\Phi) - 4\mathcal{C}_1(\Phi))}{\mathcal{A}(\Phi)}\\ + \frac{11\mathcal{C}_2^2(\Phi) - 4\mathcal{C}_1^2(\Phi) - 16\mathcal{C}_1(\Phi)\mathcal{C}_2(\Phi)}{6\mathcal{A}(\Phi)},\end{aligned} \tag{A9}$$

which justifies the choice of the initial action (1) without the $\mathcal{C}_i$ functions.

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